# Sestinas and primes

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"A sestina is a highly structured poem consisting of six six-line stanzas followed by a tercet (called its envoy or tornada), for a total of thirty-nine lines. The same set of six words ends the lines of each of the six-line stanzas, but in a different order each time; if we number the first stanza's lines 123456, then the words ending the second stanza's lines appear in the order 615243, then 364125, then 532614, then 451362, and finally 246531."[Wiki]

In other words, the the final words of the lines in a stanza can be obtained by looking at final words the previous stanza and applying the permutation $\left(\begin{smallmatrix}1&2&3&4&5&6\\6&1&5&2&4&3\end{smallmatrix}\right)$. This permutation happens to be a 6-cycle, which is why you can write six stanzas before any of the terminal words come back to where they started.

We will define a *n-tina* as a poem consisting of $n$ $n$-line stanzas (maybe followed by something) in which the final words of the lines of a stanza are obtained from the final words of the previous stanza by applying the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & \cdots \\ n & 1 & n-1 & 2 & n-2 & 3 & \cdots \end{pmatrix}, \qquad f_n(k) = \begin{cases} 2k & \text{if } k \le n/2 \\ 2n+1-2k & \text{otherwise} \end{cases}.$$

To make the poem interesting, we require this permutation to be an $n$-cycle.

A few years ago (maybe in 2002 or 2003), William Flesch gave a talk at Brandeis University entitled "Math and English" in which he pointed out the pattern that $n$-tinas only seem to exist when $2n+1$ is prime.

**Theorem.** *If $f_n$ is an $n$-cycle, then $2n+1$ is prime.*

*Proof.* Assume for the moment that $k$ is relatively prime to $2n+1$. Note that 2 is relatively prime to $2n+1$, and the product of two things relatively prime to $2n+1$ is again relatively prime to $2n+1$, so $2k$ and $2n+1-2k$ are relatively prime to $2n+1$. That is, whenever $k$ is relatively prime to $2n+1$, so is $f_n(k)$.

Since $f_n$ is an $n$-cycle, every number between 1 and $n$ is of the form $f_n^\ell(1)$ for some $\ell$. By the argument in the previous paragraph, $f_n^\ell(1)$ is always relatively prime to $2n+1$. In particular, $2n+1$ has no proper divisors between 1 and $n$, from which it follows that it is prime. □

Note that the converse is not true. For example, $f_8$ is the permutation $(1\,2\,4\,8)(3\,6\,5\,7)$, which has order 4.

## References

[Wiki] http://en.wikipedia.org/wiki/Sestina