$\frac{1}{2}$ Vakil

Today I'll give a short introduction to deformation theory. People care about this stuff for very different reasons. We'll work over the complex numbers for no good reason (except familiarity). You're interested in some kind of object (left deliberately vague). Let X be such an object. How can you deform X? (this is one question we'll have to make precise) How can you deform it infinitesimally? (i.e. formally or locally, something like power series; Zariski locally is nice, and étale locally is the most you can usually do).

Let $X_0 \to \operatorname{Spec}\mathbb{C}$ be a scheme. Then we'd like to think of $\operatorname{Spec}\mathbb{C}$ as a closed subscheme of $\operatorname{Spec} A$ for some artin local ring A, and we'd like some X over $\operatorname{Spec} A$ nice so that $X_0 = X \times_A \mathbb{C}$. "Nice" will always mean flat (perhaps with other things, this is kind of mysterious, but is the answer to all your prayers).

Sample application: suppose you're lucky and there is some scheme $\mathcal{M}$ that parameterizes your objects (again, we'll have to make this precise). You make this precise by the notion of a *fine moduli space*. This means that every time you have a nice (flat plus a bit more) family $\mathcal{X} \to B$, it is equivalent to a morphism $B \to \mathcal{M}$. This should be functorial, so that if we base change by $B' \to B$,

$$\begin{tikzcd}
\mathcal{X}' \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\
B' \arrow[r] & B
\end{tikzcd}$$

then we get a factorization $B' \to B \to \mathcal{M}$. Take this as the definition of a map to $\mathcal{M}$. Yoneda's lemma will tell us that this determines $\mathcal{M}$ uniquely up to unique isomorphism.

What if you're not so lucky that you have a fine moduli space? The old answer is that you take something called the *coarse moduli space*. This is a scheme which *best* parameterizes your objects. The more modern perspective is that you have a moduli problem which is trying hard to be a scheme, but can't be. You're being told by the moduli problem that you should generalize your notion of a space. Depending on the strength of your stomach, you'll generalize "space" to be algebraic spaces, Deligne-Mumford stacks, or Artin stacks. For now, let's assume $\mathcal{M}$ is a scheme (but everything will work ok if you at least have a DM stack).

In general, $\mathcal{M}$ is bad. Given a point in the space, you'd like to know if it is smooth, etc. You'd like to find a formal neighborhood of the point. Is it smooth/unibranched? Let's say that you know that it is with 7 variables and cut out by 5 equations; then you know that the dimension of each component is at least 2 (even a little more). Let's say that you have an object X that you're interested in, but you can only get your hands on some other object X', then you could prove that thing for X' and prove that it is invariant under deformations and that X' is connected to X. You could find a map $\mathcal{M} \to \mathcal{M}'$ and ask questions about this morphism.

Example $\frac{1}{2}$.1 (Deligne-Mumford 1969)**.** The moduli space of genus g curves is irreducible. First of all, they didn't use the coarse moduli space, as would have been tradition. They got a DM stack $\mathcal{M}_g$. The goal is to show that $\mathcal{M}_g$ is connected and smooth. How do you show that it is connected? Show that there is one point to which all other points are connected. Let's make the problem harder by compactifying to get $\overline{\mathcal{M}}_g$. Then you can pick some really degenerate curve as your base point. ◇

Let's say for now that there is a fine moduli space, and you're interested in some nasty point x. Let $(R, \mathfrak{m})$ be the local ring. Then we can't quite get our hands on this local ring, but we'll get the information of $\hat{R} = \varprojlim R/\mathfrak{m}^n$, which contains all the information about the tangent space for example.

Example $\frac{1}{2}$.2. Let's say that $\mathcal{M}$ is the cuspidal cubic $y^2 = x^3$ in the plane and we're looking at the origin. Then $R = \mathbb{C}[x, y]/(y^2 - x^3)$. The first thing we see is $\mathbb{C}[x, y]/(x, y)^2$ which is two-dimensional (the tangent space is called the first order deformation space). The next thing that we see is $\mathbb{C}[x, y]/((x, y)^3, y^2 = 0)$, then we get $\mathbb{C}[x, y]/((x, y)^4, y^2 - x^3)$. As you keep going, you can show that you won't get more equations. ◇

Something is called the obstruction space. If the obstruction space is zero, then you get that $\mathcal{M}$ is smooth.

There are three brothers: aut/def/ob. If X is a smooth variety, then Def $= H^1(X, T_X)$, Ob $= H^2(X, T_X)$. Note that Aut $= H^0(X, T_X)$ is the set of vector fields, the set of infinesimal automorphisms of X. Why stop here? Well somebody probably knows what H^3 means, but we'll stick to these for now.

Exercise $\frac{1}{2}$.1. Consider $\mathcal{M}_g$ for $g \geq 2$, and consider $\overline{\mathcal{M}}_g$. What should these three guys be? what if X is singular? ◀

One lesson is that the tangent sheaf is less natural than the cotangent sheaf, so let's say $def = \mathrm{Ext}^1(\Omega_X, \mathcal{O}_X)$.

Exercise $\frac{1}{2}$.2. Fix X and deform something on X. ◀

The derived category should be lurking in the background somewhere.

Example $\frac{1}{2}$.3. Manifolds in $\mathbb{CP}^n$. Let $\pi : X \hookrightarrow \mathbb{P}^n$. Then $def(X \hookrightarrow \mathbb{P}^n) = H^0(X, \mathcal{N}_{X/\mathbb{P}^n})$. In this case, *ob* is H^1 and $aut = H^{-1} = 0$. To compute, consider the exact sequence

$$0 \to T_X \to \pi^* T_{\mathbb{P}^n} \to \mathcal{N}_{X/\mathbb{P}^n} \to 0.$$

Then you get

$$0 \to H^0(T_X) \to H^0(\pi^* T_{\mathbb{P}^n}) \to H^0(N_{X/\mathbb{P}^n}) \to H^1(T_X) \to H^1(\pi^* T_{\mathbb{P}^n}) \to H^1(N_{X/\mathbb{P}^n})$$

these should be $aut(X)$, $def\pi$ (fixing X and deforming π) , $def(X \hookrightarrow \mathbb{P}^n)$, $def(X)$, $ob(\pi)$, $ob(X \hookrightarrow \mathbb{P}^n)$. There should also be some auts (which are zero). Up to the obs, you can understand every morphism here:

$$[[\bigstar\bigstar\bigstar 0 \to aut \cdots def \cdots ob]]$$

◇

You can generalize!

Many sources for deformation theory, but Illusie's two volume thing on the (co)tangent stuff is the one place where it is very united. This two week period is a really great opportunity to learn deformation theory.

1 Lieblich

Two organizational things: (1) there will be problems assigned after lunch (these will show up in later lectures), (2) your group list and location will be posted after you get back from lunch.

The next two weeks will be fleshing out what Ravi just said. How should we even think about what a moduli space is? Let's make a list of potential moduli problems so that we can keep coming back to them.

0. varieties,
1. curves of genus g,
2. line bundles on some X,
3. maps between X and Y,
4. closed subschemes of some fixed X (say $\mathbb{P}^n$),
5. subspaces of a fixed vector space.

This only scratches the surface of things you can study with these techniques. The 5th one is not obviously algebro-geometric.

Vaguely, the points of $\mathcal{M}_i$ should correspond to objects of flavor i (where $0 \leq i \leq 5$). More general question: how should *any* scheme be described?

1. (absolute) give an affine chart
2. (relative) describe how the scheme relates to all other schemes.

Analogy: functions. How to describe $f \in C^\infty([0,1])$? (1) you could say what $f(x)$ is for all x. (2) there is a pairing of functions: if $g \in C^\infty([0,1])$, then you can produces $\int_0^1 fg \in \mathbb{R}$. Let $L_f : C^\infty[0,1] \to \mathbb{R}$ be pairing with f.

Theorem 1.1. *L defines a linear embedding* $C^\infty[0,1] \to \mathrm{Hom}(C^\infty[0,1], \mathbb{R})$.

In particular, you can define a functional and then ask if it comes from a function. "shoot first, ask questions later".

It turns out that one can do exactly the same thing in any category. The replacement for the pairing is $X, Y \mapsto \mathrm{Hom}(X, Y)$, which is a set.

f function	X object
$g \mapsto \int fg = L_f(g)$	$Y \mapsto \mathrm{Hom}(X,Y) = h_X(Y)$
$C^\infty[0,1] \to \mathrm{Hom}(C^\infty[0,1], \mathbb{R})$	$\mathcal{C} \mapsto \mathrm{Fun}(\mathcal{C}^\circ, \mathbf{Set})$
linear injectivity	functor fully faithful

If $Z \to Y$, then composition defines a map $h_X Y \to h_X Z$. So h_X is a contravariant functor from $\mathcal{C}$ to **Set**. This is the same thing as a covariant functor $\mathcal{C}^\circ$ to **Set**. Don't worry about this too much, it's just convention, not content.

Example 1.2. Let $\mathcal{C} = \mathbf{Sch}_{\mathbb{Z}}$. Then we have a number of functors we understand well, such as $h_{\mathbb{A}^1}(Y) = \Gamma(Y, \mathcal{O}_Y)$. The functoriality is clear (given by pullback of functors). We also have $\mathbb{G}_m = \mathbb{A}^1 \smallsetminus \{0\}$, $h_{\mathbb{G}_m}(Y) = \Gamma(Y, \mathcal{O}_Y^\times)$. There are some other spaces where you get some more interesting stuff.

$h_{\mathbb{P}^n}(Y) = \{\mathcal{O}_Y^{n+1} \twoheadrightarrow \mathcal{L} | \mathcal{L} \text{ invertible sheaf on } Y\}/\cong$. Everything is a moduli space: it represents the functor it represents. ◇

$\mathrm{Fun}(\mathcal{C}^\circ, \mathbf{Set})$ is actually a category whose objects are functors and whose morphisms are natural transformations.

For example, there is a natural transformation $h_{\mathbb{G}_m} \to h_{\mathbb{A}^1}$, given by $\Gamma(Y, \mathcal{O}_Y^\times) \to \Gamma(Y, \mathcal{O}_Y)$. This is induced by the injection $\mathbb{G}_m \to \mathbb{A}^1$. Because Hom is a bifunctor, any morphism $X \to X'$ gives a natural transformation $h_X \to h_{X'}$.

We have an analog of bump functions which recover the function.

Lemma 1.3 (Yoneda). $h_- : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^\circ, \mathbf{Set})$ *is a fully faithful embedding of categories.*

So it wasn't a coinsidence that the map from $h_{\mathbb{G}_m} \to h_{\mathbb{A}^1}$ came from a (unique!) map $\mathbb{G}_m \to \mathbb{A}^1$. Thinking in terms of functors is easier in many cases.

The proof is confusing because there is so little to do.

Proof. (Faithfulness) Let $f, g : X \to Y$ in $\mathcal{C}$. Then you get $h_f, h_g : h_X \to h_Y$. Show that $h_f = h_g \Rightarrow f = g$. $h_f(T) : h_X(T) = \mathrm{Hom}(T, X) \to \mathrm{Hom}(T, Y) = h_Y(T)$. There is a universal point! Consider $T = X$, then $h_f(\mathrm{id}_X) = f \in \mathrm{Hom}(X, Y)$, so $f = h_f(\mathrm{id}_X) = h_g(\mathrm{id}_X) = g$.

(Fullness) Given $F : h_X \to h_Y$, we want to show that $F = h_f$ for some $f : X \to Y$. The universal point is universal: if $\phi : T \to X$, then we get $h_X(X) \to h_X(T)$, sending id_X to ϕ.

$$\begin{tikzcd}
h_X(X) \arrow[r, "F(X)"] \arrow[d] & h_Y(X) \arrow[d] \\
h_X(T) \arrow[r, "F(T)"] & h_Y(T)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{id}_X \arrow[r] \arrow[d] & f \arrow[drr] & & \\
\phi \arrow[r] & F(T)(\phi) & = & h_f(\phi) = \phi \circ f
\end{tikzcd}$$

□

h_X is called the *functor of points of* X. TO see why this is true, if you were an old italian, then $h_X(\operatorname{Spec} \mathbb{C}) = X(\mathbb{C})$ would be the variety X to you. Yoneda's lemma tells you how to make arguments like "here is what the map of varieties does on points".

Example 1.4. $\mathbb{G}_m \to \mathbb{G}_m$ given by $x \mapsto x^2$. Well, $\mathbb{G}_m(\mathbb{C}) = \mathbb{C}^\times$. This map really works on all points and it gives us a natural transformation $h_{\mathbb{G}_m} \to h_{\mathbb{G}_m}$. The italians were right about a lot of stuff. ◇

It is true that every scheme is given by its functor of points, but not every functor comes from a scheme. For now, we'll think of all functors as being very floppy geometric objects.

Philosophy. Schemes are not the right thing to work with. Maybe we should just work with all functors. Think of a functor as some kind of generalized space, and think of schemes as some distinguishded class of spaces. Instead of $F : \mathcal{C}^\circ \to \mathbf{Set}$, just think F. To look at the internal structure of F, we can evaluate it on schemes to get $F(T)$.

Exercise 1.3 (Yoneda's lemma, really)**.** There is a natural bijection $\operatorname{Hom}(h_T, F) \xrightarrow{\sim} F(T)$. ◀

Definition 1.5. A functor F is called *representable* if there is $X \in \mathcal{C}$ such that $F \cong h_X$. ◇

Note that the fiber product of schemes is defined in terms of its functor of points, and the question is whether it exists.

Slogan: anything we can do with sets, we can do with set-valued functors.

For example, if F and G are set-valued functors with natural transformations to H, then we can define $(F \times_H G)(T) = F(T) \times_{H(T)} G(T)$, and this really is the fiber product in the category of functors. [[★★★ exercise]]

Exercise 1.4. $h_{X \times_Z Y} = h_X \times_{h_Z} h_Y$. ◀

Let's describe the functors of points for the moduli problems we talked about earlier.

0. $\mathcal{M}_0(T) = \{X \to T$ finite presentation, flat, geometrically integral fibers$\}/\cong$.
1. $\mathcal{M}_1(T) = \{C \to T$ proper, smooth, of finite presentation$|$ fibers are curves of genus $g\}/\cong$.
2. $\mathcal{M}_2(T) = \{\mathcal{L}$ an invertible sheaf on $X \times T\}/\cong$.
3. $\mathcal{M}_3(T) = \operatorname{Hom}_T(X \times T, Y \times T)$.
4. $\mathcal{M}_4(T) = \{$ $$\begin{tikzcd} Z \arrow[r, hook] \arrow[dr] & X \times T \arrow[d] \\ & T \end{tikzcd}$$ $|Z$ is T-flat closed immersion (of finite presentation) of $X \times T\}/\cong$ (here the isomorphisms have to respect the immersions).
5. (for this one, let's work over $\mathbb{C}$) $\mathcal{M}_5(T) = \{W \subseteq V \otimes \mathcal{O}_T |$ coker is locally free $\}$.

This is a reasonable list of what should be the functors of points.

We should think about which functors we should really allow to be geometric, and whether these functors fall into that class.

1 Olsson

Overview of what we'll do in the next two weeks. Ravi talked about deformation spaces, obstruction spaces, and automorphisms. We'll take a historical approach.

Week 1:

1. basic definitions
2. examples
3. obstruction spaces
4. more examples

Week 2: towards those two volumes of Illusie. By the end of week 2, we should make it seem reasonable for this theory of the cotangent complex to exist. Before that, there was some other volume of Grothendieck where he introduced a different approach based on the Picard category

5. Picard category
6. Picard stack
7. truncated cotangent complex
8. overview of the cotangent complex in full generality, at least to the point where you can find the answers to your questions in Illusie.

Today we'll talk about tangent spaces from a functorial point of view.

Motivation: Let k be an algebraically closed field, and let X be a scheme of finite type over k. Let $x \in X(k)$ be a closed point. The *tangent space of X at x* is the dual of the k-vector space $\mathfrak{m}_x/\mathfrak{m}_x^2$, where $\mathfrak{m}_x$ is the maximal ideal of $\mathcal{O}_{X,x}$. Usually, X is only given to us as a functor, not as a space, so we need to think about this differently (what is the local ring at the point?). The first step to understanding infinitesimal structure of moduli spaces is to generalize this definition.

Dual numbers. Say R is a ring and I is an R-module. Then the ring of dual numbers $R[I]$ is $R \oplus I$ as a group, with multiplication law $(r,i)(r',i') = (rr', r'i + ri')$. The basic diagram is

$$\begin{tikzcd}
(r,i) \arrow[r] & r
\end{tikzcd}$$

$$\begin{tikzcd}
(r,0) & R[I] \arrow[r, two heads, "\pi"] & R \\
r \arrow[u] & R \arrow[u] \arrow[ur, "\text{id}"'] &
\end{tikzcd}$$

Remark 1.1. $R[I]$ is functorial in I. That is, if you have a module homomorphism $g : I \to J$, then you get a map $R[I] \to R[J]$ by $(r, i) \mapsto (r, g(i))$. This is compatible with the projection and R-algebra structure (i.e. it is a map of diagrams above). ◇

Remark 1.2. If $I = R$, then we'll write $R[I] = R[\varepsilon]$. This really is $R[\varepsilon]/\varepsilon^2$ as a ring. ◇

Remark 1.3. We did this for a ring and a module, but you can do it for sheaves of rings and sheaves of modules. If $\mathcal{O}$ is a sheaf of rings on a topological spaces X, and if I is an $\mathcal{O}$-module, then we can define $\mathcal{O}[I]$. ◇

In particular, if X is a scheme and I is a quasi-coherent $\mathcal{O}_X$-module, then we get a ringed space $X[I]$, defined as the topological space X together with the sheaf of rings $\mathcal{O}_X[I]$.

Exercise 1.5. Show that $X[I]$ is a scheme. We will have

$$\begin{tikzcd} X \arrow[r, hook, "cl"] \arrow[dr, equal] & X[I] \arrow[d] \\ & X \end{tikzcd}$$

◀

Relationship with derivations. Let $A \to R$ be a ring homomorphism, and let M be an R-module. An *A-derivation from R to M* is an A-linear map $\partial : R \to M$ satisfying the Liebniz rule $\partial(xy) = x\partial y + y\partial x$. This gives us an R-module $Der_A(R, M)$.

A-**alg**$/R$ will be the category of pairs (C, f) where C is an A-algebra and $f : C \to R$ is a map of A-algebras. A morphism $(C, f) \to (C', f')$ is an A-algebra morphism $g : C \to C'$ which respects the maps to R.

Lemma 1.4. *For any A-derivation $\partial : R \to I$, the induced map $R \to R[I]$, given by $x \mapsto x + \partial x = (x, \partial x)$, is a morphism in A-**alg**$/R$, and the induced map $Der_A(R, I) \to \mathrm{Hom}_{A\text{-}\mathbf{alg}/R}(R, R[I])$ is bijective.*

It looks like we're taking something simple and making it complicated, but this is a good warm-up. Here R is viewed as an element of A-**alg**$/R$ by

$$\begin{tikzcd} R \arrow[r, "\mathrm{id}"] & R \\ A \arrow[u] \arrow[ur] & \end{tikzcd}$$

Proof. Let $s : R \to R[I]$ be a morphism in A-**alg**$/R$, then since it has to be compatible with projection to R, it must be of the form $x \mapsto (x, \delta(x))$ for some $\delta(x)$. It also has to be A-linear and it has to be a map of algebras. Since it is A-linear, we get that $\delta(x) = 0$ whenever x is in the image of A. Compatibility with multiplication says that for $x, y \in R$, $(xy, \delta(xy)) = (x, \delta x)(y, \delta y) = (xy, x\delta y + y\delta x)$, so δ is a derivation. ☐

[[★★★ Ishai: bracketing I is right adjoint to Ω. $\mathrm{Hom}_{Mod_R}(\Omega_{R/A}, I) \cong \mathrm{Hom}_{A\text{-}\mathbf{alg}/R}(R, R[I])$]]

Remark 1.5. Say $(f : C \twoheadrightarrow R) \in A\text{-}\mathbf{alg}/R$ and $I = \ker f$ is square zero. Then any section $s : R \to C$ over A induces an isomorphism

$$\begin{tikzcd}
R[I] \arrow[rr, "(r,i)\mapsto s(r)+i", "\sim"'] \arrow[dr] & & C \arrow[dl] \\
& R &
\end{tikzcd}$$

You get this from the 5-lemma:

$$\begin{tikzcd}
0 \arrow[r] & I \arrow[r] \arrow[d, equal] & R[I] \arrow[r] \arrow[d] & R \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & I \arrow[r] & C \arrow[r] & R \arrow[r] & 0
\end{tikzcd}$$

◇

Special case: $C = (R \otimes_A R)/J^2$ where $J = \ker(R \otimes_A R \to R)$, with $I = J/J^2$. Then we can define a section $s : R \to C$ by $x \mapsto x \otimes 1$. (This J/J^2 is one way to define Kähler differentials.) s induces an isomorphism $(R \otimes_A R)/J^2 \cong R[\Omega^1_{R/A}]$. This gives us that $Der_A(R, \Omega^1_{R/A})$ is in canonical bijection with sections of the diagonal map $(R \otimes_A R)/J^2 \to R$.

Question: what is the universal derivation $d : R \to \Omega^1_{R/A}$?

The usual definition is that you take $\Omega^1_{R/A} = J/J^2$ and you define $dx = x \otimes 1 - 1 \otimes x$, but we've messed things around a bit. We want some s_d so that $s_d(x) = (x, dx)$.

$$\begin{tikzcd}
R[\Omega^1_{R/A}] \arrow[r] & (R \otimes_A R)/J^2 \\
R \arrow[u, "s_d"] &
\end{tikzcd}$$

The map we want is $x \mapsto 1 \otimes x + (1 \otimes x - x \otimes 1) \mapsto (x, 1 \otimes x - x \otimes 1) \in R[\Omega^1_{R/A}]$. So the answer is $x \mapsto 1 \otimes x$.

Exercise 1.6. Check this carefully. ◀

Tangent space of a functor. Let Mod_R be the category of finitely generated R-modules, and let $H : Mod_R \to \mathbf{Set}$ be a functor (all my functors will be covariant) which commutes with finite products (the canonical map $H(I \times J) \to H(I) \times H(J)$ is an isomorphism for every I and J).

Proposition 1.6. *H factors canonically as*

$$\begin{tikzcd}
Mod_R \arrow[rr, "H"] \arrow[dr, "\overline{H}"'] & & \mathbf{Set} \\
& (R\text{-}modules) \arrow[ur, "forget"'] &
\end{tikzcd}$$

Sketch Proof. Additive structure: $H(I) \times H(I) \cong H(I \times I) \xrightarrow{H(\sigma)} H(I)$ where $\sigma : I \times I \to I$ is the sum map on I.

Multiplicative structure: for $f \in R$, we have $f \cdot - : H(I) \xrightarrow{H(f\cdot -)} H(I)$ □

Exercise 1.7. Check that this works (i.e. that this is really a module structure on $H(I)$). ◀

Start with a ring homomorphism $A \to R$. Then A-**alg**$/R$ has finite products. If we have C and C', then check that $C \times_R C'$ is the product.

$$\begin{tikzcd}
C \times_R C' \arrow[r] \arrow[d] & C' \arrow[d, "f'"] \\
C \arrow[r, "f"] & R
\end{tikzcd}$$

Lemma 1.7. *The functor* $Mod_R \to A$-**alg**$/R$, *given by* $I \mapsto (R[I], \pi : R[I] \to R)$, *commutes with finite products.*

Proof. It is enough to consider two modules. Let I and J be modules. We need to check that the natural map $R[I \times J] \to R[I] \otimes_R R[J]$ is an isomorphism. The point is that you have maps to R, and you check that both of them have kernels $I \times J$. □

Corollary 1.8. *If* $F : A$-**alg**$/R \to$ **Set** *such that for* $I, J \in Mod_R$, *the map* $F(R[I] \times_R R[J]) \to F(R[I]) \times F(R[J])$ *is an isomorphism. Then for any* $I \in Mod_R$, *the set* $F(R[I])$ *has a canonical* R*-module structure.*

The reason is that $F(R[I])$ is the image of I under the composite $Mod_R \xrightarrow{R[-]} A$-**alg**$/R \xrightarrow{F}$ **Set**, which commutes with finite products.

Definition 1.9. Let $F : A$-**alg**$/R \to$ **Set** be a functor respecting (some) products (as above), then the *tangent space* of F, denoted T_F, is the R-module $F(R[\varepsilon])$. ◇

Remark 1.10. We don't need all of A-**alg**$/R$, we just need a subcategory that contains the image of Mod_R. It is enough for F to be defined on the full subcategory $\mathcal{C} \subseteq A$-**alg**$/R$ closed under finite products and containing all the $R[I]$. So R might be a field and we consider local artinian rings with residue field R. ◇

We didn't really need to take Mod_R to be finitely generated stuff for this talk.

1 Osserman

This lecture series will be about functions of Artin rings, and specifically, representability and Schlessinger's criterion.

So far, we've seen two extremes of studying a scheme by looking at maps to the scheme. On the one hand, you can recover X from its functor of points (this is Yoneda's lemma). On the other hand, we saw that if you want to study locally, you can get the tangent space just by looking at maps from $k[\varepsilon]$. We're going to talk about something in between. It will still be quite local. It will be looking at maps $\operatorname{Spec} A \to X$, where A is a local artin ring, with image some $x \in X$. We'll see that this is much more on the local side than on the global side. We'll talk about what kind of data you can expect to get out of these kinds of maps.

From a moduli perspective, we're studying families over $\operatorname{Spec} A$, with some fixed restriction to $\operatorname{Spec} k$ (we'll typically fix the residue field k). Remember that the underlying topological space of $\operatorname{Spec} A$ is just one point, so you might think that not a lot can go on. Topologically, that's right. Everything is happening on the level of nilpotents in the sheaf of rings. Such things are called *infinitesimal thickenings* of the fixed thing over $\operatorname{Spec} k$. The data obtained in this way is the "complete local ring" at the point (at least in the case when the moduli space is a scheme). Let's make this more precise.

Recovering complete local rings. (Temporary) notation: we'll say that $Art(k)$ is the category of local artin rings with residue field k (where the morphisms respect the map to the residue field). Given a locally noetherian scheme X and $x \in X$ with $k = k(x)$, let $F_{X,x} : Art(k) \to \mathbf{Set}$ be given by $(A \to k) \mapsto \{f : \operatorname{Spec} A \to X | f \circ (\operatorname{Spec} k \hookrightarrow \operatorname{Spec} A) = (x : \operatorname{Spec} k \to X)\}$. This is just looking at maps from artin local rings with image $x \in X(k)$. What kind of data is encoded in this functor?

Proposition 1.1. *Given X locally noetherian and $x \in X$, the canonical map $\operatorname{Spec} \hat{\mathcal{O}}_{X,x} \to X$ induces an isomorphism of functors $F_{\operatorname{Spec} \hat{\mathcal{O}}_{X,x}} \to F_{X,x}$, and any complete local ring R with residue field k with a map $\operatorname{Spec} R \to X$ which induces a bijection like this is canonically isomorphic to $\hat{\mathcal{O}}_{X,x}$.*

Remark 1.2. This last part of the statement is anticipating the notion of prorepresentability. ◇

[[★★★ when can we say that a topological space is complete?]]

Proof. The first statement is equivalent to saying that any map from $\operatorname{Spec} A$ (with A artin local) with image x factors uniquely through $\operatorname{Spec} \hat{\mathcal{O}}_{X,x} \to X$. It is an easy exercise that it factors uniquely through $\operatorname{Spec} \mathcal{O}_{X,x} \to X$ (this is true for any local ring A), so it is enough to show that any map of rings $\mathcal{O}_{X,x} \to A$ factors uniquely through $\hat{\mathcal{O}}_{X,x}$. Because A is artin, the image of the maximal ideal $\mathfrak{m}_x$ is nilpotent, so a map $\mathcal{O}_{X,x} \to A$ factors uniquely through $\mathcal{O}_{X,x}/\mathfrak{m}_x^n$ for large enough n, so it factors uniquely through $\hat{\mathcal{O}}_{X,x}$ by the universal property of $\hat{\mathcal{O}}_{X,x} = \varprojlim \mathcal{O}_{X,x}/\mathfrak{m}_x^n$.

For the second part of the proposition, the point is that $\mathcal{O}_{X,x}/\mathfrak{m}_x^n$ and $R/\mathfrak{m}_R^n$ are artin rings for all n. Using a Yoneda-style trick (we'll see this precisely tomorrow), we construct compatible maps $R \to \mathcal{O}_{X,x}/\mathfrak{m}_x^n$ and $\hat{\mathcal{O}}_{X,x} \to R/\mathfrak{m}_R^n$. Then because these are complete local rings, we get isomorphisms $R \cong \hat{\mathcal{O}}_{X,x}$. □

Remark 1.3. What data is in $\hat{\mathcal{O}}_{X,x}$?

1. The dimension of X at x (this is already more information than you got from the tangent space).

2. The "singularity type" of X at x. This is something very similar to a local ring of an analytic space.

3. e.g. Cohen Theorem: if X is smooth over k (with $k = k(x)$) of dimension n, then $\hat{\mathcal{O}}_{X,x} \cong k[[x_1, \ldots, x_n]]$.

4. e.g. $y^2 = x^3 - x^2$ in the plane at the origin, then the complete local ring is the same as if you'd just taken the union of the axes (the complete local ring at zero of $y^2 = x^2$). In particular, the local ring can be an integral domain even if the complete local ring is not.

   [[★★★ picture?]]

5. e.g. $y^2 = x^3$ at zero, then you get $k[[x,t]]/(y^2 - t^3) \ncong k[[s]]$ (even though you have a homeomorphism ... the other examples have been topological in nature, now we see that we get more information than that). ◇

<u>The functors of interest</u>. We work in a relative setting. We'll fix Λ a complete local noetherian ring of residue field k, and we'll work with artinian local Λ-algebras (we'll denote this category by $Art(\Lambda, k)$; we only allow morphisms which respect the Λ structure).

Nonstandard terminology: A *predeformation functor* is a covariant functor $F : Art(\Lambda, k) \to \mathbf{Set}$ such that $F(k) = *$ (the one point set). Roughly, these arise by considering famlies over $\operatorname{Spec} A$ restricting to a fixed object over $\operatorname{Spec} k$. If you started off with a global moduli functor, it is tempting to say that you can get a predeformation functor by restriction (i.e. by choosing a fixed object over k and restricting attention to artin rings). In fact you could, but this doesn't always work well. We'll see this tomorrow.

2 Lieblich

Yesterday we were starting to think of functors as generalized spaces. Today, we'll try to find some subcategory of the category of all functors which is better for geometry.

Let X be a scheme, then we get the functor $h_X = \operatorname{Hom}(-, X)$. This functor has a nice property. For any Y, consider the association sending an open subscheme $U \subseteq Y$ to $h_X(U)$. This is a sheaf in the Zariski topology on Y. There is a more compact way to write this. If $\{U_i \subseteq Y\}$ is an open cover of Y, then

$$\begin{tikzcd} h_X(Y) \arrow[r, "a"] & \prod_i h_X(U_i) \arrow[r, shift left, "b"] \arrow[r, shift right, "c"'] & \prod_{i,j} h_X(U_i \cap U_j) \end{tikzcd}$$

is an exact sequence (i.e. a is injective, and $\operatorname{im} a = \{\alpha | b(\alpha) = c(\alpha)\}$).

There is a problem: the Zariski topology is not very "geometric". Serre's work in FAC convinced us that the Zariski topology was just as good as the analytic topology. However, when you think about things like the fundamental group, the Zariski topology is bad. Serre pointed some of these things out, but didn't know how to fix them. Grothendieck's observation was that there is an abstract categorical notion of a topology, which comes from the following observation.

Observation: let X be a topological space, then its open sets form a category *Open*(X) (the objects are $U \subseteq X$ open, and the morphisms are inclusions of open sets). Note that $|\operatorname{Hom}(U, V)| \le 1$ (there is at most one inclusion $U \hookrightarrow V$). A presheaf is exactly a contravariant functor from *Open*(X) (check it). However, this is not enough to tell us what sheaves are. For sheaves, we have to remember more: we have to remember what the open coverings are (this information is *not* contained in the category *Open*(X)). So we retain the information of when a set of inclusions $\{V_i \hookrightarrow U\}$ is an open covering. Grothendieck's observation is that we can generalize this to categories which don't come from topological spaces in this way. To do this, we have to figure out exactly what it is about coverings that we want.

Essential properties of coverings:

1. $\{U \subseteq U\}$ is a covering,
2. If $\{V_i \subseteq U\}$ is a covering of U and $W \subseteq U$, then $\{V_i \cap W \subseteq W\}$ is a covering of W, and
3. If $\{W_{ij} \subseteq V_i\}$ are coverings and $\{V_i \subseteq U\}$ is a covering, then $\{W_{ij} \subseteq U\}$ is a covering.

Definition 2.1. Given a category $\mathcal{C}$, a *Grothendieck topology on* $\mathcal{C}$ is a collection of sets of arrows $\{V_i \to U\}$ called *coverings of* U, such that

1. any isomorphism is a covering,
2. If $\{V_i \to U\}$ is a covering of U and $W \to U$, then the products $V_i \times_U W$ exist for each i and $\{V_i \times_U W \subseteq W\}$ is a covering of W, and

3. If $\{W_{ij} \to V_i\}$ are coverings and $\{V_i \to U\}$ is a covering, then $\{W_{ij} \to V_i \to U\}$ is a covering.

A *site* is a category with a Grothendieck topology. ◇

Ravi: are there cases in real life where you don't have all products? Max: I don't think so… Martin? Martin: I guess I could make one. Max: I think that means that there aren't any in real life.

Example 2.2 (Small Zariski site)**.** Let X be a scheme, then X_{Zar} is the site of open subschemes as before. The objects are open immersions $U \hookrightarrow X$ and the arrows are X-morphisms. (Note that we've thrown in all open immersions, but whatever.) ◇

Example 2.3 (Big Zariski site)**.** X_{ZAR} has objects X-schemes (i.e. the underlying category is $\mathbf{Sch}\,/X$). The coverings are collections of X-morphisms $\{Y_i \xrightarrow{\phi_i} Z\}$ such that each ϕ_i is an open immersion and $\bigcup \phi_i(Y_i) = Z$.

Consider the point Y as an $X = \mathbb{A}^1$-scheme. If you think about the presheaf h_Y on the small Zariski site, then you get nothing (no open subsets factor through a point), but it is non-trivial presheaf on the big site. ◇

Example 2.4 (Small étale site)**.** X_{et} has objects étale morphisms $Z \to X$ and morphisms are X-morphisms. The coverings are sets of X-morphisms $\{Y_i \xrightarrow{\phi_i} Z\}$ such that $\bigcup \phi_i(Y_i) = Z$ (note that the ϕ_i are automatically étale). You should think of the étale site as retaining analytic information. ◇

Example 2.5 (Big étale site)**.** X_{ET} has underlying category $\mathbf{Sch}\,/X$. Coverings are sets $\{Y_i \xrightarrow{\phi_i} Z\}$ so that each ϕ_i is étale and $\bigcup \phi_i(Y_i) = Z$. ◇

Example 2.6 (fppf site (faithfully flat locally of finite presentation))**.** X_{fppf} has underlying category $\mathbf{Sch}\,/X$. Coverings are sets of X-morphisms $\{Y_i \xrightarrow{\phi_i} Z\}$ such that each ϕ_i are flat and locally of finite presentation so that the ϕ_i are jointly surjective.

You may also want to think of this as X_{FPlPF}, but don't really. ◇

Note that these topologies have been getting finer and finer.

Remark 2.7. If you try to make a "small fppf" site, you don't have all the fiber products you want. ◇

There is also something even finer called the fpqc topology, but we won't talk about it.

Definition 2.8. Given a site $\mathcal{C}$, a *sheaf of sets* on $\mathcal{C}$ is a functor $F : \mathcal{C}^\circ \to \mathbf{Set}$ such that for all coverings $\{Y_i \to Z\}$, the sequence

$$\begin{tikzcd} F(Z) \arrow[r] & \prod_i F(Y_i) \arrow[r, shift left] \arrow[r, shift right] & \prod_{i,j} F(Y_i \times_Z Y_j) \end{tikzcd}$$

is exact. ◇

Note that intersections are exactly fiber products. Note also that you do not require $i \neq j$. You need to consider $i = j$ because fiber products of a single map with itself can be interesting (consider $\operatorname{Spec} \mathbb{Q}[\sqrt{2}] \to \operatorname{Spec} \mathbb{Q}$.)

Now given a functor on the category of schemes, we can ask if it is a sheaf in the various topologies. Being a Zariski sheaf is easier than being an étale sheaf, which is easier than being an fppf sheaf.

Theorem 2.9 (Grothendieck)**.** *For any X-scheme S, the functor $h_S : \mathbf{Sch}_X^\circ \to \mathbf{Set}$ is an fppf sheaf.*

(By the way, it turns out that h_S is even an fpqc sheaf.) Fix a covering $\{Y_i \to Z\}$. Then we want to show that

$$\begin{tikzcd} h_S(Z) \arrow[r] & \prod_i h_S(Y_i) \arrow[r, shift left] \arrow[r, shift right] & \prod_{i,j} h_S(Y_i \times_Z Y_j) \end{tikzcd}$$

is exact. Let's start with a baby case first, where everything is affine and there is only one Y. So we have a faithfully flat map $\operatorname{Spec} B \to \operatorname{Spec} A$, and let $S = \operatorname{Spec} C$. Then we have the diagram

$$\begin{tikzcd} \operatorname{Hom}(C, A) \arrow[r] & \operatorname{Hom}(C, B) \arrow[r, shift left] \arrow[r, shift right] & \operatorname{Hom}(C, B \otimes_A B) \end{tikzcd}$$

which is

$$\operatorname{Hom}(C,\ \begin{tikzcd} A \arrow[r] & B \arrow[r, shift left] \arrow[r, shift right] & B \otimes_A B \end{tikzcd})$$

Lemma 2.10. *The sequence* $\begin{tikzcd} A \arrow[r] & B \arrow[r, shift left, "b \mapsto b\otimes 1"] \arrow[r, shift right, "b \mapsto 1 \otimes b"'] & B \otimes_A B \end{tikzcd}$ *is exact.*

Proof. It is exactly the same as showing that $A \to B \xrightarrow{b \mapsto b\otimes 1 - 1 \otimes b} B \otimes_A B$ is exact. Consider the special case when we have a section $\sigma : B \to A$. Then we get a section $B \otimes_A B \to B$ given by $b \otimes c \mapsto \sigma(b)c$. Then to show that $b \otimes 1 = 1 \otimes b$ implies $b \in A$, note that applying this section we get $\sigma(b) = b$, and $\sigma(b) \in A$.

Observe that to prove that $0 \to A \to B \to B \otimes_A B$ is exact, it is enough to take a faithfully flat base change $A \to D$ and show that after tensoring with D, the sequence is exact (this is what faithfully flat means). Take $D = B$, then you are back to the case when you have a section. (By the way, you should check that when you base extend the original sequence, you get the right new sequence.) ☐

Ok, so we've done the baby case. Let's sketch how to do it in general. There is a simplifying lemma that makes it easier to check that things are fppf sheaves.

Lemma 2.11. *A functor $F : \mathbf{Sch}_X^\circ \to \mathbf{Set}$ is an fppf sheaf if and only if*

1. *F is a Zariski sheaf, and*
2. *for all $U = \operatorname{Spec} B \to \operatorname{Spec} A = V$ fiathfully flat of finite presentation, the sequence*
$$F(V) \to F(U) \to F(U \times_V U)$$
is exact.

That is, you only have to check the sheaf axiom for certain covers.

Corollary 2.12. *If S is affine, then h_S is an fppf sheaf.*

Sketch of general case. Let $S_i \subseteq S$ be an affine covering of S, and let $U \to V$ be an fppf covering with U and V affine.

$$\begin{tikzcd} h_S(V) \arrow[r] & h_S(U) \arrow[r, shift left] \arrow[r, shift right] & h_S(U \times_V U) \end{tikzcd}$$

Let $U \to S$ such that the two maps $U \times_V U \rightrightarrows U \to X$ agree [[★★★ what?]]. By some flatness and locally of finite presentation magic, there is a map of topological spaces $|V| \to |S|$ such that $|U| \to |V| \to |S|$ corresponds to $U \to S$.

Pull back $S_i \subseteq S$. Then we have

$$\begin{tikzcd} U_i \times_{V_i} U_i \arrow[r, shift left] \arrow[r, shift right] & U_i \arrow[r] \arrow[dr] & V_i \arrow[d, dashed] \\ & & S_i \end{tikzcd}$$

where S_i is affine, so the dashed arrow exists by the affine case. Thus, by the sheaf condition, we get $V \to S$ as desired. ☐

2 Olsson

Recall that we have a map of rings $A \to R$, and we formed the category A-**alg**$/R$, which is the category of diagrams

$$\begin{tikzcd}
C \arrow[r, "f"] & R \\
A \arrow[u] \arrow[ur] &
\end{tikzcd}$$

where C is an A-algebra. If $F : A\text{-}\mathbf{alg}/R \to \mathbf{Set}$ is a functor such that for any modules I and J over R, the natural map $F(R[I \oplus J]) \to F(R[I]) \times F(R[J])$ is an isomorphism, then we got a tangent space $T_F = F(R[\varepsilon])$. In fact, we saw that for every I, $F(R[I])$ is an R-module. Recall that the module structure is given by $+\colon F(R[\varepsilon]) \times F(R[\varepsilon]) \cong F(R[\varepsilon_1, \varepsilon_2]/(\varepsilon_1^2, \varepsilon_2^2, \varepsilon_1\varepsilon_2)) \xrightarrow{\varepsilon_i \mapsto \varepsilon} F(R[\varepsilon])$. and $f \times -\colon F(R[\varepsilon]) \to F(R[\varepsilon])$ is induced by $R[\varepsilon] \xrightarrow{a+b\varepsilon \mapsto a+fb\varepsilon} R[\varepsilon]$. Today we'll do some examples where we'll compute this tangent space.

<u>Problem 1</u>. Let R be a ring and let $g\colon X \to \operatorname{Spec} R$ be a separated smooth morphism. Consider the functor $\operatorname{Def}_X : \mathbb{Z}\text{-}\mathbf{alg}/R = \mathbf{Alg}/R \to \mathbf{Set}$ given by $\operatorname{Def}_X(C \xrightarrow{f} R)$ = the set of isomorphism classes of cartesian diagrams

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "g"'] & X_C \arrow[d, "g_C"] \\
\operatorname{Spec} R \arrow[r] & \operatorname{Spec} C
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & & X'_C \arrow[dl, dashed, "h"'] \arrow[ddl, bend left, "g'_C"] \\
X \arrow[r] \arrow[d, "g"'] \arrow[urr, bend left] & X_C \arrow[d, "g_C"'] & \\
\operatorname{Spec} R \arrow[r] & \operatorname{Spec} C &
\end{tikzcd}
\tag{1}$$

with g_C smooth. A morphism of diagrams is a dashed arrow making the diagram on the right commute.

You should be thinking of $\operatorname{Spec} R \to \operatorname{Spec} C$ as a nilpotent thickenning. Then Def_X is the set of infinitesimal extentions of X.

Remark 2.1. If $C = R[I]$ for some R-module I, then any morphism h as in (1) is an isomorphism. This is because we're requiring g_C and g'_C to be smooth. All the underlying topological spaces are the same, so all the information is contained in the sheaves. We have the following diagram of sheaf on $|X|$

$$\begin{tikzcd}
0 \arrow[r] & I \otimes_R \mathcal{O}_X \arrow[r] \arrow[d, equal] & \mathcal{O}_{X_C} \arrow[r] \arrow[d] & \mathcal{O}_X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & I \otimes_R \mathcal{O}_X \arrow[r] & \mathcal{O}_{X'_C} \arrow[r] & \mathcal{O}_X \arrow[r] & 0
\end{tikzcd}$$

The middle morphism must be an isomorphism by the 5 lemma. [[★★★ We used flatness to get $I \otimes_C \mathcal{O}_{X_C} \cong I \otimes_R \mathcal{O}_X$]] ◇

We'll see this over and over again, where you are working on a fixed topological space and you just work with the sheaves.

Proposition 2.2. *For any* $I, J \in R$-**mod***,* $\mathrm{Def}_X(R[I\oplus J])) \to \mathrm{Def}_X(R[I])\times\mathrm{Def}_X(R[J])$.

Proof. In Brian's lecture 3. ☐

[[★★★ Ok, everybody hold still for a 4 second exposure]][[★★★ one more please]]

How to compute the tangent space T_{Def_X}, or more generally, the module $\mathrm{Def}_X(R[I])$? There is a fancy version and a hands-on version. We'll do the hands-on version, and maybe do the fancy version next week.

Special case: X is affine. For this we need some facts (which should have been in the lecture on smoothness):

1. $\mathrm{Def}_X(R[I])$ consists of one element. This follows from the jacobien criterion for smoothness. Why is there a lifting? You can write $X = \operatorname{Spec} R[x_1, \ldots, x_r]/(f_1, \ldots, f_\ell)$ and try to lift the equations and then check smoothness with the jacobien criterion. In fact, $X[I] \to \operatorname{Spec} R[I]$ is a smooth lifting. The reason (given the fact that there is some smooth lifting): write $X = \operatorname{Spec} A$ and let $X \hookrightarrow X'$ be a smooth lifting. So
$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr] \arrow[dd] & X' \arrow[d, dashed] \\ & X' \arrow[d] \\ \operatorname{Spec} R \arrow[r] & \operatorname{Spec} R[I] \end{tikzcd}$$
The dashed arrow exists by the formal criterion for smoothness. Now doing the same kind of argument with sheaves as we did before, we see that this dashed arrow is an isomorphism (check that some kernels agree). Another way to do this is to do some hard algebra and show that if $R \to B$ is smooth, then so is $R[I] \to B[I]$. This actually follows from the fact that $B[I] = B \otimes_R R[I]$ and smoothness is preserved by base change.

2. For any deformation
$$\begin{tikzcd} X \arrow[r, hook] \arrow[d] & X' \arrow[d] \\ \operatorname{Spec} R \arrow[r] & \operatorname{Spec} R[I] \end{tikzcd}$$
the set of maps $h : X' \to X'$ as in (1) is in canonical bijection $H^0(X, T_X \otimes I)$. This is the universal property of differentials. If you have
$$\begin{tikzcd} T_0 \arrow[r, hook, "j"] \arrow[d, "f_0"'] & T \arrow[d, dashed, "f"] \\ S & Y \arrow[l] \end{tikzcd} \qquad \begin{tikzcd} T_0 \arrow[r, "f_0"] \arrow[d, hook, "j"'] & Y \arrow[d] \\ T \arrow[r, "f"] & S \end{tikzcd}$$

where j is a closed immersion defined by a square zero ideal J. The set of dashed arrows is a pseudo-torsor under $\mathcal{H}om(f_0^*\Omega^1_{Y/S}, J)$. This is a universal property of Ω^1.

Definition 2.3. *pseudotorsor*: either no arrow exists, or if there is an arrow, then there is a simply transitive action of $\mathcal{H}om(f_0^*\Omega^1_{Y/S}, J)$ on the set of arrows. ◇

The fact that there is no canonical bijection when there is a dashed arrow.

$$\begin{tikzcd}
T \arrow[rrd, "{(f_1,f_2)}"] & & \\
T_0 \arrow[u, hook] \arrow[r] & Y \arrow[r] & Y \times_S Y \arrow[d] \\
 & & S
\end{tikzcd}$$

Let's consider everything affine, so $Y = \operatorname{Spec} B$, $T = \operatorname{Spec} A$, $T_0 = \operatorname{Spec} A_0$, $S = \operatorname{Spec} V$, so we have the diagram

$$\begin{tikzcd}
A \arrow[d, two heads] & & \\
A_0 & B \arrow[l] & B \otimes_V B/I_\Delta^2 \arrow[l, "\Delta"'] \arrow[llu, "f_1 \otimes f_2"'] \\
 & & V \arrow[u]
\end{tikzcd}$$

You can add the $/I_\Delta^2$, so you get a map $A_0 \otimes_B \Omega^1_{Y/S} = I_\Delta/I_\Delta^2 \to J$, and you have to check that something is a bijection.

[[★★★ this goes somewhere?]]

$$\begin{tikzcd}
 & X' \arrow[d, dashed] \\
X \arrow[ru, hook, "j"] \arrow[r, "f_0"] \arrow[rd] & X \arrow[d] \\
 & \operatorname{Spec} R[I]
\end{tikzcd}$$

For the general case $X \to \operatorname{Spec} R$, this also shows that $(X[I] \to \operatorname{Spec} R[I]) \in \operatorname{Def}_X(R[I])$ (because smoothness is a local condition, so we can reduce to the case where X is affine).

In general, choose a covering $X = \bigcup_i U_i$ with each U_i affine. How can we build a smooth lifting? Choose for each i a smooth lifting $U'_i \to \operatorname{Spec} R[I]$ (there is only one such thing). Let $\mathcal{U}$ denote the covering $\{U_i\}$. We'd like to patch these liftings together to a lifting on X. Well, a smooth lifting $X \hookrightarrow X'$ is really the same thing as a surjection $\mathcal{O}_{X'} \to \mathcal{O}_X$ with kernel $I \otimes_R \mathcal{O}_X$ of sheaves of rings on $|X|$. The $U'_i \to \operatorname{Spec} R[I]$ correspond to surjections $\mathcal{O}_{U'_i} \to \mathcal{O}_{U_i}$ with kernel $I \otimes_R \mathcal{O}_{U_i}$. We want to glue these together, so we have to look at intersections $U_{ij} = U_i \cap U_j$. Here, we get a diagram

$$\begin{tikzcd}
& U'_i|_{U_{ij}} \arrow[dd, dashed, "\exists", "\wr"'] \\
U_{ij} \arrow[ur, hook] \arrow[dr, hook] & \\
& U'_j|_{U_{ij}}
\end{tikzcd}$$

The restriction makes sense because the underlying topological spaces are the same. These are two elements of $\operatorname{Def}_{U_{ij}}(R[I])$. Since we assumed separated, there is an isomorphism between these two. Another way to see this is that they are both the trivial deformation automatically.

For every i and j, pick an isomorphism $\sigma_i : U'_i \to U_i[I]$. Note that any other choice of σ_i is given by composing with an automorphism of $U_i[I]$, which is given by $H^0(U_{ij}, T_X \otimes I)$.

How to specify X'. Pick a σ_i for each U_i, and note that any two such differ by a section of $T_X \otimes I$. We'd like these two match up. There is an obstruction for the σ_i to glue to an isomorphism $X' \xrightarrow{\sim} X[I]$. Define $x_{ij} : U_{ij}[I] \xrightarrow{\sigma_j^{-1}|_{U_{ij}}} U'_{ij} \xrightarrow{\sigma_i|_{U_{ij}}} U_{ij}[I]$. So we have that $x_{ij} \in H^0(U_{ij}, T_X \otimes I)$ (since the U_{ij} are affine). Note that if we got all this from an isomorphism $X' \xrightarrow{\sim} X[I]$, then all the x_{ij} are zero.

Lemma 2.4. *$x_{ik} = x_{ij} + x_{jk}$ in $H^0(U_{ijk}, T_X \otimes I)$.*

Proof.

$$\begin{tikzcd}
U_{ijk}[I] \arrow[r, "\sigma_k^{-1}"] \arrow[rr, bend left, "x_{jk}"] \arrow[rrrd, "x_{ik}"'] & U'_{ijk} \arrow[r, "\sigma_j"] & U_{ijk}[I] \arrow[r, "\sigma_j^{-1}"] \arrow[rd, "x_{ij}"'] & U'_{ijk} \arrow[d, "\sigma_i"] \\
& & & U_{ijk}[I]
\end{tikzcd}$$

□

Corollary 2.5. *The $\{x_{ij}\}$ define a Čech cocyle. That is, $[X'] \in \check{H}^1(X, T_X \otimes I) = H^1(X, T_X \otimes I)$ (this is where you use affine).*

Theorem 2.6. *The map $\operatorname{Def}_X(R[I]) \to H^1(X, T_X \otimes I)$ given by $X' \mapsto [X']$ is an R-module isomorphism.*

The surjectivity is given by tweaking the x_{ij}. The injectivity is something about choosing a fixed lifting to begin with. The sticky point is to check that this is an R-module morphism.

Recap: on each affine piece you get a unique lifting (up to non-unique isomorphism). We have a fixed lifting $X[I]$. Comparing to any other lifting, you get some Čech cocyle. This cohomology class is zero exactly when the guys actually patch.

That is, $[X'] = 0$ implies that there is a $\partial_i \in H^0(U_i, T_X \otimes I)$ such that $\sigma_i - \sigma_j = x_{ij} = \partial_j - \partial_i$.

Ravi: if not smooth, then we still see that the locally trivial deformations are parameterized by H^1 of some $\mathcal{H}om$.

2 Osserman

Don't forget that Ravi is talking at 7:15pm in 60 Evans.

Today we'll do some examples of the deformation functors that we'll be considering. For "nice" global moduli functors, it works well to simply restrict to $Art(\Lambda, k)$ to obtain predeformation functors.

Example 2.1 (Deformations of a closed subscheme)**.** You'll need good hypotheses to get good behavior, but for now we won't have any restrictions. Let X_Λ be a scheme over $\operatorname{Spec}\Lambda$, and write X for $X_\Lambda|_{\operatorname{Spec} k}$ where $\operatorname{Spec} k$ is the closed point of $\operatorname{Spec}\Lambda$. Suppose we have a closed subscheme $Z \subseteq X$. Then we define a functor $\operatorname{Def}_{Z,X} \colon Art(\Lambda, k) \to \mathbf{Set}$ by $A \mapsto \{Z_A \subseteq X_\Lambda|_{\operatorname{Spec} A}$ closed subscheme, flat over A such that $Z_A|_{\operatorname{Spec} k} = Z$ (actual equality as closed subschemes)$\}$.[[★★★ restriction is pullback]]

This is exactly what you get if you start with a global deformation functor and restrict to the category $Art(\Lambda, k)$. ◇

If you take $\Lambda = k$, then you get the moduli of deformations of Z in X. If you had a subscheme of a fiber of some family, then you would need to take Λ to be bigger.

Sometimes, simple restriction of functors doesn't work so well.

Example 2.2 (Deformations of a scheme)**.** Fix some X over k. If you restrict a moduli functor, you'd get $A \mapsto \{$ schemes flat over A restricting to X over the closed point$\}$. Here we do something a little different. Define Def_X by $A \mapsto \{(X_A, \phi)|X_A$ is flat over $\operatorname{Spec} A$ and
$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[d] & X_A \arrow[d] \\ \operatorname{Spec} k \arrow[r] & \operatorname{Spec} A \end{tikzcd}$$
is cartesean$\}/\cong$ (here an isomorphism of a pair is an isomorphism $X_A \xrightarrow{\sim} X'_A$ respecting ϕ and ϕ'). Note that if we naïvely restricted functors, we'd still get a predeformation functor, but its behavior is worse. The problem is that if there are automorphisms of X which do not extend to X_A.

This is the first indication that for moduli problems involving automorphisms, functors to sets don't capture everyting.

By the time we're talking about categories fibered in groupoids, we'll be able to get everything by this kind of restriction. I (Brian) will post some notes explaining some more things about the restriction predeformation functor is bad ◇

Example 2.3 (Deformations of a quasi-coherent sheaf)**.** Fix a scheme X_Λ over $\operatorname{Spec}\Lambda$ and set $X = X_\Lambda|_{\operatorname{Spec} k}$. Also fix a quasi-coherent sheaf $\mathcal{E}$ on X. Define $\operatorname{Def}_{\mathcal{E}}$ by $A \mapsto \{(\mathcal{E}_A, \phi)|\mathcal{E}_A$ is a quasi-coherent sheaf on $X_\Lambda|\operatorname{Spec} A$ which is flat over A and $\phi : \mathcal{E}_A \to \mathcal{E}$ inducing an isomorphism $\mathcal{E}_A \otimes_A k \xrightarrow{\mathcal{E}}\}/\cong$. Again, we have to add the data of ϕ because quasi-coherent sheaves aren't ever equal, they are only isomorphic, and they typically have non-trivial automorphisms. ◇

Remark 2.4. The point of flatness is that if we don't require it, then you could have a family which is just supported on $\operatorname{Spec} k$. To talk about a family over $\operatorname{Spec} A$, you'd like it to live over all of $\operatorname{Spec} A$. The flatness gives you some kind of continuity in the fibers of the map. You'll see flatness everywhere (or it will be implicit if you don't see it). You'll have to accept that flatness is not a condition that can be completely understood from a geometric point of view. $\diamond$

Prorepresentability and hulls. We don't simply ask for the functor to be representable. Remember the example, where a complete local ring is a limit of artin rings, but is not itself an artin ring. Ravi: why not just work with the category of complete local rings instead of artin rings? Brian: I think artin rings are technically easier, and [[★★★ something something see warning below?]].

Definition 2.5. Given $F : \mathit{Art}(\Lambda, k) \to \mathbf{Set}$, let $\widehat{\mathit{Art}}(\Lambda, k)$ be the category of complete local noetherian Λ-algebras. (It is true that every such thing is an inverse limit, but not every inverse limit is noetherian) We define $\hat{F} : \widehat{\mathit{Art}}(\Lambda, k) \to \mathbf{Set}$ by $\hat{F}(R) := \varprojlim F(R/\mathfrak{m}^n)$. We say that F is *prorepresentable* if $\hat{F}$ is representable. $\diamond$

Warning 2.6 ($\hat{F}$ doesn't have the intuitive meaning you might think)**.** If we start with a global moduli problem, $\hat{F}$ is not necessarily obtained by simply considering families over R. That is, even if you have an element of $\varprojlim F(R/\mathfrak{m}^n)$, then you can't always patch them together to get something over R. This is the issue of effectivizability, which we'll talk about next week. $\lrcorner$

It turns out that there will be global things where all the local guys are prorepresentable, but the global thing is not representable, or something.

Definition 2.7. Given $F, F' \colon \mathit{Art}(\Lambda, k) \to \mathbf{Set}$, a morphism $f : F \to F'$ is *(formally) smooth* if for every surjection $A \twoheadrightarrow B$ in $\mathit{Art}(\Lambda, k)$, the map $F(A) \to F(B) \times_{F'(B)} F'(A)$ is surjective. [[★★★ check that this really does restrict to the formal criterion for smoothness in the case when F and F' are representable.]] $\diamond$

Recall that T_F, the tangent space of F, is $F(k[\varepsilon])$ (where $k[\varepsilon]$ has the standard Λ structure on k and the trivial action on ε).

Notation: Given $R \in \widehat{\mathit{Art}}(\Lambda, k)$, then let $h_R : \widehat{\mathit{Art}}(\Lambda, k) \to \mathbf{Set}$ be the functor of points of $\operatorname{Spec} R$, given by $\operatorname{Hom}_{\widehat{\mathit{Art}}(\Lambda,k)}(R, -)$. Note that this is the contravariant Yoneda embedding. Further, define $\overline{h}_R : \mathit{Art}(\Lambda, k) \to \mathbf{Set}$ to be the restriction of h_R.

We have the following replacement for isomorphisms.

Definition 2.8. Let F be a predeformation functor. The pair (R, η) with $\eta \in \hat{F}(R)$ is a *hull* for F if (1) the induced map $\overline{h}_R \to F$ is smooth, and (2) the induced map $T_{\overline{h}_R} \to T_F$ is an isomorphism. $\diamond$

If F were prorepresentable, then these conditions actually imply that $\overline{h}_R \to F$ is an isomorphism. [[★★★ btw, check that a morphism of functors induces a morphism of tangent spaces *respecting the module structure*]]

It is not true that two hulls of a functor are uniquely isomorphic, but they are (non-uniquely) isomorphic.

Proposition 2.9. *If* (R,η) *and* (R',η') *are hulls for* F*, then they are isomorphic (as pairs).*

Exercise 2.1. Prove it. ◀

Schlessinger's criterion. This criterion for prorepresentablity is a bit opaque, but it makes sense from the point of view of categories fibered in groupoids. It is extemely useful, even if it is opaque.

Definition 2.10. A surjective map $f : A \to B$ in $Art(\Lambda,k)$ is a *small thickenning* if $\ker f \cong k$, or equivalently, $\mathfrak{m}_A \cdot \ker f = 0$ and $\ker f$ is principal. ◇

Remark 2.11. It is easy to check that any surjection in $Art(\Lambda,k)$ can be factored as a series of small thickennings. Thus, when we prove that something is true for small thickennings, it will be true for all surjections by an inductive argument. ◇

Given a pair of maps $A' \to A$ and $A'' \to A$, we get a map $(*)$ $F(A' \times_A A'') \to F(A) \times_{F(A)} F(A'')$.

Theorem 2.12 (Schlessinger's criterion)**.** *If* F *is a predeformation functor, consider the following conditions:*

(H1) $(*)$ *is surjective whenever* $A'' \twoheadrightarrow A$.

(H2) $(*)$ *is bijective when* $A'' = k[\varepsilon]$ *and* $A = k$.

(H3) T_F *is finite dimensional.*

(H4) $(*)$ *is bijective when ever* $A' = A'' \twoheadrightarrow A$.

(we can replace "surjection" by "small thickenning" by the way) (H1-H3) is equivalent to F *having a hull, and (H1-H4) is equivalent to* F *being prorepresentable. Note that (H2) is automatic when something from Martin's lecture.*

$2\frac{1}{2}$ Vakil. The space that wanted to be a scheme.

You're the jury. We'll gives pros and cons for it being a scheme, and you can decide at the end.

You have some family of objects $\mathcal{X} \to B$ over a base scheme B. You say what are nice families, and the statement is that a nice family like this is the same thing as a map $B \to \mathcal{M}$. Now that you understand Yoneda's lemma, you know that knowing all the maps $B \to \mathcal{M}$ for all B, then you have determined $\mathcal{M}$ uniquely, if it exists (and there is no reason for it to exist in general). Today we'll talk about $\mathcal{M}_3$, the moduli space of genus three curves. A smooth genus three curve over B is a map $\mathcal{X} \to B$ which is smooth projective of relative dimension 1, with geometrically connected fibers of genus 3. Is there such a scheme $\mathcal{M}_3$?

Evidence that YES.

- From Martin's talks, we know about tangent spaces. Given $[C] \in \mathcal{M}_3$, we can find $T_{[C]}$. This is a 6-dimensional vector space by Martin's talk today and the stuff on Riemann-Roch that you've done.
- Soon, in Brian's talks, we'll get the entire deformation space, which will tell us that this is formally smooth.
- what is a vector bundle (on a functor)? The idea is that if you have a vector bundle $V \to \mathcal{M}_3$, then you have

$$\begin{tikzcd}
C \arrow[d] & V_B = \pi^*V \arrow[dl] & & V \arrow[dl] \\
B \arrow[rr, "\pi"'] & & \mathcal{M}_3 &
\end{tikzcd}$$

  so you should get a vector bundle on B as well as a curve C, and this vector bundle should be natural in the usual way. That is, a vector bundle on $\mathcal{M}_3$ is the same thing as a recipe for producing a vector bundle on B given a curve over B.

Example $2\frac{1}{2}$.1. There is a rank 3 vector bundle $\mathbb{E}^3 \to \mathcal{M}_3$ called the *Hodge bundle.* Given a curve $\pi : C \to B$, you show by cohomology of base change that $\pi_*\Omega^1_C$ is a rank three vector bundle on B, and that this bundle is natural.

Martin: the sections of this is the functor of points of the total space of the vector bundle. ◇

Similarly, you can take the determinant of $\mathbb{E}$ to get a line bundle on $\mathcal{M}_3$. We can make sense of the Picard group of $\mathcal{M}_3$. Mumford has something where he motivates stacks by working out the picard group of the moduli space of elliptic curves. Similarly, you can define coherent sheaves, etc.

More evidence:

- $\mathcal{H}_3 \subseteq \mathcal{M}_3$. some genus 3 curves are hyperelliptice (they admit 2-to-1 covers of $\mathbb{P}^1$). This $\mathcal{H}_3$ is a 5-dimensional Cartier divisor on the 6-dimensional $\mathcal{M}_3$ ($3g-3$ by RR).

  If you have any family $C \to B$ of genus 3 curves, then there should be some scheme-theoretical closed locus $H \hookrightarrow C$ which is the biggest possible closed subscheme over which $C|_H$ is hyperelliptic. We should feel that this H is given by $B \times_{\mathcal{M}_3} \mathcal{H}_3$, so the fact that we always find these H's is evidence that this $\mathcal{M}_3$ and $\mathcal{H}_3$ are schemes.

- $\mathcal{M}_3$ has a nice finite étale cover by a scheme called $\mathcal{M}_3[100]$ (genus 3 curves with 100-torsion of the jacobian ... assume 100 is invertible in the base ... of course it only matters that 100 is a big number). Let's pretend that we know that $\mathcal{M}_3[100]$ is a nice scheme. You expect $\mathcal{M}_3[100] \to \mathcal{M}_3$ to be finite because there are only finitely many ways to get level 100 structure on a curve. We can also argue that the map is étale, and it is clearly surjective. This allows you to get information about $\mathcal{M}_3$, like compute its tangent spaces . . . similarly, étale maps preserve deformation spaces.

  You can do even more. If you want to know what are all coherent sheaves on $\mathcal{M}_3$, it will be those $\mathcal{M}_3[100]$ quasi-coherent sheaves which satisfy descent.

- The fiber products

$$\begin{tikzcd}
X \times_{\mathcal{M}_3} Y \arrow[rr] \arrow[dd] \arrow[dr] & & Y \arrow[dd] \\
& X \times Y \arrow[ur] \arrow[dl] & \\
X \arrow[rr] & & \mathcal{M}_3
\end{tikzcd}$$

  You have a family of curves on X and one on Y, then $X \times_{\mathcal{M}_3} Y$ has to be *two* families of curves over $X \times Y$.

  Exercise $2\frac{1}{2}$.2. $X \times_{\mathcal{M}_3} Y$ should parameterize isomorphisms between the two families. Figure out exactly why this should be the fiber product. ◀

  This parameter space is called the Isom scheme.

The Isom functor. suppose over B you have two families of curves C_1 and C_2. Occasionally, the fibers are isomorphic, and the isom functor is supposed to give you the subscheme of B where this happens?

Theorem $2\frac{1}{2}$.2 (Grothendieck). $\underline{\mathrm{Isom}}_B(C_1, C_2)$ *is a scheme if* C_1 *and* C_2 *are projective and flat over* B. *(it doesn't matter that these are families of curves. . . they can be anything)*

The proof is surprisingly naïve, but we won't go into it.

At this point, surely your convinced that $\mathcal{M}_3$ is a scheme, but here is one more piece of evidence. Let $\mathcal{M}_{3,1}$ be the space of genus 3 curves with 1 marked point [[★★★ exercise]]. We get a morphism $\mathcal{M}_{3,1} \to \mathcal{M}_3$. How can we say this without saying that they are schemes, we just need to produce a natural recipe to start with a family of curves with a marked point and get a family of curves, which is really easy, no matter how you defined $\mathcal{M}_{3,1}$. In fact, this morphism is projective and smooth. We can say this without them being schemes. All we really need to check is that for every map $B \to \mathcal{M}_3$, the fiber product is projective and smooth. [[★★★ something is wrong with Hartshorne's definition of projective ... what is it?]] If this is true for any base change, then it must be that $\mathcal{M}_{3,1} \to \mathcal{M}_3$ is projective and smooth. So, given a family of curves on B, verify that there is a natural way to put a family of curves of genus 3 with marked point *on the family* C. [[★★★ exercise]]

The only black box that we've introduced so far is that Isom is a scheme.

Now we're half way through the movie, and now we turn to evidence that

$\mathcal{M}_3$ is not a scheme. In fact, it is not even a sheaf (even in the Zariski topology!). Max pointed out that schemes are sheaves. That just means that you can understand maps to a scheme locally. Fix a scheme Y. You get sheaf on Y, $U \mapsto \operatorname{Hom}(U, X)$. You can restrict maps to smaller subsets of U. If you have to maps to X which agree on an open cover, then they must be the same map. Further, if you have maps from open sets which agree on overlaps, then you can glue to get a big map. This means that h_X is a sheaf.

Now let's show that $\mathcal{M}_3$ is not a sheaf in the Zariski topology. Let Y be a the scheme which is two $\mathbb{P}^1$'s glued at two points (called the *banana curve*). Let's describe two maps to $\mathcal{M}_3$, i.e. two families of curves on Y. Fix a genus 3 curve C with a non-trivial automorphism σ. The first family is the trivial family $C \times Y$. The other family is obtained by taking the trivial family over the cut-open banana (where the two $\mathbb{P}^1$'s are only glued on one point) and glue the two C's together over the other point via σ. This is a non-trivial family, but it is trivial when you throw out either one of the two nodes. So too bad, $\mathcal{M}_3$, you're not a sheaf, so you're not a scheme.

So let's appeal the case. There will be a surprise witness (Grothendieck!). All the evidence still applies. At the very least, $\mathcal{M}_3$ is a nice functor (right?), and this should make you reasonably happy. What makes this space have these nice properties?

Grothendieck topologies. As we learned from Max today, this is a topology on a category. We're secretly thinking "open sets" on X. We have the additional data of coverings, satisfying three axioms (isomorphisms are coverings, you can refine coverings, and coverings pull back along morphisms). The fancy name for such a thing is a *site*. The fancy name for the category of sheaves on a site is *topos*.

Consider $\mathbb{A}^1_{\mathbb{C}}$ (which is just $\mathbb{C}$) with the classical topology. Consider the function $\sqrt{t}$. We'll that's not a function, but on some small enough open set U, it makes sense. Similarly, $\sqrt[3]{t-1}$ makes sense on some set V, so their sum makes sense on $U \cap V$. In the Zariski topology these functions don't make sense on any open set, but in the étale topology, they do make sense on some "open sets". In particular, we can take U to be $\operatorname{Spec}\mathbb{C}[u] \smallsetminus 0$ where $u^2 = t$, and similarly, you can find an étale map on which the other function makes sense. This is the sense in which the étale topology better mimicks the analytic topology.

How to save $\mathcal{M}_3$. The problem was automorphisms. Just as being in love means never having to say "I'm sorry", working with categories means never saying "equals". I like the curve $x^7 + y^7 + z^7 = 0$, and Martin likes the curve $a^7 + b^7 + c^7 = 0$ on $\mathbb{P}^2$. Well these are the same curve, right? No, they're isomorphic. If I want to tell Martin about a point on my curve so that he can admire it, I have to tell him which point it is on his curve, so we have to keep track of all the isomorphisms.

When I think of a category, I think of dots and arrows. We're going to write down the category of families of genus three curves:

[[★★★ picture of a fibered category with sometimes more than one isomorphism and sometimes no morphisms in the fibers, with the functor to **Sch**]]

This has a natural functor to schemes where you just remember the base scheme. You want to toss in some base changes. By the way, a category in which all morphisms are isomorphisms is called a groupoid. We're going to throw in some maps between families. These should map down to morphisms of the bases, and we require that these maps be fiber squares.

Later, you'll see a definition of a *category fibered in groupoids*. It will look a little weird, but just keep in mind that this is the picture.

Here are some properties you'll notice. (1) You can pull back families, (2) when you pull back and pull back again, you get something *isomorphic* to the pullback along the composition (this is enough to ensure that the fibers are groupoids).

Now we're ready to define the word which strikes the most fear in people.

Definition $2\frac{1}{2}$.3. A category fibered in groupoids is a *stack* if

1. $\underline{\mathrm{Isom}}_X(\xi_1, \xi_2) : \mathbf{Sch}_X \to \mathbf{Set}$ is a sheaf (in the étale topology, say). This comes for free in our case [[★★★ because of something]]
2. "objects glue". Again, we won't go into this formally. If you know a family on one open set and on another and you specify an isomorphism on the overlap, then you get a big family. ◇

(Fiber products over stacks are kind of weird.)

Definition $2\frac{1}{2}$.4. A morphism of stacks $\mathcal{M} \to \mathcal{M}'$ is *representable* if for all morphisms from a scheme $X \to \mathcal{M}'$, the fiber product $X \times_{\mathcal{M}'} \mathcal{M}$ is a scheme. ◇

This is the $\mathcal{M}_{3,1} \to \mathcal{M}_3$ example, but formalized a bit. For any $X \to \mathcal{M}_3$, the family of curves over X is a scheme.

We like this notion because remember how we defined projective and smooth. Any notion which behaves well under base change makes sense for representable morphisms!

Exercise $2\frac{1}{2}$.3. Let $\mathcal{M}$ be a stack. $\Delta_{\mathcal{M}} : \mathcal{M} \to \mathcal{M} \times \mathcal{M}$ is representable if and only if every morphism from a scheme $X \to \mathcal{M}$ is representable. (hint: use this fiber diagram

$$\begin{tikzcd}
U \times_X V \arrow[r] \arrow[d] & U \times_Y V \arrow[d] \\
X \arrow[r, "\Delta"] & X \times_Y X
\end{tikzcd}$$

This is true in any category when you have maps $X \to Y$ and $U, V \to X$.) ◀

Definition $2\frac{1}{2}$.5. A stack $\mathcal{M}$ is *Deligne-Mumford* if

- $\Delta_{\mathcal{M}}$ is representable, quasi-compact, and separated (these notions make sense because it is representable ... you have to show that this Isom guy is quasi-compact and separated, so this is where Grothendieck is the surprise witness).
- There is an étale cover by a scheme. ◇

Now you just need to read Grothendieck to learn about Isom and check that it is quasi-compact and separated.

Why is this a good definition? Some of it is clear, like having an étale cover by a scheme so that you can get some things to work. The other conditions are more technical and you only appreciate them when you need to prove anything.

The thing that is a little surprising is that we're now in the étale topology. Why was it forces upon us? Roughly, a DM stack is "étale locally a scheme" . . . something which is Zariski locally a scheme is just a scheme, so we needed a finer topology to get anything interesting.

Ok, now forget everything I just said and let the ideas be motivated in the lectures.

3 Lieblich

Yesterday we showed that a scheme is an fppf sheaf. Today we'll say some more things about Grothendieck topologies. This will not be in the direction of saying which sheaves should be admissible spaces, but about the topologies themselves.

We're going to talk about descent theory, which is a fancy way to say "gluing". In the Zariski topology, we know that we can glue things together when they're defined on an open cover. What about in other topologies?

Let X be a scheme, and let $\{U_i \to X\}$ be a covering. Let $\mathcal{F}_i$ be a quasi-coherent sheaf on U_i. Recall that to glue these to a sheaf on X, we need isomorphisms $\phi_{ij} : \mathcal{F}_j|_{U_i \cap U_j} \xrightarrow{\sim} \mathcal{F}_i|_{U_i \cap U_j}$ such that $\phi_{ij}\phi_{jk} = \phi_{ik}$.

Definition 3.1. Suppose $f \colon X' \to X$ is faithfully flat and quasi-compact (fpqc). Let $X'' = X' \times_X X'$, with projections $p_1, p_2 \colon X'' \to X'$. Let $p_{ij} : X' \times_X X' \times_X X' \to X' \times_X X'$ be the projections. Say $\mathcal{F}'$ is a quasi-coherent sheaf on X'. A *descent datum* is an isomorphism $\phi : p_1^*\mathcal{F}' \xrightarrow{\sim} p_2^*\mathcal{F}'$ such that $p_{23}^*\phi \circ p_{12}^*\phi = p_{13}^*\phi$. That is, the following hexagon commutes.

$$\begin{tikzcd}
 & p_{13}^*p_2^*\mathcal{F}' \arrow[dl, equal] & p_{13}^*p_1^*\mathcal{F}' \arrow[l, "p_{13}^*\phi"'] \arrow[dr, equal] & \\
p_{23}^*p_2^*\mathcal{F}' & & & p_{12}^*p_1^*\mathcal{F}' \arrow[dl, "p_{12}^*\phi"] \\
 & p_{23}^*p_1^*\mathcal{F}' \arrow[ul, "p_{23}^*\phi"] \arrow[r, equal] & p_{12}^*p_2^*\mathcal{F}' &
\end{tikzcd}$$

[[★★★ fix so that the arrows to to the right, please]] ◇

If the sheaf $\mathcal{F}'$ were a pullback of a sheaf $\mathcal{F}$ on X, then for example the fibers of the sheaf $\mathcal{F}'$ should agree with the fibers of $\mathcal{F}$. But there could be many points over the same point, so we should have an identification of different fibers of $\mathcal{F}'$ sitting over the same point downstairs.

Definition 3.2 (Alternative)**.** A descent datum on $\mathcal{F}'$ consists of an isomorphism $\phi_{t_1,t_2} : t_1^*\mathcal{F}' \xrightarrow{\sim} t_2^*\mathcal{F}'$ for all $t_1, t_2 \in X'(T)$ for any fixed $T \in \mathbf{Sch}_X$ such that $\phi_{t_2,t_3}\phi_{t_1,t_2} = \phi_{t_1,t_3}$. ◇

Once consequence of this, by the way, is that $\phi_{t,t} = \mathrm{id}$ (because $\phi_{t,t}\phi_{t,t} = \phi_{t,t}$).

Note that if $\mathcal{F}' = f^*\mathcal{F}$, then there is a natural descent datum. The $\phi_{t_1,t_2} : t_1^*f^* \xrightarrow{\sim} t_2^*f^*\mathcal{F}$ are the natural isomorphisms $t_1^*f^* \cong (ft_1)^* = (ft_2)^* \cong t_2^*f^*$. One can check that this is actually a descent datum. We will call this descent datum ($f^*\mathcal{F}$,can).

You can put all these descent data together into a category.

Definition 3.3. The *category of descent data for* f, $\mathcal{D}_f$, is the category of pairs $(\mathcal{F}', \phi)$ where $\mathcal{F}'$ is a quasi-coherent sheaf on X' and ϕ is a descent datum. The morphisms are maps $\psi : \mathcal{F}_1' \to \mathcal{F}_2'$ such that the following diagram commutes.

$$\begin{tikzcd}
p_1^*\mathcal{F}_1' \arrow[r, "\phi_1"] \arrow[d, "p_1^*\psi"'] & p_2^*\mathcal{F}_1' \arrow[d, "p_2^*\psi"] \\
p_1^*\mathcal{F}_2' \arrow[r, "\phi_2"] & p_2^*\mathcal{F}_2'
\end{tikzcd}$$

◇

The paragraph immediately before this definition tells us that we have a functor $\tilde{f}^* : \mathbf{Qcoh}(X) \to \mathcal{D}_f$ given by $\mathcal{F} \mapsto (f^*\mathcal{F},\text{can})$.

Definition 3.4. f is a *descent morphism* if $\tilde{f}^*$ is fully faithful. f is an *effective descent morphism* if $\tilde{f}^*$ is an equivalence. ◇

Roughly, f is a descent morphism if whenever sheaves glue, so do morphisms. It is effective if sheaves glue.

Theorem 3.5 (Grothendieck)**.** *If $f : X' \to X$ is fpqc, then f is an effective descent morphism for quasi-coherent sheaves.*

There is a more general result which says that for arbitrary f, if f has a section, then it is an effective descent morphism.

Theorem 3.6 (Girand/Grothendieck)**.** *If f has a section, then it is an effective descent morphism (for quasi-coherent sheaves, or for anything else).*

Proof. Let $\sigma\colon X \to X'$ be the section of $f\colon X' \to X$. We should show that $\tilde{f}^*$ is fully faithful and essentially surjective (i.e. every object of $\mathcal{D}_f$ is isomorphic to $(f^*\mathcal{F},\text{can})$ for some $\mathcal{F}$).

(faithful) $\tilde{f}^*$ is clearly faithful because $\sigma^*f^* = \text{id}$ (i.e. $\tilde{f}^*$ has a left inverse).

(full) A map $(\mathcal{F},\phi) \xrightarrow{\psi} (\mathcal{F}',\phi')$ is equivalent to picking, for each $t \in X'(T)$ (this is a point *over* X), an isomorphism $t^*\mathcal{F} \xrightarrow{\phi_t} t^*\mathcal{F}'$ such that the diagram square commutes. We have a point $\sigma_T \in X'(T)$ given by $T \to X \xrightarrow{\sigma} X'$. Then we get that the diagram

$$\begin{tikzcd}
\sigma^*\mathcal{F} \arrow[r, "\psi_\sigma=\sigma^*\psi"] \arrow[d, "\phi_{\sigma,t}"] & \sigma^*\mathcal{F}' \arrow[d, "\phi'_{\sigma,t}"] \\
t^*\mathcal{F} \arrow[r, "\phi_t"] & t^*\mathcal{F}'
\end{tikzcd}$$

commutes. We can use this to “propagate” ψ_σ to determine what all the ψ_t must be for each t.

(essentially surjective) Fix $(\mathcal{F},\phi) \in \mathcal{D}_f$. Say $t \in X'(T)$ for some $T \in \mathbf{Sch}_X$. The hope is that $(\mathcal{F},\phi) \cong \tilde{f}^*(\sigma^*\mathcal{F}) = (f^*\sigma^*\mathcal{F},\text{can})$. We have a map $\phi_{t,\sigma ft} : t^*\mathcal{F} \to t^*f^*\sigma^*\mathcal{F}$. We need to check that these maps respects the descent data. Given $t_1, t_2 \in X'(T)$, the diagram

$$\begin{tikzcd}
t_1^*\mathcal{F} \arrow[r, "\phi_{t_1,\sigma ft_1}"] \arrow[d, "\phi_{t_1,t_2}"'] & t_1^*f^*\sigma^*\mathcal{F} \arrow[d, "\phi_{\sigma ft_1,\sigma ft_2}=\text{can}"] \\
t_2^*\mathcal{F} \arrow[r, "\phi_{t_2,\sigma ft_2}"] & t_2^*f^*\sigma^*\mathcal{F}
\end{tikzcd}$$

commutes (because it comes from the glueing data for $\mathcal{F}$). The =can comes from the fact that $ft_1 = ft_2$. $\square$

Note that nothing had to do with quasi-coherent sheaves.

Now let's talk about the other theorem (that fpqc morphisms are effective descent morphisms).

Proof of 3.5. Consider the case when $X' = \operatorname{Spec} B$ and $X = \operatorname{Spec} A$, so $A \to B$ is a faithfully flat ring extension. Checking that $\tilde{f}^*$ is fully faithful is equivalent to checking that for any pair of A-modules M and N, the following sequence is exact.

$$\begin{tikzcd}
\mathrm{Hom}_A(M,N) \arrow[r] & \mathrm{Hom}_B(M \otimes_A B, N \otimes_A B) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, equal] & \mathrm{Hom}(B \otimes_A B)(M \otimes_A B \otimes_A B, N \otimes_A B \otimes_A B) \arrow[d, equal] \\
& \mathrm{Hom}_A(M, N \otimes_A B) & \mathrm{Hom}_A(M, N \otimes_A B \otimes_A B)
\end{tikzcd}$$

To see this, note that we have M on X, $M \otimes_A B$, and $M \otimes_A B \otimes_A B$ on X''. Note that we have two morphisms $B \to B \otimes_A B$ given by $b \mapsto 1 \otimes b$ or $b \mapsto b \otimes 1$, and this gives us an isomorphism $M \otimes_A B \otimes_A B \cong M \otimes_A B \otimes_B (B \otimes_A B) \cong M \otimes_A B \otimes_A B$. (the vertical equalities are just the universal property of tensor product).

To show that, it is enough to show that $N \to N \otimes_A B \rightrightarrows N \otimes_A B \otimes_A B$ is exact. To do this, reduce to the case when there is a section $B \to A$ (as we did yesterday). Then follow your nose. This is the magic of faithful flatness: it allows you to reduce to the case when you have a section.

Now we must show that $\tilde{f}^*$ is essentially surjective. Say that we have $(\mathcal{F}, \phi)$, where $\mathcal{F}$ is a module M, and $\phi : B \otimes_A M \xrightarrow{\sim} M \otimes_A B$ as $B \otimes_A B$-modules. Let's guess what $\mathcal{G}$ on X should be such that $\tilde{f}^*\mathcal{G} \cong (\mathcal{F},\text{can})$. It should be $N = \{m \in M | m \otimes 1 = \phi(1 \otimes m)\}$.

Observation: there is a map $\nu : N \otimes_A B \to M$ which is compatible with the descent data. We want to show that this is an isomorphism. It is enough to do so after making a faithfully flat base change, so we can reduce to the case where there is a section $B \to A$, in which case we know that descent is effective. The last thing you have to check is that when there is a section, the effectivity of descent is realized by this ν. To do that, just look at the proof that descent is effective when there is a section.

In the reduction to the affine case, you have to use full faithfullness (already shown). $\square$

3 Olsson

Obstruction theories. Let's start by summarizing yesterday's example. Start with $\pi : A' \to A$ a surjection of rings with $I = \ker \pi$ a square zero ideal (so you can think of it as an A-module). Also fix $g : X \to \operatorname{Spec} A$ a smooth separated A-scheme (separated is just to clairify the exposition). The problem is to understand liftings X' of X to $\operatorname{Spec} A'$.

$$\begin{tikzcd}
X \arrow[r, hook] \arrow[d, "g"'] & X' \arrow[d, "g'"] \\
\operatorname{Spec} A \arrow[r, hook] & \operatorname{Spec} A'
\end{tikzcd}$$

cartesian with g' smooth. Yesterday, we defined $\mathrm{Def}_X : \mathbf{Alg}/A \to \mathbf{Set}$ by sending $(f : C \to A)$ to the set of isomorphism classes smooth extensions X_C of X to $\operatorname{Spec} C$. Yesterday we showed that $T_{\mathrm{Def}_X} = H^1(X, T_{X/A})$.

When X is affine, we saw that

1. there is a lifting $X' \to \operatorname{Spec} A'$. As Brian pointed out yesterday, if A' is the dual numbers $A[I]$, then there is a section and you can base change. In general, it is a property of smoothness (which I think was in the exercises) that there exists a lifting (e.g. $A = \mathbb{F}_p$ and $A' = \mathbb{Z}/p^2$ has no section).
2. any two liftings are isomorphic, and
3. the group of automorphisms of any lifting $X' \to \operatorname{Spec} A'$ is canonically isomorphic to $H^0(X, T_X \otimes I)$. This is really the universal property of the cotangent bundle.

For general X, if you have one lifting $X' \to \operatorname{Spec} A'$, we get a bijection $\mathrm{Def}_X(A' \to A) \to H^1(X, T_X \otimes I)$. Yesterday, the fixed lifting was $X[I] \to \operatorname{Spec} A[I]$. In general, we don't have a canonical way to choose a lifting (in fact, there may not be one. But if you pick a lifting, then the argument works.

Definition of $\phi_{X'}$. Cover X by affines, so $X = \bigcup U_i$. Then let $X'' \in \mathrm{Def}_X(A')$. For each i, you have the restriction of X'' to U_i (remember that X'' and X have the same topological space, we're just restricting the structure sheaf) and the restriction of X' to X. Then there is some isomorphism $\sigma_i : U_i'' \to U_i'$. For every i, j, we get automorphisms $U_{ij}' \xrightarrow{\sigma_j^{-1}} U_{ij}'' \xrightarrow{\sigma_i} U_{ij}'$ which reduce to the identity on U_{ij}. By property 3, we see that this is given by some $x_{ij} \in H^0(U_{ij}, T_X \otimes I)$, which we verified was a Čech cocycle.

For people who know about torsors, there is a cleaner way to say it. Given X' and X'', we get a sheaf $\underline{\mathrm{Isom}}(X', X'')$ on $|X|$ given by $U \mapsto \{h : U'' \xrightarrow{\sim} U'\}$. Property 3 (and 1) says that this $\underline{\mathrm{Isom}}(X', X'')$ is a torsor under $T_X \otimes I$. There is a general fact (which you probably know for line bundles) that torsors are always classified by $H^1(X, T_X \otimes I)$ (or replace $T_X \otimes I$ by whatever the sheaf of abelian groups is).

For a general X, when is there an X' lifting it? Well, we know it exists locally, so can we find compatible patching data? Let $\{U_i\}$ be a covering of X by affines. Fix liftings U_i' of U to $\operatorname{Spec} A'$. We want to choose isomorphisms on the overlaps. For every i and j, choose $\phi_{ji} : U_i'|_{U_{ij}} \xrightarrow{\sim} U_j'|_{U_{ij}}$. If these are to patch, they should satisfy the cocycle condition $\phi_{ki} = \phi_{kj}\phi_{ji}$ on the triple intersection U_{ijk}. To measure if this commutes, define $\partial_{ijk} = \phi_{ki}^{-1}\phi_{kj}\phi_{ji} \in H^0(U_{ijk}, T_X \otimes I)$.

Lemma 3.1. *(i)* $\{\partial_{ijk}\}$ *is a Čech 2-cocycle. (ii) if* ϕ_{ji}' *is another choice of isomorphisms, giving some other* $\{\partial_{ijk}'\}$*, then the difference* $\{\partial_{ijk}^{-1}\partial_{ijk}'\}$ *is a Čech coboundary. That is, given our* $g : X \to \operatorname{Spec} A$*, there is some element* $o(g) \in H^2(X, T_X \otimes I)$*.*

Proof. draw a crazy diagram[[★★★]]. ☐

Theorem 3.2. *There exists a lifting* $X' \xrightarrow{g'} \operatorname{Spec} A'$ *of* g *if and only if* $o(g) = 0$ *in* $H^2(X, T_X \otimes I)$*.*

Summary.

(a) there is a canonical obstruction $o(g) \in H^2(X, T_X \otimes I)$ such that $o(g) = 0$ if and only if $\operatorname{Def}_X(A' \to A) \neq \varnothing$.

(b) If $o(g) = 0$, then the set of isomorphisms classes of liftings form a torsor under $H^1(X, T_X \otimes I)$.

(c) for any lifting of g, the group of automorphisms is canonically isomorphic to $H^0(X, T_X \otimes I)$.

Let's formalize the notion of an obstruction theory. Let Λ be a commutative ring. Consider a functor $F : \Lambda\textbf{-alg} \to \textbf{Set}$.

Definition 3.3. An *obstruction theory for F* consists of the following data.

1. For every morphism $A \to A_0$ of Λ-algebras with kernel a nilpotent ideal, with A_0 reduced, and $a \in F(A)$, a functor $O_a : A_0\text{-}\textbf{mod}_{ft} =$ (finite type A_0-modules) $\to A_0\text{-}\textbf{mod}_{ft}$.
2. For every diagram $A' \to A \to A_0$ of Λ-algebras and $a \in F(A)$, where $A' \to A$ and $A \to A_0$ surjective with $\ker(A' \to A) = J$ annihilated by $\ker(A' \to A_0)$, a class $o_a \in O_a(J)$ which is zero if and only if a lifts to $F(A')$.

Moreover, this should be functorial in the natural way. [[★★★ fill it in]] ◇

The setup "For every diagram $A' \to A \to A_0$ of Λ-algebras and $a \in F(A)$, where $A' \to A$ and $A \to A_0$ surjective with $\ker(A' \to A) = J$ annihilated by $\ker(A' \to A_0)$" is called a *deformation situation.*

Example. Let $j \colon X \hookrightarrow X'$ is a closed immersion defined by a square zero ideal J. Let L be a line bundle on X. Problem: understand liftings of L to a line bundle L' on X'.

There are two approaches. One is to do some patching (L is locally trivial, and you can lift the trivial guy, then worry about gluing). What do we mean by "lifting of L

to X'"? We mean a pair (L', ι), where L' is a line bundle on X' and $\iota : j^*L' \xrightarrow{\sim} L$ is an isomorphism of line bundles on X. We say that $(L', \iota) \cong (L'', \varepsilon)$ if there is a morphism of line bundles $\sigma : L' \xrightarrow{\sim} L''$ such that $\varepsilon \circ j^*\sigma = \iota$.

We have the following exact sequence of sheaves on $|X| = |X'|$

$$0 \to J \xrightarrow{g \mapsto 1+g} \mathcal{O}_{X'}^{\times} \to \mathcal{O}_X^{\times} \to 0$$

Note that the first map is additive: $(1+g)(1+f) = 1+f+g+gf = 1+f+g$ (since $J^2 = 0$). So you get a long exact sequence

$$\begin{tikzcd}
0 \arrow[r] & H^0(J) \arrow[r] & H^0(\mathcal{O}_{X'}^{\times}) \arrow[r] & H^0(\mathcal{O}_X^{\times}) \arrow[dll, rounded corners, to path={ -- ([xshift=2ex]\tikztostart.east) |- ([yshift=-1ex]\tikztostart.south) -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] \\
& H^1(J) \arrow[r] & Pic(X') \arrow[r] & Pic(X) \arrow[dll, "\partial", rounded corners, to path={ -- ([xshift=2ex]\tikztostart.east) |- ([yshift=-1ex]\tikztostart.south) -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] \\
& H^2(J) & &
\end{tikzcd}$$

Proposition 3.4. *Assume $H^0(X', \mathcal{O}_{X'}^{\times}) \to H^0(X, \mathcal{O}_X^{\times})$ is surjective.*

(a) There is an obstruction $\partial[L] = o(L) \in H^2(J)$ which is zero if and only if there exists an (L', ι).

(b) If $o(L) = 0$, then the set of isomorphisms classes of liftings (L', ι) is a torsor under $H^1(X, J)$.

(c) For any lifting, the group of automorphisms is canonically in bijection with $H^0(X, J)$.

This is where it is important that we took pairs (L, ι)

3 Osserman

Remark 3.1.

- Fiber products of rings may seem strange. We'll come back to it later.
- It happens to be the case that (H1) and (H2) are basically always satisfied if you write your predeformation functor "correctly" (i.e. somehow naturally).
- (H3) tends to be related to some kind of properness hypothesis.
- (H4) is related to presence of automorphisms. ◇

Definition 3.2. A predeformation functor F is a *deformation functor* if it satisfies (H1) and (H2). ◇

Note that (H3) doesn't make sense for an arbitrary predeformation functor, but it turns out that if (H2) is satisfied, then tangent space makes sense (it is true that (H2) implies the similar thing with I, but you could just use the construction from Martin's lecture, for which you only need the case of ε). Thus, (H3) makes sense for any deformation functor.

$\underline{\mathrm{Def}_X}$. Recall the following definition.

Definition 3.3. Given a pair $(X_A, \phi) \in \mathrm{Def}_X(A)$, an automorphism of the pair (X_A, ϕ) (or an *infinitesimal automorphism of* X) is an automorphism of X_A over A commuting with ϕ. ◇

Theorem 3.4. *Let X be a scheme over k, and let Def_X be the functor of deformations of X. Then*

1. Def_X *is a deformation functor.*
2. *If X is proper,* Def_X *satisfies (H3).*
3. Def_X *satisfies (H4) if and only if for all small thickennings $A' \to A$ and pairs $(X_{A'}, \phi)$ over A',* $\mathrm{Aut}(X_{A'}, \phi) \to \mathrm{Aut}(X_{A'}|_A, \phi|_A)$ *is surjective (i.e. every automorphism of the restriction extends). In particular, if $H^0\big(X, \mathcal{H}om(\Omega^1_{X/k}, \mathcal{O}_X)\big) = 0$, (H4) is satisfied.*

Corollary 3.5. *If X is proper, then* Def_X *has a hull. Furthermore, if $H^0\big(X, \mathcal{H}om(\Omega^1_{X/k})\big) = 0$, then* Def_X *is prorepresentable.*

Example 3.6. If X is a smooth proper curve, then it has a hull. We have that $H^0(X, T_X) = 0$ if the genus is at least 2. ◇

Lemma 3.7. *Consider the diagram of compatible ring and module morphisms*

$$\begin{tikzcd}
N \arrow[rr, "p''"] \arrow[dr, "p'"] \arrow[dd, dash] & & M'' \arrow[dr, "u''"] \arrow[dd, dash] & \\
& M' \arrow[rr, "u'"'] \arrow[dd, dash] & & M \arrow[dd, dash] \\
B \arrow[rr] \arrow[dr] & & A'' \arrow[dr] & \\
& A' \arrow[rr] & & A
\end{tikzcd}$$

with $B = A' \times_A A''$*,* $N = M' \times_M M''$*,* M *and* M'' *flat over* A' *and* A'' *(respectively), and (i)* $A'' \twoheadrightarrow A$ *with nilptent kernel, and (ii)* u' *induces* $M' \otimes_{A'} A \xrightarrow{\sim} M$ *and similarly for* u''*.*

Then N *is flat over* B *and* p' *induces* $N \otimes_B A' \xrightarrow{\sim} M$ *and similarly for* p''*. Also, in the same situation, if* L *is a* B*-module and we have maps* $q' : L \to M'$ *and* $q'' : L \to M''$ *such that* q' *induces* $L \otimes_B A \xrightarrow{\sim} M'$*, then* $q' \times q'' : L \to N$ *is an isomorphism.*

This tells you that when you have flat M' and M'' which agree over A, then you can get a flat thing over B by just taking their fiber product.

Note that this is more general than what is necessary for Schlessinger (we are not assuming that these are artin rings, they are arbitrary). You'll see today that modules are flat over artin local rings exactly when they are free; in this context, the lemma is quite easy. We're going to do this general lemma because it will be useful later.

The argument for this lemma uses the local criterion for flatness.

Exercise 3.1. Prove the lemma. ◀

Proposition 3.8. *Say* $A' \to A$ *and* $A'' \to A$ *(arbitrary rings), with* $A'' \to A$ *is surjective with nilpotent kernel and* $B = A' \times_A A''$*. Then*

1. *Given* X' *and* X'' *flat over* A' *and* A'' *(respectively) and an isomorphism* $\phi : X'|_A \xrightarrow{\sim} X''|_A$*, there exists some* Y*, flat over* B*, with maps* $\phi' \colon X' \to Y$ *and* $\phi'' \colon X'' \to Y$ *inducing isomorphisms* $X' \to Y|_{A'}$ *and* $X'' \to Y|_{A''}$ *and so that* $\phi = \phi''|_A \circ \phi'^{-1}|_A$*.*

$$\begin{tikzcd}
X' \arrow[r, "\phi'"] \arrow[d] & X' \arrow[d] \\
\operatorname{Spec} B \arrow[r] & \operatorname{Spec} A'
\end{tikzcd}$$

2. *Given* Y_1 *and* Y_2 *flat over* B*, the natural map*

$$\underline{\mathrm{Isom}}_B(Y_1, Y_2) \to \underline{\mathrm{Isom}}_{A'}(Y_1|_{A'}, Y_2|_{A'}) \times_{\underline{\mathrm{Isom}}_A(Y_1|_A, Y_2|_A)} \underline{\mathrm{Isom}}_{A''}(Y_1|_{A''}, Y_2|_{A''})$$

is a bijection.

Proof. (1) We'll construct Y on the same topological space as X'. We identify the topological spaces of X'' and $X''|_A$. We also identify these with the topological space

$X'|_A$ using ϕ. Write $i : X'|_A \to X'$ (this is the one which isn't a homeomorphism). Set $\mathcal{O}_Y(U) = \mathcal{O}_{X'}(U) \times_{\mathcal{O}_{X'|_A}(i^{-1}(U)} \mathcal{O}_{X''}(i^{-1}(U))$, so $\mathcal{O}_Y = \mathcal{O}_{X'} \times_{i_*\mathcal{O}_{X'|_A}} i_*\mathcal{O}_{X''}$.

The lemma tells us that "$\mathcal{O}_Y$" is flat over B and that it recovers $\mathcal{O}_{X'}$ and $\mathcal{O}_{X''}$ upon restriction to A' and A'' (respectively). We also check that this $\mathcal{O}_Y$ is in fact a sheaf and defines a scheme structure (which boils down to showing that the fiber product of modules commutes with localization).

(2) Is similar using the second part of the lemma. □

Sketch proof of Theorem 3.4. (1) It is just a diagram chase to show that (H1) follows from the first part of the proposition. (H2) uses the second part the second part of the proposition, and the fact that $A = k$, so the ϕ in the definition of Def_X rigidify the isomorphisms.

(3) Is similar. The only problem you could run into is that you could have some gluing automorphism over A which doesn't extend.

(2) is true for smooth proper X from Martin's lecture. Martin will prove the general statement later. □

4 Lieblich

Last couple of lectures were to convince you that Grothendieck topologies are nice and are kind of like regular topologies. Today, we'll return to our moduli problems and see how they fit in to the framework we have. Recall the numbering from the first day.

(3) $\mathrm{Hom}(X, Y)$

(4) closed subschemes of X

(5) subspaces of V

Recall that we made up some functors of points for these moduli problems. Are these functors sheaves?

We had that $h_{\mathcal{M}_3}(T) = \mathrm{Hom}_T(X_T, Y_T) = \mathrm{Hom}(X \times T, Y)$. Is this a sheaf in the étale or fppf topolgies? We proved that schemes are sheaves, so Y is a sheaf, so $h_{\mathcal{M}_3}$ is an fppf sheaf.

We had that $h_{\mathcal{M}_4}(T) = \{Z \hookrightarrow X \times T \text{ closed} \,|Z \text{ is } T\text{-flat}\}/\cong$. Giving Z is equivalent to giving $\mathcal{I}_Z \subseteq \mathcal{O}_{X\times T}$, so isomorphisms are unique when they exist. So the sheaf condition is translated into descent data on $\mathcal{I}_Z \subseteq \mathcal{O}_{X\times T}$ (that is, you don't have to worry about the cocycle condition because of uniqueness of isomorphisms). We showed that descent works for quasicoherent sheaves, so we can glue. That is, $h_{\mathcal{M}_4}$ is an fppf sheaf.

If we have a covering $\{T_i \to T\}$ and we want to look at

$$h_{\mathcal{M}_4}(T) \to \prod h_{\mathcal{M}_4}(T_i) \rightrightarrows \prod h_{\mathcal{M}_4}(T_i \times_T T_j)$$

First we descend $\mathcal{I}_Z$, and remembering that descent gives an equivalence of categories, we can descend the map as well. This is a sequence of isomorphism classes, but the fact that isomorphisms are unique tells us that it is harmless to choose representatives.

We have to show that the descended thing is flat. We know that $\mathcal{I}_Z$ is T_i-flat for all i, so we can conclude that $\mathcal{I}_Z$ is T-flat. You can use the following lemma. You'll see how to descend some properties in the exercises.

Lemma 4.1. *Assume $f\colon X' \to X$ is faithfully flat. A quasi-coherent sheaf $\mathcal{F}$ on X is X-flat (resp. finitely presented, etc.) if and only if $f^*\mathcal{F}$ is.*

We had that $h_{\mathcal{M}_5}(T) = \{W \subseteq \mathcal{O}_T \otimes V | \text{cokernel is locally free}\}/\cong$. Again, isomorphisms are unique and things are quasi-coherent sheaves, so you get the same kind of descent argument.

So we are basically using the same tricks over and over again: schemes are sheaves, and descent for quasi-coherent sheaves.

Martin: you could have defined $h_{\mathcal{M}_4}$ without requiring Z flat, and the same argument would show that it would be a sheaf, so why do we ask for flatness? Max: We'll see that being a sheaf is not quite enough to get some geometry.

The other examples were

(0) varieties

(1) curves of genus g

(2) line bundles on X.

We had that $h_{\mathcal{M}_2}(T) = \{\mathcal{L} \text{ on } X \times T\}/\cong = Pic(X \times T)$. The sheaf condition is that for a cover $\{T_i \to T\}$ the sequence

$$Pic(T) \to \prod Pic(T_i) \rightrightarrows \prod Pic(T_i \times_T T_j)$$

should be exact. Well, it isn't. The first map is not injective, and the sequence is not exact in the middle. The sequence is as un-exact as possible. Exactness on the left is: if two line bundles are locally isomorphic, are they isomorphic. Exactness in the middle is: if you have a line bundle defined locally, with isomorphisms on overlaps, can you glue them to get a global line bundle?

Example 4.2. Exactness always fails on the left. We have elements of $Pic(X \times T)$ which are not meant to be there. Choose T such that $Pic(T) \neq 0$ (exercise: you can always find such a T, over any base). Let $\mathcal{M}$ be a non-trivial invertible sheaf on T. This gives rise to $p_2^*\mathcal{M} \in Pic(X \times T)$. $\mathcal{M}$ is not trivial, but it is locally trivial in the Zariski topology. That is, we can choose an open covering $\{T_i \subseteq T\}$ so that $\mathcal{M}|_{T_i}$ is trivial. Then we find that $p_2^*\mathcal{M}$ and $\mathcal{O}_{X\times T}$ both map to the trivial element of $\prod Pic(X \times T_i)$. Exercise: check that $p_2^*\mathcal{M}$ and $\mathcal{O}_{X\times T}$ are not isomorphic. ◇

Example 4.3. In general, exactness fails in the middle. Let X be the curve over $\mathbb{R}$ given by $x^2+y^2+z^2 = 0$ in $\mathbb{P}^2_{\mathbb{R}}$. We know that $X \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{P}^1_{\mathbb{C}}$, but $X \not\cong \mathbb{P}^1_{\mathbb{R}}$ (because there are no points). So we see (by Riemann-Roch, if you like) that there are no divisors of degree 1 on X. But if we try to do descent for the covering $\operatorname{Spec}\mathbb{C} \to \operatorname{Spec}\mathbb{R}$, then this moduli problem *thinks* there should be a divisor of degree 1. We get the sequence

$$\begin{tikzcd}
Pic(X) \arrow[r] \arrow[d, equal] & Pic(X \otimes \mathbb{C}) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, equal] & Pic(X \otimes \mathbb{C} \otimes \mathbb{C}) \\
\mathbb{Z} \arrow[r, "2"] & \mathbb{Z} \arrow[r, shift left] \arrow[r, shift right] & \mathbb{Z} \times \mathbb{Z} \\
 & 1 \arrow[r, mapsto, shift left] \arrow[r, mapsto, shift right] & (1,1)
\end{tikzcd}$$

So we see that this is not exact in the middle. ◇

Something about how far the Brauer group is from being a sheaf in the Zariski topology.

The problem here was that isomorphisms are not unique. We had lots of choices for the isomorphisms $\mathcal{M}|_{T_i} \cong \mathcal{O}_{T_i}$, and this was the source of our problems.

Descent fails because we can have local line bundles $\mathcal{L}$ on $X \times T'$, with $p_1^*\mathcal{L} \xrightarrow{\sim} p_2^*\mathcal{L}$, but the cocycle condition fails (because we don't have the uniqueness to tell us that the cocycle condition is trivial).

Fix our problem. The source of this confusion is that we're working with sets. We shoe-horned ourselves into a cage (or a shoe, or whatever), when things naturally come to us as categories. Equality is too much to hope for. Let's think about categories instead of sets. We don't really want the whole category of invertible sheaves, because there are lots of weird maps. We just need to remember the isomorphisms to deal with our problem.

Definition 4.4. A *groupoid* is a category where every arrow is an isomoprhism. ◇

Starting with any category, we can forget all arrow which aren't isomorphisms, and we get a kind of "skeleton" category (this isn't the actual skeleton, which is where you take a full subcategory with one object in each isomorphism class).

Definition 4.5. A groupoid $\mathcal{C}$ is *discrete* if for every $x \in \mathcal{C}$, $\mathrm{Aut}(x) = \{\mathrm{id}_x\}$ (this is equivalent to saying that $|\mathrm{Hom}(x, y)| \leq 1$). A groupoid is *connected* if any two objects are isomorphic. ◇

A discrete groupoid can have lots of objects, but not many morphisms between objects. A group G can be thought of a groupoid with one object x with $\mathrm{Aut}(x) = G$. This is a connected groupoid with lots of automorphisms.

We have a functor $\chi : \mathbf{Set} \to \mathbf{Gpoid}$ (the arrows in **Gpoid** are functors) given by $S \mapsto$the category with one object for each element of S and no non-identity morphisms.

Lemma 4.6. *The essential image of χ is the subcategory of discrete groupoids. That is, every discrete groupoid is equivalent (as a groupoid) to the image of a set.*[1]

The nice sheaves we got from some of our moduli problems were because we got discrete groupoids.

More good things. $\mathcal{M}_2(T)$ is a groupoid, the groupoid of line bundles $\mathcal{L}$ on $X \times T$. But there is more. If you have a morphism $f\colon S \to T$, then we get a functor $\mathcal{M}_2(T) \xrightarrow{(\mathrm{id} \times f)^*} \mathcal{M}_2(X)$ given by $\mathcal{L} \mapsto (\mathrm{id}_X \times f)^*\mathcal{L}$.

Guess: $\mathcal{M}_2$ gives us a functor $\mathbf{Sch}^\circ \to \mathbf{Gpoid}$. However, there is trouble. If you have $T'' \xrightarrow{g} T' \xrightarrow{f} T$, then you have f^* and g^* and a natural isomorphism $g^*f^* \xrightarrow{\sim} (fg)^*$ which comes from the universal property of pullback (pullbacks are unique up to unique isomorphism), but it is not an equality. If we have $T''' \xrightarrow{h} T'' \xrightarrow{g} T' \xrightarrow{f} T$, then we get a big diagram of natural isomorphisms commutes.

$$\begin{tikzcd}
h^*g^*f^* \arrow[r] \arrow[d] & h^*(fg)^* \arrow[d] \\
(gh)^*f^* \arrow[r] & (fgh)^*
\end{tikzcd}$$

So $\mathcal{M}_2$ is almost as good as a functor to groupoids, but not quite, because $g^*f^* \neq (fg)^*$.

---

[1] You have to observe that the category of groupoids is a 2-category. That is, every discrete groupoid is not *isomorphic* to a set, it is only *equivalent* to a set (i.e. isomorphic up to 2-morphism).

Definition 4.7. A *fibered category (in groupoids) with cleavage* or a *pseudo-functor* over a category $\mathcal{C}$ is

1. for each $c \in \mathcal{C}$, a groupoid $F(c)$,
2. for each $f : c \to d$ in $\mathcal{C}$, a functor $f^* : F(d) \to F(c)$, and
3. for each pair $c \xrightarrow{f} d \xrightarrow{g} e$ an isomorphism $\nu_{f,g} : f^*g^* \to (gf)^*$.

such that the diagram above (using the ν's) commutes. ◇

4 Olsson

Last time we introduced the abstract notion of an obstruction theory, and gave an example. Today we'll see more examples.

Say $A' \to A$ is a surjective map of rings with square zero kernel J. Let $P' \to \operatorname{Spec} A'$ be a smooth scheme with reduction $P \to \operatorname{Spec} A$, and suppose $j : X \hookrightarrow P$ is a closed immersion over $\operatorname{Spec} A$ such that X is smooth over $\operatorname{Spec} A$.

$$\begin{tikzcd}
X \arrow[r, hook] \arrow[dr, "sm"'] & P \arrow[d] & & X' \arrow[r, hook] \arrow[dr, "sm"'] & P' \arrow[d] \\
& \operatorname{Spec} A & & & \operatorname{Spec} A'
\end{tikzcd}$$

Problem: understand how to lift the diagram on the left to the diagram on the right.

Think of P as projective space. So instead of just wanting to lift X to X', we're trying to lift it together with its embedding into projective space.

You could do this with Čech cocycles, but let's do it from a different point of view. We have the following diagram of sheaves of algebras on $|X|$. We want to fill in $\mathcal{O}_{X'}$ so that we get the kernel $J \otimes \mathcal{O}_X$

$$\begin{tikzcd}
J \otimes \mathcal{O}_X \arrow[r, dashed] & \mathcal{O}_{X'} \arrow[r, dashed, two heads] & \mathcal{O}_X \\
& j^{-1}\mathcal{O}_{P'} \arrow[r, two heads] \arrow[u, dashed] & j^{-1}\mathcal{O}_P \arrow[u, two heads] \\
J \arrow[r] & A' \arrow[r] \arrow[u] & A \arrow[u]
\end{tikzcd}$$

This is a local problem on the topological space $|X|$.

We define a sheaf $\mathcal{L}$ on $|X|$ which to any open $U \subseteq |X|$ associates the set of diagrams

$$\begin{tikzcd}
U \arrow[r, hook] \arrow[d, hook, "j"'] & U' \arrow[d, "j'"'] \arrow[dd, bend left, "sm"] \\
P \arrow[r, hook] \arrow[d] & P' \arrow[d] \\
\operatorname{Spec} A \arrow[r, hook] & \operatorname{Spec} A'
\end{tikzcd}$$

j is an immersion (need not be closed). Max argued that this is a sheaf in the flat topology. You can also see this more concretely because if there is some other U'', there is at most one arrow $U' \to U''$ filling in the diagram. We want to understand what the global sections of $\mathcal{L}$ are. A global section is a diagram like the one on the right at the top of the page.

If U is affine, we know that U' is unique up to isomorphism, so there is no data there, but there is data in the morphism j'. We want to know in what ways can we fill in the diagram

$$\begin{tikzcd}
U \arrow[r, hook] \arrow[d, hook, "j"'] & U' \arrow[d, dashed, "j'"'] \\
P \arrow[r, hook, "i"] \arrow[d] & P' \arrow[d] \\
\operatorname{Spec} A \arrow[r, hook] & \operatorname{Spec} A'
\end{tikzcd}
\qquad
\begin{tikzcd}
U \arrow[r, hook] \arrow[dr, "i\circ j"] \arrow[ddr, bend right] & U' \arrow[d, dashed] \arrow[dd, bend left] \\
& P' \arrow[d] \\
& \operatorname{Spec} A'
\end{tikzcd}$$

The answer is the universal property of differentials. The set of arrows filling in the diagram form a torsor under $\operatorname{Hom}\big((ij)^*\Omega^1_{P'/A'}, J \otimes \mathcal{O}_U\big)$. There is a filling because U is affine. We can write this as $\operatorname{Hom}(j^*\Omega^1_{P/A}, J \otimes \mathcal{O}_U) = j^*T_{P/A} \otimes_A J$. This tells us that there is an action of $j^*T_{P/A} \otimes J$ on $\mathcal{L}$. It is not a free action . . . some guys act trivially.

Define the conormal bundle $\check{\mathcal{N}}$ to be $j^*I = I/I^2$ where $I \subseteq \mathcal{O}_P$ is the ideal of X and $j\colon X \hookrightarrow P$. We have a map $I \xrightarrow{d} \Omega^1_{P/A}$, which gives us $I/I^2 = j^*I \to j^*\Omega^1_{P/A}$. In fact, we get an exact sequence (see Hartshorne)

$$0 \to I/I^2 \to j^*\Omega^1_{P/A} \to \Omega^1_{X/A} \to 0$$

which dualizes (everthing is locally free) to

$$0 \to T_{X/A} \to J^*T_{P/A} \to \mathcal{N} \to 0$$

which we can tensor with J to get

$$0 \to T_{X/A} \otimes J \to J^*T_{P/A} \otimes J \to \mathcal{N} \otimes J \to 0$$

(the sequence stays exact because it starts as a sequence of locally free sheaves)

Claim. *$T_{X/A} \otimes J$ acts trivially on $\mathcal{L}$.*

To see this, a section ∂ of $T_{X/A} \otimes J$ over U corresponds to a diagram

$$\begin{tikzcd}
& U' \arrow[d, "\partial"'] \arrow[ddd, bend left] \\
U \arrow[ur, hook] \arrow[r, hook] \arrow[d] & U' \arrow[d, "j'"] \\
P \arrow[r, hook] \arrow[d] & P' \arrow[d] \\
\operatorname{Spec} A \arrow[r, hook] & \operatorname{Spec} A'
\end{tikzcd}
\qquad
\begin{tikzcd}
& & U' \arrow[ddl, bend left] \\
U \arrow[urr, hook] \arrow[r, hook] \arrow[d] & U' \arrow[ur, dashed, "\partial"'] \arrow[d, "j'\circ\partial"'] & \\
P \arrow[r, hook] & P' &
\end{tikzcd}$$

here the two U''s (with diagram) are "equal", meaning that there is a filler arrow ∂.

So we get an action of $\mathcal{N} \otimes J$ on $\mathcal{L}$.

Proposition 4.1. *$\mathcal{L}$ is a torsor under $\mathcal{N} \otimes J$*

Torsor means that

1. for each $U \subseteq X$, there is a covering $U = \bigcup U_i$ such that $\mathcal{L}(U_i) \neq \varnothing$ for each i. (this is true because of the infinitesimal lifting property of smoothness)
2. for all $U \subseteq X$, either $\mathcal{L}(U) = \varnothing$, or the action of $(\mathcal{N} \otimes J)(U)$ on $\mathcal{L}(U)$ is simply transitive.

Sketch proof. Check that if U is affine the the action of $(\mathcal{N} \otimes J)(U)$ on $\mathcal{L}(U)$ is simply transitive. We've almost done it already. We have the sequence

$$0 \to (T_{X/A} \otimes J)(U) \to (J^*T_{P/A} \otimes J)(U) \to (\mathcal{N} \otimes J)(U) \to 0$$

(we keep exactness because U is affine, so sections are exact) $\mathcal{L}(U)$ is the set of ways to fill in the diagram.

$$\begin{tikzcd}
U \arrow[r, hook] \arrow[d, hook, "j"'] & U' \arrow[d, dashed, "j'"'] \\
P \arrow[r, hook, "i"] \arrow[d] & P' \arrow[d] \\
\operatorname{Spec} A \arrow[r, hook] & \operatorname{Spec} A'
\end{tikzcd}$$

We've seen that the ways to fill in the diagram is a torsor under the middle term and two things are the same (isomorphic) if they are related by the action of the left-most term. Thus, the quotient is a torsor under the right term. $\square$

Remember that we get uniqueness if there is an isomorphism because of

$$\begin{tikzcd}
 & & \mathcal{O}_{U''} \arrow[d, dashed, "unique"] & \\
J \otimes \mathcal{O}_U \arrow[r] & \mathcal{O}_{U'} \arrow[r] & \mathcal{O}_U \arrow[r] & 0 \\
j^{-1}(J \otimes \mathcal{O}_P) \arrow[r] \arrow[u, two heads] & j^{-1}\mathcal{O}_{P'} \arrow[r] \arrow[u, two heads] \arrow[uur, crossing over] & j^{-1}\mathcal{O}_P \arrow[r] \arrow[u, two heads] & 0
\end{tikzcd}$$

<u>General fact</u>: if G is a sheaf of abelian groups, then the set of isomorphism classes of G-torsors on $|X|$ are in (canonical) bijection with $H^1(X, G)$. You probably already know it in a special case: isomorphism classes line bundles are in bijection with $H^1(X, \mathcal{O}_X^\times)$. In our case (where $G = \mathcal{N} \otimes J$), choose a covering $X = \bigcup U_i$ with each U_i affine and an element $s_i \in \mathcal{L}(U_i)$ (we can do this because $\mathcal{L}$ is a torsor). On $U_i \cap U_j$, you get two sections, $s_i|_{U_{ij}}$ and $s_j|_{U_{ij}}$, of $\mathcal{L}(U_{ij})$. There is no reason for these to be equal, but we know that the action of $(\mathcal{N} \otimes J)$ is simply transitive on $\mathcal{L}(U_{ij})$, so there is a unique $x_{ij} \in (\mathcal{N} \otimes J)(U_{ij})$ such that $x_{ij} * s_i|_{U_{ij}} = s_j|_{U_{ij}}$. Now you check that $\{x_{ij}\}$ is a Čech 1-cocycle (this is very similar to what we did before). This gives us a class in $H^1(X, \mathcal{N} \otimes J)$. So $\mathcal{L}$ corresponds to this class $[\mathcal{L}] \in H^1(X, \mathcal{N} \otimes J)$.

$\mathcal{L}$ trivial would mean that $\mathcal{L}(X) \neq \varnothing$. This is the same as saying that $[\mathcal{L}] = 0 \in H^1(X, \mathcal{N} \otimes J)$.

Summary. We use the notation of the diagram

$$\begin{tikzcd}
X \arrow[r, hook, dashed] \arrow[d, hook, "j"'] & X' \arrow[d, dashed, "j'"'] \\
P \arrow[r, hook, "i"] \arrow[d] & P' \arrow[d] \\
\operatorname{Spec} A \arrow[r, hook] & \operatorname{Spec} A'
\end{tikzcd}$$

1. There is a canonical obstruction $o(j) \in H^1(X, \mathcal{N} \otimes J)$ whose vanishing is equivalent to existence of the lifting j' of j.

2. When $o(j) = 0$, the set of liftings j' of j form a torsor under $H^0(X, \mathcal{N} \otimes J)$. Ishai: That is, when $\mathcal{L}$ has a global section, since the action of $\mathcal{N} \otimes J$ is simply transitive, each section of $\mathcal{N} \otimes J$ gives a new section of $\mathcal{L}$. That is, this statement is actually already encoded in the first statement.

Ishai: Why should we get H^1 instead of H^2 or anything else? Martin: There is some complex governing stuff.

There are tons of things that you should be checking here. Here are some of these things.

Remark 4.2. We have the sequence

$$0 \to T_{X/A} \to j^*T_{P/A} \to \mathcal{N} \to 0$$

which induces a long exact sequence (of cohomologies over X)

$$\begin{tikzcd}
H^0(\mathcal{N} \otimes J) \arrow[r] & H^1(T_{X/A} \otimes J) \arrow[r] & H^1(j^*T_{P/A} \otimes J) \arrow[dll, rounded corners] \\
H^1(\mathcal{N} \otimes J) \arrow[r, "\delta"'] & H^2(T_{X/A} \otimes J) &
\end{tikzcd}$$

What is $\delta\bigl(o(j)\bigr)$? It is the obstruction $o(g)$, where $g\colon X \xrightarrow{j} P \to \operatorname{Spec} A$. ◇

Somehow what we need to understand questions like "what is a morphism of moduli problems" because that is clearly what we have here. We'll talk about this stuff next week.

Example 4.3. Say P is a smooth proper surfact over k and $X \subseteq P$ is a smooth rational curve with $X.X = -1$. The normal bundle is then a line bundle on X. You then find that $\deg \mathcal{N} = -1$ [Hartshorne, V.1.4.1]. This implies that $H^1(X, \mathcal{N} \otimes J) = 0$ and $H^0(X, \mathcal{N} \otimes J) = 0$.

$$\begin{tikzcd}
 & X[\varepsilon] \arrow[d, hook] & X \arrow[l, hook'] \arrow[d] \arrow[dd, bend left] \\
P \arrow[d] & P[\varepsilon] \arrow[l] \arrow[d] & P \arrow[l, hook'] \arrow[d] \\
\operatorname{Spec} k & \operatorname{Spec} k[\varepsilon] \arrow[l] & \operatorname{Spec} k \arrow[l, hook']
\end{tikzcd}$$

[[★★★ Something about $X[\varepsilon]$ unique filling in]]

Ravi: this gives a fast proof of 27 lines on any cubic surface theorem. You get 27 lines on your favorite cubic surface. Then you end up with some proper étale map between some things. It's degree at some point is 27, so it is 27 everywhere. ◇

4 Osserman

$(*)$ $F(A' \times_A A'') \to F(A') \times_{F(A)} F(A'')$

(H1) $(*)$ surjective if $A'' \to A$ is small thickenning.

(H2) $(*)$ bijective if $A'' = k[\varepsilon]$, $A = k$.

(H3) T_F is finite dimensional.

(H4) $(*)$ is bijective if $A' = A''$ and $A' \to A$ is a small thickenning

The theorem is that F has a hull if and only if $(H1 - H3)$ are satisfied, and F is prorepresentable if and only if $(H1 - H4)$ are satisfied.

Proof of Schlessinger's Theorem. By far, the hardest part is to show that $(H1-H3)$ imply that F has a hull. There are a couple of backgroud concepts that we need first. The first sheds some light on deformation functors.

Proposition 4.1. *Let F be a deformation functor (i.e. satisfies (H1) and (H2)), and $A' \to A$ a small*[1] *thickening with kernel I. Then for every $\eta \in F(A)$, when the set of $\eta' \in F(A')$ restricting to η is non-empty, it has a transitive action of $T_F \otimes_k I$. This action commutes with morphisms $F' \to F$ of functors.*

(H4) is satisfied if and only if for all small thickenings $A' \to A$ and all choices of $\eta \in F(A)$, this action is free (making the set into a $T_F \otimes I$ torsor when it is non-empty).

We've seen already in Martin's lecture that the behavior of the tangent space is similar to the behavior of infinitesimal liftings, and this proposition makes this precise.

The proof of the proposition is relatively straightforward. It is in the exercises, along with some description of how to actually construct the action.

Definition 4.2. A surjection $p\colon A' \to A$ in $Art(\Lambda, k)$ is called *essential* if for all $q : A'' \to A'$ such that pq is surjective, q is surjective. ◇

The following lemma is an exercise.

Lemma 4.3. *If p is a small thickening, then p is* not *essential if and only if it has a section.*

Example 4.4. $k[\varepsilon] \to k$ and $\mathbb{Z}/p^2 \to \mathbb{F}_p$ are both small thickening, but the first one has a section, so it is not essential. In particular, the composite $k \to k[\varepsilon] \to k$ is surjective even though $k \to k[\varepsilon]$ is not. The lemma tells us that the second guy is essential. ◇

Recall that a hull is a pair (R, ξ), where R is a complete local noetherian ring and $\xi \in \hat{F}(R)$ wuch that the induced map $\overline{h}_R \to F$ is smooth and $T_R = T_{\overline{h}_R} \to T_F$ is an isomorphism.

---

[1] Could be replaced by a weaker condition, where we allow a broader class of kernels I.

Proposition 4.5. *If (H1-H3) are satisfied, then F has a hull.*

Proof. There are two parts: first we construct the hull, then we actually prove that it is the hull.

(Construction) Let $\mathfrak{n}$ be the maximal ideal of Λ, and let $r = \dim T_F$ (which is finite by (H3)). Set $S = \Lambda[[t_1, \ldots, t_r]]$ (we know that to order 2, it is actually isomorphic to this, and we might have to kill some stuff higher up), and let $\mathfrak{m}$ be the maximal ideal of S. We'll construct R as S/J where $J = \bigcap_{i \geq 2} J_i$ and the J_i are constructed inductively.

Set $J_2 = \mathfrak{m}^2 + \mathfrak{n}S$ (recall that this is what we quotient $\mathfrak{m}$ by to get the relative tangent space S over Λ). Then we get that $S/J_2 = k[T_S^*] \cong k[T_F^*] \cong k[\varepsilon] \times_k \cdots \times_k k[\varepsilon]$ (r times). Let $R_2 = S/J_2$, and use (H2) to construct $\xi_2 \in F(R_2)$ which induces a bijection $T_{R_2} \to T_F$ (we get that $F(R_2) = F(k[\varepsilon])^r = T_F^r$, so pick a tuple which is a basis for T_F, and this should work as a ξ_2).

Suppose we have $R_{i-1} = S/J_{i-1}$, and $\xi_{i-1} \in F(R_{i-1})$. We'll choose J_i to be the minimal ideal J such that

- $\mathfrak{m}J_{i-1} \subseteq J \subseteq J_{i-1}$,
- ξ_{i-1} can be lifted to an element of $F(S/J)$.

To show that such a minimal guy exists, we must check that these conditions are preserved under arbitrary intersection. The first condition is preserved under arbitrary intersection clearly. Next we'll show that given the first condition, the second condition is also preserved under arbitrary intersection.

Note that the J's satisfying the fist condition correspond to vector subspaces of $J_{i-1}/\mathfrak{m}J_{i-1}$, which is finite dimensional (because everything is noetherian). To check that the intersection of a family of subspaces lies in the family, it suffices to check that pairwise intersections work (basically because you can only have finite chains of decreasing subspaces). So we've reduced to checking that the second condition is preserved under pairwise intersections.

Suppose J and K satisfy the two conditions. Using $J_{i-1}/\mathfrak{m}J_{i-1}$, we can replace K without changing $J \cap K$ so that $J + K = J_{i-1}$ [[★★★ how exactly is this done?]]. Now we've set ourselves up to use Schlessinger's criterion. We have that $S/J \times_{S/J_i} S/K \cong S/(J \cap K)$. By (H1), we have some element $\xi \in F\big(S/(J \cap K)\big)$ restricting to ξ_{i-1}. This shows that $J \cap K$ satisfies both conditions.

So we can set J_i to be the minimal such ideal and $\xi_i = \xi$. Then set $J = \bigcap_i J_i$ and $R = S/J$. If we set $R_i = S/J_i$, then because $\mathfrak{m}^i \subseteq J_i$, we can write $R = \varprojlim R_i$, and we get a well-defined element $\xi \in \hat{F}(R) = \varprojlim F(R_i)$ which restricts to the ξ_i. Now we have a candidate hull (R, ξ).

(Its a hull) The isomorphism $T_R \xrightarrow{T}_F$ is immediate by how we chose J_2 and ξ_2. Smoothness is harder. Fix a small thickening $p : A' \to A$, and suppose we have $\eta' \in F(A')$ such that $p(\eta') = \eta \in F(A)$ and $u \colon R \to A$ such that $u(\xi) = \eta$. We want to lift u to a map $u' \colon R \to A'$ such that $u'(\xi) = \eta'$ (this is the definition of formal smoothness).

First we'll just construct *some* lift u', and not worry about the condition. Since A is an artin ring, u factors through $R \to R_i$ for some i.

$$\begin{tikzcd}
R_{i+1} \arrow[rr, dashed] & & A' \arrow[d, "p"] \\
R \arrow[u] \arrow[urr, dashed, "u'"'] \arrow[r, two heads] \arrow[rr, bend right] & R_i \arrow[r] & A
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
S \arrow[r, "w"] \arrow[d] & R_i \times_A A' \arrow[d, "p_1"] \\
R_{i+1} \arrow[ur, dashed] \arrow[r, two heads] & R_i
\end{tikzcd}$$

To construct the lift u', it is enough to construct the map $R_{i+1} \to A'$. If there is a dashed arrow in the second diagram, then there is a section of p_1. Since we assume there is no dashed arrow, we may assume p_1 is essential, so we can choose w (by definition of essential). To get the dashed arrow, is enough for $\ker w \supseteq J_{i+1}$. This follows from (H1).

So we have some u'. We want to have $u'(\xi) = \eta'$. But we have compatible transitive actions of $T_F \otimes I \cong T_R \otimes I$ on $F(p)^{-1}(\eta)$ and $h_R(p)^{-1}(\eta) = \{h \colon R \to A' | h(\xi) = \eta)\}$. There exists $\tau \in T_F \otimes I$ sending $u'(\xi)$ to η', so we can modify u' by τ to produce the u' that we really want. ☐

$4\frac{1}{2}$ Starr. Deformation theory and a theorem of Mori.

Fix an algebraically closed field κ (*do not* assume $\kappa = \mathbb{C}$, this will be important later).

Definition $4\frac{1}{2}$.1. A *fano manifold* X over κ is a smooth proper connected variety such that $\det T_X = \breve{\omega}_X$ is ample. ◇

Example $4\frac{1}{2}$.2.

1. among curves, there is $\mathbb{P}^1$. Fano should somehow mean "positively curved".
2. among surfaces, these are called Del Pezzo surfaces. These are $\mathbb{P}^1 \times \mathbb{P}^1$ of a blowup of $\mathbb{P}^2$ at (≤ 9) general points. These are all rational.
3. If $X_d \subseteq \mathbb{P}^n$ is a smooth hypersurface of degree d, X is Fano if $d \leq n$. That is, Fano manifolds are things of relatively small degree. These are not all rational, but this was not known for a long time. ◇

Theorem $4\frac{1}{2}$.3 (Mori)**.** *Every Fano manifold X of dimension ≥ 1 is uniruled (i.e. every closed point p of X is contained in the image of some finite (i.e. non-constant) morphism $\mathbb{P}^1 \to X$).*

This is a very geometric theorem. This condition of being Fano is some geometric condition (positive curvature). However, there doesn't seem to be a purely differential geometric proof of this theorem. We'll give an algebraic proof in the case where $char\kappa = p$ and use a great trick to get the result in characteristic zero. This trick is an important technique all on its own.

Obstruction Theory (just the part we need). Let R be a local complete noetherian ring with residue field κ. Let $\mathcal{C}_R$ be the category of local artinian R-algebras with residue field κ (this forces them to be finitely generated R-algebras). An *infinitesimal extension* in this category is a surjection $q : A' \twoheadrightarrow A$ such that $\ker q = N$ is annihilated by $\mathfrak{m}_{A'}$ (so $\mathfrak{m}_{A'}N = 0$), so N is a finite dimensional κ vector space. We'll often write an infinitesimal extension as a short exact sequece (where the second map is a map of algebras). A morphism is a morphism of exact sequences

$$\begin{tikzcd}
\Sigma \arrow[d] & 0 \arrow[r] & N \arrow[r] \arrow[d] & A' \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & 0 \\
\tilde{\Sigma}' : & 0 \arrow[r] & \tilde{N} \arrow[r] & \tilde{A}' \arrow[r] & \tilde{A} \arrow[r] & 0
\end{tikzcd}$$

Let $F : \mathcal{C}_R \to \mathbf{Set}$ be a functor with $F(\kappa) = *$. A *deformation situation* is a pair $(\Sigma, X \in F(A))$. A morphism of deformation situation is a morphism of infintesimal extensions Σ's such that the X's match up. An *obstruction theory* is a pair (O, ω), where O is a finite dimensional κ vector space (giving a functor $N \mapsto N \otimes_\kappa O$), and ω (the *obstruction*) is a rule $(\Sigma, X) \rightsquigarrow \omega_{\Sigma,X} \in N \otimes_\kappa O$ which is

- suitibly natural in the deformation situation. That is, for $u : (\Sigma, X) \to (\tilde{\Sigma}, \tilde{X})$, the image of $\omega_{\Sigma,X}$ maps to $\omega_{\tilde{\Sigma},\tilde{X}}$ under the map $N \otimes_\kappa O \to \tilde{N} \otimes_\kappa O$.

- $\omega_{\Sigma,X}$ is zero if and only if X is the image of an element $X' \in F(A')$.

Exercise $4\frac{1}{2}$.1. The first condition implies the "if" part of the second condition. ◀

Example $4\frac{1}{2}$.4. Let $F = h_S$, $S = R[[x]]/I = (R[[x_1, \ldots, x_n]]/\langle f_1, \ldots, f_s\rangle)^\vee$. Then you get

$$I/I^2 \to \widehat{\Omega}_{R[[x]]/R} \otimes_{R[[x]]} S$$

where $\widehat{\Omega}$ is the free module on $R[[x]]$ generated by dx_i. So we get

$$\operatorname{Hom}_{R[[s]]}(\widehat{\Omega}_{R[[x]]/R}, \kappa) \to \operatorname{Hom}_S(I/I^2, \kappa)$$

given by $(\phi : dx_i \mapsto c_i) \mapsto (f_j \mapsto \sum_{i=1}^r \overline{\frac{\partial f_j}{\partial x_i}} c_i)$. Then we define O to be the cokernel of this map.

Why? Imagine that we have a deformation situation

$$\begin{tikzcd}
 & & R[[x]] \arrow[d] \arrow[dd, dashed, bend right, "v'"'] \arrow[ddr, bend left, "v"] & & \\
 & & S \arrow[dr, "w"] & & \\
0 \arrow[r] & N \arrow[r] & A' \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

There is a dashed arrow v' making the diagram commute, but there is more than one. Every other lift is of the form $v' + \partial$, where $\partial :$"dx_i"$\mapsto$ some element in N. More canonically, it is a map $\partial : \widehat{\Omega}_{R[[x]]/R} \to N$. Does this v' actually factor through S? This is what we care about. $v' + \partial$ factors through S if and only if the generators f_j map to zero. This $v' + \partial$ gives rise to an S-module homomorphism from $I/I^2 \to N$, given by $f_j \mapsto (v' + \partial)(f_j)$. Thus, we get a well-defined (independent of choices) element of the cokernel O.

The upshot is that the element

$$\omega \in \operatorname{Hom}_S(I/I^2, N)/\operatorname{Hom}_{R[[x]]}(\widehat{\Omega}, N) \cong (\operatorname{Hom}(I/I^2, \kappa)/\operatorname{Hom}(\widehat{\Omega}, \kappa)) \otimes_\kappa N$$

(because N is a free module) is independent of the choice of v'. This obstruction vanishes if and only if w extends to an R-algebra homomorphism $S \to A'$. This is the canonical example of an obstruction theory. This is a special case of something much more general using the cotangent complex (see Martin's later talks). ◇

Example $4\frac{1}{2}$.5. Let C be a smooth projective connected curve over κ. Let $Z \subseteq C$ be an effective Cartier divisor. Let X be a smooth κ-scheme. Let $f_0 : C \to X$ be a κ-morphism. Denote $g := f_0|_Z : Z \to X$. Here $R = \kappa$. Take the functor $F \colon A \mapsto \{f_A \colon C \times_{\operatorname{Spec}\kappa} \operatorname{Spec} A \to X \times_{\operatorname{Spec}\kappa} \operatorname{Spec} A | f_A$ is a $\operatorname{Spec} A$ morphism such that

(i) $f_A \equiv f_0$ modulo $\mathfrak{m}_A$, and (ii) $f|_{Z\times \operatorname{Spec} A} = g \times \mathrm{id}_{\operatorname{Spec} A}\}$. It is not hard to apply Schlessinger's criterion to show that this functor is prorepresentable.

Now let's make an obstruction theory. Let $O := H^1(C, f^*T_X \otimes I_Z)$. Given a deformation situation

$$0 \longrightarrow N \longrightarrow A' \longrightarrow A \longrightarrow 0$$

and $f_A\colon C_A \to X_A$ (as maps of topological spaces, $f = f_A = f_{A'}$), let $U \subseteq C$ be an affine open, so we have

$$\begin{tikzcd}
 & & X \\
U_A \arrow[r, hook] \arrow[urr, bend left, "f_A|_{U_A}"] & U_{A'} \arrow[ur, dashed, "\tilde{f}_{A,U}"'] & \\
Z_A \arrow[u, hook] \arrow[r, hook] & Z_{A'} \arrow[u, hook] \arrow[uur, bend right, "g_{A'}"'] &
\end{tikzcd}$$

By the infintesimal extension property for smooth things, you get a dashed arrow (so $\mathcal{O}_X \to f_*\mathcal{O}_{U_{A'}}$), which is not unique, but every other one differs by a derivation $\Omega_X \to N \otimes f_*\mathcal{O}_{U_A} \otimes I_Z$.

Now let $\{U_\beta\}$ be an open cover of C. For every β, choose a $\tilde{f}_{A,U_\beta}$. Then on $U_\beta \cap U_\gamma$, these don't agree, but their difference in a derivation $\Omega_X \to N \otimes f_*\mathcal{O}_{U_A} \otimes I_Z$. This gives an element

$$\omega_{\Sigma,f} \in \check{H}^i\big(C, \mathcal{H}om_{\mathcal{O}_{C_A}}(f_A^*\Omega_X, N \otimes_\kappa I_Z)\big)$$

$$H^1\big(C, \mathcal{H}om_{\mathcal{O}_C}(f^*\Omega_X, N \otimes_\kappa I_Z)\big) = N \otimes_\kappa H^1(C, f^*T_X \otimes I_Z)$$

◇

Fact: let $F = h_S$ be a prorepresentable functor on $\mathcal{C}_R$. Let $O_{\mathrm{can}} = \mathrm{Hom}(I/I^2,\kappa)/\mathrm{Hom}(\widehat{\Omega}_{R[[x]]/R},\kappa)$ (this is not canonical because it depends on the presentation $I/I^2 \to \widehat{\Omega}_{R[[x]]}$; if you choose the minimal set of generators, I is in the square of the maximal ideal; choose a minimal set of generators with a minimal set of relations, then we'll get something with the s begin the dimension of O_{can}. Let O be any other obstruction theory. Then there exists a unique κ-linear map $\psi : \mathcal{O}_{\mathrm{can}} \to O$ such that every $\omega_{\Sigma,X}$ is the image under ψ of $\omega_{\Sigma,X,\mathrm{can}}$.

Assume $I \subseteq \mathfrak{m}^2_{R[[x]]}$. If we have any

$$\begin{tikzcd}
0 \arrow[r] & I/\mathfrak{m}I \arrow[r] \arrow[d] & R[[x]]/\mathfrak{m}_{R[[x]]}I + \mathfrak{m} \arrow[r] \arrow[d, dashed] & R[[x]]/I + \mathfrak{m}^c_{R[[x]]} \arrow[d, "X"] & = & S/\mathfrak{m}^c_S \\
0 \arrow[r] & N \arrow[r] & A' \arrow[r] & A \arrow[rr] & & 0
\end{tikzcd}$$

$\mathfrak{m}^c_{A'} = 0$.

An ψ is injective. $\omega \in I/\mathfrak{m}_{R[[x]]}I \otimes_\kappa O$. If not injective, then some element in the kernel, so there is some ω which maps to zero under the map to $\kappa \otimes_\kappa O$, then something about extending under taking pushout. Since representable, it does not extend after taking pushout, otherwise you'd have a section of an infinitesimal extension, which you don't. So we have that $\ker\psi = 0$.

This has an immediate and important consequence. The point is that $I/\mathfrak{m}_{R[[x]]}I$ is a free κ vector space with basis the images of a minimal set of generators for I (by Nakayama). So when we wrote $S = R[[x_1,\dots,x_r]]/\langle f_1,\dots,f_x\rangle$, the tangent space T_{h_S} is generated by the dual basis to the dx_i. $\dim_\kappa T_F = r =$ minimal number of generators. $\dim_\kappa O \geq s =$ minimal number of relations. By the hauptidealsatz, $\dim S \geq \dim R + r - s \geq \dim R + \dim T_F - \dim O$. In particular, we get that $\dim S/\mathfrak{m}_R S \geq \dim T_F - \dim O$. If this inequality is an equality, then this is the canonical obstruction theory and the hauptidealsatz holds strictly, which tells you that S is flat over R. So if nature gives you a tangent space and obstruction space, then you get a bound on the dimension of S.

Example $4\frac{1}{2}$.6. $R = \kappa$. Curves.

$$\begin{tikzcd} C \arrow[r, "f_0"] & X \\ Z \arrow[u, hook] \arrow[ur, "g"'] & \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, hook] \arrow[rr, bend left, "f"] & C_A \arrow[r, "f_A"'] & X \\ Z \arrow[u, hook] \arrow[r, hook] & Z_A \arrow[u, hook] \arrow[ur, "g\times\mathrm{id}_{\operatorname{Spec}\kappa}"'] & \end{tikzcd}$$

$O = H^1(C, f^*T_X \otimes I_Z)$, $T_F = H^0(C, f^*T_X \otimes I_Z)$. In this case we have $\dim S \geq (h^0 - h^1)(C, f^*T_X \otimes I_Z)$ which you can compute (by Riemann-Roch) to be $\deg(f^*T_X) + \dim(X)(1 - g(C) - \deg Z)$ (We know that $\deg f^*T_X = \deg(f^* \det T_X)$). If this dimension is positive, then there actually exists a curve ... there is an actual representing object $\mathrm{Hom}(C, X; g : Z \to X)$ and the dimension $\dim_{[f_0]} \mathrm{Hom}(C, X; g : Z \to X)$ is positive, and X is quasi-projective (fix the degree of f^* of some ample sheaf $\mathcal{O}_X(1)$), so there is an affine curve in $[f_0] \in B \subseteq \mathrm{Hom}$.

$$\begin{tikzcd} & \widetilde{\overline{B}\times C} \arrow[dl, "\nu"'] \arrow[dr, "\tilde{f}"] & \\ \overline{B}\times C & B\times C \arrow[l, hook'] \arrow[r, "f"] & X \\ B\times Z \arrow[u, hook] \arrow[r, "pr_Z"] & Z \arrow[u, "g"'] & \end{tikzcd}$$

Something blowing up to get some $\mathbb{P}^1$'s and the map on fibers of $B \times C \to C$ being constant. If the degree is zero, then it is zero everywhere so all of these get contracted.

Lemma $4\frac{1}{2}$.7 (Rigidity Lemma). *If $\tilde{f}$ is regular on $\overline{B} \times \{p_i\}$, which is contractible, then $\tilde{f}$ factors through $\overline{B} \times C \xrightarrow{pr_C} C$.*

Something doesn't happen, so there have to be some points where the map is not defined. This shows that if the dimension is positive, for every something, there is a rational curve in another thing. How can we get this dimension positive? Mori showed that there is a way to to this in positive characteristic, where some problem (Riemann-Hurbritz?) goes away. Then you want to deduce the result in characteristic zero. You can take your original scheme over $\mathbb{C}$ which is defined by a finite number of equations and stuff. Then you see that your scheme is the base change of some scheme defined over $\mathbb{Z}$.

$$\begin{tikzcd}
C_{\mathbb{C}} \subseteq X_{\mathbb{C}} \arrow[r] \arrow[d] & X \supseteq C \arrow[d, "\text{proj, flat}"] \\
\operatorname{Spec} \mathbb{C} \arrow[r] & \operatorname{Spec} \mathbb{Z}[t_1, \ldots, t_r]/J
\end{tikzcd}$$

Something about getting rational curves over almost all primes. ◇

$4\frac{1}{2}$ Vakil. Murphy's Law for Deformation Spaces.

Computing deformation spaces is hard. As with moduli problems in general, once you know that some stuff exists, you can do a lot of things without getting your hands dirty. Say we're interested in some kind of object, then we have deformation theory globally, where you're trying to understand a global moduli space, and you have local stuff, where you're trying to understand a formal neightborhood of a point in a moduli space.

So what do moduli spaces actually look like. If there are no obstructions, then the moduli space is smooth. One way this can happen is for the obstruction space to vanish (sometimes the obstruction space is non-zero, but all the obstructions are zero, so they're also ok). So in general, you hope that moduli spaces are smooth.

The goal today is to show you that things are quite bad. In fact, arbitrarily bad.

Hilbert scheme. The Hilbert scheme Hilb_n parameterizes closed subschemes of $\mathbb{P}^n$ (this is a precise statement, it means that a morphism to Hilb_n is a flat family of projective spaces). This is representable by Grothendieck. Something else you've seen in Hartshorne: you know that in a flat family, the Hilbert polynomial is constant. That is, $\mathrm{Hilb}_n = \sqcup \mathrm{Hilb}_{n,p(t)}$. We also know that each $\mathrm{Hilb}_{n,p(t)}$ is a projective scheme (this is Grothendieck's theorem). The Hilbert scheme (and more generally the Quot scheme) is really useful for constructing other moduli spaces. There are not many statements that we know about the Hilbert scheme. We know that $\mathrm{Hilb}_{n,p(t)}$ is projective (Grothendieck) and connected (Hartshorne). There is a folklore statement (by Mumford?).

Murphy's Law(Harris-Morrison, p. 18): There is no geometric possibility so horrible that it cannot be fount on some Hilbert scheme $\sqcup \mathrm{Hilb}_n$.

Well, Mumford is now working in computer vision, so instead of asking him what he meant, we look at his papers. He wrote some papers on pathologies. In his Pathologies II paper, he gives an example of a curve in $\mathbb{P}^3$ of degree 14 and genus 24. He shows that the hilbert scheme of curves at this point is not smooth ... it isn't even reduced. This hilbert scheme has two components; this is in one of them, and the other one is reduced.

The curve looks very innocent, but there is a direction in which you can deform it to first order, but not more. This upset a lot of people, and since then there have been more results of this flavor.

Mumford was saying that it gets even worse in general.

Definition $4\frac{1}{2}$.1. A singularity is a pointed scheme (X, p) up to the equivalence relation generated by $(X,p) \xrightarrow{\text{smooth}} (Y,q)$ then $(X,p) \sim (Y,q)$. ◇

Two planes crossing and two lines crossing is sort of the same singularity. In the étale topology, this really just means that a singularity is the same as a singularity cross $\mathbb{A}^1$.

Definition $4\frac{1}{2}$.2. We say that *Murphy's Law* holds for a scheme (or stack) X if every singularity type (of finite type over $\mathbb{Z}$)[1] appears on it. ◇

Note that anything that satisfies Murphy's law has to have an infinite number of components and be horribly non-compact.

Theorem $4\frac{1}{2}$.3. *Murphy's law holds for*

1. *the Hilbert scheme,*
2. *the Hilbert scheme of smooth curves in projective space (the dimension of the projective space depends on how bad a singularity you want).*
3. *the Hilbert scheme of smooth surfaces in $\mathbb{P}^5$,*
4. *the Hilbert scheme of surfaces in $\mathbb{P}^4$,*
5. *Kontsevich's space of stable maps,*
6. *Chow varieties.*

Some moduli spaces are nice. In the moduli space of curves, there is no obstruction (because there is no H^2). What about the moduli space of smooth surfaces (with very ample canonical bundle). Back in the day, Kodaira developed deformation theory of surfaces. He was able to show that many surfaces are unobstructed. He said that he was surprised to hear from Mumford of an obstructed surface. You take $\mathbb{P}^3$ and blow it up along this genus 24, degree 14 curve. Let $X = Bl(\mathbb{P}^3)$ be this space. Then we have Def_X, and we have $ob(X)$, $def(X)$, and $aut(X)$. The point is that we have

$$0 \to \mathrm{Aut}\,\mathbb{P}^3 \to \mathrm{Def}(C \hookrightarrow \mathbb{P}^3) \to \mathrm{Def}(X) \to 0$$

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Aut}\,\mathbb{P}^3 \arrow[r] & \mathrm{Def}(C \hookrightarrow \mathbb{P}^3) \arrow[r] & \mathrm{Def}(X) \arrow[r] & 0 \end{tikzcd}$$

$$def$$

$$ob$$

you get some long exact sequence. Anyway, this is just supposed to show that you can make this intuitive picture precise.

Kodaira was disturbed by this, but this was a 3-fold, and he only cared about surfaces. Later, people came up with surfaces which caused trouble. Kodaira hoped that these were pathalogical cases. He conjectured that maybe: if S is of general type, $h^1(\mathcal{O}_S) = 0$, K_S is ample, then it is unobstructed. The reason for this conjecture is

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[1] Since moduli problems are usually defined over $\mathbb{Z}$, these are the only types of singularities that have a chance of appearing.

because all the counterexamples broke these conditions. It turns out that the conjecture is false.

Murphy's law also holds for the moduli space of surfaces (or anything with dim > 1) (smooth, very ample K, any nice things you like.

Consider the singularity type $5^7 = 0$. That is, we're in characteristic 5. Murphy's law says that over the integers, you have curves that are stuck in characteristic 5 to (exactly) 7th order! Serre gave the first example of something in positive characteristic which doesn't lift ot characteristic zero. Raynaud mad some other example. Now we see that you can get these things stuck in finite characteristic however you like. In some sense, these examples are really easy. Here is an example of a surface which is stuck in characteristic 2: $\mathbb{P}^2_{\mathbb{F}_2}$, the Fano plane (this is seven points and seven lines [[★★★ insert the usual picture]]). Blow up the seven points. Deforming the resulting surface is equivalent to deforming the original configuration (or something like this?). Now you have a surface stuck in characteristic 2 with this marked divisor (the seven exceptional curves). Then you take something branched over those seven curves (and maybe another one) which somehow remembers how it is a branch cover. This means that the deformation space of the branch cover is equivalent to the deformation space of the guy with the seven marked lines.

Now we see where to look for counterexamples to commen-sense things from characterisitic zero in finite characteristic. Pick something which is stuck in finite characteristic.

Plane curves. A curve in the plane is cut out by a polynomial. There is a classical question (100 years old). You want to deform the curve and keep all the singularities with the right type. Severi considered the case of deforming curves with (only) nodes. This is because every abstract curve can be represented this way (so he wanted to show that the moduli space of curves is smooth, which we now know for fancy reasons). He showed that they are unobstructed. Then he threw in cusps because surfaces are branch covers over these curves. Severi showed that this was smooth too. Somebody showed that the Severi argument wasn't quite complete, so tried to fill in the gaps. Wahl showed in his phd thesis that in fact, this moduli space is not smooth. He started with Mumford's example and did a lot of clever things to show that the same problem shows up. So Severi was wrong. In fact, he was maximally wrong: plane curves moduli space satisfies Murphy's law.

Moduli space of stable coherent sheaves satisfies Murphy's Law.

def of singularity satisfies Murphy's Law (even reasonably nice singularities: Cohen-Macaulay).

The moral is basically that bad things can happen.

Philosophy. Nature is not your friend. We know that some moduli spaces are smooth, but for cheap reasons. Anything that isn't smooth for cheap reasons, you should expect to be arbitrarily bad.

Good	Bad
curves	surfaces or higher
branced covers of $\mathbb{P}^1$	branched covers of $\mathbb{P}^2$
surfaces in $\mathbb{P}^3$	surfaces in $\mathbb{P}^4$
def of nodal	def of cuspidal

There are quit few moduli spaces that don't fit into one of these columns (good=smooth, bad=maximally bad). Something about how you should expect badness in the middle of your moduli space (not just on the boundary, which you throw in to compactify).

About the proof. We reduce some cases to others. For example, if you have a surface with some deformation space, then you can get a surface in $\mathbb{P}^5$ in such a way that the deformations in $\mathbb{P}^5$ is the same as the deformations as an abstract surface. The somewhat informal description is this. Given a surface S with some big power of the (positive) canonical bundle (2 is enough). We get $S \xrightarrow{|2K|} \mathbb{P}^5$ by picking six sections.

$$\begin{tikzcd}
 & \mathrm{Def}(S, \mathcal{L}, 6 \text{ sections}) \arrow[dl] \arrow[d] \\
\mathrm{Def}(S \hookrightarrow \mathbb{P}^5) & \mathrm{Def}(S, \mathcal{L} = K) \arrow[d] \\
 & \mathrm{Def}\, S
\end{tikzcd}$$

given a guy in the bottom, there is a lift, which is unique. Choosing six sections is smooth, and quotient by $\mathbb{C}^\times$.

Say $\pi \colon S \hookrightarrow \mathbb{P}^5$.

$$0 \longrightarrow T_S \longrightarrow \pi^* T_{\mathbb{P}^5} \longrightarrow N \longrightarrow 0$$

Then we get a long exact sequence Def $\to$ Ob, where every third term (the $\pi^* T_{\mathbb{P}^5}$ terms) vanish.

If you want a surfact in $\mathbb{P}^4$, just map it to $\mathbb{P}^4$ (so that it doesn't cross itself too badly). The ideal sheaf of the surface is a stable coherent sheaf. Something to get deformations of singularities. To get the thing for curves, you slice the surface with a hypersurface. You take the branch shadow to get the cuspidal curves. You keep getting reductions to this to reduce to the case of the moduli space of surfaces.

Mnëv's Theorem: draw m points and n lines in the plane $\mathbb{P}^2$. Fix incidence data (specify which points are on which lines). Define the insidence scheme in $(\mathbb{P}^2)^m \times (\mathbb{P}^{2*})^n$. If you think about it, it is a sequence of smooth choices, so you'd think that incidence schemes are smooth. However, this is wrong. The union of all incidence schemes satisfies Murphy's Law. This is Mnëv's theorem.

From this incidence scheme case, you can get a surface with the same deformations. Now we've reduced to Mnëv's theorem.

How to prove the theorem? You give me a bad singularity with equations. Let's make a recipe for how to encode the singularities in incidence relations. If we can encode addition, multiplication, subtraction, and equality with incidence relations.

Fact. Fix a line ℓ in $\mathbb{P}^2$, and name three points 0, 1 and ∞. Then you've identified all the points on ℓ with numbers. Then produce a configuration of lines which encode relations $x + y = z$ and another configuration which forces $y = z$ and another one which forces $x = -y$.

Now if we want to encode any equation, we can construct new points and force them to be equal, and this incidence relations force whatever equation you start with. So we win!

"$\mathbb{C}$ cares about $\mathbb{Q}$". Suppose you're a geometer over $\mathbb{C}$, so you're interested in the moduli space of complex surfaces. You have some moduli space $\mathcal{M}_{\mathbb{C}} \to \operatorname{Spec} \mathbb{C}$. Somebody else might think about the same thing over the integers. It happens that $\mathcal{M}_{\mathbb{C}}$ comes from some $\mathcal{M}$ over $\mathbb{Z}$. That is, (étale) locally, everything can be written with a finite number of variables over $\mathbb{Z}$. You could have four lines whose cross ratio is π $yx(y - x)(y - \pi x)$. This thing will never show up as a singularity over $\mathbb{C}$.

The following question is answered either by "yes" or "no", and either case would be really neat, but not knowing is not nice.

Suppose you have a "nice" object X over $\mathbb{C}$. Then the deformation space $\operatorname{Def} X$ is "defined over $\mathbb{Z}$".

If this is true: this is interesting because it would be surprising that non-algebraic objects have to care about $\mathbb{Z}$. If this is false: you just have to produce a single object which is nice whose deformation space is not defined over $\mathbb{Z}$. A single provable counterexample has not been found, and would be great to see.

Summary.

Many things satisfy Murphy's Law. These spaces are deformation spaces. The moral to take home is that the moduli spaces you care about are usually not very nice. If you're interested anything which isn't curves, you should expect badness.

5 Lieblich

Last time we talked about categories fibered in groupoids/pseudofunctors. For the rest of the lectures (at least starting tomorrow), there will be more examples.

Definition 5.1. A functor $F : \mathcal{D} \to \mathcal{C}$ is a *category fibered in groupoids (or a groupoid over $\mathcal{C}$)* if

1. for all $\beta \colon c_1 \to c_2$ in $\mathcal{C}$, and for all $d_2 \in \mathcal{D}$ with $F(d_2) = c_2$, there is some $\alpha \colon d_1 \to d_2$ such that $F(\alpha) = \beta$.
2. For all diagrams $$\begin{tikzcd} d_1 \arrow[r, "\alpha_2"] & d_2 \\ d_3 \arrow[u, dashed, "\alpha_3"] \arrow[ur, "\alpha_2"'] & \end{tikzcd}$$ over $$\begin{tikzcd} c_1 \arrow[r, "\beta_2"] & c_2 \\ c_3 \arrow[u, dashed, "\beta_3"] \arrow[ur, "\beta_2"'] & \end{tikzcd}$$, for every β_3 there is a unique α_3. ◇

Definition 5.2. Given some $c \in \mathcal{C}$, the *fiber category* $\mathcal{D}_c$ or F_c has objects $d \in \mathcal{D}$ such that $F(d) = c$ and arrows morphisms $\alpha \colon d_1 \to d_2$ such that $F(\alpha) = \mathrm{id}_c$. ◇

Definition 5.3. A *1-morphism* of categories fibered in groupoids $F_1 \colon \mathcal{D}_1 \to \mathcal{C}$ and $F_2 \colon \mathcal{D}_2 \to \mathcal{C}$ is a functor $F \colon \mathcal{D}_1 \to \mathcal{D}_2$ so that $F_2 \circ F = F_1$ (actually equal!). F is an *equivalence* if for all $c \in \mathcal{C}$ the induced functor $F_c \colon (\mathcal{D}_1)_c \to (\mathcal{D}_2)_c$ is an equivalence. [[★★★ this is actually the same as the functor F being an equivalence.]] ◇

Note that since morphisms of fibered categories are functors, we actually get that $\mathrm{Hom}_{\mathcal{C}}(F_1, F_2)$ is a groupoid (whose arrows are natural isomorphisms between functors).

Let $\mathcal{C} = \mathbf{Sch}_S$. Then we have our old friends, $\mathrm{Fun}(\mathcal{C}^\circ, \mathbf{Set})$, and our even older friends, schemes over S. Since $\mathbf{Set} \subseteq \mathbf{Gpoid}$, these old friends naturally define categories fibered in groupoids.

Example 5.4. Take $\mathcal{D}_1 = h_X$, where X is an S-scheme. It turns out that there is an equivalence of categories $\mathrm{Hom}_{\mathcal{C}}(h_X, \mathcal{D}_2) \xrightarrow{\sim} (\mathcal{D}_2)_X$. ◇

Recall that we had $\mathcal{M}_0$, the moduli of varieties. If X is a scheme, $\{X \to \mathcal{M}_0\}$ is equivalent to the category of flat families of varieties $\mathcal{V} \to X$.

Question: is the pseudofunctor thing really necessary? If we allow ourselves to think about fibered categories up to equivalence, then each pseudofunctor can be replaced by an actual functor. This may seem like it would be helpful, but it is not at all.

Are there properties that fibered categories can have that would make them seem more geometric?

Example 5.5. $X \mapsto \mathbf{Qcoh}(X)$ (the subcategory of quasicoherent sheaves in which all arrows are isomorphisms) defines a category fibered in groupoids. This fibered category has some properties which we like. We have descent theory! This is some kind of “sheafiness”. ◇

Descent Theory (gluing) in General. When you want to glue things, you have something defined on open sets with isomorphisms on overlaps and a cocycle condition on triple intersections, and you want a big object. Fix some category fibered in groupoids $\mathcal{D} \to \mathcal{C} = \mathbf{Sch}_S$, where we fix the big étale topology on $\mathbf{Sch}_S$, and we fix some cleavage.

Definition 5.6. Given a covering $\{V_i \to X\}$, we define the *category of descent data (with respect to this covering)* to be $\mathcal{D}_{\{V_i \to X\}}$ which has objects (d_i, ϕ_{ij}) where $d_i \in \mathcal{D}_{V_i}$ and $\phi_{ij} : d_i|_{V_i \times_X V_j} = pr_1^* d_i \xrightarrow{\sim} pr_2^* d_j = d_j|_{V_i \times_X V_j}$ (note that we're using the cleavage here) such that $\phi_{jk}\phi_{ij} = \phi_{ik}$ for all i, j, k. The arrows are what you think (arrows $d_i \to d_i'$ which are compatible with the ϕ_{ij} and ϕ_{ij}'). ◇

Observe that any object in $\mathcal{D}_X$ gives rise to an object d in $\mathcal{D}_{\{Y_i \xrightarrow{\psi_i} X\}}$, given by $d_i = d|_{Y_i} = \psi_i^* d$, with ϕ_{ij} the canonical isomorphism induced by $pr_1^*\psi_i^* \cong pr_2^*\psi_j^*$. The cocycle condition is built into pseudofunctors. The upshot is that you can actually get a functor $\nu_{\{Y_i \to X\}} : \mathcal{D}_X \to \mathcal{D}_{\{Y_i \to X\}}$.

Definition 5.7. $\mathcal{D}$ is a *prestack* on $\mathcal{C}$ if $\nu_{\{Y_i \to X\}}$ is fully faithful for all coverings $\{Y_i \to X\}$ (i.e. all coverings are descent morphisms). $\mathcal{D}$ is a *stack* if $\nu_{\{Y_i \to X\}}$ is an equivalence for all coverings (i.e. all coverings are effective descent morphisms). ◇

A fibered category is like a presheaf. A prestack is like a separated presheaf. A stack is like a sheaf.

Prestacks: a reinterpretation. Suppose we're given $a, b \in \mathcal{D}_X$, define a presheaf on $\mathbf{Sch}_X$ as follows: given some $f : Y \to X$, let $I(a,b)(f) = Isom_{\mathcal{D}_Y}(f^*a, f^*b)$.

Lemma 5.8 (Exercise)**.** *$\mathcal{D}$ is a prestack if and only if for all X, a, and b, $I(a,b)$ is a sheaf on X_{ET}. "Isomorphisms form a sheaf".*

Just as one can sheafify a presheaf, one can stackify a prestack. You can even stackify a fibered category (remember that fibered categories correspond to presheaves). That is, there is some stack with some universal property.

Theorem 5.9. *Given a fibered category $\mathcal{D} \to \mathcal{C}$ (where $\mathcal{C}$ has a site structure), there exists a stack $\mathcal{D}^s$ and a 1-morphism $\mathcal{D} \to \mathcal{D}^s$ over $\mathcal{C}$ such that for all stacks $\mathcal{S} \to \mathcal{C}$, the map $\mathrm{Hom}(\mathcal{D}^s, \mathcal{S}) \to \mathrm{Hom}(\mathcal{D}, \mathcal{S})$ is an equivalence of groupoids.*

Here is something we proved before.

Proposition 5.10. **Qcoh** *is a stack on $\mathbf{Sch}_{\mathbb{Z}}$ in the fpqc topology (so also in the fppf, étale, zariski topologies).*

Proposition 5.11. *Sheaves on $(\mathbf{Sch}_{\mathbb{Z}})_{ET}$ form a stack. Here $\mathbf{Sh}_T = \{$sheaves on $T_{ET}\}$. This works on any site, by the way.*

Let's look at our moduli problems. Are they stacks?

(5) Subspaces of V. This is a stack because any sheaf is a stack!

(4) Closed subschemes of X. Again, this is a sheaf so it is a stack.

(3) $\mathrm{Hom}(X, Y)$. Again, since schemes are sheaves, this is a sheaf, so this is a stack.

(2) Line bundles on X. This is *not* a sheaf, but it is a stack because of descent theory.

(1) Curves of genus g ($g \neq 1$). Essentially, you showed in the exercises that this is a stack. Ravi explained on Tuesday how this is not a sheaf (in fact, the natural target isn't even **Set**, it is **Gpoid**, so it doesn't even want to be a sheaf somehow).

(0) Varieties. This is a prestack (i.e. $\underline{\mathrm{Isom}}(X, Y)$ is a sheaf; we have descent for morphisms of schemes), but this is NOT A STACK. There is a smooth 3-fold X (not quasi-projective) over $\mathcal{C}$ with a descent datum relative to the cover $\operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{R}$ which does not descend. For smooth surfaces, this can't happen.

The last example is kind of funny. A scheme X is a sheaf, so a family $X \to T$ is a sheaf on T_{ET}, but this obstructed descent datum says that we can't glue this together *as schemes*, but we can glue them as sheaves. That is we have a sub-prestack **Sch** $\subseteq$ **Sh** which isn't a substack. Why not take the stacky closure of **Sch** inside of **Sh**. We'll put off the definition until next time.

5 Olsson

The goal for this week is quite different from last week. I want to at least give a feeling for the cotangent complex, but just defining it will be opaque, so we'll take the historical approach, which is through the Picard stack.

The idea is that in each example we had, we had some kind of nice structure with H^0, H^1, and H^2. We had additive structure floating around. You kind of expect to have complexes lurking somewhere. We'll try to make precise the notion of a morpism of deformation problems.

Definition 5.1. A *Picard category* (maybe an abelian group category would be a better term) is a groupoid P with the following extra structure.

(a) A functor $+\colon P \times P \to P$.

(b) An isomorphism of functors

$$\begin{tikzcd} P \times P \times P \arrow[r, "\mathrm{id}\times +"] \arrow[d, "+\times\mathrm{id}"'] & P \times P \arrow[d, "+"] \\ P \times P \arrow[r, "+"'] \arrow[ur, phantom, ""] & P \arrow[from=2-1, to=1-2, Rightarrow, shorten=12pt, "\sigma"] \end{tikzcd}$$

That is, $\sigma_{x,y,z} : (x+y)+z \xrightarrow{\sim} x+(y+z)$.

(c) A natural transformation

$$\begin{tikzcd} P \times P \arrow[rr, "\text{flip}"] \arrow[dr, "+"'] & \arrow[d, phantom, ""] & P \times P \\ & P \arrow[ur, "+"'] & \arrow[from=1-1, to=1-3, Rightarrow, shift right=4, shorten=30pt, "\tau"] \end{tikzcd}$$

That is, $\tau_{x,y} : x+y \xrightarrow{\sim} y+x$.

Such that

0. for all $x \in P$, the functor $P \to P$, $y \mapsto x+y$ is an equivalence.

1. (Pentagon axiom)

$$\begin{tikzcd} & (x+y)+(z+w) \arrow[dl, "\sigma_{x,y,z+w}"'] \arrow[dr, "\sigma_{x+y,z,w}"] & \\ x+(y+(z+w)) \arrow[d, "\sigma_{y,z,w}"'] & & ((x+y)+z)+w \arrow[d, "\sigma_{x,y,z}"] \\ x+((y+z)+w) \arrow[rr, "\sigma_{x,y+z,w}"] & & (x+(y+z))+w \end{tikzcd}$$

2. $\tau_{x,x} = \mathrm{id}$ for all $x \in P$

3. for all $x, y \in P$, $\tau_{x,y} \circ \tau_{y,x} = \mathrm{id}$
4. (Hexagon axiom)

$$\begin{tikzcd}
 & x+(y+z) \arrow[r, "\tau"] \arrow[dl, "\sigma"'] & x+(z+y) \arrow[dr, "\sigma"] & \\
(x+y)+z \arrow[dr, "\tau"] & & & (x+z)+y \arrow[dl, "\tau"] \\
 & z+(x+y) \arrow[r, "\sigma"] & (z+x)+y &
\end{tikzcd}$$

◇

Now check that anything you'd want to commute commutes.

Example 5.2. Let X be a scheme, and let $\mathcal{P}ic(X)$ be the groupoid of line bundles on X with group structure given by tensor product. ◇

Brian: why not have flip commute with + on the nose? Martin/Ishai: That would already fail for line bundles.

Now let's give the example which is the primordial deformation problem.

Let $f\colon X \to Y$ be a morphism of schemes, and let I be a quasi-coherent $\mathcal{O}_X$-module. An *I-extension of X over Y* is a diagram

$$\begin{tikzcd}
X \arrow[r, hook, "j"] \arrow[d, "f"'] & X' \arrow[dl, "f'"] \\
Y &
\end{tikzcd}$$

where j is a square-zero thickening, together with an isomorphism $\iota\colon I \to \ker(\mathcal{O}_{X'} \to \mathcal{O}_X)$. This is sort of like what we were studying before. Let $\underline{\mathrm{Exal}}_Y(X, I)$ be the category of I-extensions of X over Y.

Remark 5.3. Exal means extensions of algebras.

$$\begin{tikzcd}
I \arrow[r, hook] & \mathcal{O}_{X'} \arrow[r, dashed] & \mathcal{O}_X \\
 & f^{-1}\mathcal{O}_Y \arrow[u, dashed] \arrow[ur] &
\end{tikzcd}$$

◇

Remark 5.4. If $A \to B$ is a morphisms of sheaves of algebras on a topological space T and I is a B-module, then we get a category $\underline{\mathrm{Exal}}_A(B, I)$. ◇

Remark 5.5. $\underline{\mathrm{Exal}}_Y(X, I)$ is a groupoid. To see this, let h be a morphism.

$$\begin{tikzcd}
 & X'_2 \arrow[d, "h"] \arrow[dd, bend left] \\
X \arrow[ur, hook] \arrow[r, hook] \arrow[dr] & X'_1 \arrow[d] \\
 & Y
\end{tikzcd}$$

Then we have

$$\begin{tikzcd}
I \arrow[r, "\iota_1", "\sim"'] & \ker(\mathcal{O}_{X'_2} \to \mathcal{O}_X) \\
I \arrow[u, "\mathrm{id}"] \arrow[r, "\iota_2", "\sim"'] & \ker(\mathcal{O}_{X'_1} \to \mathcal{O}_X) \arrow[u]
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & I \arrow[r] \arrow[d, equal] & \mathcal{O}_{X'_2} \arrow[r] & \mathcal{O}_X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & I \arrow[r] & \mathcal{O}_{X'_2} \arrow[r] \arrow[u, "\wr"] & \mathcal{O}_X \arrow[r] & 0
\end{tikzcd}$$

◇

Remark 5.6. If $U \subseteq X$ is an open subset, then there is a restriction functor $\underline{\mathrm{Exal}}_Y(X, I) \to \underline{\mathrm{Exal}}_Y(U, I|_U)$. The point is that we think of everything as being sheaves of algebras. ◇

Remark 5.7. Let $u \colon I \to J$ be a map of $\mathcal{O}_X$-modules. Then there is a functor $u* : \underline{\mathrm{Exal}}_Y(X, I) \to \underline{\mathrm{Exal}}_Y(X, J)$. To see this

$$\begin{tikzcd}
I \arrow[r, hook, "\iota"] & \mathcal{O}_{X'} \arrow[r, "\pi"] & \mathcal{O}_X \\
 & f^{-1}\mathcal{O}_Y \arrow[u] \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & X'_u \arrow[d] \arrow[ddr, bend left] & \\
X \arrow[ur, hook, "J"] \arrow[r, hook, "I"] \arrow[drr, bend right] & X' \arrow[dr] & \\
 & & Y
\end{tikzcd}$$

Let $\mathcal{O}_{X'_u} = \mathcal{O}_{X'} \otimes J = \mathcal{O}_{X'}[J]/\{(i, -u(i)) | i \in I\}$ "$= (\mathcal{O}_{X'} \oplus J)/I$". You really have $I \xrightarrow{(\iota,-u)} \mathcal{O}_{X'} \oplus J$.

Then define $u* : X' \mapsto X'_u$. ◇

Lemma 5.8. *If I and J are two quasi-coherent $\mathcal{O}_X$-modules, then we have that the functor $(pr_1*, pr_2*) \colon \underline{\mathrm{Exal}}_Y(X, I \oplus J) \to \underline{\mathrm{Exal}}_Y(X, I) \times \underline{\mathrm{Exal}}_Y(X, J)$ is an equivalence of categories.*

Brian essentially wrote this lemma a couple of days ago. The proof was basically on the homework. This is like verifying (H2) is a more general setting, except it is pure algebra. If you're getting confused in the lectures to follow, you don't loose much if you just think about the case where you're working over a point (i.e. where Y is a point).

Now we're in business because we can define an additive structure. Let $\Sigma : I \oplus I \to I$ be the summation map. Then we define

$$+\colon \underline{\mathrm{Exal}}_Y(X,I) \times \underline{\mathrm{Exal}}_Y(X,I) \xrightarrow{\sim} \underline{\mathrm{Exal}}_Y(X, I \oplus I) \xrightarrow{\Sigma *} \underline{\mathrm{Exal}}_Y(X,I).$$

Now there is a horrible number of things to check; you have to construct σ and τ, which will be the obvious canonical isomorphisms, and you have to check that all those diagrams commute.

Question: why does it matter what algebra structure we put in the middle. Martin: you can't reconstruct the algebra structure just from the structure sheaf; you really need the derivations.

Let's do another example. This may look trivial, but there will be a theorem tomorrow which says that this is essentially all examples.

Let $f\colon A \to B$ be a homomorphism of abelian groups. Define $\mathcal{P}_f$ to be a category whose objects are elements $x \in B$ and a morphism $x \to y$ is an element $h \in A$ with $f(h) = y - x$. You can check that this gives a good category. Note that we really mean that a morphism is a specific h; you can have many morphisms from x to y (in particular, if you have one morphism, you can change it by any element of the kernel of f). You can add objects just by adding them in B, and you can define addition of morphisms in the obvious way. $+\colon \mathcal{P}_f \times \mathcal{P}_f \to \mathcal{P}_f$ is given by $(x, y) \mapsto x +_B y$, and $\big((x,y) \xrightarrow{(h,g)} (x',y')\big) \to \big(x +_B y \xrightarrow{h+_A g} x' +_B y'\big)$. Now we're in luck because our groups are abelian, so $f(h+g) = f(h) + f(g) = (x'-x) + (y'-y) = (x'+y') - (x+y)$.

Now we want to sheafify the whole story, for which we'll need stacks. Let T be a topological space (or a site, for that matter).

Definition 5.9. A *Picard (pre)stack over* T is a (pre)stack $\mathcal{P}$ in groupoids with morphisms of stacks $(+, \sigma, \tau)$ such that for all $U \subseteq T$, the fiber $(\mathcal{P}(U), +, \sigma, \tau)$ is a Picard category. (The $+$ is defined as $\mathcal{P} \times_T \mathcal{P} \to \mathcal{P}$.) ◇

Brian: is imposing the various axioms on all the fibers equivalent to imposing them on the whole category? Martin: yes, I think it is equivalent.

Example 5.10. $\mathcal{Pic}(-)$ defines a Picard stack on $|X|$. ◇

Example 5.11. $\underline{\mathrm{Exal}}_Y(-, I)$ gives a Picard stack on $|X|$. ◇

Example 5.12. If $f\colon A \to B$ is a homomorphism of sheaves of abelian groups ona topological space T, we then get a Picard *prestack* $pch(A \to B)$ on $|T|$. This is the same as our morphism of abelian groups case, but we take sections of everything. ◇

The notion of a Picard stack is supposed to generalize the notion of abelian groups, so we should be able to do the usual things we like to do with abelian groups.

Definition 5.13. Let T be a topological space and let $\mathcal{P}_1$ and $\mathcal{P}_2$ be Picard stacks over T. A *morphism* $\mathcal{P}_1 \to \mathcal{P}_2$ is a pair (F, ι), where $F\colon \mathcal{P}_1 \to \mathcal{P}_2$ is a morphism of stacks and $\iota\colon (x+y) \xrightarrow{\sim} F(x) + F(y)$ is an isomorphism of functors, such that

$$\begin{tikzcd}
F(x+y) \arrow[r, "\iota"] \arrow[d, "F(\tau)"'] & F(x)+F(y) \arrow[d, "\tau"] \\
F(y+x) \arrow[r, "\iota"] & F(y)+F(x)
\end{tikzcd}$$

$$\begin{tikzcd}
F((x+y)+z) \arrow[r, "\iota"] \arrow[d, "F(\sigma)"'] & F(x+y)+F(z) \arrow[r, "\iota"] & (F(x)+F(y))+F(z) \arrow[d, "\sigma"] \\
F(x+(y+z)) \arrow[r, "\iota"] & F(x)+F(y+z) \arrow[r, "\iota"] & F(x)+(F(y)+F(z))
\end{tikzcd}$$

◇

Given Picard stacks $\mathcal{P}_1$ and $\mathcal{P}_2$, we get a Picard stack $\mathrm{HOM}(\mathcal{P}_1, \mathcal{P}_2)$ (the stack of morphisms of Picard stacks, on an open set U, it returns the groupoid of morphisms from $\mathcal{P}_1|_U$ to $\mathcal{P}_2|_U$). This has

- an identity element
- kernels
- a tensor product.

5 Osserman

(∗) $F(A' \times_A A'') \to F(A') \times_{F(A)} F(A'')$

(H1) (∗) surjective if $A'' \to A$ is small thickenning.

(H2) (∗) bijective if $A'' = k[\varepsilon]$, $A = k$.

(H3) T_F is finite dimensional.

(H4) (∗) is bijective if $A' = A''$ and $A' \to A$ is a small thickenning

The theorem is that F has a hull if and only if $(H1 - H3)$ are satisfied, and F is prorepresentable if and only if $(H1 - H4)$ are satisfied.

Later, we'll go throught why you should expect this theorem to be true, but for now, let's just finish the proof.

Last time, we showed that (H1-H3) gives us a hull. All that is left to check is that if we have a hull, then (H1-H3), and that H4 is equivalent to prorepresentable given a hull.

Suppose F has a hull (R, ξ). Then (H3) is trivial, because part of the definition of a hull is that $T_R \cong T_F$, and since R is noetherian, it has a finite dimensional tangent space.

Now suppose we have $p' : A' \twoheadrightarrow A$ and $p'' : A'' \to A$ in $Art(\Lambda, k)$ with p' surjective. Then for (H1) we want to show that (∗) is surjective. Suppose $\eta' \in F(A')$ and $\eta'' \in F(A'')$ both restricting to $\eta \in F(A)$. We know from the definition of a hull that the map $\bar{h}_R \to F$ is smooth, so it is surjective (by one of the exercises), so there exists some $u' \colon R \to A'$ such that $u'(\xi) = \eta'$. [[★★★ we can get \emph{ξ} to hit η'? yes by exercises?]]. Similarly, we can get $u'' \colon R \to A''$ so that $u''(\xi) = \eta''$. Set $\zeta = (u' \times_u u'')(\xi) \in F(A' \times_A A'')$. This lifts (η', η'') such that $p'' \circ u'' = p' \circ u'$, [[★★★ this is where we use surjectivity of p''?]] and thus proves (H1). Recap: choose u' by surjectivity, then we need to find u'' lifting the same map, which is where we use the smoothness and the surjection.

For (H2), assume $A = k$ and $A'' = k[\varepsilon]$, and we want (∗) to be injective (we already know it is surjective). Suppose you have some $\zeta \in F(A' \times_A A'')$ which restricts to a given (η', η''). Suppose you have another element $v \in F(A' \times_A A'')$ which also restricts to (η', η''). Then we want to show that $v = \zeta$. Keeping the same $u' \colon R \to A'$, we apply smoothness to the map $A' \times_k k[\varepsilon] \to A'$ to obtain $q'' \colon R \to k[\varepsilon]$ such that $(u' \times q'')(\xi) = v$. By definition $q'', u'' \in T_R$. Because $T_R \cong T_F$ and because we had $(u' \times u'')(\xi) = \zeta$, we have that $u''(\xi) = \zeta|_{A''} = v|_{A''} = q''(\xi)$, so we conclude that $u'' = q''$, so $\zeta = v$. This shows (H2), so we're done.

Now we've shown that if F has a hull, then (H1-H3) are satisfied. Now assume (H1-H4) as satisfied. Then F has a hull (R, ξ), and we want to show that that hull proprepresents F, i.e. that for all artin rings A, there is a bijection $h_R(A) \xrightarrow{\sim} F(A)$. We already know from smoothness (from the definition of a hull) that we have surjectivity, so we just need to show injectivity. We prove this by induction on the length of A.

Recall the proposition that if you have a small thichening, then (H4) is equivalent to the action of the tangent space making something into a pseudo-torsor. Let $p : A' \to A$ be a small thickening with kernel I, and suppose $h_R(A) \xrightarrow{\sim} F(A)$, then we want to conclude that $h_R(A') \xrightarrow{\sim} F(A')$. For all $\eta \in F(A) \cong h_R(A)$, we have that the sets $h_R(p)^{-1}(\eta)$ and $F(p)^{-1}(\eta)$ are both pseudo-torsors under $T_F \otimes I \cong T_R \otimes I$. By functoriality, this is a compatible action. But we know that $h_R(p)^{-1}(\eta) \twoheadrightarrow F(p)^{-1}(\eta)$, induced by the map $h_R(A') \twoheadrightarrow F(A')$, so they are either both empty or they are both have a simple transitive action of the same group, so they must be in bijection. Since this holds for all $\eta \in F(A)$, we have a bijection $h_R(A') \xrightarrow{\sim} F(A')$. Since every artin ring has finite length, we can break every map into a series of small thickenings, and in the case of $A = k$, both sets are just a point. Thus, by induction, R prorepresents F.

Finally, if F is prorepresentable, then $(*)$ is always bijective because $A' \times_A A''$ is a categorical fiber product in $\widehat{Art}(\Lambda, k)$. In particular, (H4) holds. This completes the proof of Schlessinger's criterion.

Example 5.1 (Deformations of a quotient sheaf)**.** Let X_Λ be a scheme over Λ, and let $\mathcal{E}_\Lambda$ be a quasi-coherent sheaf on X_Λ. As usual, we write X and $\mathcal{E}$ for the restrictions to k. Fix a quasi-coherent quotient $\mathcal{E} \twoheadrightarrow \mathcal{F}$. Define the predeformation functor corresponding to this quotient as $\mathrm{Def}_{\mathcal{F},\mathcal{E}} : A \mapsto \{\mathcal{E}_\Lambda|_A \twoheadrightarrow \mathcal{F}_A$ restricting to $\mathcal{E} \twoheadrightarrow \mathcal{F}$ after tensoring with $k | \mathcal{F}_A$ flat over $A\}$. Note that we never have automorphisms to worry about; we could even have a notion of equality of quotients (given by equality of kernels). The significance of this fact is that this predeformation functor will always satisfy (H4), so it will have a hull if and only if it is prorepresentable. ◇

Theorem 5.2. $\mathrm{Def}_{\mathcal{F},\mathcal{E}}$ *is always a deformation functor (i.e. satisfies (H1) and (H2)) and always satisfies (H4). If* X_Λ *is proper and* $\mathcal{E}$ *is coherent, then* $\mathrm{Def}_{\mathcal{F},\mathcal{E}}$ *also satisfies (H3), so it is prorepresentable.*

Remark 5.3. If you've seen Quot schemes, you know that this is the local version of a Quot scheme. The hypotheses for representablity of the Quot scheme are stronger . . . you need projective instead of just proper. Grothendieck expressed a great deal of frustration about this fact that you need this projectivity. Here we see that the local behavior is still scheme-like under the properness hypothesis. You might imaging that there is some larger class of spaces for which the Quot functor is representable when X is proper. This is the notion of algebraic spaces. ◇

Sketch Proof of Theorem. Given $A' \to A$ and $A'' \to A$, with $\mathcal{F}_{A'}$ and $\mathcal{F}_{A''}$ both restricting to $\mathcal{F}_A$ on A. Set $B = A' \times_A A''$. If we want a quotient on B, the natural thing to do is to take $\mathcal{F}_B = \mathcal{F}_{A'} \times_{\mathcal{F}_A} \mathcal{F}_{A''}$. We get a [surjection] $\mathcal{E}_B = \mathcal{E}_\Lambda|_B \to \mathcal{F}_B$. To see this, we get $\mathcal{E}_B \to \mathcal{E}_{A'} \times_{\mathcal{E}_A} \mathcal{E}_{A''} \twoheadrightarrow \mathcal{F}_B$. If you'd assumed $\mathcal{E}$ is flat over Λ, the first map would be an isomorphism, but even without flatness, the composite map is still a surjection; there is some checking to do, but it isn't hard. This gives us (H1); in fact, we've actually constructed an inverse to $(*)$, so we get (H2) and (H4) as well.

As Martin showed, the tangent space to $\mathrm{Def}_{\mathcal{F},\mathcal{E}}$ is given by $H^0(X, \mathcal{H}om(\mathcal{G}, \mathcal{F}))$, where $\mathcal{G}$ is the kernel of the map $\mathcal{E} \twoheadrightarrow \mathcal{F}$. [[★★★ exercise]] If we assume $\mathcal{E}$ is coherent and X_Λ proper, then the quotient and kernel are coherent, so this H^0 is a finite dimensional vector space, so we have (H3). ☐

Corollary 5.4. *Suppose X_Λ is a scheme over Λ with restriction X to k, and $Z \subseteq X$, then $\mathrm{Def}_{Z,X}$ is a deformation functor and satisfies (H4). If X_Λ is proper over Λ, then (H3) is satisfied, so $\mathrm{Def}_{Z,X}$ is prorepresentable.*

Proof. Set $\mathcal{E}_\Lambda = \mathcal{O}_{X_\Lambda}$. Then closed subschemes are exactly quasi-coherent quotients of this structure sheaf. ☐

Example 5.5. Suppose we're given X_Λ and Y_Λ schemes over Λ. Suppose we're also given a morphisms $f\colon X \to Y$, then we have $\mathrm{Def}_f : A \mapsto \{f_A : X_\Lambda|_A \to Y_\Lambda|_A$ over A restricting to f on $k\}$. Then we have the following result.

Corollary 5.6. *If X_Λ and Y_Λ are locally of finite type over Λ and X_Λ is flat over Λ and Y_Λ is separated over Λ, then Def_f satisfies (H1), (H2), (H4). If X_Λ and Y_Λ are proper over Λ, then you also get (H3). Martin: you probably don't need Y_Λ to be proper. Brian: that's probably true.*

The proof goes by looking at the graph of the morphism. The separated hypothesis comes in so that the graph is a closed subscheme. The flatness implies that the deformations of the graph are flat. You also need flatness to get that when you deform the graph, you still get a morphisms. EGA: you can check if a morphism is an isomorphism as long as the base is flat. ◇

6 Lieblich

Yesterday we got stacky. We saw that most of our moduli problems were stacks. Today, we'll talk about what it means for a stack to be a geometric object. Let S be a scheme. We'll ultimately consider stacks on the big étale site on $\mathbf{Sch}_S$.

Recall $\mathbb{P}^n$. There are two competing ways to describe $\mathbb{P}^n$.

1. $h_{\mathbb{P}^n}(T) = \{\mathcal{O}_T^{n+1} \twoheadrightarrow \mathcal{L} | \mathcal{L} \text{ invertible on } T\}/\cong$.
2. "$\mathbb{P}^n = (\mathbb{A}^{n+1} \smallsetminus \{0\})/\mathbb{G}_m$" where $\mathbb{G}_m(T) = \Gamma(T, \mathcal{O}_T^\times)$ (or $\mathbb{G}_m = \operatorname{Spec} \mathbb{Z}[t, t^{-1}]$, if you like). We put this in quotes because we don't really know what that means.

Let's pretend that the second definition makes sense. $\mathbb{G}_m$ acts freely on $\mathbb{A}^{n+1} \smallsetminus \{0\}$, so the natural map $\mathbb{A}^{n+1} \smallsetminus \{0\} \to \mathbb{P}^n$ should be a $\mathbb{G}_m$-torsor. Then we can pull back.

$$\begin{tikzcd}
T \arrow[r, "\mathbb{G}_m\text{-eq}"] \arrow[d, "\mathbb{G}_m\text{-tors}"'] & \mathbb{A}^{n+1} \smallsetminus \{0\} \arrow[d] \\
A \arrow[r] & \mathbb{P}^n
\end{tikzcd}$$

Equivariance means that for all $t \in T$ and $\alpha \in \mathbb{G}_m$, $f(\alpha t) = \alpha f(t)$

The following proposition is an exercise.

Proposition 6.1. *There is a natural equivalence of categories, {relatively affine X-schemes with $\mathbb{G}_m$-action and $\mathbb{G}_m$-equivariant morphisms} is equivalent to {$\mathbb{Z}$-graded quasi-coherent $\mathcal{O}_X$-modules with graded morphisms}$^\circ$.*

The idea: given $f\colon Y \to X$ relatively affine, we get a $\mathbb{G}_m$-action on $f_*\mathcal{O}_Y$ over X. This breaks up as a sum of eigen-sheaves indexed by the characters of $\mathbb{G}_m$ (which are $\mathbb{Z}$). A character is a map $\mathbb{G}_m \to \mathbb{G}_m$, and each one is given by $t \mapsto t^n$.

Example 6.2. The action of $\mathbb{G}_m$ on $\mathbb{A}_X^{n+1} = \mathcal{S}pec_X\, \mathcal{O}_X[x_1, \ldots, x_{n+1}]$, where we have a grading by total degree, and this is the grading given by the action of $\mathbb{G}_m$. That is, if $x_i \mapsto tx_i$, then a monomial of degree d is multiplied by t^d. Ravi: how do you know you shouldn't be multiplying by t^{-1} instead of t? Max: ok, you're right; it should be. The usual action is $t \cdot (a_1, \ldots, a_{n+1}) = (t^{-1}a_1, \ldots, t^{-1}a_{n+1})$. We'll use this other action so that I don't get all my signs wrong. Hopefully, it will all work out. ◇

Say $T \to X$ is a $\mathbb{G}_m$-torsor. We know that this thing is relatively affine (because $\mathbb{G}_m$ is affine). By this proposition, this corresponds to some graded sheaf of algebras on X. Which graded sheaves of algebras correspond to $\mathbb{G}_m$-torsors?

Proposition 6.3. *Given a $\mathbb{G}_m$-torsor $T \to X$, there is an invertible sheaf $\mathcal{L}$ on X such that $T \cong \mathcal{S}pec_X \bigoplus_{i\in\mathbb{Z}} \mathcal{L}^{\otimes i} \subseteq \mathcal{S}pec_X \bigoplus_{i\geq 0} \mathcal{L}^{\otimes i} = \overline{T}$.*

Proof. fppf (or étale, or whatever) locally on X, $T \cong \mathcal{S}pec_X\, \mathcal{O}_X[x, x^{-1}]$ with the $\mathbb{G}_m$ action corresponding to the natural grading. A descent datum is a graded isomorphism $\mathcal{O}[x, x^{-1}] \xrightarrow{\sim} \mathcal{O}[x, x^{-1}]$. Note that each graded piece has the form $x^i\mathcal{O}$. Something about gluing and you get the thing you want. You can see that all the invertible sheaves you get are all powers of the same fixed $\mathcal{L}$. ☐

Ravi: a $\mathbb{G}_m$-torsor is trivial locally in the fppf topology, but invertible sheaves are trivial Zariski locally, what is going on? Max: Magic, it turns out that the torsors are Zariski-locally trivial. This is Hilbert theorem 90 when you interpret étale cohomology as Galois cohomology.

A $\mathbb{G}_m$-equivariant map

$$\begin{tikzcd}
\mathcal{S}pec_X \bigoplus \mathcal{L}^i \arrow[r, phantom, "="] & T \arrow[r] \arrow[d, hook] \arrow[dr] & \mathbb{A}^{n+1} \smallsetminus \{0\} \arrow[d, hook] & \\
\mathcal{S}pec_X \bigoplus_{i\geq 0} \mathcal{L}^i \arrow[r, phantom, "="] & \overline{T} \arrow[r] & \mathbb{A}^{n+1} \arrow[r, phantom, "="] & \mathcal{S}pec_X\, \mathcal{O}_X[x_1, \dots, x_{n+1}]
\end{tikzcd}$$

We get $\mathcal{O}_X[x_1, \dots, x_{n+1}] \to \bigoplus_{i\geq 0} \mathcal{L}^i$ a graded morphism of sheaves of algebras. This gives you $\mathcal{O}_X^{n+1} \twoheadrightarrow \mathcal{L}$ (surjective because else it's image in some fiber is zero, so the point doesn't map into the complement of 0). Conclusion: the functor of point tells us that in fact $\mathbb{A}^{n+1} \smallsetminus \{0\} \to \mathbb{P}^n$ is a $\mathbb{G}_m$-torsor.

Sitting in your brains should be the fact that $h_{\mathbb{P}^n_k}$ is $\{W \subseteq V | \dim W = n-1\}$ where $\dim V = n$.

Suppose we have a group scheme G and a scheme X, with a right action of G on X. We would love to make a quotient X/G such that $X \to X/G$ is a G-torsor. But there are many funny actions which could be non-free. The moral of the story so far is that the following definition is reasonable.

Definition 6.4. The *quotient stack* $[X/G]$ has a fiber category over Y whose objects are pairs $(T \to Y, \phi)$, where $T \to Y$ is a G-torsor and $\phi : T \to X$ is G-equivariant, and the arrows are maps of G-torsors $\psi : T \to T'$ respecting the G-equivariant map to X. It follows, by the way, that ψ must be an isomorphism. ◇

Note that there is a natural map $\nu : X \to [X/G]$ given by the trivial torsor $X \times G \to X$, with G-equivariant map $X \times G \mapsto X$ given by the action of G on X (it follows from the axioms of an action that this map is G-equivariant).

Claim. *ν makes X into a G-torsor over $[X/G]$.*

Proof. This is roughly equivalent to saying that whenever you pull back, you get a G-torsor.

$$\begin{tikzcd}
Y \times_{[X/G]} X \arrow[r] \arrow[d] & X \arrow[d] \\
Y \arrow[r] & {[X/G]}
\end{tikzcd}$$

But what the heck is a fiber product of two schemes over a stack? ☐

Definition 6.5. Given morphisms of stacks $\mathcal{X} \xrightarrow{\alpha} \mathcal{Z} \xleftarrow{\beta} \mathcal{Y}$, we define the fiber product $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ to have fibers $(\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y})_T = \{(x, y, \phi) | x \in \mathcal{X}_T, y \in \mathcal{Y}_T, \phi : \alpha(x) \xrightarrow{\sim} \beta(y)$ an arrow in $\mathcal{Z}_T\}$. A morphism $(x, y, \phi) \to (x', y', \phi')$ is a pair of morphisms $x \to x'$ and $y \to y'$ such that the diagram

$$\begin{tikzcd}
\alpha(x) \arrow[r] \arrow[d, "\phi"'] & \alpha(x') \arrow[d, "\phi'"] \\
\beta(y) \arrow[r] & \beta(y')
\end{tikzcd}$$

◇

Exercise: check that this fiber product is a stack.

Question: if you're working in the flat topology, should you assume G is a flat group scheme? Max: not necessarily, but then it will not be clear that what you get is an algebraic stack.

Example 6.6. Let $\nu : X \to [X/G]$ be the natural map, and let $U \to Y$ be a G-torsor with G-equivariant map $U \to X$. Let's try to compute $(X \times_{[X/G]} Y)_T$.

1. $x \in X(T)$, $y \in Y(T)$
2. $\nu(x) \in [X/G]$ corresponds to the pullback of $X \leftarrow X \times G \to X$ to $T \leftarrow T \times G \to X$, and $\beta(y) \in [X/G]$, $\begin{tikzcd} y^*U \arrow[r] \arrow[d] & U \arrow[r] & X \\ T \end{tikzcd}$

Now ϕ: $\begin{tikzcd} G \times T \arrow[r] \arrow[d] & y^*U \arrow[r] \arrow[dl] & X \\ T \end{tikzcd}$, which is a choice of a point of $U(T)$.

The isoms: $(x, y, \phi) \xrightarrow{\sim} (x', y', \phi')$ has to be a pair of isomorphisms $x \to x'$ and $y \to y'$. Since X and Y are schemes, there are only equalities, which implies that $\phi = \phi'$. Thus, the only isomorphisms are equalities.

So $\begin{tikzcd} X \times_{[X/G]} Y \arrow[r] \arrow[d, phantom, "\wr\|"] & Y . \\ U \arrow[ur, "G\text{-torsor}"'] \end{tikzcd}$ ◇

Now go home and try to do this for various actions you might know. If the action is free, then you won't get anything stacky.

Example 6.7. Say $X \xrightarrow{f} \mathcal{Z}$ and $Y \xrightarrow{g} \mathcal{Z}$ are schemes over a stack $\mathcal{Z}$. Then $(X \times_{\mathcal{Z}} Y)_T$ has objects (α, β, ϕ) where $\alpha \in X(T)$, $\beta \in Y(T)$, and ϕ: $f \circ \alpha \xrightarrow{\sim} g \circ \alpha$. This is equivalent to $(f \circ pr_1) \circ (\alpha \circ \beta) \xrightarrow{\sim} (f \circ pr_2) \circ (\alpha \times \beta)$.

$$\begin{tikzcd}
T \arrow[r, "\alpha{,}\beta"] & X \times Y \arrow[r, "pr_2"] \arrow[d, "pr_1"'] & Y \arrow[d, "g"] \\
& X \arrow[r, "f"] & \mathcal{Z}
\end{tikzcd}$$

What we have is a map $X \times_{\mathcal{Z}} Y \to X \times Y$. If you think carefully, you'll see that this map is a very concrete thing. You'll see that $X \times_{\mathcal{Z}} Y \to X \times Y$ is $\underline{\mathrm{Isom}}(pr_1^* f, pr_2^* g)$. Note that as before, we don't get any morphisms in the fibers because you can't have morphisms between points of schemes, so it is not surprising that the fiber product is fibered in sets. ◇

Definition 6.8. We'll say that a morphism of stacks $\mathcal{X} \to \mathcal{Y}$ is *representable (by schemes)* if for all maps $T \to \mathcal{Y}$ from a scheme T, $\mathcal{X} \times_{\mathcal{Y}} T$ is equivalent to the fibered category associated to a scheme. ◇

There is one other example you should think about, which is the most stacky exmaple. Consider $[*/G]$ for a group G. The fiber category over a scheme T is supposed to consist of G-torsors over T with an equivariant map to a point (which doesn't add any data). This is $BG = [*/G]$, the fiber category of G-torsors. For example, $* \to [*/(\mathbb{Z}/2)]$ is a finite étale map of degree 2.

6 Olsson

Recall some stuff from last time. We're talking about Picard stacks. Recall that if you have a topological space T, then a Picard stack is a stack $\mathcal{P}$ with a functor $+$, an assisiativity isomorphism σ, and a commutativity isomorphism τ.

Consider a two term comlex $K^\bullet \in C^{[-1,0]}(T)$ concentrated in degrees -1 and 0 $K^{-1} \xrightarrow{d} K^0$. Associated to such a guy, we get a Picard prestack $pch(K)$. The objects over U are elements $x \in K^0(U)$, and a morphism $x \to y$ in the fiber over U is an element $z \in K^{-1}(U)$ such that $dz = y - x$. The additive structure is induced by the additive sturcture on K^0 and K^1. There was this exercise (which I image nobody did) that you can stackify to get $ch(X)$. That is, if $\mathcal{P}$ is a Picard stack, then $\mathrm{HOM}_{picstks}(ch(K), \mathcal{P}) \to \mathrm{HOM}_{picprestks}(pch(K), \mathcal{P})$ is an equivalence of categories.

Brian: if you forget the Picard structure, is the stackification the stackification of the underlying stack, or does the Picard structure do something? Martin: it turns out that you get the same thing.

Remark 6.1. The map $pch(K) \to ch(K)$ is fully faithful (more generally, this is true for stackification). Then it is enough (to answer Brian's question) to show that every object is locally in the image, which is true. ◇

Don't worry too much about the stackification. We'll see that sometimes you get a stack from the complex, and you don't have to stackify.

Remark 6.2. If we have a morphism of complexes $f\colon K_1^\bullet \to K_2^\bullet$, then we get an induced morphism of Picard stacks $ch(f) : ch(K_1) \to ch(K_2)$. To see this, it is enough to get the morphism for the prestacks (by the universal property). So for every object $x \in K_1^0$, you need to get an object K_2^0, which the map f gives you. Similarly, you get the functor on morphisms.

If we have $f_1, f_2\colon K_1^\bullet \to K_2^\bullet$, then you can have a homotopy $h\colon K_1^0 \to K_2^{-1}$ between them such that for every x, $f_1(x) - f_2(x) = dh(x)$, and $f_1^{-1} - f_2^{-1} = hd$. Normally a homotopy is a map in each degree, but here since our complexes are concentrated in degree's -1 and 0, we only get one non-zero map h. Anyway, this gives is an isomorphism of morphisms $ch(h)\colon ch(f_1) \to ch(f_2)$. Again, it is enough to check on the prestack level. That is, for every object $x \in pch(K_1)$, an isomorphism $ch(f_1)(x) \to ch(f_2)(x)$. For each $x \in K_1^0$, we want some $z \in K_2^{-1}$ such that $dz = f_2(x) - f_1(x)$. Our homotopy gives us this data (exercise: check this). ◇

Lemma 6.3. *If K^{-1} is a flasque sheaf, then the prestack $pch(K)$ is a stack.*

Proof. We have a map $\pi\colon pch(K) \to ch(K)$, which we want to show is an equivalence. We have that it is fully faithful, so we just need to show that it is essentially surjective. Let $U \subseteq T$ be open and suppose $x \in ch(K)_U$. We need to produce a section of K^0 over U so that x is isomorphic to that. Let $\mathcal{L}$ be the sheaf on U given by $V \mapsto \{(y, \ell) | y \in K^0(V), \ell\colon \pi(y) \xrightarrow{\sim} x|_V \text{ in } ch(K)_V\}$. We want to say that $\mathcal{L}$ has a section over U.

Claim: $\mathcal{L}$ is a $K^{-1}|_U$-torsor. Reason: suppose (y', ℓ') is a second section of $\mathcal{L}$ over V. Then you get an isomorphism $\pi(y) \xrightarrow{\ell} x|_V \xrightarrow{\ell'^{-1}} \pi(y')$, which must be (unique by full faithfulness) some element $z \in K^{-1}(V)$. But we know that K^{-1}-torsors are classified by $H^1(U, K^{-1}) = 0$, so any K^{-1}-torsor is trivial, so $\mathcal{L}$ is the trivial K^{-1}-torsor. In particular, it has a section.

David Brown: why does $\mathcal{L}$ locally have a section? Martin: you know that any object in $ch(K)$ is locally in the image of the morphism $pch(K) \to ch(K)$. Ravi: this is similar to the fact that any section of a sheafification is locally a section of the presheaf. Martin: yes, the way you form the stackification is as the stack of descent data, so this is built into the construction. □

Remark 6.4 (a)**.** Consider the sheaf associated to the presheaf $U \mapsto$the set of isomorphism classes in $ch(K)_U$. It is $K^0/\operatorname{im}(K^{-1} \to K^0) = \mathcal{H}^0(K^\bullet)$ (remember that there is sheafification going on when you do the quotient). ◇

Remark 6.5 (b)**.** What is the automorphism group of an object $x \in ch(K)_U$? It is $\mathcal{H}^{-1}(K^\bullet)$, the kernel of $d\colon K^{-1} \to K^0$. This is because $\operatorname{Aut}(x) = \{z \in K^{-1}(U) | dz = x - x = 0\}$. ◇

Corollary 6.6. *If $f\colon K_1^\bullet \to K_2^\bullet$ is a quasi-isomorphism of complexes, then the morphism of Picard stacks $ch(f)\colon ch(K_1) \to ch(K_2)$ is an equivalence.*

You have to check that if you have a morphism of stacks which induces an isomorphism on these associated sheaves and on automorphism groups.

Lemma 6.7. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of stacks over some site, and let $\overline{f}: X \to Y$ be the corresponding morphism of sheaves of isomorphism classes. Assume that $\overline{f}$ is an isomorphism, and that for every $U \subseteq T$ and $x \in \mathcal{X}_U$, the map of sheaves $\operatorname{Aut}_{\mathcal{X}}(x) \to \operatorname{Aut}_{\mathcal{Y}}(f(x))$ is an isomorphism. Then f is an equivalence.*

Sketch proof. Given $x, y \in \mathcal{X}_U$, we want to check that $\underline{\operatorname{Isom}}_{\mathcal{X}}(x, y) \to \underline{\operatorname{Isom}}_{\mathcal{Y}}(f(x), f(y))$ $(*)$ to be an isomorphism (this would imply that when you take sections of these sheaves over U, you get full faithfulness).

(injectivity) suppose $\alpha, \beta\colon x \to y$ with $f(\alpha) = f(\beta)\colon f(x) \to f(y)$. This means that $\alpha^{-1} \circ \beta \in \ker(\operatorname{Aut}_{\mathcal{X}}(x) \to \operatorname{Aut}_{\mathcal{Y}}(f(x)))$, which we assumed was zero. [[★★★ both $\underline{\text{Isom}}$s are pseudo-torsors under the automorphism sheaves]]

(surjectivity) suppose $\sigma\colon f(x) \to f(y)$. It is enough to show that σ is in the image locally. The point is that x and y map to the same thing in X ($f(x)$ and $f(y)$ are isomorphic, so we get local isomorphisms between x and y). Locally, let $\tau : x \to y$ be an isomorphism. Then $\sigma^{-1} \circ f(\tau)$ is an automorphism of $f(x)$, so we can tweak our choice of τ to make it agree with σ.

(essential surjectivity) Let $y \in \mathcal{Y}$. Shrinking our space T, we can assume $y \in \mathcal{Y}_T$. We want to show that it comes from an object in $\mathcal{X}$. Since $\overline{f}$ is an isomorphism, there is some covering $T = \bigcup U_i$ and (x_i, ℓ_i) with $x_i \in \mathcal{X}_{U_i}$ and $\ell_i\colon f(x_i) \xrightarrow{\sim} y|_{U_i}$ in $\mathcal{Y}_{U_i}$. We'd like to glue them together. We have descent for objects in $\mathcal{X}$. On U_{ij}, there is a unique isomorphism $\sigma_{ij}\colon x_i|_{U_{ij}} \to x_j|_{U_{ij}}$ such that the diagram commutes

$$\begin{tikzcd}
f(x_i)|_{U_{ij}} \arrow[r, "f(\sigma_{ij})"] \arrow[d, "\ell_i"'] & f(x_j)|_{U_{ij}} \arrow[dl, "\ell_j"] \\
y|_{U_{ij}} &
\end{tikzcd}$$

because of full faithfulness already shown. Now we have that $\sigma_{ij} \circ \sigma_{jk}, \sigma_{ik}\colon x_i|_{U_{ijk}} \to x_k|_{U_{ijk}}$ are both equal to the unique morphism

$$\begin{tikzcd}
f(x_i)|_{U_{ijk}} \arrow[rr, dashed, "\exists!"] \arrow[dr, "\ell_i"'] & & f(x_k)|_{U_{ijk}} \arrow[dl, "\ell_k"] \\
& y|_{U_{ijk}} &
\end{tikzcd}$$

Now we get a global guy x together with descent data for an isomorphism to y, so you get an isomorphism to y. □

Let $\tilde{C}^{[-1,0]}(T) \subseteq C^{[-1,0]}(T)$ be the full subcategory of complexes $K^{-1} \to K^0$ with K^{-1} injective.

Theorem 6.8. *The ch construction induces an equivalence of 2-categories $\tilde{C}^{[-1,0]}(T) \to$(Picard stacks over T).*

Corollary 6.9. *The category (Picard stacks, isomorphism classes of morphisms) is equivalent to $\mathcal{D}^{[-1,0]}(T)$ (derived category with complexes concentrated in degrees -1 and 0).*

Remark 6.10. Why are we doing all this? We saw the example of Exal and some other deformation problems, which come to us as Picard stacks. So there is always some two term complex sitting in the background controlling everything. This will be the first approximation of the cotangent complex. ◇

Let's sketch a proof of the Theorem, or at least state the lemmas involved.

Lemma 6.11. *Suppose $\mathcal{P}$ is a Picard stack over T, suppose $\{U_i\}$ is a collection of open subsets, and $k_i \in \mathcal{P}(U_i)$ for each i. Define $K = \bigoplus_i \mathbb{Z}_{U_i}$ ($\mathbb{Z}_{U_i} = j_!\mathbb{Z}$ where $j\colon U_i \hookrightarrow T$; the stalks are zero outside of U_i and $\mathbb{Z}$ for points in U_i). Then there exists a morphism $f\colon ch(0 \to K) \to \mathcal{P}$ (note that the first guy is just the sheaf K; there are no morphisms) and isomorphisms $\sigma_i\colon F(1 \in \mathbb{Z}_{U_i}(U_i)) \xrightarrow{\sim} k_i$ (i.e. you can make consistent choices) and the data $(F, \{\sigma_i\})$ is unique up to unique isomorphism.*

To prove this lemma, you need to use all the complicated axioms of a Picard stack. We won't prove it.

Lemma 6.12. *Let $\mathcal{P}$ be a Picard stack over T. Then there exists a complex $K^\bullet \in C^{[-1,0]}(T)$ and an isomorphism $ch(K) \xrightarrow{\sim} \mathcal{P}$.*

Note that we can always replace a complex quasi-isomorphically so that the first term is injective, and once we know full faithfullness, this will tells us what we want with the $\tilde{C}^{[-1,0]}(T)$.

Example 6.13. $\mathcal{P}ic(X)$ is $ch(\mathcal{O}_X^\times \to 0)$. Exercise! ◇

Proof. Choose data

- $\{U_i \subseteq T\}_{i \in I}$
- for all i some $k_i \in \mathcal{P}(U_i)$

such that for all $V \subseteq T$ and $k \in \mathcal{P}_V$, there is a covering $V = \bigcup V_j$ such that $k|_{V_j} \cong k_i$ for some i with $V_j \subseteq U_i$. Define $K^0 = \bigoplus \mathbb{Z}_{U_i}$. Then we get a map $ch(0 \to K^0) \to \mathcal{P}$ by the previous lemma. This is essentially surjective because all objects are in the image, but the morphisms are all wrong.

Define $K^{-1}(V) = \{(x, \ell) | x \in K^0(V), \ell\colon F(0) \xrightarrow{\sim} F(x)\}$ and define $K^{-1} \to K^0$ by $(x, \ell) \mapsto x$. We define $(x, \ell) + (x', \ell') = (x + x', ?)$. We set $?\colon F(0) \xrightarrow{\sim} F(0) + F(0) \xrightarrow{\ell + \ell'} F(x) + F(x') \cong F(x + x')$. What is an arrow $x \to x'$ in $pch(K^{-1} \to K^0)$? It is a pair $(x' - x, \ell)$ where $\ell\colon F(0) \xrightarrow{\sim} F(x' - x)$. This gives us a map

$$\begin{tikzcd}
F(0) + F(x) \arrow[r, "\sim"] \arrow[d, equal] & F(x' - x) + F(x) \arrow[d, equal] \\
F(x) \arrow[r] & F(x')
\end{tikzcd}$$

□

6 Osserman

Dimensions of hulls. Mori used a lower bound on the dimension of a space of morphisms (given in terms of the tangent and obstruction spaces) as a key technical tool to prove some really amazing theorems on the existence of rational curves on various varieties. We saw this in Jason Starr's talk. This was basically the start of the minimal model program (more generally). Today, we'll focus on an exposition of this key technical result.

You may recall that Jason talked about a universal obstruction space. We'll do a less informative, but shorter version. We need some background on obstruction theories first.

Definition 6.1. A morphism $\pi\colon A' \to A$ in $Art(\Lambda, k)$ is a *thickening* if it is surjective and $\ker \pi$ is annihilated by the maximal ideal $\mathfrak{m}_{A'}$ of A'. That is, $\ker \pi$ actually have the structure of a k vector space. ◇

In fact, this is exactly the sort of thing that showed up in Martin's definition of a deformation situation (we take $A_0 = k$). This is the typical situation that showes up with obstruction theories. Unfortunately, we need to use a slightly different definition of an obstruction theory from what Martin said.

Definition 6.2. Given a predeformation functor F, an *obstruction theory for* F is a k-vector space V, and for all thickenings $\pi\colon A' \to A$ and all elements $\eta \in F(A)$, an element $ob(\eta, A') \in V \otimes_k \ker \pi$, satisfying the following conditions.

1. $ob(\eta, A) = 0$ if and only if there is some $\eta' \in F(A')$ which restricts to $\eta \in F(A)$.
2. If $$\begin{tikzcd} A' \arrow[rr] \arrow[dr] & & A \\ & B \arrow[ur] & \end{tikzcd}$$ is an intermediate thickening with $\ker(A' \to A) = I$, $\ker(A' \to B) = J$, $ob(\eta, B)$ is induced by $ob(\eta, A')$ by the natural map $V \otimes I \to V \otimes (I/J)$. ◇

Theorem 6.3. *Suppose F has a hull (R, ξ) and an obstruction theory with values in some vector space V. Then* $\dim \Lambda + \dim T_F - \dim V \le \dim R \le \dim \Lambda + \dim T_F$*. If Λ is regular (which is usually the case) and if the first inequality is an equality, then R is a complete intersection inside* $\Lambda[[t_1, \dots, t_r]]$.

This is a great theorem. As we saw in Martin's lectures, there are many examples where you can write down the tangent space and the obstruction space.

To prove the theorem, we'll reduce to the prorepresentable case.

Lemma 6.4. *Suppose $f\colon F_1 \to F_2$ is a smooth morphism of predeformation functors and suppose we have an obstruction theory for F_2 taking values in some vector space V. Then we obtain an obstruction theory for F_1 taking values in the same V.*

Proof. Given a thickening $A' \to A$ and some $\eta \in F_1(A)$, set $ob(\eta, A') := ob(f(\eta), A')$. By smoothness, this satisfies the first condition of an obstruction theory. The second condition follows from the definition of a morphism of functors. □

Proof of Theorem. The lemma reduces to the case where $F = \overline{h}_R$ is prorepresentable since by definition of a hull, $\overline{h}_R \to F$ is smooth, so checking the condition is equivalent to checking it for the functor represented by R.

Let $d = \dim T_R$. Schlessinger, in the proof of his theorem, constructs R as a quotient S/J of the power series ring $S = \Lambda[[t_1, \ldots, t_d]]$. It is enough to prove that J can be generated by $\dim V$ elements. The upper bound already follows from the fact that $R \cong S/J$. Now by the hauptidealsatz, we know that the dimension drops by at most one each time, so it is enough to show that there are at most $\dim V$ of them. If the drop in the dimension is exactly right, then we have a complete intersection.

By the Artin-Rees lemma, we know that $J \cap \mathfrak{m}_S^n \subseteq J\mathfrak{m}_S$ for some sufficiently large n. Set $A' = \Lambda[[t_1, \ldots, t_r]]/(\mathfrak{m}_S J + \mathfrak{m}_S^n)$ and set $A = \Lambda[[t_1, \ldots, t_r]]/(J + \mathfrak{m}_S^n)$. This give us a thickening $0 \to I \to A' \to A \to 0$. From the definitions, $I = (J + \mathfrak{m}_S^n)/(\mathfrak{m}_S J + \mathfrak{m}_S^n) = J/\mathfrak{m}_S J$. We have a natural quotient map $R = S/J \to A$. From this, we get an object in $\xi_A \in \overline{h}_R(A)$ (just by definition), so we get an obstruction $ob(\xi_A, A') \in V \otimes I$ to lifting it to A'. The main idea underlying the whole construction is that you want to show something about J; what you show is that there are some elements (at most $\dim V$ of them) that you can kill in order to kill the obstruction. We can write $ob(\xi_A, A') = \sum_{j=1}^{\dim V} v_j \otimes \overline{x}_j$, where the v_j form a basis for V and the $\overline{x}_j$ are the images of some elements $x_j \in J$. Our goal to to show that the x_j generate J. By Nakayama's lemma, it is enough to show that the $\overline{x}_j$ generate $I = J/\mathfrak{m}_S J$.

Consider the ring $B = A'/(\overline{x}_j)$, which is an intermediate quotient $A' \to B \to A$. We know that B surjects on to A with some kernel I'. We get some obstruction $ob(\xi_A, B) \in V \otimes I'$ to lifting ξ_A to B. By the functoriality condition, this obstruction is the image of the previous obstruction $ob(\xi_A, A') = \sum v_j \otimes \overline{x}_j$, so it is zero! So we know that we have a lift $R \to B$.

$$\begin{tikzcd}
S = \Lambda[[t_1, \ldots, t_d]] \arrow[r] \arrow[d, dashed] & R \arrow[d] \arrow[dr] & \\
S = \Lambda[[t_1, \ldots, t_d]] \arrow[r] & B \arrow[r] & A
\end{tikzcd}$$

We want to show that $J \subseteq \mathfrak{m}_S J + (x_i) + \mathfrak{m}_S^n = \ker(X \to B)$ because this will show what we want. By looking at kernels, we see that this is equivalent to saying that the diagram above commutes if we take the dashed arrow to be the identity map. We can choose some $\phi : S \to S$ making the diagram commute just by choosing appropriate values for the t_i, and use the existence of this map to give a direct argument for the containment we want. By definition, ϕ commutes with the two maps to A, so it is the identity modulo $\ker(S \to A) = J + \mathfrak{m}_S^n$. In particular, ϕ is the identity on $\mathfrak{m}_S/\mathfrak{m}_S^2$ (we took J to have the minimal number of elements possible, so T_R is the same as T_S). From the exercises (hulls are unique up to isomorphism) ϕ is an isomorphism. Then $\phi^{-1}(J) \subseteq J + \mathfrak{m}_S^n$, so $J \subseteq \phi(J) + \phi(\mathfrak{m}_S^n) = \phi(J) + \mathfrak{m}_S^n$. By commutativity of the square, we have that $\phi(J) \subseteq \mathfrak{m}_S J + (x_i) + \mathfrak{m}_X^n$, so $J \subseteq \phi(J) + \mathfrak{m}_S^n \subseteq \mathfrak{m}_S J + (x_i) + \mathfrak{m}_S^n$, as desired. □

Example 6.5. Jason Starr told us about one nice example. Say X and Y are smooth varieties, and $f\colon X \to Y$. Consider Def_f. It is a fact that the tangent space T_{Def_f} is given by $H^0(X, f^*T_Y)$ (Brian: have we done this? Martin: this is just the universal property of differentials, isn't it?) and there is an obstruction theory in $H^1(X, f^*T_Y)$. Since Y is smooth, T_Y is locally free. If X is a curve, then $h^0 - h^1$ of f^*T_Y is the Euler characteristic $\chi(f^*T_Y)$, which is computed by Riemann-Roch, so you can get your hands on it. $\diamond$

Example 6.6. Deformations of a smooth surface X. We have that the tangent space is $H^1(X, T_X)$ and there is an obstruction theory in $H^2(X, T_X)$. If we understand $H^0(X, T_X)$ (which we often do), then we can compute $h^1 - h^2$ of T_X by computing the Euler characteristic $\chi(T_X)$, which you can get from Riemann-Roch for surfaces. If X has finite (or discrete) automorphism group (in characteristic 0), then we know that $H^0(X, T_X) = 0$ (because we know that sections of T_X parameterize infinitesimal automorphisms of X). $\diamond$

7 Lieblich

Today we're finally ready to think about geometry.

Algebraic Stacks. The vague hippy question for today is "what is geometry?" We've talked about gluing and topology so far. One answer: it is the local structure (on top of topology).

Example 7.1. Let F be a sheaf on S_{fppf} (for S a scheme). When is this thing a scheme? The claim is that F is a scheme if and only if there exists a scheme U and a map $U \xrightarrow{a} F$ (of functors) which is Zariski-locally an isomorphism. That is, there exists an open covering $\{G_i \subseteq F\}$ such that for each i, there exists $U_i \subseteq U$ open so that $U_i \xrightarrow{\sim} G_i \hookrightarrow F$.

Today I really want to emphasize the map a, which is sort of like an atlas. Really, a is a *uniformization.* ◇

A sheaf with a uniformization should be a nice geometric object.

Definition 7.2 (temporary)**.** An *étale algebraic space* over S is a sheaf F on S_{ET} such that there exists a scheme U and a surjective étale representable morphism $U \to F$. Similarly, an fppf algebraic space is an fppf sheaf with a surjective fppf representable map from a scheme. Martin: you need to add some hypotheses; $F \to F \times F$ must be quasi-affine, or something like this. ◇

Max: we'll always assume (1) F is locally of finite presentation over S (given the next hypotheses, we can just require U to be locally of finite presentation), (2) $F \to F \times_S F$ is representable and of finite type (so quasi-affine). (3) all diagonals are of finite type.

The condition of representablity is somehow saying that the map carries geometric information.

Theorem 7.3 (Artin)**.** *(Martin: if the diagonal of finity type) An fppf algebraic space is an étale algebraic space.*

Ravi: do people use different definitions for algebraic spaces. Martin: the answer is no . . . look in Artin's papers. Max: Yes, Artin makes a very weak definition, but all the things in nature will satisfy any hypotheses you want. All these should be different: DM stacks, Artin stacks, and algebraic stacks [[★★★ really, what's the difference between the last two?]].

You saw on the exercises that there exists a smooth 3-fold T over $\mathbb{C}$ with a descent datum with respect to the cover $\operatorname{Spec}\mathbb{C} \to \operatorname{Spec}\mathbb{R}$ which is not effective. BUT there does exist a sheaf $\overline{T}$ over $\mathbb{R}$ such that $\overline{T} \otimes \mathbb{C} = T$, and $T \to \overline{T}$ is a finite étale map. You shouldn't throw out $\overline{T}$ just because it started life over the wrong base.

More examples: group quotients, contractions, etc. You can't do all of these constructions in schemes, but you can in algebraic spaces.

What about stacks? What does it mean to uniformize a stack in some topology? You have to keep track of objects and isomorphisms.

Definition 7.4. A stack $\mathcal{X}$ on S_{ET} is a *Deligne-Mumford stack (or DM stack)* if

1. $\mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is representable (by schemes), quasi-compact, and separated, and
2. there exists an étale surjection $X \to \mathcal{X}$ from a scheme X. ◇

Note that (1) actually implies that *any* map from a scheme to $\mathcal{X}$ is representable. These are what Deligne and Mumford called algebraic stacks.

(1) says: for all $f \colon T \to \mathcal{X}$ and $g \colon T' \to \mathcal{X}$, then $\underline{\mathrm{Isom}}(pr_1^* f, pr_2^* g) \to T \times T'$ is a quasi-compact separated map of schemes.

(2) says: if X is our cover and p is a point which (say) factors through X, then the product $X \times_{\mathcal{X}} p \to p$ is étale with section.

$$\begin{tikzcd}
X \times_{\mathcal{X}} p \arrow[r] \arrow[d] & X \arrow[d] \\
p = \operatorname{Spec} \bar{k} \arrow[r] \arrow[u, dashed, bend left] \arrow[ur, dashed] & \mathcal{X}
\end{tikzcd}$$

Sitting in $X \times_{\mathcal{X}} p$ is the automorphism sheaf $\mathrm{Aut}(p)$. Since $X \times_{\mathcal{X}} p \to p$ is étale, so is $\mathrm{Aut}(p) \to p$. This happens if and only if there are no (non-trivial) infinitesimal automorphisms of the object parameterized by p (thinking of $\mathcal{X}$ as a moduli space). Thus, when we saw infinitesimal automorphisms in some of our moduli problems, those moduli spaces couldn't possibly have been DM.

What about the things that came up where we did have lots of infinitesimal automorphisms, like $B\mathbb{G}_m$. This is what leads to the definition of an Artin stack.

Definition 7.5. An *Artin stack* on S_{ET} is a stack $\mathcal{X}$ such that

1. $X \to \mathcal{X} \times_S \mathcal{X}$ is representable (by algebraic spaces)[1], quasi-compact, and separated. (Martin: there is only one place where people assume separatedness of the diagonal)
2. there exists a scheme X and a smooth surjection $X \to \mathcal{X}$.

◇

Max: these conditions will always be satisfied in real life. Martin: No, the moduli of projective varieties. Max: that's a bad moduli problem.

You could have used fppf instead of smooth. Would you get a different notion?

Theorem 7.6 (Artin)**.** *If $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable (by algebraic spaces), quasi-compact, and there is an fppf surjection from a scheme, then there is a smooth cover by a scheme (i.e. $\mathcal{X}$ is an Artin stack).*

How do we know if a moduli problem is an Artin stack?

---

[1] If you don't like this, imagine schemes, and you'll only get a slightly smaller class of stacks.

Proposition 7.7. *Suppose $\mathcal{M}$ is a moduli stack (locally of finite presentation) such that* $\underline{\mathrm{Isom}}$ *sheaves are representable (by algebraic spaces), quasi-compact, and separated. Then $\mathcal{M}$ is Artin if and only if there is a surjection $X \to \mathcal{M}$ from a scheme which is formally smooth.*

$$\begin{tikzcd}
X \arrow[r] & \mathcal{M} \\
\overline{Y} \arrow[u] \arrow[r, hook] & Y \arrow[u] \arrow[ul, dashed]
\end{tikzcd}$$

Theorem 7.8 (Artin)**.** *An Artin stack (with all the hypotheses so far) $\mathcal{X}$ is DM if and only if $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is unramified if and only if no object has non-trivial infintesimal automorphisms.*

There is still the question: how would you ever show that something is an Artin stack?

Example 7.9. Let $S = \operatorname{Spec} \mathbb{C}$ (or $\operatorname{Spec} \mathbb{Z}$, if you like), and let $\mathcal{M}_{1,1}$ be the stack of elliptic curves. The fiber of $\mathcal{M}_{1,1}$ over T is $\{ \mathcal{E} \underset{\sigma}{\overset{\pi}{\rightleftarrows}} T \mid \pi$ is proper and smooth with genus 1 connected fibers, and σ a section$\}$. The claim is that $\mathcal{M}_{1,1}$ is a DM stack.

The condition on $\underline{\mathrm{Isom}}$ sheaves is not so bad. Ravi told us how to get isoms for higher genus curves using some methods. The saving grace here is that there are not infinitesimal automorphisms. That is because infinitesimal automorphisms of $(E, p) \to \operatorname{Spec} \bar{k}$ is $H^0(E, T_E) = H^0(E, \mathcal{O}_E) = \bar{k}$, except that we have to fix that point p, so it is $H^0(E, \mathcal{O}_E(-p)) = 0$. The claim is that this is enough to show that $\mathcal{M}_{1,1}$ is an Artin stack.

To prove this, we must find a formally smooth family $B \to \mathcal{M}_{1,1}$. That is, we must find a family of elliptic curves $\mathcal{E} \rightleftarrows B$ which knows everything.

When we're young, we learn that if you take three times a point on an elliptic curve, that gives an embedding of the elliptic curve into $\mathbb{P}^2$. We do something like this in families . . .

The idea is to uniformize by the family of plane cubics, but we also have to keep track of the point. Since plane cubics in the plane are given by an equation, it shouldn't be too bad to get a moduli space.

(1) There is a scheme U representing the functor $T \mapsto$

$$\begin{tikzcd}
C \arrow[r, hook] \arrow[d] & \mathbb{P}^2_T \arrow[dl] \\
T &
\end{tikzcd}$$

smooth families of cubic curves. If T is a point, it is clear what we mean by "cubic curve". If T is bigger, there should be an étale cover $T' \to T$ such that

$$\begin{tikzcd}
C_{T'} \arrow[r, hook] \arrow[d] & \mathbb{P}^2_{T'} \arrow[dl] \\
T' &
\end{tikzcd}$$

, $C_{T'}$ is the vanishing locus of a section of $\mathcal{O}_{\mathbb{P}_{T'}}(3)$.

Proof of (1). Take the universal cubic

$$\begin{tikzcd} \left(\sum_{i+j+k=3} \alpha_{i,j,k} X^i Y^j Z^k\right) \arrow[r, phantom, "\subseteq"] & \mathbb{A}^{10} \times \mathbb{P}^2 \arrow[d] \\ & \mathbb{A}^{10} \end{tikzcd}$$

such that $\widetilde{U} \subseteq \mathbb{A}^{10}$ parameterizing smooth cubics and U is the image of $\widetilde{U}$ in $\mathbb{P}^9 \leftarrow \mathbb{A}^{10} \smallsetminus \{0\}$.

(2) There exists a scheme P ($\to U$) representing the functor $T \mapsto \{$

$$\begin{tikzcd} \mathcal{E} \arrow[r, phantom, "\subseteq"] \arrow[d] & \mathbb{P}^2_T \arrow[dl] \\ T \arrow[u, bend left, "\sigma"] \end{tikzcd}$$

pointed smooth cubics$\}$.

(3) The point is that you want to map P down to $\mathcal{M}_{1,1}$, but we are missing something. We want to now force $\mathcal{O}(1)|_{\mathcal{E}} = \mathcal{O}(3\sigma)|_{\mathcal{E}}$. Call the result P'.

(4) There is an action of PGL_3 on P' coming from choosing coordinates on $\mathbb{P}^2$.

(5) $[P'/PGL_3] \cong \mathcal{M}_{1,1}$.

This proves that $\mathcal{M}_{1,1}$ is an Artin stack. ◇

Question: why not just use Hilbert scheme? Max: [[★★★ I didn't catch this]]

When you start learning GIT, remember to do everything with stacks.

7 Olsson

Yesterday I left off in the middle of a proof. We're working over a topological space T. Recall that we're trying to prove the following theorem.

Theorem 7.1. *$ch\colon \tilde{C}^{[-1,0]}(T) \to$(Picard stacks) is an equivalence of 2-categories.*

We gave a sketch yesterday.

Lemma 7.2. *If $\mathcal{P}$ is a Picard stack, then there exists a complex $K^\bullet \in C^{[-1,0]}(T)$ and an equivalence $ch(K) \xrightarrow{\sim} \mathcal{P}$.*

This was a big construction, but you do it and it works.

Lemma 7.3. *Let $K^\bullet, L^\bullet \in C^{[-1,0]}(T)$ and let $F\colon ch(K) \to ch(L)$ be a morphism of Picard stacks. Then there exists a quasi-isomorphism $k\colon K' \to K$ and a morphism $\ell\colon K' \to L$ such that $F \simeq ch(\ell) \circ ch(k)^{-1}$ (we saw already that quasi-isomorphisms of complexes induce equivalences of Picard stacks).*

$$\begin{tikzcd} & K' \arrow[dl, "k"', "qism"] \arrow[dr, "\ell"] & \\ K & & L \end{tikzcd} \qquad \begin{tikzcd} & ch(K') \arrow[dl, "ch(k)"', "\sim"] \arrow[dr, "ch(\ell)"] & \\ ch(K) \arrow[rr, "F"'] & & ch(L) \end{tikzcd}$$

In particular, if $K \in \tilde{C}^{[-1,0]}(T)$, then any morphism $F\colon ch(K) \to ch(L)$ is isomorphic to $ch(f)$ for some morphism of complexes $f\colon K^\bullet \to L^\bullet$. This is because if K^{-1} is injective, then quasi-isomorphisms from it are actually invertible up to homotopy.

Sketch Proof. Choose data $\{(U_i, k_i, \ell_i, \sigma_i)\}$ such that

(a) $U_i \subseteq T$ is open,

(b) $k_i \in K^0(U_i)$, $\ell_i \in L^0(U_i)$, $\sigma_i\colon F(k_i) \xrightarrow{\sim} \ell_i$.

(c) the map $K'^0 := \bigoplus_{i\in I} \mathbb{Z}_{U_i} \to K^0$ is surjective.

Now I have to tell you what is K'^{-1} and what are the maps. Define $K'^{-1} = K^{-1} \times_{K^0} K'^0$ (you don't have much choice because you want the map from (c) to be surjective and the map to be a quasi-isomorphism). Now we define $\ell : K' \to L$. $\ell^0 : K'^0 \to L^0$ is induced by the maps $\mathbb{Z}_{U_i} \to L^0$ given by the ℓ_i. $\ell^{-1} : K'^{-1} \to L^{-1}$ is given by sending $\big(v, (U_i, k_i, \ell_i, \sigma_i)\big) \in K^{-1}$ to the unique element $t \in L^{-1}$ such that

$$\begin{tikzcd} F(0) \arrow[r, "F(v)"] \arrow[d, "\simeq"'] & F(k_i) \arrow[d, "\sigma_i"] \\ 0 \arrow[r, "t"] & \ell_i \end{tikzcd}$$

The σ_i define an isomorphism $\sigma : F \xrightarrow{\sim} ch(\ell) \circ ch(k)^{-1}$. There are a bunch of things to check, but we won't do it here. □

Lemma 7.4. *Let $K_1^\bullet, K_2^\bullet \in \tilde{C}^{[-1,0]}(T)$. For two morphisms of complexes $f_1, f_2\colon K_1^\bullet \to K_2^\bullet$ with associated morphisms $F_1, F_2\colon ch(K_1) \to ch(K_2)$ and any isomorphism $H : F_1 \xrightarrow{\sim} F_2$, there exists a unique homotopy $h\colon K_1^0 \to K_2^{-1}$ such that $H = ch(h)$.*

Sketch Proof. We saw before that there is no stackification when our complexes come from $\tilde{C}$. The idea is this. If $k \in K_1^0$ is a section, then we have $F_1(k) \xrightarrow{H} F_2(k)$, which is the same thing as a section $h(k) \in K_2^{-1}$ such that $dh(k) = f_2(k) - f_1(k)$. Now you have to check that the axiom for H to be a morphism of Picard stacks is exactly what makes h into a homotopy. ☐

Instead of explaining what is an equivalence of 2-categories, we'll just say that the content of the theorem is exactly the three lemmas.

This is supposed to lead up to the cotangent complex. Let's make a preliminary definition.

Definition 7.5 (preliminary)**.** Let $f\colon X \to S$ be a morphism of schemes. The *truncated tangent complex* (this is non-standard terminology) denoted $\tau_{\leq 1}\mathbb{T}_{X/S}[1] \in \tilde{C}^{[-1,0]}(|X|)$ is the complex with $ch(\tau_{\leq 1}\mathbb{T}_{X/S}[1]) \xrightarrow{\sim} \underline{\mathrm{Exal}}_S(X, \mathcal{O}_X)$. ◇

One problem is that "is" doesn't make sense, so you have to choose the isomorphism as well. More Problems:

(a) this doesn't see the $\mathcal{O}_X$-module structure. One could go define more general Picard stacks …

(b) Not the full cotangent complex. This is kind of meaningless because I haven't justified why you need anything more

Let's try to compute it.

Proposition 7.6. *Let $j\colon X \hookrightarrow S$ be a closed immersion defined by an ideal I. Then $\tau_{\leq 1}\mathbb{T}_{X/S}[1]$ is quasi-isomorphic to $\mathcal{N}_{X/S}$, where $\mathcal{N}_{X/S} = \mathcal{H}om(j^*I, \mathcal{O}_X)$.*

Proof. $\underline{\mathrm{Exal}}_S(X, \mathcal{O}_X)$ was the category of thickenings of X with square zero ideal sheaf isomorphic to $\mathcal{O}_X$:

$$\begin{tikzcd}
X \arrow[r, hook, "\mathcal{O}_X"] \arrow[d, hook, "j"'] & X' \arrow[dl] \\
S &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathcal{O}_X \arrow[d, equal] & \mathcal{O}_{X'} \arrow[l, two heads] & \mathcal{O}_X \cdot \varepsilon \arrow[l, hook'] \\
\mathcal{O}_X & j^{-1}(\mathcal{O}_S/I^2) \arrow[l, two heads] \arrow[u] & j^*I = j^{-1}(I/I^2) \arrow[l, hook'] \arrow[u, "\partial\in\mathcal{N}_{X/S}"']
\end{tikzcd}$$

Since X' comes with a map to S, we have a map $j^{-1}\mathcal{O}_S \to \mathcal{O}_{X'}$, which must send $j^{-1}I$ to $\mathcal{O}_X \cdot \varepsilon$, so it must send $j^{-1}I^2$ to zero, so it must factor through $j^{-1}(\mathcal{O}_S/I^2)$, giving the middle vertical arrow [[★★★ how to say this cleanly? maybe add remark about "universality" of the first order neighborhood $\mathcal{O}_S/I^2$ of X among square zero thickenings.]]. Then we get an element $\partial \in \mathcal{N}_{X/S} = \mathcal{H}om(j^*I, \mathcal{O}_X)$. [[★★★ how do we know that the element X' of $\underline{\mathrm{Exal}}$ has no automorphisms?]] ☐

Proposition 7.7. *Let $f\colon X \to S$ be a smooth morphism. Then $\tau_{\le 1}\mathbb{T}_{X/S}[1] \cong T_{X/S}[1]$ (this means $T_{X/S} \to 0$ in degrees -1 and 0).*

Proof. We already know that $\mathcal{H}^0(\tau_{\le 1}\mathbb{T}_{X/S}[1]) = 0$.

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, hook] & X \arrow[d] \\
X' \arrow[r] \arrow[ur, dashed] & S
\end{tikzcd}$$

identifies X' with $X[\mathcal{O}_X \cdot \varepsilon]$. So any two guys are locally isomorphic. That means that the complex is quasi-isomorphic to $\mathcal{H}^{-1}(\tau_{\le 1}\mathbb{T}_{X/S}[1] = T_{X/S}[1]$. ☐

Proposition 7.8. *Suppose we have a commutative diagram*

$$\begin{tikzcd}
X \arrow[r, hook, "j"] \arrow[d, "f"'] & P \arrow[dl, "g"] \\
S &
\end{tikzcd}$$

with g smooth and j an immersion. Then $\tau_{\le 1}\mathbb{T}_{X/S}[1]$ is quasi-isomorphic to the complex $(j^*T_{P/S} \to \mathcal{N}_{X/P})$. Remember that if I is the ideal of X in P, then we get $I/I^2 \xrightarrow{d} j^*\Omega^1_{P/S}$.

Sketch Proof. If $z\colon j^*I \to \mathcal{O}_X$ (this is a section of $\mathcal{N}_{X/P}$), we should get an extension, and given a section of $j^*T_{P/S}$ should give an isomorphism of extensions and then the some equivalence of the resulting Picard stacks. So given z, form

$$\begin{tikzcd}
j^*I \arrow[r, "z"] \arrow[d, hook] & \mathcal{O}_X \cdot \varepsilon \arrow[d, dashed] \arrow[dr, bend left, "0"] & \\
j^{-1}(\mathcal{O}_P/I^2) \arrow[r, dashed] \arrow[rrd, bend right, two heads] & \mathcal{O}_{X'} \arrow[ul, phantom, "\lrcorner", very near start] \arrow[dr, dashed] & \\
& & \mathcal{O}_X
\end{tikzcd}$$

Consider the (noncommutative!) diagram

$$\begin{tikzcd}
& X_{z'} \arrow[d, dashed, "h"'] \arrow[dd, bend left, "f'"] \\
X \arrow[ur, hook, "\iota'"] \arrow[r, hook, "\iota"] \arrow[dr] & X_z \arrow[d, "f"'] \\
& P
\end{tikzcd}$$

given $z, z' \in \mathcal{N}_{X/P}$ then $f \circ h$ and f' are two maps filling the following diagram

$$\begin{tikzcd}
X \arrow[r, hook] \arrow[d] & X_{z'} \arrow[d] \arrow[dl, dashed] \\
P \arrow[r] & S
\end{tikzcd}$$

So $f \circ h - f' \in j^* T_{P/S}$.

The upshot: there is a fully faithful functor from the prestack $pch(j^* T_{P/S} \to \mathcal{N}_{X/P}) \to \underline{\mathrm{Exal}}_S(X, \mathcal{O}_X)$. Now from a lemma we proved last time, to check that the induced morphism of stacks is an equivalence, it is enough to check that any object of $\underline{\mathrm{Exal}}_S(X, \mathcal{O}_X)$ is locally in the essential image of the functor and that the induced functor on automorphisms is an isomorphism. First consider X affine

$$\begin{tikzcd}
X \arrow[rr, hook] \arrow[dr] \arrow[ddr] & & X' \arrow[dl] \arrow[ddl] \\
& P \arrow[d] & \\
& S &
\end{tikzcd}$$

□

(a) we have an issue of choice of factorization of f. If your thing is quasi-projective or something then that's not too big a deal. It is a nice exercise to show that if you choose two factorizations you get the same complex up to quasi-isomorphism.

(b) a factorization need not exist. Question: are there provable examples, assuming finite type. Brian: I think there are examples (for non-stupid reasons) on Ravi's webpage.

Everything we care about is on the topological space of X. We don't actually care about the object P. There is a substitute for an embedding into a smooth guy.

Replacement for factorization. We have $f\colon X \to S$. Think about sheaves of $f^{-1}\mathcal{O}_S$-algebras. There is a forgetful functor F to sheaves of sets. The point is that F has a left adjoint, which sends a sheaf of sets Ω to the free algebra $f^{-1}\mathcal{O}_S\{\Omega\}$ (the braces notation is just to not conflict with dual numbers). Question: can $f^{-1}\mathcal{O}_S$ be any sheaf of algebras. Martin: Yes.

Idea: if $X = \operatorname{Spec} A$ and $S = \operatorname{Spec} B$, choose elements $f_i \in A$ generating A as a B-algebra such that $B[x_i] \to A$, given by $x_i \to f_i$. $f^{-1}\mathcal{O}_S\{\Omega\}$ on U is $f^{-1}\mathcal{O}_S(U)[x_i]_{i \in \Omega(U)}$ and then sheafify. Think about $f^{-1}\mathcal{O}_S \to \mathcal{O}_X$. When we do this free algebra construction, this is a good enough approximation to an embedding into a smooth guy. We want a diagram

$$\begin{tikzcd}
\mathcal{O}_X & f^{-1}\mathcal{O}_S\{\Omega\} \arrow[l, two heads] \\
f^{-1}\mathcal{O}_S \arrow[u] \arrow[ur] &
\end{tikzcd}$$

How do we choose Ω? (a) one way is to take $\Omega = F(\mathcal{O}_X)$. (b) another way is to choose open sets $j_i \colon U_i \hookrightarrow |X|$ and sections $f_i \in \mathcal{O}_X(U_i)$ and take $\Omega = \coprod_i j_{i!}\{*\}$ where $*$ is the punctual sheaf. Repitition is allowed in the U_i. (in fact, I think that (a) is an example of (b)).

Now in all the diagrams, you should be able to replace $\mathcal{O}_P$ by I/I^2.

Definition 7.9. The truncated tangent complex of f is the complex

$$\mathcal{H}om(\Omega^1_{f^{-1}\mathcal{O}_X\{F(\mathcal{O}_X)\}/f^{-1}\mathcal{O}_S}, \mathcal{O}_X) \to \mathcal{H}om(I/I^2, \mathcal{O}_X). \qquad \diamond$$

Now we see that this is a complex of $\mathcal{O}_X$-modules and you didn't have to choose some stuff.

7 Osserman

Effictivity and Algebraization. Now that we're done with Schlessinger's criterion, you might think you know everything about functors on Artin rings. The lesson for today is : "don't get cocky".

Two remaining questions:

1. (Effectivity) Suppose F is a deformation functor which we got from a global moduli problem, $R \in \widehat{Art}(\Lambda, k)$, and $\eta \in \hat{F}(R)$, when does η actually come from a family over $\operatorname{Spec} R$ from the original problem.

   Ravi: what does it buy you to know the answer to this question. Brian: you'd like to say that when something looks like it corresponds to a family, then it actually is a family. That is, when you can find a formal family, you'd like to be able to get an actual family.

2. (Algebraization) Suppose we're in the same situation and the answer to the above question is "yes", so we have a family over $\operatorname{Spec} R$. Then you might say that instead of constructing the family over R (which is some complete local noetherian ring), you'd like to construct a family over a curve or whatever (whatever you took the complete local ring of). When is this induced from an "algebraic object", e.g. from something over R', of finite type over the base?

So if you have a compatible series of n-th order thickenings of a curve, when can you extend it to the complete local ring, and when can you extend this to an actual neighborhood. In practice, it is actually the first problem which is likely to fail.

Effectivity. There is no general positive answer, but there is one tool that you use to prove all the positive results. The main tool to obtain positive results is Grothendieck's existence theorem. In this special case, you can phrase it as follows.

Theorem 7.1 (Grothendieck's existence theorem). *Suppose $f\colon X \to \operatorname{Spec} A$ is proper with A a complete local noetherian ring. Let $A_n = A/\mathfrak{m}_A^n$ and $X_n = X \otimes_A A_n$. Given $\{\mathcal{F}_n\}$ a compatible collection of coherent sheaves on X_n, then there exists a coherent sheaf $\mathcal{F}$ on X inducing all of them.*[1]

This is exactly a positive result to the effectivity question when you're dealing with coherent sheaves on a proper scheme. A natural question to ask is: what about the moduli of abstract schemes? The answer is: No! There are very concrete examples you can write down. It is "yes" for curves, but beyond that, it starts to fail very badly. Specifically, it fails for K3 surfaces (smooth projective surfaces with trivial canonical class and $H^1(X, \mathcal{O}_X) = 0$). In this case, if you look at Def_X, it looks like you should

---

[1] In the background lecture, you probably saw a more general result. There is an equivalence of categories between inverse limits of coherent sheaves on the A_n and the category of coherent sheaves on X.

have a 20-dimensional moduli space (and in the analytic category, you do get a 20-dimensional moduli space), but only a 19-dimensional moduli space of effectivizable deformations. [[Martin: you don't mean that there is a closed locus which classifies K3 surfaces which are algebraic. That is, you can't tell infinitesimally if a deformation is algebraic. If you fix a polarization on X, then if there is a polarization, then it is unique, but you could choose different polarizations.]] Brian: This 19-dimensional sub-thing could be (is?) the union of countably many 19-dimensional closed sub-things which are algebraic.

So it looks kind of bad, but not all hope is lost. Instead of working with abstract moduli of varieties, you work with moduli of polarized varieties (i.e. you have a choice of an ample line bundle). It is a consequence of Grothendieck's theorem (the equivalence of categories version) is that you get effectivizablity for the moduli of polarized (projective) varieties.

You can imagine that if instead of choosing an ample line bundle, you take a high power so that it is ample, then you're looking at the moduli of closed subschemes of projective space. These correspond to coherent sheaves. Then using the equivalence of categories, you get stuff. This gives you effectizability for the Quot functor.

Brian (Conrad?): it might be worth noting that Grothendieck's existence theorem is useful when A is not local, such as $\mathbb{Z}[[x]]$.

Algebraization. Artin considers (uni)versal families and proves a positive result quite generally. Here he uses his earlier approximation theorems. This requires: the base S must be of finite type either over a field or over an excellent Dedekind domain (basically thown in to allow us to work with mixed characteristic varieties … $\mathbb{Z}$ is an excellent Dedekind domain; so is anything of finite type over $\mathbb{Z}$).

By the way, this is an area where there is a lot of competing terminology. We've been using Schlessinger's terminology.

Definition 7.2. Let $F\colon \mathbf{Sch}_S \to \mathbf{Set}$ be a contravariant functor. We say that F is *locally of finite presentation* over S if for all filtering projective systems of affine schemes $Z_\lambda \in \mathbf{Sch}_S$, we have $\varinjlim F(Z_\lambda) = F(\varprojlim Z_\lambda)$. ◇

If you want to prove something about non-noetherian rings, you write your ring as a limit of noetherian rings and then use this finite presentation business. Why this definition? EGA tells us that if $F = h_X$ for some $X \in \mathbf{Sch}_X$, then this is equivalent to $X \to S$ being locally of finite presentation.

Notation: say F is a deformation functor, then (R, ξ) (with $\xi \in \hat{F}(R)$ and $R \in \widehat{Art}(\Lambda, k)$) is *smooth* over F if the induced map $\overline{h}_R \to F$ is smooth. I believe this is what Artin calls *versal*. This is just the definition of hull, without the condition that the map on tangent spaces is an isomorphism.

Theorem 7.3. *Suppose* $F\colon \mathbf{Sch}_S \to \mathbf{Set}$ *is a contravariant functor which is locally of finite presentation over* S*, and* $\eta_0 \in F(k)$*, where we're given some* $\operatorname{Spec} k \to S$ *of finite type, with image* $s \in S$*. Let* R *be a complete local noetherian algebra over* $\mathcal{O}_{S,s}$ *with residue field* k*, and suppose we have* $\xi \in F(R)$ *which induces* η_0 *over* k *(you can think* *of it as being a deformation of η_0), and with (R, ξ) smooth over the local deformation functor corresponding to η_0. Then there exists some scheme X of finite type over S, a closed point $x \in X$, and $\eta \in F(X)$ with an isomorphism $\hat{\mathcal{O}}_{X,x} \xrightarrow{\sim} R$ such that η maps to $\xi_n \in F(R/\mathfrak{m}_R^{n+1})$ for all n.*

This says essentially that we have this universal family over this complete noetherian local ring. Under this mild hypothesis of being locally of finite presentation, you can get something. Martin: don't you want to fix an n? Brian: This isn't the approximation theorem; you can tweak the map as n increases and pass to a limit.

One of the confusing points is that we get ξ_n for all n, but not ξ. In general, This doesn't imply that η maps to ξ unless ξ is uniquely determined by the truncations ξ_n. I think in any real moduli problem, ξ will be determined by the ξ_n.

Note that we snuck the effectivity in because we assumed already that we have this $\xi \in F(R)$. This is the real hypothesis in the theorem.

Theorem 7.4. *In the situation of the previous theorem, suppose that ξ is uniquely determined by the ξ_n. Then the triple (X, x, η) is unique up to étale morphisms. That is, if (X', x', η') also satisfies the conditions, then there is a third triple (X'', x'', η'') and étale maps*

$$\begin{tikzcd}
 & (X'', x'', \eta'') \arrow[dl, "et"'] \arrow[dr, "et"] & \\
(X, x, \eta) & & (X', x', \eta')
\end{tikzcd}$$

8 Lieblich

Today we'll talk about Artin's representability theorem. We're going to whack a couple of our moduli problems with it. Let's fix a couple of things to begin with. S is a scheme locally of finite type over an excellent Dedekind scheme (like Spec of a field or a finitely generated $\mathbb{Z}$-algebra). $\mathcal{F}$ is a stack on S_{ET}, locally of finite presentation, which means that if $A = \varinjlim A_i$ is a limit of rings, then $\varinjlim \mathcal{F}_{\operatorname{Spec} A_i} \to \mathcal{F}_{\operatorname{Spec} A}$ is an equivalence of categories (to define a limit of groupoids, follow your nose). Brian explained yesterday that if you start with $\operatorname{Spec} k \xrightarrow{x} \mathcal{F}$, then if x admits an effective versal formal deformation, then there exists a family $\operatorname{Spec} k \to X \xrightarrow{f} \mathcal{F}$ such that f is "formally smooth at x" with X of finite type over X. This means that

$$\begin{tikzcd}
\overline{Y} \arrow[r] \arrow[d, hook] & X \arrow[d] \\
Y \arrow[r] \arrow[ur, dashed] & \mathcal{F}
\end{tikzcd}$$

if $\overline{Y}$ and Y are local artin schemes with the point of $\overline{Y}$ maps to x, then a lift exists. Now if we can make X smooth in a neighborhood of x, then we have part of a smooth cover of $\mathcal{F}$.

Content:

1. Schlessinger stuff which produced a versal formal deforamtion (in fact, gave us a hull). This is purely infinitesimal.
2. Formal to effective. This is like the Grothendieck existence theorem, which has more than infinitesimal information. This is roughly an étale local exististence statement.

There are some conditions that help us link up the deformation and obstruction theories.

Notation: Given $X \to S$ and $a \in \mathcal{F}_X$, then let $\mathcal{F}_a$ be a [[★★★ stack]] groupoid for each $(f\colon X \to Y)$ which has objects $\alpha\colon a \to b$ such that the image of α in S_{ET} is f (this is $\{(b \in \mathcal{F}_Y, \phi\colon a \xrightarrow{\sim} f^*b)\}$ if you choose a cleavage). We also get a functor $\overline{\mathcal{F}}_a$, the functor of isomorphism classes of $\mathcal{F}_a$.

$\underline{(\mathrm{S}1')}$ Given infinitesimal thickenings $A' \twoheadrightarrow A \twoheadrightarrow A_0$ with $\ker(A' \to A)$ an A_0-module.

$$\begin{tikzcd}
 & B \arrow[d] \arrow[dr, two heads] & \\
A' \arrow[r, two heads] & A \arrow[r, two heads] & A_0
\end{tikzcd}$$

with A_0 reduced. Let $a \in \mathcal{F}_{\operatorname{Spec} A} = \mathcal{F}(A)$. (S1′) is that $(\mathcal{F}_a)(A' \times B) \to \mathcal{F}_a(A') \times \mathcal{F}_a(B)$ is an equivalence of categories.

$\underline{(S2)}$ Let $a_0 = a|_{\operatorname{Spec} A_0}$. Martin showed us that for an A_0-module M of finite type, we $\overline{\text{have}}$ $\mathcal{F}_a(A_0[M]) = D_{a_0}(M)$. (S2) is the condition that $D_{a_0}(M)$ is an A_0-module.

Suppose that given an obstruction theory (à la Martin). $A \twoheadrightarrow A_0$ an infitesimal extension and $a \in \mathcal{F}(A)$, we get $O_a \colon A_0\text{-}\mathbf{mod}_{ft} \to A_0\text{-}\mathbf{mod}_{ft}$ such that for all $A' \to A \to A_0$ with $\ker(A' \to A) = M$ an A_0-module, then $o_a(A') \in O_a(M)$ such that $o_a(A') = 0$ if and only if a lifts to A'.

These conditions are stronger than Schlessinger's criteria (which is the special case over a field).

In addition, given $A \twoheadrightarrow A_0$ infinitesimal extension with A_0 reduced, assume:

(4.1)(i) Etale localization: if $A \to B$ is étale, then the natural maps are isomorphisms: $D_{a_0}(M_0 \otimes B_0) \xleftarrow{\sim} D_{a_0}(M_0) \otimes B_0$ for $B_0 = A_0 \otimes_A B$, $M_0 \in A_0\text{-}\mathbf{mod}_{ft}$. Also, $O_{b_0}(M_0 \otimes B_0) \xleftarrow{\sim} O_{a_0}(M_0) \otimes B_0$ where $b_0 = a_0|_{B_0}$. All tensor products are over A_0.

(4.1)(ii) Completion: If $\mathfrak{m} \subseteq A_0$ is a maximal ideal, then $D_{a_0}(M) \otimes \hat{A}_0 \xrightarrow{\sim} \varprojlim D_{a_0}(M/\mathfrak{m}^n M)$ is an isomorphism. Conrad: they might not be isomorphic at any finite level, it is only an isomorphism when you complete.

(4.1)(iii) Constructability: There exists a dense set of closed points $p \in \operatorname{Spec} A_0$ such that $D_{a_0}(M) \otimes k(p) \xrightarrow{\sim} D_{a_0}(M \otimes k(p))$ and $O_{a_0}(M) \otimes k(p) \xrightarrow{\sim} O_{a_0}(M \otimes k(p))$ are isomorphisms.

Conrad: in this whole discussion, $\operatorname{Spec} A_0$ is finite type over S.

Theorem 8.1 (Artin)**.** *Given $\mathcal{F}$, O satisfying (S1′), (S2), and (4.1), if $f \colon X \to \mathcal{F}$ with $X \to S$ finite type and f formally smooth at $x \in X$, then there exists $U \subseteq X$ with $x \in U$ such that $f|_U \colon U \to \mathcal{F}$ is formally smooth.*

Proposition 8.2 (Artin)**.** *$\mathcal{F}$ is an Artin stack locally of finite type over S if*

1. *$\mathcal{F} \to \mathcal{F} \times \mathcal{F}$ is representable (by algebraic spaces), quasi-compact, and separated,*
2. *(S1′) and (S2) hold,*
3. *if $(\hat{A}, \mathfrak{m})$ is a complete local noetherian ring over S, then $\mathcal{F}(\hat{A}) \to \varprojlim \mathcal{F}(\hat{A}/\mathfrak{m}^n)$ is an equivalence, and*
4. *D and O satisfy (4.1).*

(1) and (2) are exactly what we need for algebraization and (3) gives you Grothendieck's existence theorem, and (4) is [[★★★ something]]

Example 8.3. $\mathcal{M}_g$ ($g > 1$), the stack of curves of genus g. The objects are $\mathcal{C} \to T$ proper smooth with geometrically connected fibers of genus g.

1. $\mathcal{M}_g \to \mathcal{M}_g \times \mathcal{M}_g$ Grothendieck or [one second]
2. Brian showed us that we have Schlessinger's criterion

3. Grothendieck's existence theorem. Since $g > 1$, the canonical sheaf is ample, so some high power is very ample, so then you can use Grothendieck's existence theorem for subschemes of projective space.

4. Starting with $\mathcal{C} \to T = \operatorname{Spec} A \hookrightarrow \operatorname{Spec} A'$, $M = \ker(A' \to A)$, then $O_{\mathcal{C}}(M) = H^2(\mathcal{C}_{A_0}, T_{\mathcal{C}_{A_0}/A_0} \otimes M)$ and $D_{a_0}(M) = H^1(\mathcal{C}_{A_0}, T_{\mathcal{C}_{A_0}/A_0} \otimes M)$ A_0-module. (i) compatible with étale base change $A_0 \to B_0$ (this is in Hartshorne). (ii) is less obvious, you look in EGAIII or Hartshorne (theorem on formal functions). (iii) constructability follows from cohomology of base change, which is also in Hartshorne. $f : \mathcal{C}_0 \to \operatorname{Spec} A_0$. We want to show that $R^i f_*(T_{\mathcal{C}_{A_0}/A_0} \otimes M) \otimes k(p) \to H^i(\mathcal{C}_p, T_{\mathcal{C}_p/p} \otimes M)$ is an isomorphism. Martin: you don't have to worry about this dense open set business because of [[★★★ something]]. Max: this follows from cohomology of base change.

So by Artin's theorem, you get that $\mathcal{M}_g$ is an Artin stack. From yesterday's result, there are no non-trivial infinitesimal automorphisms ($H^0(\mathcal{C}, T)$). This implies that $\mathcal{M}_g$ is a DM stack. You can see that H^2 vanishes, so $\mathcal{M}_g$ will be smooth, and maybe you can compute the dimension. Other than that, we haven't said much of anything about this space. ◇

You can make the list even better.

Theorem 8.4 (Artin). *$\mathcal{F}$ is an Artin stack locally of finite type over S if*

1. *(S1′) and (S2) hold and if $a_0 \in \mathcal{F}(A_0)$ and M is a finite A_0-module, then $\operatorname{Aut}^{inf}_{a_0}(A_0[M])$ is a finite A_0-module.*

2. *if $(\hat{A}, \mathfrak{m})$ is a complete local noetherian ring over S, then $\mathcal{F}(\hat{A}) \to \varprojlim \mathcal{F}(\hat{A}/\mathfrak{m}^n)$ is an equivalence, and*

3. *D, O, and $\operatorname{Aut}^{inf}_{a_0}(A_0[M])$ satisfy (4.1).*

4. *If ϕ is an automorphism of a_0 such that $\phi = \mathrm{id}$ at a dense set of points of $\operatorname{Spec} A_0$, then $\phi = \mathrm{id}$. (this is something about the diagonal of the diagonal; that it is separated)*

5. *[[Fact: (1-4)⇒ $\mathcal{F} \to \mathcal{F} \times \mathcal{F}$ is representable and separated]] $\mathcal{F} \to \mathcal{F} \times \mathcal{F}$ is quasi-compact (we need representability for this to make sense).*

If you find yourself in a dark alley with a stack, do this. Conrad: it is sometimes hard to check these directly, but you can sometimes relate your problem to another thing which is a stack.

Martin: you should say that this is an if and only if.

8 Olsson. The cotangent complex; an overview

Recall where we left off. Let $f\colon X \to S$ be a morphism of schemes. Then

$$\begin{tikzcd}
GF(\mathcal{O}_X) \arrow[r, phantom, "="] & f^{-1}\mathcal{O}_S\{F(\mathcal{O}_X)\} \arrow[r, two heads, "\pi"] & \mathcal{O}_X \\
& f^{-1}\mathcal{O}_S \arrow[u] \arrow[ur] &
\end{tikzcd}$$

The forgetful functor F, from $f^{-1}\mathcal{O}_S$-algebras to sheaves of sets has an adjoint G. We get the complet $\tau_{\geq -1}L_{X/S} = (I/I^2 \to \Omega^1_{GF(\mathcal{O}_X)/f^{-1}\mathcal{O}_S} \otimes \mathcal{O}_X)$ where $I = \ker \pi$.

Theorem 8.1. *For any quasi-coherent $\mathcal{O}_X$-module I, $ch\big(\tau_{\leq 0}(\mathbf{R}\mathcal{H}om(\tau_{\geq -1}L_{X/S}, M)[1])\big) \cong \underline{\mathrm{Exal}}_S(X, M)$. We call $\tau_{\geq -1}L_{X/S}$ the* truncated cotangent complex*.*

If you're familiar with it, you get $L_{X/S}$, the full cotangent complex as follows (don't worry about this too much, we'll list the imporant propoerties soon). Given $n \geq 0$, we have $\mathcal{A}_n = GFG\cdots GFGF(\mathcal{O}_X)$ ($n+1$ copies of GF) which is an $f^{-1}\mathcal{O}_S$-algebra and there is a surjection $\mathcal{A}_n \twoheadrightarrow \mathcal{O}_X$ from the adjunction. This $\mathcal{A}_\bullet$ is really a simplicial $f^{-1}\mathcal{O}_S$-algebra, so we have $d_i\colon \mathcal{A}_{n+1} \to \mathcal{A}_n$ and $s_j\colon \mathcal{A}_n \to \mathcal{A}_{n+1}$. You have $GF \xrightarrow{a} \mathrm{id}$ and $\mathrm{id} \xrightarrow{b} FG$. How do we get the d's and s's? $\mathcal{A}_2 = GFGFGF(\mathcal{O}_X)$ and $\mathcal{A}_1 = GFGF(\mathcal{O}_X)$, so we can get $d_0 = aGFGF\colon GFGFGF\mathcal{O}_X \to GFGF\mathcal{O}_X$, and get $d_1 = GFaGF$ and $d_2 = GFGFa$. Similarly, we get maps the other way using b. The compatibility conditions of a simplicial object follow from the adjunction trivially. $L_\bullet := \Omega_{\mathcal{A}_\bullet/f^{-1}\mathcal{O}_S} \otimes \mathcal{O}_X$ is a simplicial $\mathcal{O}_X$-module, so you get $\tilde{L}_2 \substack{\Longrightarrow\\[-1em]\longrightarrow} \tilde{L}_1 \Longrightarrow \tilde{L}_0$. Taking alternating sums of these d_i's, and the result is called the *cotangent complex* (again, the fact that this is a complex follows from the fact that it comes from a simplicial module). All this stuff is very general. This construction is not very enlightening, but there is one imporant point.

Remark 8.2. This is an actual complex of flat $\mathcal{O}_X$-modules. Often people like to think of complexes as just living in the derived category, but you really have it. ◇

Max: morally, you're trying to compute the derived functor of Ω^1.

We want to actually solve some deformation problems. Here are some properties.

(i) $\mathcal{H}^i(L_{X/S})$ is quasi-coherent and coherent if S is locally noetherian and f is of finite type. We sort of saw this when we had an embedding into something smooth.

(ii) Suppose

$$\begin{tikzcd}
X' \arrow[r, "u"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
Y' \arrow[r, "v"] & Y
\end{tikzcd} \qquad (*)$$

Then there is a base change morphism $u^*L_{X/Y} \to L_{X'/Y'}$. If $(*)$ is cartesian and tor-independent (e.g. if either v or f is flat), then this base change morphism is a quasi-isomorphism and $f'^*L_{Y'/Y} \oplus u^*L_{X/Y} \to L_{X'/L}$ is a quasi-isomorphism.

(iii) This is the key point where you need more than just the truncated guy. If $X \xrightarrow{f} Y \xrightarrow{g} Z$ then there is a distinguished triangle $f^*L_{Y/Z} \to L_{X/Z} \to L_{X/Y} \to f^*L_{Y/Z}[1]$. To get that third arrow, you'll have to invert some map up to homotopy, so this triangle only lives in the derived category.

(iv) $\tau_{\geq -1}L_{X/Y}$ is equal (not just quasi-isomorphic) to our earlier $\tau_{\geq -1}L_{X/Y}$.

Remark 8.3 (a)**.** It is always true that $\mathcal{H}^0(L_{X/Y}) = \Omega^1_{X/Y}$. This actually follows from our earlier discussion. ◇

Remark 8.4 (b)**.** If f is smooth, then $L_{X/Y} \to \Omega^1_{X/Y}$ is a quasi-isomorphism. ◇

Remark 8.5 (c)**.** If $X \hookrightarrow Y$ is a closed immersion (defined by ideal I) which is a local complete intersection, then $L_{X/Y} = I/I^2[1]$. ◇

These are the basic things that people actually know about the structure of the cotangent complex. Magically, it can be used to solve all sorts of things.

Theorem 8.6 (Illusie)**.** *$ch\big(\tau_{\geq -1}(\mathbf{R}\mathcal{H}om(L_{X/Y}, I))[1]\big) \simeq \underline{\mathrm{Exal}}_Y(X, I)$. This implies that $\mathrm{Ext}^1(L_{X/Y}, I) \simeq \mathrm{Exal}_Y(X, I)$. So $\mathrm{Ext}^0(L_{X/Y}, I) = \mathrm{Hom}(\Omega^1_{X/Y}, I)$ is the automorphism group of any*

$$\begin{tikzcd} X \arrow[rr, hook, "I"] \arrow[dr] & & X' \arrow[dl] \\ & Y & \end{tikzcd}$$

(This is the universal property of differentials).

Even if you don't read Illusie, you should at least be able to use this result.

Problem: Given

$$\begin{tikzcd} X_0 \arrow[r, dashed, "i"] \arrow[d, "f_0"'] & X \arrow[d, dashed, "f"] \\ Y \arrow[r, hook, "j"] \arrow[d] & Y \arrow[dl] \\ S & \end{tikzcd}$$

with j a closed immersion defined by a square zero ideal J. The problem is to fill in the diagram as indicated with i square zero such that $f_0^*J \xrightarrow{\sim} \ker(\mathcal{O}_X \to \mathcal{O}_{X_0})$. There are no flatness or finite type hypotheses.

Solution: use $X_0 \to Y_0 \to Y$ to get the distinguished triangle $f_0^*L_{Y_0/Y} \to L_{X_0/Y} \to L_{X_0/Y_0} \to f_0^*L_{Y_0/Y}[1]$. This gives us

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Ext}^0(L_{X_0/Y_0}, f_0^*J) \arrow[r] & \mathrm{Ext}^0(L_{X_0/Y}, f_0^*J) \arrow[r] & \mathrm{Ext}^0(f_0^*L_{Y_0/Y}, f_0^*J) \arrow[dll, rounded corners] \\ & \mathrm{Ext}^1(f_0^*L_{Y_0/Y}, f_0^*J) \arrow[r] & \mathrm{Ext}^1(f_0^*L_{Y_0/Y}, f_0^*J) \arrow[r] & \mathrm{Ext}^1(f_0^*L_{Y_0/Y}, f_0^*J) \arrow[dll, rounded corners, "\partial"] \\ & \mathrm{Ext}^2(L_{X_0/Y_0}, f_0^*J) & & \end{tikzcd}$$

We can rewrite these terms to get

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ext}^0(L_{X_0/Y_0}, f_0^*J) \arrow[r] & \operatorname{Ext}^0(L_{X_0/Y}, f_0^*J) \arrow[r] & 0 \arrow[dll] \\
 & \operatorname{Ext}^1(L_{X_0/Y_0}, f_0^*J) \arrow[r] & \underline{\operatorname{Exal}}_Y(X_0, f_0^*J) \arrow[r] & \operatorname{Hom}(f_0^*J, f_0^*J) \arrow[dll, "\partial"'] \\
 & \operatorname{Ext}^2(L_{X_0/Y_0}, f_0^*J) & &
\end{tikzcd}$$

(The complex $f_0^*L_{Y_0/Y}$ has no degree 0 term and $H^{-1}(f_0^*L_{Y_0/Y}) = f_0^*J$, which is how you get the 0 and Hom in the right column) We want an object in $\underline{\operatorname{Exal}}_Y(X_0, f_0^*J)$ whose image in $\operatorname{Hom}(f_0^*J, f_0^*J)$ is the identity. What is the obstruction?

Theorem 8.7. *(i) There exists an obstruction* $o(f_0) = \partial(\mathrm{id}) \in \operatorname{Ext}^2(L_{X_0/Y_0}, f_0^*J)$ *whose vanishing is necessary and sufficient for a solution to the problem. (ii) If this obstruction is zero, then the set of isomorphism classes of solutions form a torsor under* $\operatorname{Ext}^1(L_{X_0/Y}, f_0^*J)$*. (iii)* $\operatorname{Aut} = \operatorname{Ext}^0(L_{X_0/Y_0}, f_0^*J)$*.*

The point is that if you take the truncated guy, you can only compute up to Ext^1. Conrad: is the proof difficult? Martin: very difficult. Cocycles are great if you have smooth thing, but otherwise, you have to use this machinery.

Max: repeating the thing with cocycles. Cocycles are nice when things are not locally obstructed, but this is what you need when there are local obstructions.

The next thing is a hint that the Artin theorem from Max's talk is an if and only if.

Problem:

$$\begin{tikzcd}
X_0 \arrow[r, hook, "i"] \arrow[d, "f_0"] \arrow[dd, bend right=60, "h_0"'] & X \arrow[d, dashed, "f"'] \arrow[dd, bend left=60, "h"] & I \\
Y_0 \arrow[r, hook, "j"] \arrow[d, "g_0"] & Y \arrow[d, "g"'] & J \\
Z_0 \arrow[r, hook, "k"] & Z & K
\end{tikzcd}$$

The three horizontal embeddings have ideals I, J and K. The problem is to find a dashed arrow.

Theorem 8.8 (Illusie)**.** *There is a canonical class* $o(f_0) \in \operatorname{Ext}^1(f_0^*L_{Y_0/Z_0}, I)$ *such that* f *exists if and only if* $o(f_0) = 0$*. If* $o(f_0) = 0$*, then the set of maps* f *is a torsor under* $\operatorname{Ext}^0(f_0^*L_{Y_0/Z_0}, I)$*.*

The first thing says that this is an obstruction theory (except for finite type). In some situation (said before), the cohomology is coherent, so the Ext^1 is finite. In the same situation, Ext^0 is finite. This tells you that what you want to hold for Artin's theorem actually holds. Let's sketch the proof now. Assume $g^*K \xrightarrow{\sim} J$.

We have classes $e(X) \in \mathrm{Ext}^1_{\mathcal{O}_{X_0}}(L_{X_0/Z}, I)$ and $e(Y) \in \mathrm{Ext}^1_{\mathcal{O}_{Y_0}}(L_{Y_0/Z}, J)$. There is a map $\mathrm{Ext}^1_{\mathcal{O}_{X_0}}(L_{X_0/Z}, I) \to \mathrm{Ext}^1_{\mathcal{O}_{X_0}}(f_0^* L_{Y_0/Z}, I)$; let $e(X) \mapsto z_X$. There is also a map

$$\begin{tikzcd}
\mathrm{Ext}^1_{\mathcal{O}_{Y_0}}(L_{Y_0/Z}, J) \arrow[r] & \mathrm{Ext}^1_{\mathcal{O}_{X_0}}(f_0^* L_{Y_0/Z}) \arrow[d] \\
 & \mathrm{Ext}^1_{\mathcal{O}_{X_0}}(f_0^* L_{Y_0/Z}, I)
\end{tikzcd}$$

Let $e(Y) \mapsto z_Y$. After unwinding stuff, you see that there is a filler arrow f exactly when $z_X = z_Y$. To prove this, you'll have to work with sheaves instead of schemes . . . [[★★★ missed it]]

You get a distinguished triangle $h_0^* L_{Z_0/Z} \to f_0^* L_{Y_0/Z} \to f_0^* L_{Y_0/Z_0} \to$ which gives an exact sequence

$$\begin{tikzcd}
 & & 0 = \mathrm{Ext}^0(h_0^* L_{Z_0/Z}, I) \arrow[dll, rounded corners] & & \\
\mathrm{Ext}^1(f_0^* L_{Y_0/Z_0}, I) \arrow[r] & \mathrm{Ext}^1(f_0^* L_{Y_0/Z}, I) \arrow[r] & \mathrm{Ext}^1(h_0^* L_{Z_0/Z}, I) & = & \mathrm{Hom}(h_0^* K, I)
\end{tikzcd}$$

You get that $o(f_0) := z_X - x_Y$.

8 Osserman. Groupoid Perspective.

So far I've avoided talking about categories fibered in groupoids. Today, we'll see that we should have been using categories fibered in groupoids all along. We'll be talking about the condition (S1$'$) from Max's talk.

One nice property: when we work with categories fibered in groupoids, we can restrict naturally from global problems to local problems and actually get the right result. This issue of taking isomorphic things versus taking things with a particular isomorphism doesn't arise. For example, we can specify pairs (X_A, ϕ) where X_A is flat over A and $\phi : X \to X_A$ induces an isomorphism on restriction to k.

Definition 8.1. A *category cofibered in groupoids* over some category $\mathcal{C}$ is a category fibered in groupoids over $\mathcal{C}^\circ$. ◇

Definition 8.2. A groupoid is *trivial* if there is exactly one morphism between any two objects (i.e. if it is equivalent to the one object category with only the identity map). "The" trivial groupoid is any groupoid which is trivial. ◇

Remark 8.3. Artin uses (S1$'$). Rim uses "homogeneous groupoids". ◇

Definition 8.4. A category $\mathcal{S}$ cofibered in groupoids over $Art(\Lambda, k)$ is a *deformation stack* if $\mathcal{S}_k$ is trivial and for all $A' \to A$ and $A'' \twoheadrightarrow A$, we have

(i) for all $\eta_1, \eta_2 \in \mathcal{S}_{A' \times_A A''}$, the natural map $Mor_{A'} \times_A A'')(\eta_1, \eta_2) \to Mor_{A'}(\eta_1|_{A'}, \eta_2|_{A'}) \times_{Mor_A(\eta_1|_A, \eta_2|_A)} Mor_{A''}(\eta_1|_{A''}, \eta_2|_{A''})$ is a bijection, and

(ii) given $\eta' \in \mathcal{S}_{A'}$, $\eta'' \in \mathcal{S}_{A''}$ and an isomorphism $\phi : \eta'|_A \to \eta''|_A$, there exists some $\zeta \in \mathcal{S}_{A' \times_A A''}$ inducing η', η'', and ϕ upon restriction. ◇

Note that (i) is like a sheaf condition and (ii) is like a descent condition. This is like a more uniform version of the Schlessinger criteria.

Given $\mathcal{S}$, we write $F_{\mathcal{S}} : Art(\Lambda, k) \to \mathbf{Set}$ for the functor of isomorphism classes of objects of $\mathcal{S}$.

Proposition 8.5. *Let $\mathcal{S}$ be a deformation stack. Then $F_{\mathcal{S}}$ is a deformation functor.*

Proof. First we see that $F_{\mathcal{S}}(k)$ is the one point set because we assumed the fiber $\mathcal{S}_k$ is trivial (so there is a single isomorphism class). Note that (H1) follows immediately from (ii). (H2) follows from condition (i). More generally, we get injectivity of $(*)$ as long as $A = k$ since $Mor_A(\eta_1|_A, \eta_2|_A)$ has exactly one element. In general, the failure of injectivity would arise from an isomorphism over A' and another one over A'' which restrict to different isomorphisms over A. ☐

Remark 8.6. How do we know that this condition which implies Schlessinger's criteria isn't unreasonably strong? Although being a deformation stack is formally stronger than satisfying (H1) and (H2), it seems that in practice that any proof of (H1) and (H2) is really a proof that your functor is of the form $F_{\mathcal{S}}$ with $\mathcal{S}$ a deformation stack. As an example, verify that our proof that Def_X is a deformation functor actually proves the deformation stack conditions. ◇

Let me give you an actual mathematical statement (which is in one of Martin's papers).

Lemma 8.7 (Olsson, Crystaline sohomology of stacks and Hyodo-Kato cohomology, Lemma 1.4.4)**.** *If $\mathcal{S}$ is the local deformation problem at a point of an Artin stack, then $\mathcal{S}$ is a deformation stack.*

You can check this lemma in a fairly straightforward way.

Remark 8.8. The argument for the lemma directly involves the asymmetry ($A' \to A$ need not be surjective, but $A'' \to A$ is surjective) because you have to use the formal criterion for smoothness applied to a smooth cover by a scheme ($A'' \to A$ has nilpotent kernel). ◇

More good properties of deformation stacks.

- Given $A' \to A$ with kernel I and $\eta \in \mathcal{S}_A$, then $\{(\eta' \in \mathcal{S}_{A'}, \phi) | \phi : \eta'|_A \xrightarrow{\sim} \eta\}/\cong$ is a pseudo-torsor over $T_{\mathcal{S}} \otimes I = T_{F_{\mathcal{S}}} \otimes I$. In the context of deformation functors, we had an action of $T \otimes I$, which may have had fixed points (it was a torsor if (H4) was satisfied).
- Given $A' \to A$, $\eta' \in \mathcal{S}_{A'}$, $\phi \in \mathrm{Aut}(\eta'|_A)$, then $\{\phi' \in \mathrm{Aut}(\eta') | \phi'|_A = \phi\}$ is a torsor over $\mathrm{Aut}(\zeta_\varepsilon) \otimes I$, where ζ_ε is the trivial deformation over $k[\varepsilon]$. It is a funny quirk that $\mathrm{Aut}(\zeta_\varepsilon)$ has a composition, and you get another additive structure when you make it a module, and the two agree. In particular, you get that the composition in $\mathrm{Aut}(\zeta_\varepsilon)$ is commutative.

Proposition 8.9. *If $\mathcal{S}$ is a deformation stack, then $\mathcal{F}_{\mathcal{S}}$ satisfies (H4) if and only if for all $A' \twoheadrightarrow A$ and all $\eta' \in \mathcal{S}_{A'}$, the natural map $\mathrm{Aut}(\eta') \to \mathrm{Aut}(\eta'|_A)$ is surjective.*

The proof is only a few lines. The idea is that (H4) tells you that an object over $A' \times_A A'$, then it is determined by its restriction to each factor. The only concern is if you have isomorphisms on each factor, they may restrict to different automorphisms over A, but the surjectivity helps you out. [[★★★ work out details]]

In fancier language, in a global setting, (H4) is satisfied if and only if the Isom functor is smooth at the identity. Remember that $\mathrm{Aut}(\eta')$ means automorphisms in the deformation problem, so the automorphism has to restrict to the identity. That is, we are only looking at infinitesimal automorphisms. The Isom functor doesn't have to be smooth everywhere.

Why deformation *stacks*? That is, why all these fiber products of rings?

Lemma 8.10. *If $B \hookrightarrow B' \times B''$ and $q'(\ker(q'')$ is an ideal, then diagrams on the left are equivalent to diagrams on the right.*

$$\begin{tikzcd}
A' \times_A A'' \arrow[r] \arrow[d] & A'' \arrow[d, two heads] & & B \arrow[r, "q'"] \arrow[d, two heads, "q''"'] & B' \arrow[d] \\
A' \arrow[r] & A & \leftrightsquigarrow & B'' \arrow[r] & B' \otimes_B B''
\end{tikzcd}$$

(Here $A' \times_A A'' = B$*, etc. these are really the same rings and all these maps are the same).*

(1) Note that the condition $B \hookrightarrow B' \times B''$ is equivalent to saying that $\operatorname{Spec} B' \sqcup \operatorname{Spec} B'' \twoheadrightarrow \operatorname{Spec} B$ (scheme theoretically surjective). (2) $\otimes$ corresponds to fiber product of schemes (i.e. "intersections", from the point of view of descent theory).

Now at least the descent condition looks like a descent condition, but the covers are really weird. What is still dissatisfying is that if you impose descent for these square, it is not equivalent to imposing descent on the Grothendieck topology generated by these squares. This is basically because scheme theoretic surjectivity is not preserved by base change.

$8\frac{1}{2}$ Hacking

Slides for this talk are available here [[★★★ add url]]

Constructing surfaces of general type by deformation theory. Work of Y. Lee and J. Park. `math.AG/0609072` Last night, a sequel to this prepring was posted.

Motivation. $k = \mathbb{C}$.

Curves C. There is only one topological invariant (the genus g). For surfaces X, consider the underlying topological 4-manifold. The main invariant is the intersection form $Q = \cup \colon H^1(X, \mathbb{Z}) \times H^2(X, \mathbb{Z}) \to H^4(X, \mathbb{Z}) \cong \mathbb{Z}$. This is a unimodular symmetric bilinear form (it's determinant is ± 1).

Theorem $8\frac{1}{2}$.1 (Freedman)**.** *Assume $\pi_1(X) = 0$ (and that X admits a differentiable structure). Then X is uniquely determined up to homeomorphism by Q.*

Theorem $8\frac{1}{2}$.2 (Donaldson)**.** *There exists a topological 4-manifold with infinitely many non-isomorphic differentiable structures (his examples were elliptic surfaces) (This is really quite surprinsing, because it is never true for* $\dim \neq 4$ *(caveat: you have to fix a Pontryagan class in higher dimensions)).*

Problem: Classify topological types of surfaces of general type. Recall that a surface is of *general type* if $\omega_X^{\otimes N}$ defines a birational map for $N \gg 0$. If you know about classification of surfaces, you know that we understand all surfaces very explicitly except surfaces of general type.

Assume again that $\pi_1(X) = 0$. Hodge theory implies that the signature of Q is $(2h^{2,0} + 1, h^{1,1} - 1)$. Here $h^{2,0} = h^0(K_X)$. Consider the simplist case $h^{2,0} = 0$. Then $Q = \begin{smallmatrix} 1 & 0 \\ 0 & -1_n \end{smallmatrix}$ for some n. By Freedman, we know that $X = Bl^n\mathbb{P}^2$ (homeomorphic to blowup of n points in $\mathbb{P}^2$). A topologist will write this as $X \cong \mathbb{P}^2_{\mathbb{C}} \# \overline{\mathbb{P}}^2_{\mathbb{C}} \# \cdots \# \overline{\mathbb{P}}^2_{\mathbb{C}}$ (a $\mathbb{P}^2_{\mathbb{C}}$ connect sum with n others with reverse orientation).

Severi ($\tilde{1}920$): does there exist a surface of general type homeomorphic to $\mathbb{P}^2$.

Yau '76: No. We get $c_1^2 = 3c_2$ from being homeomorphic to $\mathbb{P}^2$, then $X = B/\Gamma$, where $B \subseteq \mathbb{C}^2$ is the unit ball and Γ is a group which acts on B discretely. In this case, we see that B is the universal cover of X, so $\pi_1(X) = \Gamma \neq 0$ (Γ is infinite, else X is not compact).

Barlow '82 (student of Miles Reid). There is a surface of general type homeomorphic to $\mathbb{P}^2 \# (\overline{\mathbb{P}}^2)^{\# 8}$.

Theorem $8\frac{1}{2}$.3 (Y.Lee and J.Park '06)**.** *There is a surface of general type homeomorphic to $\mathbb{P}^2 \# (\overline{\mathbb{P}}^2)^{\# 8}$.*

Idea of the proof: Assume there is such an X. Consider the mofuli space M of deformations of X. From Starr's lecture, we have

$$\dim M \geq h^1(T_X) - h^2(T_X) = \chi(T_X)$$

(these are the tangent space and obstruction space of the problem). We have that $h^0(T_X) = 0$, because general type imlies that there are no infinitesimal automorphisms.

(Eisenbud: this is a local estimate; you have to know that there is a moduli space. Hacking: yes, by what we've been doing, you know that it is a DM stack)

Riemann-Roch: $\chi(E) = \deg(ch(E).td(X)_{\dim X}$. This implies that $\chi(T_X) = (7c_1^2 - 5c_2)/6 = (7 \cdot 2 - 5 \cdot 10)/6 = -6$. This means that if M is non-empty, it's dimension is at least 6.

M is not compact, so compactify by addition points to singular surfaces at the boundary. There is a natural way of doing this using the miinimal model program. So construct a singular surface Y corresponding to a point of the boundary which has the right numerical invarients and prove that there exists a smoothing.

Max: how do you control the topological type of the smoothing. Hacking: using the so-called Milnor fiber.

Local model. Notation: define the singularity type $\frac{1}{r}(1, a) := \mathbb{C}^2/\mu_5$ where a $\zeta \in \mu_r$ acts as $(x, y) \mapsto (\zeta x, \zeta^a y)$. We'll always assume $(a, r) = 1$ (i.e. free in codimension 1). Consider $Y = \frac{1}{n^2}(1, na - 1)$.

Smoothing: $Z = \frac{1}{n}(1, -1) = (uv = w^n) \subseteq \mathbb{C}^3$ mapping to $Y = \frac{1}{n}(1, na-1) = (uv = w^n) \subseteq \frac{1}{n}(1, -1, a)$. Here $u = x^n, v = y^n, w = xy$.

Take $\mathcal{Y} = (uv = w^n + t) \subseteq \frac{1}{n}(1, -1, a) \times \mathbb{C}_t^1$. Note that K_y is $\mathbb{Q}$-cartier.

Milnor Fibre of a smoothing of a singularity. Think of a quadratic cone being smoothed to a hyperboloid; look at a little ball around the singularity as you do the smoothing. The Milnor fibre M' of a smoothing is $\mathcal{Z} = (uv = w^n + t)$ of Z.

Brieskorn: there is a simultaneous resolution of family of ADE singularities (after finite base change).

(1) This implies that the Milnor fiber looks like a wedge of S^2s ($n-1$ of them) (it's a string of $\mathbb{P}^1$s).

(2) Milnor fibre M of $Y \subseteq \mathcal{Y}$. $M' \xrightarrow{\mu_n} M$ étale implies that $\pi_1(M) = \mathbb{Z}/n\mathbb{Z}$ and $e(M') = ne(M)$.

There are a few general statements about Milnor fibres. M has homotopy type of a CW complex of real dimension 2 (using Morse theory). This gives us that M is has the rational homology of a ball. That is, $H_*(M, \mathbb{Q}) = H_*(B, \mathbb{Q})$.

Globally, we're replacing a singularity C in Y by the Milnor fibre M in X so that topologically, there is nothing going on away from the singularity. Mayer-Vietoris gives us that

$$\begin{aligned} e(X) &= e(X^\times) + e(M) - e(\partial M) \\ e(Y) &= e(Y^\times) + e(C) - e(\partial M) \\ e(C) &= e(M) = 1 && (C \text{ cone}) \end{aligned}$$

This implies that $e(X) = e(Y)$.

π_1 $\partial M = S^3/(\mathbb{Z}/n\mathbb{Z})$ "lens space". $\pi_1(\partial M) \twoheadrightarrow \pi_1(M)$ ($\mathbb{Z}/n^2\mathbb{Z} \twoheadrightarrow \mathbb{Z}/n\mathbb{Z}$). Using Van Kampen, we get that $\pi_1(X) = \pi_1(X^\times) *_{\pi_1(\partial M)} \pi_1(M)$, which then must have a surjection from $\pi_1(X^\times) \cong \pi_1(Y^\times)$, so it is enough to show that $\pi_1(Y^\times) = 0$.

Smoothability. Martin covered this yesterday. Let X be a surface with isolated singularities. Let $L_X = L_{X/k}$ be the cotangent complex.

Consider artinian rings

$$0 \to I \to A' \to A \to 0$$

of the form

$$0 \to kt^{n+1} \to k[t]/t^{n+2} \to k[t]/t^{n+1} \to 0$$

$$\begin{tikzcd}
X \arrow[r, hook, dashed] \arrow[d] & X' \arrow[d, dashed] \\
\operatorname{Spec} A \arrow[r, hook] & \operatorname{Spec} A'
\end{tikzcd}$$

The obstruction to getting X' is in $\operatorname{Ext}^2(L_X, \mathcal{O}_X)$ and if the obstruction is zero, then the extensions forma torsor under $\operatorname{Ext}^1(L_X, \mathcal{O}_X)$.

Local-to-global. $H^p(\mathcal{E}xt^q) \Rightarrow \operatorname{Ext}^{p+q}$ spectral sequence gives the exact sequence

$$\begin{tikzcd}
0 \arrow[r] & H^1(\mathcal{E}xt^0) \arrow[r] & \operatorname{Ext}^1 \arrow[r] & H^0(\operatorname{Ext}^1) \arrow[dll] \\
& H^2(\mathcal{E}xt^0) \arrow[r] & \operatorname{Ext}^2 &
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & H^1(T_X) \arrow[r] & \operatorname{Ext}^1(L_X, \mathcal{O}_X) \arrow[r] & H^0(\operatorname{Ext}^1(L_X, \mathcal{O}_X)) \arrow[dll] \\
& H^2(T_X) \arrow[r] & \operatorname{Ext}^2(L_X, \mathcal{O}_X) &
\end{tikzcd} \qquad (\dagger)$$

Claim: if $H^2(T_X) = 0$, then every infinitesimal deformation of singularities is induced by a deformation of X.

PS: first order - by $(\dagger)$, higher order - spectral sequence. Gives $\operatorname{Ext}^2(L_X, \mathcal{O}_X) = H^0(\mathcal{E}xt^2(L_X, \mathcal{O}_X))$. So if there is a local lift, there is a global lift. Now liftings form a torsor under

$$\operatorname{Ext}^1(L_X, \mathcal{O}_X) \twoheadrightarrow H^0(\mathcal{E}xt^1(L_X, \mathcal{O}_X)$$

global *local*

This proves the result.

Warning $8\frac{1}{2}$.4. If X is a surface of general type. Then often $H^2(T_X) \neq 0$. Reason: Serre duality, and the canonical bundle is basically ample because general type. For example $X = X_d \subseteq \mathbb{P}^3$ and $d > 0$ then $H^2(T_X) \neq 0$.

However, X may still have unobstructed deforamtions For example, take our X_d. Then the deformation problem of deforming X embedded in $\mathbb{P}^3$ is unobstructed. Exercise: This actually surjects onto the deformation problem of deforming X as an abstract variety, except for $d = 4$, which is when we have K3 surfaces (which can have non-algebraic deformations). ⌟

Now let's talk about the actual construction of Y. First of all, Y is rational. It is surprising that if you allow singularities, the canonical bundle can have sections! Second thing: Y is constructed using Kodaira's theory of elliptic fibrations.

Picture: C cubic in $\mathbb{P}^2$, a line ℓ through two flex points p, q (changing concavity), then third point r is also a flex. B a conic tangent to C at the first two points, and A a line though the third one. Condier the pencil of cubics generated by $A+B$, C. Blow up base points p, q, r 3 times to get elliptic fibration (see slide for picture now). $Z \xrightarrow{p} \mathbb{P}^1$.

Degenerate fibres: $\tilde{E}_6$, $\tilde{A}_1$ and $Z \times$ (nodal cubic) for C general. Blow up 18 times to get $g : \tilde{Z} \to Z$ to get the other picture in the slides. $h : \tilde{Z} \to Y$.

Martin: I'm getting a little lost, can you give some intuition for why you blow up 18 times. Hacking: [[★★★]]

Note: A chain of $\mathbb{P}^1$'s with self intersection at most -2 contracts to a cyclic quotient singularity.

Why is it of general type? $K_{\tilde{Z}} = g^*K_Z + E$ E effective and g-exceptional. $K_{\tilde{Z}} = h^*K_Y - F$, F effective and h-exceptional. This gives you that $h^*K_Y = g^*K_Z + E + F = E + F - f$ (f fibre of $\tilde{Z} \to \mathbb{P}^1$ chosen cleverly). So $h^*K_Y \sim D$ effective. Now check h^*K_Y is *nef* (numerically eventually free; looks like a base-point-free linear system) (i.e. $h^*K_Y.C \geq 0$ for all curves C). Only need to check for C in the support of D. $K_Y^2 = 2$ from Noether's formula (see slide for calculation). We still need some stuff to vary well in families. We need to know that there is some power of K which is Cartier in the family (not just the special fiber). This will work by the construction.

$H^2(T_X) = 0$.

Lemma $8\frac{1}{2}$.5. *Y surface with cyclic quotient singularities $\pi\colon \tilde{Z} \to Y$ minimal resolution, E exceptional locus. If $H^2(T_{\tilde{Z}}(-\log E)) = 0$ (vector fields tangent to E) then $H^2(T_Y) = 0$.*

By Serre duality, some stuff and use elliptic fibration $\tilde{Z} \to \mathbb{P}^1$ to get vanishing.

Ok, so now we have an infinitesimal thickening. Can we effectivize? Use the Grothendieck existence theorem. Yes: lift a line bundle … the obstruction lies in H^2 of a structure sheaf, which is zero.

Martin: can you always effectivize a family where K is nef?

$8\frac{1}{2}$ Conrad

§1. Discussion of discrete Galois modules. F is a field (usually of characteristic zero, maybe a number field) and F_x/F is a fixed separable closure, with Galois group $G_F = Gal(F_s/F) = \mathrm{Aut}(F_s/F) = \varprojlim Gal(F'/F)$ (F'/F finite); this is a profinite group, so it gets a topology making it compact. The Krull correspondence tells us that that closed subgroups correspond to intermediate fields, and open subgroups (which are always closed) correspond to finite subextensions.

Example $8\frac{1}{2}$.1. Let V be a commutative group scheme of finite type over F. Then G_F acts on the abelian group $V(F_s) = M$. If V is quasi-projective (in $\mathbb{P}^N_F$, then G_F acts on the coordinates. Note that any $m\colon \operatorname{Spec} F_s \to V$ factors through some finite Galois extension $\operatorname{Spec} F' \xrightarrow{m'} V$. Thus, the G_F action on any $m \in V(F') \subseteq V(F_S)$ (subgroup) have stabilizer $Gal(F_S/F') \subseteq G_F$ (open subgroup). Thus, any element is fixed by an open subgroup ◇

Example $8\frac{1}{2}$.2. Suppose $X \to \operatorname{Spec} F$ is separated of finite type and $\ell \neq charF$. Then $H^i_{et}(X_{F_s}, \mathbb{Z}/\ell^n\mathbb{Z})$ has a natural action of G_F. It turns out that this cohomology is the limit of cohomologies of the finite guys, so again G_F acts with open stabilizers. It is very important that you're working with finite coefficients. ◇

Definition $8\frac{1}{2}$.3. A *discrete G_F-module* is a G_F-module M such that every element of the module has an open stabilizer in G_F. ◇

Note that any finite set of elements has an open stabilizer obtained by intersecting.

Example $8\frac{1}{2}$.4. A G_F-module M with $\#M < \infty$ is discrete if and only if G_F acts on M through some finite quotient $Gal(F'/F)$. That is, $G_{F'} \subseteq G_F$ acts trivally on M. ◇

Remark $8\frac{1}{2}$.5. If Γ is any profinite group, then we can have the same discussion. For example, $\Gamma = \mathbb{Z}_p$, $GL_n(\mathbb{Z}_p)$, $\pi_1^{et}(X, x)$, etc. ◇

Example $8\frac{1}{2}$.6. Suppose E is an elliptic curve over F, and suppose $N \in \mathbb{Z}^+$ so that $char(F) \nmid N$. Then $E[N] = E[N](F_s) \simeq \mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$ (choosing a basis), which G_F acts on throught the finite quotient $Gal(F(E[N])/F)$ (the extension generated by the coordinates of the N-torsion points). Call this $\rho_{E,N}\colon G_F \to \mathrm{Aut}(E[N]) \cong GL_n(\mathbb{Z}/N\mathbb{Z})$ (choosing a basis), which has a big open kernel, so this is continuous. Now as we allow N to be more and more divisible, we'll lose discreteness but retain continuity.

Let $N = p^r$ with p prime not equal to $char(F)$. Fact:

$$\begin{tikzcd}
E[p^{r+1}] \arrow[d, two heads] \arrow[r, phantom, "\simeq"] & \mathbb{Z}/p^{r+1} \times \mathbb{Z}/p^{r+1} \arrow[d, "\text{reduction}"] \\
E[p^r] \arrow[r, phantom, "\simeq"] & \mathbb{Z}/p^r \times \mathbb{Z}/p^r
\end{tikzcd}$$

So we can choose compatible maps $G_F \to GL_2(\mathbb{Z}/p^n)$. This is a compatible family of deformations of $\rho_{E,p}$.

If $F = \mathbb{C}$ and $E = \mathbb{C}^2/\Lambda$. Then $E[N] = \frac{1}{N}\Lambda/\Lambda \cong \Lambda/N\Lambda$. Then $E[N \cdot d] \xrightarrow{\cdot d} E[N]$ is surjective and given by the usual quotient $\Lambda/Nd\Lambda \to \Lambda/N\Lambda$.

$T_p(E)$, the p-adic Tate module is $\varprojlim E[p^r]$, which is $\mathbb{Z}_p \times \mathbb{Z}_p$. If $F + \mathbb{C}$ and $E = \mathbb{C}/\Lambda$, then $T_pE = \varprojlim \Lambda/p^r\Lambda = \Lambda \otimes_{\mathbb{Z}} \mathbb{Z}_p = H_1(E, \mathbb{Z}_p)$. You get that G_F acts continuously and $\mathbb{Z}_p$-linearly on this $T_p(E)$, so we get $\rho_{E,p^\infty} \colon G_F \to GL_2(\mathbb{Z}_p) \subseteq M_2(\mathbb{Z}_p)$ (open). This is continuous with closed (but not open) kernel. That is, $T_p(E)$ is in general not a discrete Galois module. ◇

Arithmetic application. Suppose F is a finite field of size q and choose $char(F) \neq= \ell$. We get $\rho_{E,\ell^\infty} \colon G_F \to GL_2(\mathbb{Z}_\ell) = \mathrm{Aut}_{\mathbb{Z}_\ell}(T_\ell E)$. We have an element $\phi = Frob_{F,q}$ in G_F, which acts by $t \mapsto t^q$. The characteristic polynomial of the ϕ-action is $X^2 - a_E X + q$, where a_E is the number of points of $E(F)$ minus $q+1$, which is an integer.

The point is that characteristic polynomials of interesting actions tends to encode a lot of information.

Example $8\frac{1}{2}$.7. Say E is an elliptic curve over $\mathbb{Q}_p$ given by $y^2 = x^3 + ax + b$ and E' another one. Look at the $G_{\mathbb{Q}_p}$-action on $E[p^2]$, $E'[p^2]$. $G_{\mathbb{Q}_p} \rightrightarrows GL_2(\mathbb{Z}/p^2)$. Suppose $|a' - a|, |b' - b| \ll 1$. Fact: if you take $|a' - a|, |b' - b| \ll 1$, then $\rho_{E,p^7} \simeq \rho_{E',p^7}$ as $G_{\mathbb{Q}_p}$-modules. So if you choose the bases compatibly, what you have is "two p-adic deformations of the same mod-p^7 representation". ◇

§2. Cohomology. Now let's talk about the kind of cohomology theory you need to study these things. Take Γ to be a profinite group (e.g. G_F). We want to think about Γ-**mod**, the category of discrete Γ-modules (this is not $\mathbb{Z}[\Gamma]$-**mod** because of discreteness).

Exercise $8\frac{1}{2}$.1. Γ-**mod** has enough injectives. ◀

We'll be interested in the functor Γ-**mod** $\to$ **Ab** given by $M \mapsto M^\Gamma = \{m \in M | \gamma m = m \text{ for all } \gamma \in Ga\}$. This is a left exact functor.

Definition $8\frac{1}{2}$.8. Define $H^*(\Gamma, -)$ to be the derived functors of the functor $-^\Gamma$. ◇

Remark $8\frac{1}{2}$.9. You can compute this using "continuous cochains". ◇

Let's describe H^1.

Example $8\frac{1}{2}$.10. $H^1(\Gamma, M) = Z^1(\Gamma, M)/B^1(\Gamma, M)$ where $B^1(\Gamma, M) = \{\Gamma \to M, \gamma \mapsto \gamma m_0 - m_0\}$. Note that since m_0 has an open stabilizer, this is continuous (but not a homomorphism) (it factors through $\Gamma/Stab(m_0)$. $Z^1(\Gamma, M) = \{\Gamma \xrightarrow{c} M \text{ continuous} | c(\gamma_1\gamma_2) = \gamma_1 \cdot c(\gamma_2) + c(\gamma_1)\}$.

Say Γ acts trivially, so $B^1(\Gamma, M) = 0$ and $Z^1(\Gamma, M) = \mathrm{Hom}_{cont}(\Gamma, M)$. ◇

Remark $8\frac{1}{2}$.11. Given $\phi\colon \Gamma \to \Gamma'$ continuous, then any discrete Γ'-module can be viewed as a discrete Γ-module, so Γ'-**mod** $\to$ Γ-**mod**. Likewise, given any $M' \in \Gamma'$-**mod**, we have that $M'^{\Gamma'} \subseteq M'^{\Gamma}$. This induces a natural maps $H^*(\Gamma', M') \to H^*(\Gamma, M')$ given by composition with ϕ on the level of cochains. ◇

Example $8\frac{1}{2}$.12. Let F'/F be a field extension (like $\mathbb{Q}_p/\mathbb{Q}$), then we have

$$\begin{tikzcd}
F_s \arrow[r, hook, "\text{choose}"] \arrow[d, dash] & F'_s \arrow[d, dash] \\
F \arrow[r] & F'
\end{tikzcd}$$

which defines $G_{F'} \to G_F$ which is well defined up to conjugation (since you can choose a different map on separable closures and $H^*(\Gamma, M') \to H^*(\Gamma, M')$ is *invariant* under the conjugation action by Γ'. Thus, $H^*(G_F, M) \to H^*(G_{F'}, M)$ is canonical and is the pullback map along $\operatorname{Spec} F' \to \operatorname{Spec} F$. ◇

This has many of the nice properties that you're used to, but is really a different beast.

Example $8\frac{1}{2}$.13. $F = \mathbb{Q}$, and consider $H^1(G_{\mathbb{Q}}, \mathbb{Z}/2) = \operatorname{Hom}_{cont}(G_{\mathbb{Q}}, \mathbb{Z}/2)$, which as a set is $\{\mathbb{Q}[\sqrt{d}]\}$ where d is square free (the additive structure is multiplication of the d's up to squares (d could be 1)). The continuity is very important; $G_{\mathbb{Q}}$ has lots of index 2 subgroups which aren't open. This set is not finite dimensional over $\mathbb{Z}/2$ because it is infinite.

We want to work with a quotient of $G_{\mathbb{Q}}$ subject to restricted ramification. ◇

What should replace G_F? Let F be a number field (so $[F : \mathbb{Q}] < \infty$). For a finite extension F'/F, we have

$$\begin{tikzcd}
\operatorname{Spec} \mathcal{O}_{F'} \arrow[d] & \textit{picture} \\
\operatorname{Spec} \mathcal{O}_F & \textit{picture with points}
\end{tikzcd}$$

has some ramification points. We want to only consider those finite extensions which are ramified inside Σ, a fixed finite set of closed points of $\operatorname{Spec} \mathcal{O}_F$. That is, we replace G_F with $\pi_1^{et}(\operatorname{Spec} \mathcal{O}_{F,\Sigma})$. That is, we work with $G_{F,\Sigma} = Gal(F_\Sigma/F)$, where F_Σ is the compositum of all the finite F'/F which are unramified outside of Σ. This is like considering the fundamental group of a surface with some punctures (but a fixed finite number of punctures).

Basic algebraic number theory tells us that

Theorem $8\frac{1}{2}$.14 (Tate). *If M is a finite discrete $G_{F,\Sigma}$-module, then (1) $H^i(G_{F,\Sigma}j, M)$ is finite (as a set) for all i (= 0 for $i > 2$ if $\#M$ is odd), and (2) If $[L : \mathbb{Q}_p] < \infty$ then $H^i(G_L, M)$ is finite for M finite (and 0 for $i > 2$) for all i.*

btw, I'm secretly allowing ramification at infinity.

Example $8\frac{1}{2}$.15. If $F = \mathbb{Q}$ and $\Sigma = \{2, 3, 7\}$, then $H^1(G_{\mathbb{Q},\Sigma}, \mathbb{Z}) = \{\mathbb{Q}[\sqrt{d}] | d$ squarefree $d|42\}$ which is finite. ◇

§3. Deformations. Historically, the motivation was that Hida constructed certain representations $\rho\colon G_{\mathbb{Q},\Sigma} \to GL_2(\mathbb{Z}_p[[x]])$ such that $x \mapsto (1+p)^k - 1$ for $k \geq 2$ and these gave interesting representations $\rho_k\colon G_{\mathbb{Q},\Sigma} \to GL_2(\mathbb{Z}_p)$. For representation that come from geometry, there will be good ramification. There were some reps coming from modular something and Hida realized that they were specializations of a single one.

Consider $\overline{\rho}\colon \Gamma \to GL(V_0)$ continuous with Γ profinite and V a finite dimensional vector space over some finite field k. Define $\hat{\mathcal{C}}_k$ to be the category of complete local noetherian rings with residue field k (this is the coefficient ring $\Lambda = W(k)$ Witt vectors; if $k = \mathbb{F}_p$, then $\Lambda = \mathbb{Z}_p$). A *lifting* of $\overline{\rho}$ to $A \in \hat{\mathcal{C}}_k$ is a pair (V_A, θ) where V_A is a finite free A-module equipped with a continuous action $\rho\colon \Gamma \to GL(V_A)$, and $\theta\colon V_A/\mathfrak{m}_A V_A \xrightarrow{\sim} V_0$ as $k[\Gamma]$-modules.

Say $(V_A, \theta) \sim (V'_A, \theta')$ if there exists an isomorphism $V_A \cong V'_A$ as $A[\Gamma]$-modules such that mod $\mathfrak{m}_A$ carries θ to θ' (i.e. respects identification with V_0). A *deformation* of $\overline{\rho}$ is an equivalence class of lifts. It is important to distinguish between liftings and deformations. Liftings have automorphisms (say $A^\times$).

Matrix meaning: a lifting $\rho\colon \Gamma \xrightarrow{cont} GL_N(A)$ so that $\rho \equiv \overline{\rho}\text{-}\mathbf{mod}\mathfrak{m}_A$ and $\rho \sim \rho'$ means that $\rho = M \cdot \rho' \cdot M^{-1}$ where $M \in GL_N(A)$ and $M \equiv 1\text{-}\mathbf{mod}\mathfrak{m}_A$.

Warning $8\frac{1}{2}$.16. This notion of deformation is totally unrelated to geometric deformation. If $E \to S$ is a family of elliptic curves with S p-adic variety, then for any $s \in S(\mathbb{Q}_p)$ we get $\rho_s\colon G_{\mathbb{Q}_p} \to GL_2(\mathbb{Z}_p)$, but these don't come from a single representation $G_{\mathbb{Q}_p} \to GL_2(\mathbb{Z}_p[[x_1, x_2, \ldots]])$. ⌟

Definition $8\frac{1}{2}$.17. $\mathrm{Def}_{\overline{\rho}}\colon \hat{\mathcal{C}}_k \to \mathbf{Set}$ is $A \mapsto \{$deformations of $\overline{\rho}$ to $A\}$ (not liftings!). This is clearly a functor by pushing forward matrix coefficients ($V_A \rightsquigarrow A' \otimes_A V_A$). ◇

Exercise $8\frac{1}{2}$.2. $\mathrm{Def}_{\overline{\rho}}(k[\varepsilon]) = H^1(\Gamma, \mathrm{End}_k(V_0))$ where Γ acts by conjugation on elements of $\mathrm{End}_k(V_0)$. ◀

"proof": given $\overline{\rho}\colon \Gamma \to GL_N(k)$ and we want a lifting $\rho\colon \Gamma \to GL_N(k[\varepsilon])$. We can write $\rho(\gamma) = 1 + \varepsilon c(\gamma)\overline{\rho}(\gamma)$ where $c(\gamma) \in M_N(k)$. Check that ρ is a continuous homomorphism exactly when $c \in Z^1_{cont}(\Gamma, \mathrm{End}(V_0))$ and $\rho \sim \rho'$ exactly when $c - c' \in B^1(\Gamma, \mathrm{End}(V_0))$.

If we want a good deformation theory, that H^1 had better have some good finiteness properties.

Theorem $8\frac{1}{2}$.18 (Mazur)**.** *If* $\dim H^1(\Gamma, \mathrm{End}(V_0)) < \infty$ *(this is automatic in interesting situations by some theorems of Tate), then* $\mathrm{Def}_{\overline{\rho}}$ *satisfies (H1-H3). If* $\mathrm{End}_k(V_0) = k$

(e.g. $\overline{\rho}$ irreducible), then (H4) holds. So you get a universal deformation $\overline{\rho}^{univ}\colon \Gamma \to GL_N(\mathcal{R}_{\overline{\rho}}^{univ})$*. That is,*

$$\begin{tikzcd}
\Gamma \arrow[r, "\overline{\rho}^{univ}"] \arrow[rr, bend left, "\rho"] \arrow[dr, "\overline{\rho}"'] & GL_N(\mathcal{R}) \arrow[r] \arrow[d] & GL_N(A) \arrow[dl] \\
 & GL_N(k) &
\end{tikzcd}$$

carries $\overline{\rho}^{univ}$ *to* ρ *up to 1-unit matrix conjugation!*

To do anything with this, you want to impose more conditions than just being unramified outside a finite set.