Introduction to toric varieties 1

Definition. A *toric variety* is a normal variety X with a dense open torus $(\mathbb{C}^\times)^n \cong T \subseteq X$ so that the action of the torus on itself extends to an action on X.

$$\begin{tikzcd} T \times X \arrow[r] & X \\ T \times T \arrow[u] \arrow[r] & T \arrow[u] \end{tikzcd}$$ ◇

Let $M = \mathrm{Hom}_{\mathsf{Gp}}(T, \mathbb{C}^\times)$ be the character lattice of T, and let $N = \mathrm{Hom}_{\mathsf{Gp}}(\mathbb{C}^\times, T)$ be the group of 1-parameter subgroups. If we identify T with $(\mathbb{C}^\times)^n$, we can identify M with $\mathbb{Z}^n$, where $m = (m_1, \dots, m_n) \in \mathbb{Z}^n$ acts by $\chi^m(t_1, \dots, t_n) = t_1^{m_1} \cdots t_n^{m_n} \in \mathbb{C}^\times$. We can identify N with $\mathbb{Z}^n$, where $u = (u_1, \dots, u_n)$ gives $\lambda^u(t) = (t^{u_1}, \dots, t^{u_n})$.

We have a pairing $\langle , \rangle \colon M \times N \to \mathbb{Z}$, given by $\chi \circ \lambda(t) = t^{\langle \chi, \lambda \rangle}$: $\mathbb{C}^\times \xrightarrow{\lambda} T \xrightarrow{\chi} \mathbb{C}^\times$. This is a perfect pairing. You can check that $\langle m, u \rangle = \langle \chi^m, \lambda^u \rangle = m_1 u_1 + \cdots + m_n u_n$. What this does is restrict a character of T to a character of the 1-parameter subgroup (which is a copy of $\mathbb{C}^\times$).

[[★★★ questions about T, M, N?]]

Now we can try using the torus to "sniff out" the rest of the variety. There are two ways to do this sniffing: using characters and using 1-parameter subgroups.

We can try to see which 1-parameter subgroups "extend to X". That is, for each 1-parameter subgroup λ, we can inquire about the limit point $\lim_{t \to 0} \lambda(t)$.

Example. Which characters of $\mathbb{C}^\times$ extend to $\mathbb{C}$? The characters are of the form $t \mapsto t^k$ for $k \in \mathbb{Z}$, and they extend exactly when $k \geq 0$. ◇

Example. Consider $X = \mathbb{C}^2 \supseteq (\mathbb{C}^\times)^2 = T$, with the usual action $(t_1, t_2) \cdot (x, y) = (t_1 x, t_2 y)$. Let's compute the limit points of the 1-parameter subgroups.

$$\lim_{t \to 0} \lambda^{(a,b)}(t) = \begin{cases} (0,0) & a, b > 0 \\ (1,0) & a = 0, b > 0 \\ (0,1) & a > 0, b = 0 \\ (1,1) & a = b = 0 \\ DNE & \text{otherwise} \end{cases}$$ ◇

This gives us a picture in N, called the *fan of* X. $\mathbb{C}^2$ gives a relatively boring fan.

Exercise. *Compute the fan for* $\mathbb{P}^2$.

The cones of dimension k correspond to T-orbits of *codimension* k in the following way. If σ is a cone of dimension k, the stabilizer of the corresponding limit point is the subgroup generated by all the 1-parameter subgroups in σ (this happens to be $\bigcap_{m \in \sigma^\perp} \ker \chi^m$, where $\sigma^\perp = \{m \in M | \langle m, u \rangle = 0 \text{ for all } u \in \sigma\}$). To see that this subgroup stabilizes the limit point, note that each of the 1-parameter subgroups stabilize the limit point. So the stabilizer of the point is of dimension k, so the orbit is a torus of dimension $n - k$.

Definition. The T-invariant subvariety D_σ associated to a cone σ in the fan is the closure of the corresponding T-orbit. ◇

Note that $D_\sigma \subseteq D_\tau$ if and only if $\tau \subseteq \sigma$. The D_σ are themselves toric varieties; you can see their fans by forgetting any maximal cones not containing σ an then "projecting out" the cone σ (this is projecting the pointwise stabilizer out of the orbit to get the smaller torus). Note that X is (set theoretically) a disjoint union of tori.

I claim that the fan completely determines X. If you only had the fan, how might you go about producing X? Well, you'd like to find a cover of X by open affine subvarieties. To specify an affine subvariety you need to specify the regular functions on it. Any regular function will restrict to a regular function of T (if the subvariety contains T). So a good question is, "which regular functions (characters) on T extend to the different T-orbits?" Well, if λ is a 1-parameter subgroup with limit point p, and χ is a character which extends to the T-orbit of p, then the induced character $\chi|_\lambda$ is a character of $\mathbb{C}^\times$ which extends to $\mathbb{C}$. The characters of $\mathbb{C}^\times$ are given by $z \mapsto z^k$, and such a character extends to $\mathbb{C}$ exactly when $k \geq 0$. The induced character on $\lambda \cong \mathbb{C}^\times$ is given by $z \mapsto z^{\langle \chi, \lambda \rangle}$.

So for a closed cone σ in the fan of X, one might come up with the following procedure. Consider the *dual cone* $\sigma^\vee = \{m \in M | \langle m, u \rangle \geq 0$ for all $u \in \sigma\}$. The ring $\mathbb{C}[\sigma^\vee] \subseteq \mathbb{C}[M]$ is the ring of regular functions on some open subvariety X_σ of X. Note that if τ is a face of σ, then $\sigma^\vee \subseteq \tau^\vee$, so $X_\tau \subseteq X_\sigma$ is an open subvariety. This gives the familiar construction of X, given by gluing the X_σ together along the open subsets corresponding to shared faces. Let X_Σ or $X(\Sigma)$ be the toric variety associated to the fan Σ.

Example. Actually compute $\mathbb{P}^1$ explicitly. ◇

[[★★★ questions?]]

Toric varieties form a category.

Definition. A *morphism of toric varieties* $f\colon X \to Y$ is a morphism of varieties which is a group homomorphism on the underlying tori (this implies that the actions play nice).

$$\begin{tikzcd}
T_X \times X \arrow[rr] \arrow[dr] & & X \arrow[dr] & \\
& T_Y \times Y \arrow[rr] & & Y \\
T_X \times T_X \arrow[uu] \arrow[rr] \arrow[dr] & & T_X \arrow[uu] \arrow[dr] & \\
& T_Y \times T_Y \arrow[uu] \arrow[rr] & & T_Y \arrow[uu]
\end{tikzcd}$$
◇

Not surprisingly, this is equivalent to a linear map of lattices $N_X \to N_Y$ such that each cone in Σ_X lands inside of some cone in Σ_Y.

Example. $\mathbb{A}^1$ mapping to some toric variety is exactly picking out a 1-parameter subgroup which has a limit point. ◇

You can check that the product of fans $\Sigma \times \Sigma'$ (made up of cones $\sigma \times \sigma'$) yields the toric variety $X(\Sigma) \times X(\Sigma')$.

Now we're ready for some examples.

Example. $\mathbb{A}^n$, $\mathbb{P}^n$, $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{A}^2 \smallsetminus \{0\}$, $\mathbb{P}^1 \times \mathbb{P}^1 \smallsetminus$stuff, $\mathbb{P}^1 \times \mathbb{A}^1$ ◇

To get some more examples, let me tell you about blow-ups.

Blowing up. If σ is an n-dimensional simplicial cone, generated (over $\mathbb{R}$) by $v_1, \dots, v_n$, then the blowup of X_σ at the point D_σ is given by $X(\Sigma)$, where Σ consists of all the cones generated by subsets of $\{v_0, \dots, v_n\}$ not containing $\{v_1, \dots, v_n\}$, where $v_0 = \sum_{i \geq 1} v_i$. [[★★★ I'm pretty sure this is right in general. It is definitely right in the case where σ is non-singular.]] Note that the new ray is a $\mathbb{P}^{n-1}$ (you can see the fan of the ray by "looking down the ray"). Maybe you can guess how to blow up subvarieties.

If σ is not simplicial, I don't think it is possible to put the structure of a toric variety on the blowup of X_σ at D_σ.

Theorem 1. *X_Σ is proper if and only if the support of Σ is all of N. More generally, If $f\colon N' \to N$ is a morphism of lattices compatible with fans Σ' and Σ (each cone in Σ' is sent into a cone of Σ), then the induced toric morphism $X(\Sigma') \to X(\Sigma)$ is proper if and only if $f^{-1}(|\Sigma|) = |\Sigma'|$.*

It is clear that if there is a 1-parameter subgroup $\lambda \in N$ which has a limit point (is in $|\Sigma|$), and a 1-parameter subgroup $\lambda' \in N'$ lying over λ which does not have a limit point, then the map cannot be proper.

Theorem 2. *X_σ is smooth if and only if σ is generated by a subset of a basis for N. In particular, $X(\Sigma)$ is non-singular if and only if each cone in Σ is generated by a subset of a basis for N.*

If σ is generated by a subset of a basis for N, then X_σ is of the form $\mathbb{C}^k \times (\mathbb{C}^\times)^{n-k}$. If σ is not generated by a subset of a basis for N, then $\sigma^\vee$ will have the same problem, so the generators of $\mathbb{C}[\sigma^\vee]$ will have relations. These relations produce singularities.

Example. A_1 and A_2 singularities. ◇

Theorem 3. *If σ is simplicial (with generators spanning a sublattice N' of N), then X_σ is the quotient of a smooth affine toric variety (a $\mathbb{C}^k \times (\mathbb{C}^\times)^{n-k}$) by an action of the finite abelian group N/N'. In particular, if all the cones in Σ are simplicial, then $X(\Sigma)$ is an orbifold.*

[[★★★ explanation]]

Example. Variety of singular 2×2 matrices. Resolutions of the singularity ◇

2 Introduction to toric varieties 2

References: Fulton and Cox.

Plan for last time:

- Defined a toric variety X as normal variety with dense open torus T so that the action of T on itself extends to an action on X. $M = \mathrm{Hom}_{\mathsf{Gp}}(T, \mathbb{C}^\times)$ is the character lattice (think of characters as rational functions on X) and $N = \mathrm{Hom}_{\mathsf{Gp}}(\mathbb{C}^\times, T)$ is the lattice of 1-parameter subgroups.
- We used the 1-parameter subgroups to "sniff out" the rest of X via limit points ... this produced the fan of X. I argued that the fan of X determines X.
- I talked about the category of toric varieties. We learned about morphisms of toric varieties (they are linear morphisms of lattices compatible with the fans), and I told you that a product of fans corresponds to the product of varieties.
- I showed you how to blow up a toric variety at (certain) torus invariant points.
- how to tell if $X(\Sigma)$ is proper.
- how to tell if a toric variety is non-singular (check that each affine piece is non-singular). I hinted at how to resolve singularities (you break up singular cones into a bunch of non-singular cones), but I didn't tell you about the singularities that show up on toric varieties. Come to Dustin's talk about that.

Plan for this time: The goal of these talks is to show you that toric varieties are easy to work with, so you should have them in your bag of examples (you might want to come to the Monday seminar to learn more). Last time, we talked about properness and smoothness (and touched on resolutions of singularities); what other things might you want to know about a variety?

- The divisor class group. This is the free abelian group on irreducible closed subvarieties modulo those that come from rational functions. I'll show you how to compute this.
- The Picard group. I'll show you how to compute this, and we'll see how to work with invertible sheaves a bit. This will lead us to the polytope description of toric varieties.
- Intersection theory. I'll tell you how to compute the Chow ring of a non-singular projective toric variety.
- Topological properties of X. I won't talk about this today, but you can check Fulton if you're interested. For example, if $d_n > 0$, $\pi_1(X) = 0$. For a smooth proper toric variety X, if d_k is the number of k-cones, the Betti numbers are $\beta_{2k} = \sum_{i=k}^{n}(-1)^{i-k} d_{n-i}$, the Euler characteristic is d_n.
- Sheaf cohomology. I won't talk about this, but Dan and Tony will talk about cohomology of line bundles in the Monday seminar.

Warmup for divisors. Recall from last time that the following sets are in bijection

- cones σ in the fan Σ,
- limit points $\lim(\sigma)$ of one parameter subgroups
- T-orbits $T \cdot \lim(\sigma)$, and
- irreducible T-invariant subvarieties of X $D_\sigma = \overline{T \cdot \lim(\sigma)}$.

If $\chi \in M$ is a character of T, then it is a rational function on X, and the divisor associated to it is supported on $X \smallsetminus T = D_1 \cup D_2 \cup \cdots \cup D_d$, where the D_i are the irreducible components of $X \smallsetminus T$, so we have

$$div(\chi) = \sum_{i=1}^{d} a_i^\chi D_i = \sum_i \langle \chi, u_i \rangle D_i.$$

This gives us d functions $M \to \mathbb{Z}$, given by $\chi \mapsto a_i^\chi$. It is easy to see that they are linear (multiplying rational functions adds their divisors), so they are given by elements u_i of N. These u_i are the rays of the fan of X (you can check this on the affine open subsets X_σ).

Theorem 4. *If Σ is not contained in a proper subspace of $N_\mathbb{R}$ (this is just to make the first arrow injective), the following sequence is exact:*

$$0 \to M \xrightarrow{(a_i)} \mathbb{Z}^{\Sigma(1)} \to A^1(X) \to 0.$$

Example. For the A_1 singularity, we get the sequence

$$0 \to \mathbb{Z}^2 \xrightarrow{\left(\begin{smallmatrix}1 & 0\\ 1 & 2\end{smallmatrix}\right)} \mathbb{Z}^2 \xrightarrow{(1\ 1)} \mathbb{Z}/2 \to 0. \qquad \diamond$$

Now on to the Picard group. In general, if X is a normal integral scheme there is an injection $Pic(X) \hookrightarrow A^1(X)$. In many situations (like if X is smooth), $A^1(X) \cong Pic(X) \cong_{X \text{ integral}} CaCl(X)$, but let's look at the difference between these. If I start with a Weil divisor D, then I can construct a fractional ideal sheaf $\mathcal{O}(D) \subseteq \mathcal{K}$ whose sections are rational functions f such that $div(f) \geq -D$.

Example. Consider the divisor $D = D_1$ on the A_1 singularity. We get a picture in M of all the characters χ such that $div(\chi) \geq -D$. [[★★★ if you restrict your attention to torus invariant divisors, you can just look at characters]] But notice that this fractional ideal is really just $\mathcal{O}_X = \mathcal{O}(0)$. There are no sections that realize the equality $div(\chi) = -D$. Notice, by the way, that $D = 2D_1$ does not have this problem. $\diamond$

This fact that there is no rational function realizing $div(\chi) = -D$ indicates that D_1 is a Weil divisor which is not a Cartier divisor. Think of the Cartier divisors are those which you can recover from their fractional ideal sheaf (I might call this the *invertible sheaf* associated to D), or you can remember that the Cartier divisors are those of the form $div(\chi)$ on each affine piece X_σ. It turns out that a Weil divisor $\sum a_i D_i$ is a Cartier divisor if and only if for each maximal cone σ, there is a $\chi_\sigma \in M$ such that for all $u_i \in \sigma$ (rays in the fan), $\langle \chi_\sigma, u_i \rangle = -a_i$.

Exercise. *Show that Σ is simplicial if and only if for any Weil divisor D, some multiple of D is a Cartier divisor.*

Theorem 5. *If Σ isn't contained in a proper subspace,*

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, equal] & Div_T(X) \arrow[r] \arrow[d, hook] & Pic(X) \arrow[r] \arrow[d, hook] & 0 \\ 0 \arrow[r] & M \arrow[r] & \mathbb{Z}^{\Sigma(1)} \arrow[r] & A^1(X) \arrow[r] & 0 \end{tikzcd}$$

Moreover, $Pic(X)$ is a free abelian group.

Ok, so now you know how to compute the divisor class group $A^1(X)$ and the Picard group $Pic(X)$. Let's learn a little about how to work with invertible sheaves. If $\mathcal{L} \cong \mathcal{O}(D)$ is an invertible sheaf (corresponding to some divisor since X is integral), we can identify its sections on the affine pieces of X. Intersecting the wedges we get gives us a basis for the space of global sections of $\mathcal{O}(D)$. Call the resulting polytope P_D.

Example. The anti-canonical divisor on the first Hirzebruch surface. $\diamond$

Theorem 6. *The line bundle $\mathcal{O}(D)$ is ample if and only if the normal fan to P_D is the original fan of X. In particular, a toric variety X is projective if and only if its fan is the normal fan to some polytope.*

I don't have time to explain why you should believe this theorem (somehow $\mathcal{O}(D)$ has "enough sections" to see all the information in X; it has to generate the sheaf, separate points and separate tangent vectors), but it gives us a new way to look at toric varieties.

Remark. Remember that an invertible sheaf $\mathcal{L}$ on a projective variety X is ample if and only if $X \cong Proj\big(\bigoplus \Gamma(X, \mathcal{L}^{\otimes n})\big)$ (this is almost the definition of ample if you think about it real hard). $\diamond$

So we could give a new construction. Given a lattice polytope $P \subseteq M$, let S be the graded semigroup which is the cone on $P \times \{1\} \subseteq M \times \mathbb{Z}$ (graded by height). Then $\mathbb{C}[S]$ is a graded commutative ring, and we can define $X(P) = Proj(\mathbb{C}[S])$. By the remark, this $X(P)$ is exactly the toric variety you'd get from the dual fan to P.

You can translate everything we've learned into the polytope description. For example, blowing up is not chopping off a vertex of the polytope. One nice feature of the polytope description is that the dimensions are "right". The torus invariant divisors show up as codimension 1 faces, and they have the right combinatorial properties (you can see their intersections).

Theorem 7. *If $X = X(\Sigma)$ is a non-singular projective toric variety, it's Chow ring is $A^*(X) = H^*(X) = \mathbb{Z}[D_1, \dots, D_d]/I$, where I is the ideal generated by all (i) $D_{i_1} \cdots D_{i_k}$ where $\{v_{i_j}\}_{1 \leq j \leq k}$ are not in a cone of Σ, and (ii) $\sum_{i=1}^d \langle \chi, u_i \rangle D_i$ for $\chi \in M$. For (i), it is enough to consider sequences without repeats (duh), and for (ii), it is enough to consider χ in some fixed basis of M.*

Example. For projective space $\mathbb{P}^n$, we have the fan generated by $\{e_1, \dots, e_n, -e_1 - \cdots - e_n\}$, so the only relation of type (i) is $D_1 \cdots D_{n+1}$, and the relations of type (ii) are of the form $D_i - D_{n+1}$ for $i \leq n$. So we see that the Chow ring is $\mathbb{Z}[x]/x^{n+1}$. $\diamond$