MATH 242 - SYMPLECTIC GEOMETRY
LECTURE NOTES

ANTON

Disclaimer

These are notes from Math 252, Fall 2005, taught by professor Weinstein. I try to make them accurate and correct, but they are still full of errors and typos and logical gaps. Send comments/corrections/whatever to `anton@math.berkeley.edu`. Specifically, don't hesitate to send me an email to remind me to post the latest notes.

Lecture 1

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GSI(sorta):Christian Bloman?

orbit method in representation theory.

symplectic geometry and algebraic geometry. moment maps. Attiya.

90s. conformal field theory. Syberg-Witten theory.

Poisson bracket ... lie algebra structure. Leads to Poisson geometry. Mechanics - Poisson brackets are classical limit of commutators.

00s. Generalized complex geometry, which includes complex and symplectic geometry.

Weekly reading assignments - try to look at it before the week starts, and make questions. Long list of weekly homework; you only have to hand in about three each week, due on the following Thursday. Term paper - sample papers on website; the idea is to write a survey of the topic of your choice. In October, we will have a session with librarian to learn how to search for stuff. There is no final exam (no exams at all, in fact).

There should be an email list on bspace, where you can send questions. Until that is set up, send questions directly to Alan (answers sent to whole class unless otherwise specified).

The book really is required. There is a website for corrections. The author also has a paper on the arXiv which is kind of a sequel to the book. Please understand everything in the reading.

Symplectic linear algebra:

Bilinear forms: V vector space over a field (almost always $\mathbb{R}$ or $\mathbb{C}$), but you could work over other fields or rings.

A *bilinear form* $B : V \times V \to k$ (sometimes to $\mathbb{R}$ if $k = \mathbb{C}$) is bilinear. In the case $k = \mathbb{C}$, we also have *hermitian*[1], which means $B(ax, by) = a\bar{b}B(x, y)$ for complex numbers a and b, and $B(x, y) = \overline{B(y, x)}$. We can talk about *symmetric* forms ($B(x, y) = B(y, x)$), *skew symmetric*.

In super algebra, symmetric and skew symmetric things are the same thing. Hermitian and skew hermitian are closely related.

Dualization:If B is a bilinear form on V, it induces a linear map $\tilde{B} : V \to V^*$ given by $\tilde{B}(x)(y) = B(x, y)$. This is the notion of *dualization*. We say B is *non-degenerate* if $\tilde{B}$ is an isomorphism. Note that this gives a way to get B from $\tilde{B}$, as well as the other way.

Orthogonality: (symmetric, skew-symmetric, hermitian). Two vectors are orthogonal if $B(x, y) = 0$. If $W \subseteq V$, we define $W^\perp = \{x \in V | B(x, y) = 0 \ \forall y \in W\}$. In particular, $V^\perp = \ker \tilde{B}$, which is zero if B is non-degenerate. In the finite-dimensional case, the converse is true. In the infinite-dimensional case, it turns out that many of the symplectic structures are *weakly non-degenerate* (i.e. $\tilde{B}$ injective, but not surjective). In the finite-dimensional case, non-degeneracy ensures $(W^\perp)^\perp = W$

Assume finite dimensional, non-degenerate. Then $\dim W + \dim W^\perp = \dim V$. In particular, if $W = W^\perp$, then $\dim W = \frac{1}{2}\dim V$ (and so $\dim V$ must be even). Such W are called *lagrangian* in the skew-symmetric non-degenerate case (which is exactly *symplectic*).

$W \subseteq V$ is *isotropic* is $W \subseteq W^\perp$. In the skew-symmetric case, if $\dim W = 1$, then W is isotropic because $B(v, v) = -B(v, v)$, so $B(v, v) = 0$. Isotropy implies that $\dim W \leq \frac{1}{2}\dim V$. Any isotropic space is contained in a maximal isotropic space (which are lagrangians!). W is *coisotropic* if $W^\perp \subseteq W$. That is, if $W^\perp$ is isotropic.

By the way, symplectic implies even dimension!

Examples:

(a) Let E be any vector space, and let $V = E \oplus E^*$ (this is even-dimensional). Define $\Omega_\pm : V \times V \to k$ by $\Omega_\pm((x, p), (x', p')) = \langle x, p' \rangle \pm \langle x', p \rangle$ ([2]). When is this non-degenerate? $\tilde{\Omega}_\pm : V \to V^*$, sending $E \oplus E^*$ to $E^* \oplus E^{**}$ given by

$$\Omega_\pm(x, p)(x', p') = p'(x) \pm p(x')$$

so $\tilde{\Omega}_\pm(x, p) = (\pm p, i(x))$, where $i : E \to E^{**}$ is the natural map. In the finite-dimensional case, we can identify $i(x)$ with x because i is an isomorphism. E is called *reflexive* if i is an isomorphism, in which case $\Omega_\pm$ is non-degenerate. When E is finite-dimensional, note that Ω_- is symplectic! This is in some sense *the* example.

In general, if you assume choice, i is always injective, so $\Omega_\pm$ is always weakly non-degenerate.

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[1] When some name becomes part of an adjective, you stop capitalizing

[2] Here, $\langle x, p \rangle = p(x)$.

In many infinite-dimensional cases, we often require that maps be continuous (in some sense or another). Then this gives some sort of topological dualization.

(b) Let E be a vector space. B any bilinear form on E, giving $\tilde{B} : E \to E^*$. Look at the graph of $\tilde{B}$, which is a (linear!) subspace of $E \oplus E^*$. We can ask, "when is this graph lagrangian (w.r.t $\Omega_\pm$)?" It has a chance because it is half dimensional.

$$\begin{aligned}\tilde{\Omega}_\pm((x, \tilde{B}(x)), (y, \tilde{B}(y))) &= \tilde{B}(y)(x) \pm \tilde{B}(x)(y)\\ &= B(y, x) \pm B(x, y)\end{aligned}$$

So the graph is lagrangian for Ω_+ $[\Omega_-]$ if and only if B is skew-symmetric [symmetric].

We think of lagrangian subspaces of $(E \oplus E^*, \Omega_{+[-]})$ as "generalized" skew-symmetric [symmetric] forms on E.

From the graph of $\tilde{B}$, you can read off properties of B. For example, B is non-degenerate (in the finite dimensional case) if the graph doesn't intersect E. A graph never intersects E^*.

A lagrangian subspace of $(E \oplus E^*, \Omega_+)$ is called a *Dirac structure* on E. Special case is a skew-symmetric form. And a special case of that is a symplectic structure. Note that E and E^* are themselves lagrangian.

(c) Operations on spaces with bilinear forms. $(V, B) \rightsquigarrow (V, -B)$ preserves type ... we often write this $V \to \bar{V}$, called the *opposite symplectic structure*.

Structure on $(V_1, B_1) \oplus (V_2, B_2)$ defined by $((x_1, x_2), (y_1, y_2)) \mapsto B_1(x_1, x_2) + B_2(y_1, y_2)$. We can add spaces with the same kind of structure.

In $\bar{V}_1 \oplus V_2$, given a linear map $L : V_1 \to V_2$, when is its graph lagrangian?

$$((x, L(x)), (y, L(y))) \mapsto -B_1(x, y) + B_2(L(x), L(y)) = 0.$$

The graph of L is isotropic if and only if L "preserves 'inner' product". It is lagrangian if and only if L is an isomorphism of B-spaces (spaces with non-degenerate bilinear form).

Lagrangian subspaces of $\bar{V}_1 \oplus V_2$ as "generalized" B-space isomorphisms of V_1 to V_2.

In the symmetric case, these are isometries. If the space is of the form $E \oplus E^*$ (which it always is), then we get a correspondence between isometries and skew-symmetric forms. If $V_1 = V_2$, isometries form the orthogonal group, and skew-symmetric forms form the lie group of that group!

In the symplectic case, we will get a correspondence between some other things.

LECTURE 2

Office hours:

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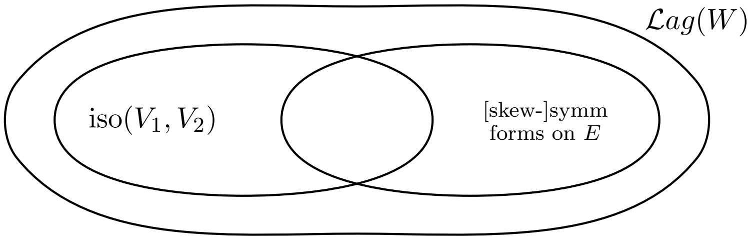
Questions from last time or from the reading:

(a) If we have V_1, V_2, then isomorphisms between V_1 and V_2 correspond to lagrangians in $\bar{V}_1 \oplus V_2$. Lagrangians in $E \oplus E^*$ (with $\Omega_{+[-]}$) correspond to skew-symmetric forms [symmetric forms] on E.

How do we put these together? If $W = \bar{V}_1 \oplus V_2 \cong E \oplus E^*$. Then there are two big subsets of $\mathcal{L}ag(W)$, one corresponding to isomorphisms $V_1 \to V_2$, and the other corresponding to forms on E.



If V has a (non-degenerate) "pure"[3] bilinear form B, then we can talk about $\mathcal{L}ag(V) \subseteq Gr_{1/2}(V)$ = half-dimensional subspaces, and it is always a submanifold. To see that $Gr_{1/2}(V)$ is a manifold, note that it is homogeneous under the action of $GL(V)$. That is, we have a map $GL(V) \twoheadrightarrow Gr_{1/2}(V)$, where the kernel is stuff like $\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)$.

Theorem 2.1 (Witt's Theorem)**.** Aut (V, B) *acts transitively on* $\mathcal{L}ag(V)$*.*

So we can show that $\mathcal{L}ag(V)$ is a manifold. If we take the case $V = E \oplus E^*$, the elements of $\mathcal{L}ag(V)$ not intersecting E^* are bilinear forms (skew or symmetric, depending on $\Omega_\pm$). Thus, we have a bijection between such forms and an open subset of $\mathcal{L}ag(V)$.

Thus, in the skew case, $\mathcal{L}ag(V)$ has dimension $\frac{n(n+1)}{2}$, and in the symmetric case, it is $\frac{n(n-1)}{2}$.

(b) Suppose we have V, W, X all pure of the same type. And say $L_1 \subseteq \bar{V} \times W$ and $L_2 \subseteq \bar{W} \times X$ lagrangians. Then we can form $L_2 \circ L_1 \subseteq \bar{V} \times X$. It is the set of all $(v, x) \in \bar{V} \times X$ such that there is a $w \in W$ such that $(v, w) \in L_1$ and $(w, x) \in L_2$. This kind of composition is defined for arbitrary relations. It is not hard to see that if L_1 and L_2 are linear, then so is $L_2 \circ L_1$. What is amazing is that if if L_1 and L_2 are lagrangian, then so is $L_2 \circ L_1$.*

*Optional Problem: prove this

Furthermore, in $\bar{V} \times V$, the diagonal, Δ_V, is lagrangian (clear), and it is the identity under composition (when defined[4]).

Thus, we have a category. The objects are all pure, non-degenerate vector spaces, and $\mathrm{Hom}(V, W) = \mathcal{L}ag(\bar{V} \times W)$ (and it is the empty set if V and W have different types). Two things are isomorphic in this category if and only if they are isomorphic in the usual sense. We have added some homomorphisms.

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[3] Either skew-symmetric or symmetric.

[4] That is, when the domains and ranges match up.

Example: What are the homomorphisms between 0 and V? They are just lagrangians in $\bar{0} \times V = V$, and $\mathrm{Hom}(V, 0) = \mathcal{L}ag(\bar{V})$.

When is $\mathrm{Hom}(V, W)$ non-empty? Well, V and W have to have the same type. Then the question is, "when does $\bar{V} \times W$ have lagrangians?"

Lemma 2.2. *In the skew case, $\mathcal{L}ag(anything) \neq \varnothing$.*
In the symmetric case, if over $\mathbb{C}$, same as skew symmetric. If we are over $\mathbb{R}$, we have lagrangians only if the signiture[5] *is zero.*

You can take a lagrangian in $\bar{V}$ and a lagrangian in W, you can cross them. When you take direct products, signatures add, so in the real symmetric case, you need to have $sign(V) = sign(W)$.

Differentiable Manifolds

All this linear algebra should be thought of as taking place in the tangent spaces of manifolds. All our manifolds are C^∞.

2-forms: $\omega \in \Omega^2(M) = \Gamma(\wedge^2 T^*M)$ is the set of 2-forms on M. We can make $\tilde{\omega} : TM \to T^*M$, given by $\tilde{\omega}(v)(w) = \omega(v, w)$, which will be a smooth map of bundles. This is also sometimes written $\tilde{\omega}(v) = i_v\omega = v\lrcorner\omega$. In fact, $i_v : \wedge^p T_x^*M \to \wedge^{p-1} T_x^*M$ for $v \in T_xM$ by putting v in for the first entry.

In local coordinates $(x^1, \ldots, x^m)$, we can write

$$\omega = \frac{1}{2}\omega_{ij}(x) dx^i \wedge dx^j$$

where $\omega_{ij} = -\omega_{ji} = \omega\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$.

$\omega \in \Omega^2(M)$ is *non-degenerate* if $\tilde{\omega}$ is invertible, and we say it is *presymplectic* if $d\omega = 0$ (i.e. ω closed). We say it is *symplectic* if it is both.

Example: On $\mathbb{R}^{2n}$, with coordinates $(q^1, \ldots, q^n, p_1, \ldots, p_n)$, take $\omega_n = dq^i \wedge dp_i := \sum_i dq^i \wedge dp_i$. It is clear that ω is presymplectic. In matrix form, it is $\left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$.

Lemma 2.3. *ω is non-degenerate if and only if $\omega^{\dim M/2} = \omega \wedge \cdots \wedge \omega$ is nowhere zero in $\Omega^{top}(M)$.*

In our case, we have $(\omega_n)^n = n!\, dq^1 \wedge dp_1 \wedge \cdots \wedge dq^n \wedge dp_n$, which is almost the canonical volume form (which is $dq^1 \wedge \cdots \wedge dq^n \wedge dp_1 \wedge \cdots \wedge dp_n$).

The *symplectic volume* is either $\frac{1}{n!}\omega^n$ or $(-1)^{\text{something}}\frac{1}{n!}\omega^n$, according to convention ... generally the first one ($2n = \dim M$). Not every manifold can support a symplectic structure ... it has to be even dimensional, and orientable! Look at $[\omega] \in H^2_{DR}(M, \mathbb{R})$. We have that $[\omega]^n = [\omega^n]$. If M is compact, then

$$\int_M \omega^n > 0$$

so then ω is not exact. Thus, M must have some cohomology in degree 2 too. You also have to have a non-degenerate 2-form, which is equivalent to an *almost complex* structure (i.e., a map $J : TM \to TM$ such that $J^2 = -I$).

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[5] Number of positive eigenvalues minus the number of negative ones.

It took a while to find an example satisfying these conditions, but not having a symplectic structure. Take the connected sum $\mathbb{C}P^2\#\mathbb{C}P^2\#\mathbb{C}P^2$. This manifold has lots of cohomology in degree 2 with the right squares, and it has an almost complex structure, but it doesn't have a symplectic structure (some invariant is non-zero ... Sieberg-Witten stuff). There is a theorem that any symplectic manifold locally looks like the example we gave above (with ω_n on $\mathbb{R}^{2n}$). Note that

$$d\omega = \frac{1}{2}\frac{\partial \omega_{ij}}{\partial x^k} dx^k \wedge dx^i \wedge dx^j.$$

The vanishing of this expression doesn't ensure the vanishing of $\frac{\partial \omega_{ij}}{\partial x^k}$. It turns out that it is zero if and only if

$$\frac{\partial \omega_{ij}}{\partial x^k} + \frac{\partial \omega_{jk}}{\partial x^i} + \frac{\partial \omega_{ki}}{\partial x^j} = 0$$

There is a slightly more general version which says that $d\omega = 0$ and $\tilde{\omega}$ has constant rank (i.e. if the matrix $\omega_{ij}(x)$ has constant rank), then in suitable coordinates $(q^1, \dots, q^r, p_1, \dots, p_r, \lambda^1, \dots, \lambda^s)$, we have

$$\omega = dq^i \wedge dp_i.$$

In these coordinates, the kernel of $\tilde{\omega}$ is the span of the $\frac{\partial}{\partial \lambda^i}$. In general, we have $\ker \tilde{\omega} \subseteq TM$. If ω has constant rank, these spaces all have the same dimension, and they define a smooth sub-bundle of TM. In this case, one can prove that

$$d\omega = 0 \Rightarrow \ker \tilde{\omega} \text{ is involutive}$$

which implies (by the Frobenius theorem) that $\ker \tilde{\omega}$ is tangent to a "characteristic" foliation of M. In this picture, the λ^is are the coordinates (locally) along the foliation, and the p_is and q^is are transverse to the foliation. We can get back to the symplectic case by throwing away the λ^is.

LECTURE 3

Questions from last time:

- Foliations
- de Rham cohomology
- Lagragian subspaces as morphisms. If V and W are of the same type, and the signatures are the same, then there is an injection $Iso(V, W) \hookrightarrow \mathcal{L}ag(\bar{V} \times W) = \mathrm{Hom}(V, W)$. If $V = W$, then $\mathrm{Aut}\, V \hookrightarrow \mathcal{L}ag(\bar{V} \times V)$. This is a semigroup with identity, which contains $\mathrm{Aut}\, V$ as a subgroup. In fact, these are just the invertible elements. What can we say about this semigroup? e.g., say $V = \mathbb{R}^3$ Euclidean, which gives $O(3)$ as automorphisms, which is an open subset of $\mathcal{L}ag(\bar{V} \times V)$, and it is closed since it is compact. I think $O(3)$ is all of $\mathcal{L}ag(\bar{V} \times V)$. To show this, we need to show that no lagrangian subspace intersects the V axis.

  If $V = \mathbb{R}^2$ symplectic, then the automorphisms are $SL_2(\mathbb{R})$ (non-compact), and $\mathcal{L}ag(\mathbb{R}^4)$ look like 2×2 symmetric matrices. The automorphism group sits in $lag(\mathbb{R}^4)$ as an open dense subset.

  Warning: The product on $\mathcal{L}ag(\bar{V} \times V)$ is not, in general continuous!

really?

Optional: analyze this multiplication

C. Sabot: somehow related lagrange grassmanian to probability, and modified this so that the multiplication is continuous.

- The non-symplectic example $(\mathbb{C}P^2)^{\#3}$ ... more? I dunno.

Important things from the reading:

On the cotangent bundle T^*X of any manifold, there is a natural structure ... coordinates (x^i, ξ_i). $\alpha = \xi_i dx^i$ and $\omega = -d\alpha = dx^i \wedge d\xi_i$. If ϕ is any 1-form (section of cotangent bundle), $\phi^*\alpha = \phi$, and this characterizes α. This α is called the *canonical 1-form* or the *Liouville form*. You can see from local coordinates that ω gives a symplectic structure.

$T_{(x,0)}T^*X \cong T_xX \oplus (T_xX)^*$, which will always have a natural symplectic structure Ω_-. This works along the zero section, but elsewhere, you don't have a canonical identification with a space and its dual, so you have to go through the canonical 1-form. There are some nice ways to characterize the canonical 1-form, but it is not so easy to see the 2-form.

In Chapter 3, we talk about symplectic maps, and lagrangian submanifolds of symplectic manifolds. Let (M, ω) be symplectic. Then say $N \overset{i}{\looparrowright} M$ is *an immersion* if Ti injects into $T_{pt}M$. We say i is [co]isotropic [lagrangian] if $(T_xi)(T_xN) \subseteq T_{i(x)}M$ is. That is, i isotropic if $i^*\omega = 0$, etc.

What we called $\bar{V} \times V$ before is now $\bar{M} \times M$, where $\bar{M}$ is the same manifold, but with the sign of the symplectic structure changed. Then $(\bar{M} \times M, -\pi_1^*\omega + \pi_2^*\omega)$, where the π_i are the natural projections. We can also look at $\bar{M}_1 \times M_2$, where M_1 and M_2 are different. Remember that the graphs of morphisms $V_1 \to V_2$ were lagrangians in $\bar{V}_1 \times V_2$. The same is true here. If $f : M_1 \to M_2$, then the graph of f is lagrangian in $\bar{M}_1 \times M_2$ if and only if $f^*\omega_2 = \omega_1$ and $\dim M_1 = \dim M_2$, which implies that f is an immersion (otherwise, $f^*\omega_2$ would be degenerate). All of this implies that f is a local diffeomorphism. We call such a map a local symplectomorphism, and if it is a global difeomorphism, then it is called a symplectomorphism. The conclusion is that the graph of f is lagrangian if and only if f is a local symplectomorphism. Now we can look at $\mathcal{L}ag(\bar{M}_1 \times M_2)$, the set of lagrangian submanifolds. Inside of it are the local symplectomorphisms $Locsymp(M_1, M_2$ (in particular, the symplectomorphisms). We have that $\mathrm{Aut}\,(M, \omega) \hookrightarrow \mathcal{L}ag(\bar{M}_1 \times M_2$. Ok, can we compose lagrangian submanifolds? Well, you don't always get a manifold (this relates to the non-continuity problem we talked about earlier). It still makes sense to think of the elements of $\mathcal{L}ag(\bar{M}_1 \times M_2)$ as generalized morphisms, and call it "Hom"(M_1, M_2) in some category. Note that even if the dimensions don't match, we can have some lagrangian submanifolds. In particular, if $M_1 = pt$, then "Hom"$(pt, M) = \mathcal{L}ag(M)$. So in this category, the role of points is played by lagrangian submanifolds.

pictures

The other interesting case in the vector space case was $E \oplus E^*$. This corresponds to asking about the lagrangian submanifolds of T^*X. Well, there are two kinds of submanifolds of T^*X. There are the fibers, along which the canonical 1-form vanishes, an therefore so does the 2-form. The other kind of submanifold is the graph of a 1-form. When is it lagrangian?

Let's call the 1-form ϕ.

$$\phi \text{ a lagrangian immersion} \Leftrightarrow 0 = \phi^*(-d\alpha) = -d(\phi^*\alpha) = -d\phi$$

So ϕ must be closed. Locally, we must have $\phi = dS$ where $S \in C^\infty(X)$. We can change S by any locally constant function. Thus, lagrangian sections of T^*X are more or less $C^\infty(X)/\text{consts}$. Of course, if X is not simply connected, then there should be some correction (since not all closed 1-forms are of the form dS).

Thus, we should think of $\mathcal{L}ag(T^*X)$ as "generalized functions". We also see these in probability ... e.g. the Dirac delta distribution should correspond to a fiber of the bundle. What about as lagrangian section? Some references: Bates -W. - Lectures on the geometry of quantization.; Guillemin-Sternberg - Geometric asymptotics.

Take the example $X = \mathbb{R}$. Take a lagrangian subspace of T^*X, then this has the form $\xi = ax$ for some a, which corresponds to the closed 1-form $a\,dx = d(\frac{1}{2}ax^2)$. In the case $f = \frac{\partial s}{\partial x}$, we have the corresponding something $e^{\frac{i}{2}ax^2}$. As $x \to \infty$, this oscillates faster and faster, and as $a \to \infty$, the smooth part gets scrunched in to the origin.

something is funny here.

More on generalized functions. Identify the function $u(x)$ with the linear functional

$$\psi \mapsto \int u(x)\psi(x)\,dx$$

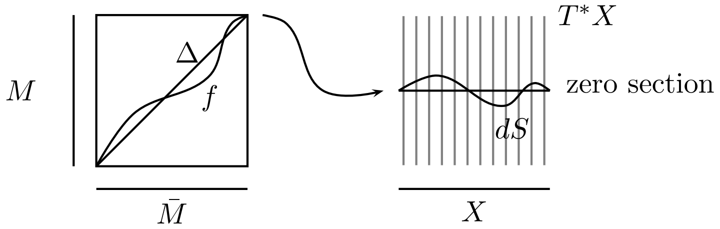
where ψ (the "test function") ranges over $C_c^\infty(\mathbb{R})$. Thus, when we talk about $e^{\frac{i}{2}ax^2}$ we should think about how it acts on test functions. Well if ψ is supported away from the origin, $e^{\frac{i}{2}ax^2}\psi$ integrates to something very small. You can check that if $\text{Supp}\psi$ doesn't contain the origin, then $\int e^{\frac{i}{2}ax^2}\psi = o(\frac{1}{a^N}$ for $N > 0$. If the origin is in the support of ψ, then we can check that

$$\sqrt{a}\int e^{\frac{i}{2}ax^2}\psi(x)\,dx \to \sqrt{\pi}(sign\ a)\psi(0)$$

"Geometric WKB prescription" Identify the graph of $\text{im}(dS) \in \mathcal{L}ag(T^*X)$ with the generalized function(s) $const \cdot e^{iS}$. This const also eliminates the problem that we can change S by a constant. Then this correspondence extends in a reasonable way, associating to more general lagrangian submanifolds on T^*X distributions on X.

We think of the lagrangian submanifold as a state. This is the beginning of the association of classical states and quantum mechanics (or in general, analysis on X with geometry on T^*X).

Another remark: in linear algebra, if you had $V \oplus \bar{V}$, you could sometimes identify it with an $E \oplus E^*$. In this way, you get some identification of $\text{Aut}\,V$ with $Symm(E)$. Now let's try it on manifolds. Suppose you could identify $\bar{M} \times M$, as a symplectic manifold, with T^*X. Then we get some identification between $\text{Aut}\,M$ and generalized functions on X, $C^\infty(X)$. The function associated to an automorphism is called the *generating function* of the morphism. In fact, there are different ways to pick T^*X, or the identification. This gives different flavors of generating functions.



If you identify the diagonal on $\bar{M} \times M$ with the zero section on T^*X, then you get an identification of X with M. And then a symplectomorphism close to the identity gives a closed 1-form. The points on the diagonal (fixed points of f) go to points on the zero section (critical points of S).

Take the example where $M = S^2$ with the usual symplectic structure. There is a theorem like the Lefshetz fixed point theorem that an automorphism close to the identity, then there are two fixed points (with multiplicity). Now suppose you have an area-preserving map on M (i.e. a symplectic map). Then this would correspond to some dS, where S is a function on S^2. The section S must have two distinct critical points (a max and a min). Thus, an area-preserving map close to the identity has at least two (geometric) fixed points. This is the simplest case of the *Arnol'd conjecture*: if you have a symplectomorphism close to the identity, then the number of fixed points is the same as the number of critical points of some function.

The technology for solving the Arnol'd conjecture developes into ways of analyzing intersections of lagrangian submanifolds, which is what lead to *Floer homology.*

LECTURE 4

Today's lecture was given by Christian.

GENERATING FUNCTIONS

So far we have $Sym(M_1, M_2) \hookrightarrow \mathcal{L}ag(\bar{M}_1 \times M_2)$. The idea of generating functions is this. You start with a function on $M_1 \times M_2$, and by some differentiation process, you get a lagrangian manifold, and then you check if it corresponds to a symplectomorphism. Starting with a function and getting a morphism is easy (because it is differentiation), and the other way is hard.

Let's say that $M = T^*X$ and $\mu \in \Omega^1(X)$. Then we have seen that $\text{im}\mu \in \mathcal{L}ag(M)$ if and only if $d\mu = 0$. Thus, we can just start with a zero form (a function), and differentiate it to get a lagrangian manifold.

Now consider the case where $M_1 = T^*X_1$ and $M_2 = T^*X_2$. Then we have $\overline{T^*X_1} \times T^*X_2 \overset{\sigma}{\leftarrow} T^*(X_1 \times X_2)$ given by $\sigma((x_1, x_2), (\xi_1, \xi_2)) = ((x_1, -\xi_1), (x_2, \xi_2))$. This is called the Schwartz transform. Now we can consider $f \in C^\infty(X_1 \times X_2)$, then $\text{im}df \in \mathcal{L}ag(\bar{M}_1 \times M_2)$. How do we check if this is the graph of some symplectomorphism, ϕ? We must have that $\phi(x_1, -\xi_1) = (x_2, \xi_2)$, which happens if and only if $\xi_1 = -d_1 f, \xi_2 = d_2 f$[6]

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[6] These are the natural projections of df.

That is,

$$\xi_{1i} = -\frac{\partial f}{\partial x_1^i}(x_1, x_2)$$
$$\xi_{2i} = \frac{\partial f}{\partial x_2^i}(x_1, x_2)$$

By the implicit function theorem, this is locally solvable for x_2 when $\det \left| \frac{\partial^2 f}{\partial x_1^i \partial x_2^i} \right| \neq 0$. It is hard to tell when we can solve this globally ... you have to check it separately.

Fibre-preserving diffeomorphisms: A symplectomorphism

$$\begin{tikzcd} T^*X \arrow[r, "\phi"] \arrow[d, "\pi"] & T^*X \arrow[d, "\pi"] \\ X \arrow[r, "\psi"] & X \end{tikzcd}$$

is fibre-preserving if this diagram commutes for some ψ.

Examples:

(1) Take $\psi : X \to X$ diffeomorphism, and let $\phi = T^*\psi$. How do we see this is a symplectomorphism? The canonical 1-form does not depend on coordinates.

(2) *Fibre translation.* Take $\psi = \mathrm{Id}_X$, and let $\phi : (x, \xi) \mapsto (x, \xi + \mu(x))$, where $\mu \in \Omega^1(X)$. Then you can check that $\phi^*\alpha = \alpha + \pi^*\mu$ (you can see this using the local description of α):
$\xi_i(\alpha) = \frac{\partial}{\partial x} \lrcorner \alpha$.

$$\omega = -d\alpha \overset{!}{=} \phi^*\omega = \phi^*(-d(\alpha + \pi^*\mu)) = \omega + \pi^* d\mu \Leftrightarrow d\mu = 0 \ (\mu = df)$$

(3) Take $\phi : T^*X \to T^*X$ a symplectomorphism which is fibre-preserving. Then this induces $\psi : X \to X$. We can check that

$$\phi = \underbrace{(\phi \circ T^*\psi)}_{\text{of type 1}} \circ \underbrace{T^*\psi^{-1}}_{\text{of type 2}}$$

(4) Consider $M_1 = (T^*X, \omega = -d\alpha)$ and $M_2 = (T^*X, \omega_B = \omega + \pi^*B$ where $B \in \Omega^2(X)$. Then ω_B is symplectic if and only if B is closed. To check that this is non-degenerate, check

$$\omega_{Bij} = \begin{array}{c} \begin{array}{cc} \frac{\partial}{\partial x^i} & \frac{\partial}{\partial p_i} \end{array} \\ \begin{pmatrix} B_ij & I \\ -I & 0 \end{pmatrix} \end{array}$$

   (1) $\omega \mapsto \phi^* = \omega$
   (2) $\omega \mapsto \phi^*\omega = \omega + \pi^* d\mu$ is a symplectomorphism if and only if there is a $\mu \in \Omega^1(X)$ such that $B = d\mu$. This is not always possible when $H^2(X, \mathbb{R}) \neq 0$. (This μ is often called A)

(5) If the manifolds are the same, then $\mathrm{Aut}\,(M, \omega) \hookrightarrow \mathcal{L}ag(\bar{M} \times M)$. The first is a Lie group, so we can study the Lie algebra. The Lie algebra of a lie group G is T_eG as a vector space. Take $t \mapsto \phi_t$ to be

a smooth curve in Aut (M,ω) such that $\phi_0 = \mathrm{Id}_M$. By smooth we mean that $\phi : M \times \mathbb{R} \to M$ is smooth. $\frac{d}{dt}\phi_t(x)|_{t=0} = \sqrt{x}$, then

$$0 = \frac{d}{dt}(\phi_t^* \omega - w)|_0 = \mathscr{L}_v \omega = 0.$$

A vector field with $\mathscr{L}_v\omega = 0$ is called *symplectic*. The collection of symplectic vector fields is the lie algebra of Aut $(\bar{M} \times M)$.

Is $\mathscr{L}_{[v,w]}\omega = 0$? $\mathscr{L}_v = d \circ i_v + i_v \circ d$ and $i_{[v,w]} = \mathscr{L}_v i_w - i_w \mathscr{L}_v$. Thus we compute

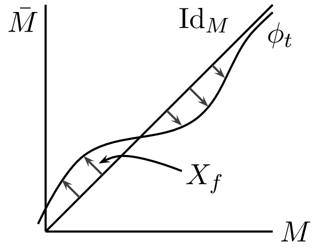
$$\begin{aligned}
\mathscr{L}_{[v,w]}\omega &= (d \circ i_{[v,w]} + i_{[v,w]} \circ d)\omega \\
&= d(\mathscr{L}_v i_w - i_w \mathscr{L}_v \lrcorner)\omega \\
&= d(\mathscr{L}_v i_w \omega) \\
&= \mathscr{L}_v(d i_w \omega) && (\mathscr{L}_v d = d\mathscr{L}_v) \\
&= \mathscr{L}_v(\mathscr{L}_w - i_v \circ d)\omega = 0
\end{aligned}$$

so this really is a lie algebra. Note that $\mathscr{L}_v\omega = 0$ if and only if $d(i_v\omega) = 0$ (e.g., for $i_v\omega = df$ for a function f). Recall that we have $\tilde{\omega} : TX \to T^*X$ ... note that $\tilde{\omega}(v) = i_v\omega$. Thus, we can solve $\tilde{\omega}(v) = df$, so $X_f = v = \tilde{\omega}^{-1}(df)$. This is called *the hamiltonian vector field generated by f*.

Locally, you can always find $f \in C^\infty(U_X)$, so symplectic vector fields are often called *locally hamiltonian*.

So we have the following picture:

$$f \xrightarrow{\text{“differ”}} X_f \xrightarrow{\text{integrate}} flow\,\phi_t \longrightarrow graph\,\phi_t \in \mathcal{L}ag(\bar{M} \times M)$$



What are the generators of these lagrangian submanifolds?

A glimpse of Hamilton-Jacobi theory: From mechanics: Call the base manifold Q, with coordiantes q^i, called the configuration space. Then T^*Q is phase space, with coordinates $q^i, p_i(\alpha) = \frac{\partial}{\partial q^i} \lrcorner \alpha$ (the second is *momentum*). Then we have $H \in C^\infty(T^*Q)$, which we usually think of as energy. Then the flow, ϕ_t, generated by X_H is given by $\phi_t : (q^i, p_i) \mapsto (\bar{q}^i, \bar{p}_i)$. Then we can think about the generating functions of the lagrangian submanifolds given by the ϕ_ts. Call them $S : Q \times Q \times \mathbb{R} \to \mathbb{R}$, $S = S(q, \bar{q}, t) = \int_q^{\bar{q}} \mathscr{L}\, dt$ (this is integrating the Lagrangian) ... more or less $H = qp - \mathscr{L}$. Then conservation of energy says what? $X_H H = 0$ (follows from ω being skew-symmetric).

Thus, $E = H(q,p) = H(\bar{q}^i, \bar{p}_i) = [[H(d_2S) = E]]$ since

$$p_i = -\frac{\partial S}{\partial q^i}$$
$$\bar{p}^i = \frac{\partial S}{\partial \bar{q}^i}$$

The boxed equation is what you have to solve to get at S. If you are lucky, then it will be some partial differential equation. If you're not lucky, then you'll have some arbitrary function of derivatives.

Lecture 5

`www.math.ist.utl.pt/~acannas/Books/symplectic.pdf` [or ps] has corrected references. Also, there is a nice article on the arXiv at `---.math.SG/0505366`.

$$\frac{d}{dt}(f_t^*\omega_t) = f_t^*\frac{d\omega_t}{dt} + \left(\frac{d}{dt}(f_t^*)\right)\omega_t$$

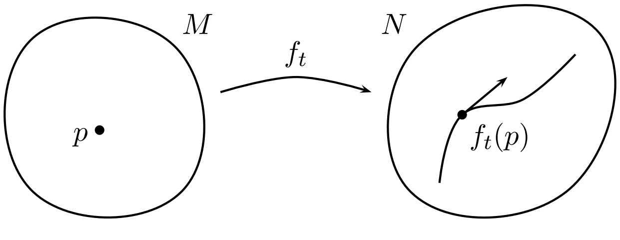
You can think of f_t^* as a matrix and ω_t as a vector. And $\frac{df_t}{dt} = v_t \circ f_t$ (these are vectors along the paths $f_t(p)$). If you have a family of maps $f_t : M \to N$. It turns out that you can write

$$\left(\frac{d}{dt}(f_t^*)\right)\omega_t = f_t^*\mathscr{L}_{v_t}\omega_t$$

so

$$(*) \qquad \frac{d}{dt}(f_t^*\omega_t) = f_t^*\left(\frac{d\omega_t}{dt} + i_{v_t}d\omega_t + d(i_{v_t}\omega_t)\right)$$

by the Cartan magic formula.



If ξ is a vector field along $f : M \to N$, then we have an interior product/pullback operator $i_\xi : \Omega^{\cdot}(N) \to \Omega^{\cdot-1}(M)$ given by $i_\xi\omega(p)(v_1, \dots, v_{k-1}) = (\xi(p), (Tf)(v_1), \dots)$.

If $f_t : M \to N$ a family of maps, then [[box this]]

$$\frac{d}{dt}f_t^*\omega_t = f_t^*\frac{d\omega_t}{dt} + d(i_{\frac{df_t}{dt}}\omega_t) + i_{\frac{df_t}{dt}}d\omega_t$$

Meditate on this for a while.

This idea was used by Moser, and then used all over the place. Moser used this in the case where M is compact, and ω_t is a family of symplectic forms such that $[\omega_t] \in H^2(M, \mathbb{R})$ is constant. Then (M, ω_t) are all symplectomorphic. Trying to solve the equation $g_t^*\omega_0 = \omega_t$, it's a mess, but $f_t^*\omega_t = \omega_0$ is nice. To solve, let $f_0 = \mathrm{Id}$ and $0 = (*)$, so

$$\frac{d\omega_t}{dt} + d(i_{v_t}\omega_t) = 0$$

is what you want (Moser's equation). Solve for ω_t's and then find the f_t's. The compactness of M is essential for the second step (you have to integrate a time-dependent vector field). To solve Moser's equation

$$d(\tilde{\omega}_t(v_t)) = -\frac{d\omega_t}{dt}$$

($\tilde{\omega}_t$ isomorphism), you only have to solve an equation of the form

$$d\alpha_t = -\frac{d\omega_t}{dt}$$

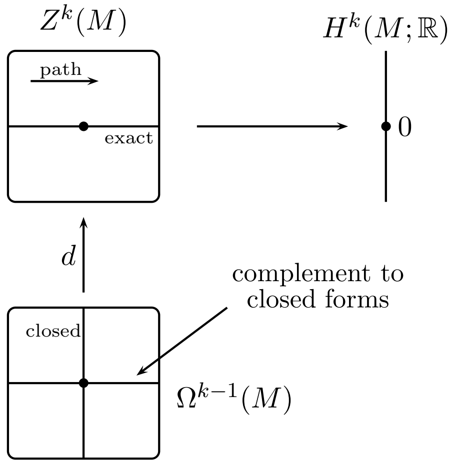
where α_t are 1-forms. Then we have

$$d\left(\frac{d\omega_t}{dt}\right) = \underbrace{\frac{d}{dt}(d\omega_t)}_{0}$$

Consider the 2-sphere, and let $\omega_t = e^t\omega_0$. Then the total area of the sphere is changing, so you cannot get them to pull back to each other. The condition that the cohomology class is constant forces the volume to remain the same. $\omega_{t+h} - \omega_t = d\theta_h$, so when you divide by h and let $h \to 0$, then we should get $\frac{d\omega_t}{dt} = d(\lim \theta_h/h)$, but it is not clear that $\lim \theta_h/h$ behaves. We can take care of this another way:

$$\int_C \frac{d\omega_t}{dt} = \frac{d}{dt}\int_C \omega_t = 0$$

Let $Z^k(M)$ be the collection of closed k-forms on M. You have to show that the exact forms are a closed sub-blah.



We've solved for a particular α_t, now we have to make it smooth with respect to t.

Suppose we know that $\omega_t = \omega_0 + t(\omega_1 - \omega_0)$ are symplectic. Then $\frac{d\omega_t}{dt} = \omega_1 - \omega_0$, so we only have to solve once. But in going on this straight line, we might stop being symplectic (the condition of being non-degenerate may fail). Since the collection of non-degenerate 2-forms is a dense open subset of all forms, so you can take a tubular neighborhood of some path, then take a bunch of straight steps staying inside of the space of symplectic forms, always staying within the neighborhood. This shows that if there is some path that works, it can be replaced by a piece-wise linear path.

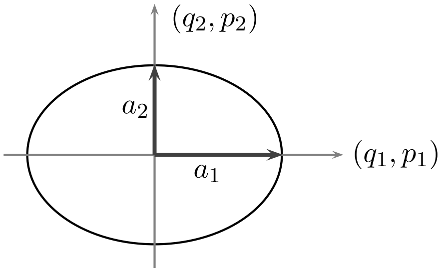
Thus, you cannot deform symplectic structure within the same cohomology class and get something new. If we are trying to classify symplectic structures, then we have a continuous invariant: cohomology class. If you only move a little in cohomology, then you stay symplectic [[why?]], so symplectic structure is not too rigid.

non-compact case: Take the plane with a disk. Take a cylinder, and a half cylinder, which have the same area, but are not symplectomorphic. *Greene-Wu*: given two symplectic structures with the same orientation and the same total volume, they are equivalent (by an end-preserving diffeomorphism) if and only if either

- both have the same finite area, or
- both have infinite area, and both are finite on the same set of ends.

The proof is a version of Moser's method. The cohomology part is trivial (a non-compact 2-d manifold has $H^2 = 0$), but you have to worry about running off your manifold when you solve for v_t.

In 4 dimensions or higher, consider (q_1, q_2, p_1, p_2)



The volume of the ellipse is proportional to $a_1^2 a_2^2$. If you change the volume, you change the symplectic structure. If you change the a_i's around so as to preserve volume, can you retain the symplectic structure? Or if you take a rectangle of the same volume? *Gromov*: $\max(a_1, a_2)$ is a symplectic invariant, and therefore, so is the minimum. The argument is based on the following. Take a much larger symplectic manifold (cylinder)

Then if $\max(b_1, b_2) > a_1$ (from some other ellipsoid), there is no symplectic embedding of $E(b_1, b_2)$ into $E(a_1, \infty)$ (the cylinder). This is Gromov's *non-squeezing theorem*, proven with psuedo-holomorphic curves stuff. This stuff goes under the general notion of "symplectic capacity".

Question: are the a_is invariants of $E(a_1, \dots, a_n)$ (in dimension $2n$)?

It turns out that the same method Moser used can be used to prove local things (in particular, the Darboux theorem). For the moment, consider two symplectic manifolds, (M_1, ω_1) and (M_2, ω_2) with submanifolds N_1, N_2, respectively.

If $f : N_1 \to N_2$, then are there neighborhoods so that f extends to a symplectomorphism. In the case where $N_2 = pt$, we have the Darboux theorem, and in the case $N_2 = M_2$, we have Moser's theorem.

If $i_j : N_j \to M_j$ are the inclusions, then a necessary condition is that (1)$f^*(i_2^*\omega_2) = i_1^*\omega_1$. $g^*\omega_2 = \omega_1$. If g extends, then $g \circ i_1 = i_2 \circ f$. And another necessary condition if that (2)$\dim M_1 = \dim M_2$.

Theorem 5.1 (Givental). *This is sufficient locally in N.*

That is, the pullback form is the only local invariant. This is not the case in Riemannian geometry ... you have have isometric morphisms of submanifolds which do not extend to isometric maps on neighborhoods.

Sufficient condition to get it for all of N_1, N_2:

$$\begin{tikzcd}
TN_1 \arrow[r, "Tf"] \arrow[d, hook] & TN_2 \arrow[d, hook] \\
T_{N_1}M_1 \arrow[d, hook] & T_{N_2}M_2 \arrow[d, hook] \\
TM_1 & TM_2
\end{tikzcd}$$

The condition is that Tf extends to $T_{N_1}M_1$ as a morphism of symplectic vector bundles. In the case where the Ns are points, it says that you have to find symplectic isomorphism between the tangent spaces at the points, which you can always do. In fact, if N_1 and N_2 are lagrangian, then you can always do this. But in any symplectic manifold M, if you have a lagrangian N, you can look at the cotangent bundle of N with the zero section. Since M locally looks like $\bar{N} \times N$, we win.

LECTURE 6

Let $H \subseteq TM$ is a codimension 1 sub-bundle. In dimension 2, this is a direction field, so you can integrate it, but in higher dimensions, you need the Frobenius condition. Locally, $H = \ker \phi$ for some 1-form ϕ. $H^\perp \subseteq T^*M$ is a 1-dimensional sub-bundle (called the conormal bundle to H). Such a thing is usually trivial (except when you have möbius stuff going on). When it is trivial, H is called co-orientable (note that everything is locally co-orientable). The condition is that

$$d\phi \wedge \phi \equiv 0$$

the opposite condition is that

$$d\phi \wedge \phi \neq 0$$

everywhere. This is equivalent (in 3 dimensions) to saying that $d\phi|_{\ker \phi}$ is non-degenerate. In higher dimensions, you want $(d\phi)^{(\dim M-1)/2} \wedge \phi$ to be a volume form. This is *the contact condition.*

Note that ϕ is not determined by H. Suppose we replace ϕ by something with the same kernel, then it is of the form $f\phi$, where f is a no-where vanishing C^∞ function. Then consider $\phi \wedge (d(f\phi))^n$, where $\dim M = 2n+1$. We get $\phi \wedge (f d\phi + df \wedge \phi)^n = f^n \phi \wedge (d\phi)^n$, so this thing remains a volume element. Equivalently, look at $d\phi(v,w)$ where $v,w \in H$:

$$\begin{aligned} d\phi(v,w) &= v\cdot\phi(w) - w\cdot\phi(v) - \phi[v,w] \\ d(f\phi)(v,w) &= v\cdot(f\phi)(w) - w\cdot(f\phi)(v) - f\phi[v,w] \end{aligned}$$

but $H=\ker\phi$, so we just have $d\phi(v,w) = -\phi[v,w], d(f\phi)(v,w) = -f\phi[v,w]$, so if we have non-degeneracy in one case, we have if for the other case too. By the way, the vector field version of the Frobenius condition is that $[v,w]\in H$.

How do we get between contact and symplectic? Locally, one can prove that $\phi = dz + \sum p_i dq^i$ in some coordinates $z, q^1, \ldots, q^n, p_1, \ldots, p_n$. Notice that $d\phi = dp_i \wedge dq^i$. If you look at $H^\perp \subseteq T^*M$, we can restrict the symplectic form from T^*M to $H^\perp$. $\dim M = 2n+1$, $\dim T^*M = 4n+2$, so $\dim H^\perp = 2n+2$. On the cotangent bundle, we have the additional coordinates z^*, p_i^*, q_i^*. Then the symplectic form on the cotangent bundle is $dz\wedge dz^* + dq_i \wedge dq_i^* + dp_i \wedge dp_i^*$. Now we have to describe $H^\perp$. We can put coordinates (z, q_i, p_i, t) on $H^\perp$, where we map this point to $(z, q_1, \ldots, q_n, p_1, \ldots, p_n | t, p_1 t, \ldots, p_n t, 0, \ldots, 0)$ (what we've done is take all the same z, q_i, p_i, and t times the form ϕ). Then the pullback form on $H^\perp$ is

$$\begin{aligned} dz\wedge dt + dq^i \wedge (p_i dt + dp_i\, t) &= dz\wedge dt + (dq_i \wedge dp_i) + p_i dq^i \wedge dt \\ &= (dz \wedge p_i dq^i)\wedge dt - t(dq_i \wedge dp_i) \end{aligned}$$

When is this symplectic? Well, let's look at the highest wedge power, we get

$$(dz + p_i dq^i)\wedge dt \wedge (-t)^n (dq_i\wedge dp_i)^n = \phi\wedge dt\wedge (t^n)(d\phi)^n$$

So we have that $\dot{H}^\perp = H^\perp \smallsetminus \{\text{zero section}\}$ is a symplectic submanifold of T^*M if and only if H is contact. The pullback of the symplectic form to $\dot{H}^\perp$ is $\underbrace{\phi\wedge dt + td\phi}_{\textit{should be } d(t\phi)}$. This is called the *symplectification* or the *symplectization* of the contact manifold M. Consider

$$\begin{aligned} \mathscr{L}_{\frac{\partial}{\partial t}}\omega &= d(\frac{\partial}{\partial t}\lrcorner\omega) \\ \mathscr{L}_{t\frac{\partial}{\partial t}\lrcorner\omega} &= d(t\frac{\partial}{\partial t}\lrcorner\omega) \end{aligned}$$

so

$$\mathscr{L}_{t\frac{\partial}{\partial t}}\omega = \omega$$

$t\frac{\partial}{\partial t}$ is called the Euler vector field. The flow along it is multiplication:

$$\mathscr{L}_{t\frac{\partial}{\partial t}} f = t\frac{\partial f}{\partial t} = nf$$

says exactly that f is homogenous of degree n is some sense.

Thus, we have that ω is homogeneous of degree 1.

If we have a vector field ξ such that $\mathscr{L}_\xi\omega = \omega$ (i.e., $d(\xi\lrcorner\omega) = \omega$), then we call ξ a Liouville vector field. So you start with a contact manifold, you symplectify it to get a symplectic manifold together with a Liouville vector field. The converse is also true. Say (M,ω) is symplectic and ξ is no-where vanishing Liouville vector field, then consider $M/(\text{flow of } \xi)$. For example, take $M = T^*X$, with coordinates q^i, p_i, and $\xi = p_i \frac{\partial}{\partial p_i}$. The flow of ξ leaves q^i fixed, and multiplies the p_is by a constant. It is easy to see that $\mathscr{L}_\xi \omega = \omega$. Then we have that $T^{\cdot *}X/(\text{flow of } \xi)$ is the cotangent ray bundle. This is the same thing as the space of co-oriented hyperplanes in TM. TM is a contact manifold in a natural way.

Note that $\mathscr{L}_\xi \omega^n = n\omega^n$, so volume is expanding. Thus, you cannot have something like this on a compact manifold. Another way to see this is that $\omega = d(\xi \lrcorner \omega)$, so ω is exact, and we saw that this is impossible on a compact manifold. Another way to write this is $d(\tilde{\omega}(\xi)) = \omega$, so this is equivalent to solving $d\psi = \omega$ and then set $\xi = \tilde{\omega}^{-1}(\psi)$.

Suppose we have a symplectic manifold M with a Liouville vector field, then the claim is that this descends to a contact structure on $X = M/(\text{flow of } \xi)$. To see this, take a point $\hat{x} \in M$, which maps to $x \in X$. Look at $\langle \xi \rangle^\perp \supseteq \langle \xi \rangle$ in $T_{\hat{x}}$. So there is some hyperplane field containing ξ, so we can get something on X. What if we take some other lift $\hat{\hat{x}}$ of x. Is $\langle \xi \rangle^\perp$ invariant under the flow of ξ? Yes! Well, $\langle \xi \rangle$ is invariant; ω is invariant up to scalar multiple; hence $\langle \xi \rangle^\perp$ is invariant. Ok, now how do we show that we actually have a contact structure on X? Choose a cross section in $N \subseteq M$. On N, let $\phi = \xi \lrcorner \omega$, which is a 1-form, then $\ker \phi = \langle \xi \rangle^\perp \cap TN$ and ξ is a contact form. So we can take a cross section as a model for the quotient space. This procedure more or less tells you that M is the symplectification of N. We call M the contactization of N.

Lecture 7 - (Almost) Complex Structures

A complex structure on a (real) vector space V is a map $J : V \to V$ such that $J^2 = -I$. J cannot have any (real) eigenvalues. Such an operator induces the structure of a $\mathbb{C}$-vector space on V. Namely, we define $(a + ib) \cdot v = (a + Jb)v$. Conversely, if we have a complex structure, then we have a real structure and a map whose square is $-I$ (multiplication by i). This is actually an isomorphism of categories.

Given a real vector space, V, we can look at $V_\mathbb{C} := V \otimes_\mathbb{R} \mathbb{C} \cong V \oplus$"$iV$". We will call this *complexification*. How can we get back from $V_\mathbb{C}$ to V. In general, we can't, but on $V_\mathbb{C}$, we have a "real structure" given by an operator $V_\mathbb{C} \to V_\mathbb{C}$, namely complex conjugation. It has the properties

1. conjugate linear: $\overline{zw} = \bar{z}\bar{w}$
2. involution: it squares to the identity (so eigenvalues are ± 1)

If W is a complex vector space, then a *real structure* on W is a conjugate linear involution, c. Then we can define $W_\mathbb{R}$ to be $\{x \in W | c(x) = x\}$. One may check that $(W_\mathbb{R})_\mathbb{C} \cong W$ and $(V_\mathbb{C})_\mathbb{R} \cong V$. This yields an equivalence of categories (real vector spaces, and complex vector spaces with a real structure) when you extend it to morphisms in the obvious way.

We can extend these notions to bundles of vector spaces by saying that J or complex conjugation should vary continuously. Specifically, we can talk about a complex structure on TM, where M is some manifold. A complex structure on TM is called an *almost complex structure* on M. The reason we say "almost" is because there is more to be said: integrability.

Suppose (M_1, J_1) and (M_2, J_2) are almost complex manifolds, and we have a map $M_1 \xrightarrow{f} M_2$, then f is called *pseudo-holomorphic* if $Tf \circ J_1 = J_2 \circ Tf$. In particular, if $M_1 = M_2 = \mathbb{C}$, then a pseudo-holomorphic map is just a holomorphic map (the condition is just the Cauchy-Riemann conditions).

If we have a complex manifold, then we can identify the tangent space at a point with $\mathbb{C}^n$, and the J from $\mathbb{C}^n$ induces a complex structure on that tangent space. The compatibility of the charts ensures that this is well-defined. Does every almost complex structure come from a complex structure?

A complex manifold has lots of holomorphic functions (or call them pseudo-holomorphic if you like) on it (just compose the coordinate system with holomorphic functions on $\mathbb{C}^n$). To test if a manifold has a complex structure, we can ask if it has enough holomorphic functions. If we can find $f_1, \dots, f_n : U \to \mathbb{C}$ pseudo-holomorphic such that the induced map $U \to \mathbb{C}^n$ is a local diffeomorphism, then we have a complex structure.

Cauchy-Riemann equations: $(M, J) \supseteq U \xrightarrow{f} \mathbb{C}$ is pseudo-holomorphic if $i \circ (Tf) = Tf \circ J$. That is, for every $v \in T_xM$, $(Tf)(Jv) = i(Tf)(v)$. We can re-write this as $(Jv)f = i(v \cdot f)$, or $(Jv - iv)f = 0$.

If $(x_1, \dots, x_n, y_1, \dots, y_n)$ are coordinates on M such that $J(\frac{\partial}{\partial x_j}) = \frac{\partial}{\partial y_j}$, then this condition is just

$$\left(\left(\frac{\partial}{\partial y_j}\right) - i\left(\frac{\partial}{\partial x_j}\right)\right) f = \frac{\partial f}{\partial y_j} - i\frac{\partial f}{\partial x_j} = 0$$

On an open subset of (M, J), a function $f : M \to \mathbb{C}$ is pseudo-holomorphic if and only if f is annihilated by the elements $\{Jv - iv | v \in TM\} \subseteq T_{\mathbb{C}}M$. So in the complexified tangent bundle, $T_{\mathbb{C}}M$, of M, we have a distinguished subspace consisting of all complex tangent vectors w such that $w = Jv - iv$ for some $v \in TM$. When is $w = Jv - iv$? Look at the paragraph after next.

The sections of $T_{\mathbb{C}}M$ are called complex vector fields, and they act as derivations on $C^\infty(M, \mathbb{C})$, and you can prove that they are all the derivations.

We can look at $J_{\mathbb{C}}(Jv - iv) = J^2 - iJv = -v - iJv$, which is equal to $-i(Jv - iv)$. Thus, the set of w above is the $(-i)$-eigenspace of the operator $J_{\mathbb{C}}$. Similarly, $\{Jv + iv\}$ is the i-eigenspace of $J_{\mathbb{C}}$. This gives us a decomposition $V = V^{1,0} \oplus V^{0,1}$ with $\overline{V^{0,1}} = V^{1,0}$.

Figure 2

Thus, we can give an alternative definition: an almost complex structure on V is a complex subspace $V^{0,1} \subseteq V_{\mathbb{C}}$ such that $V_{\mathbb{C}} = \overline{V^{0,1}} \oplus V^{0,1}$. We've shown that an almost complex structure gives you this, and given such a splitting, we can do something.

We can extend the bracket operation $[\cdot, \cdot]$ on real vector fields to complex vector fields. As derivations, we have $[X, Y] = XY - YX$. Or, if you like,

$$[A + iB, C + iD] = [A, C] - [B, D] + i([B, C] + [A, D]).$$

If we have an almost complex manifold (M, J), we can think of it as $(M, T^{0,1}M)$, which is a complex sub-bundle of $T_{\mathbb{C}}M$, such that $T^{0,1}M \oplus$

$\overline{T^{0,1}M} = T_{\mathbb{C}}M$. A function is J-holomorphic if and only if $v \cdot f = 0$ for all sections v of $T^{0,1}M$.[7]

If $v \cdot f = 0$ and $w \cdot f = 0$, then $[v, w] \cdot f = 0$. On $\mathbb{C}^n$, $T^{0,1}\mathbb{C}^n$ is spanned by all things of the form $\frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y^j}\right) = \frac{\partial}{\partial \bar{z}_j}$, where $z^j = x^j + iy^j$. A tangent vector is in $T^{0,1}\mathbb{C}^n$ if and only if it annihilates all homomorphic functions. All this shows that if J comes from a complex structure, then the sections of $T^{0,1}M$ are closed under the bracket operation.

On the other hand, an algebraic calculation (exercise) shows that $\Gamma(T^{0,1}M)$ is closed under bracket if and only if $\mathcal{N}_J \equiv 0$.

Theorem 7.1 (Newlander-Nirenberg)**.** *$\Gamma(T^{0,1}M)$ closed under bracket implies that J is integrable (i.e. it comes from a complex structure).*

This is similar to the Frobenius theorem. It says that some sub-bundle of the tangent bundle is a foliation if it is closed under bracket.

LECTURE 8 - MORE

It's never too soon to start thinking about the term paper.

In addition to $T_{\mathbb{C}}M$, we have $T^*_{\mathbb{C}}M$, the sections of which are complex 1-forms. We will write $dz^j = dx^j + idy^j$. For a (complex) basis for $T^*_{\mathbb{C}}(\mathbb{R}^{2n}$ (at a point), we may take either d^j_x, dy^j, or $dz^j, d\bar{z}^j$.

Note that we have two notions of dual for a complexified vector space, but they are naturally isomorphic:

$$\mathrm{Hom}_{\mathbb{C}}(V_{\mathbb{C}}, \mathbb{C}) = (V_{\mathbb{C}})^* \cong (V^*)_{\mathbb{C}} = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}.$$

We can now consider

$$\wedge^k T^*_{\mathbb{C}}M = \wedge^k(\underbrace{T^{*1,0}_M}_{dz^j} \oplus \underbrace{T^{*0,1}_M}_{d\bar{z}^j}) = \bigoplus_{p+q=k} \wedge^p T^{*1,0}_M \otimes \wedge^q T^{*0,1}_M$$

This splitting is naturally attached to the almost complex structure. On the level of sections, we have $\Omega^k_{\mathbb{C}}(M)$. By the way, $(\wedge^k_{\mathbb{R}}V)_{\mathbb{C}} \simeq \wedge^k_{\mathbb{C}}(V_{\mathbb{C}})$, and there is a $d_{\mathbb{C}}$ taking k-forms to $(k+1)$-forms. In fact, the complex de Rham cohomology is the regular de Rham cohomology tensored with $\mathbb{C}$.

When we have an almost complex manifold, we can write

$$\Omega^k_{\mathbb{C}}(M) = \bigoplus_{p+q=k} \Omega^{p,q}(M)$$

where $\Omega^{p,q}(M)$ are forms with p dz^j's and q $d\bar{z}^j$'s (this is cheating, since we don't actually have z's and $\bar{z}$'s). We can always choose a basis of vector fields $(\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n)$ such that $J(\xi_j) = \eta_j$. Then we get a new basis $\bar{\alpha}_j = \frac{1}{2}(\xi_j + i\eta_j)$ and $\alpha_j = \frac{1}{2}(\xi_j - i\eta_j)$. Then the corresponding basis in the dual space are $\bar{\theta}^j$ and θ^j. Then a typical element $\omega \in \Omega^{p,q}(M)$ is of the form

$$\omega = \sum (function)\theta^{i_1} \wedge \cdots \wedge \theta^{i_p} \wedge \bar{\theta}^{i_p} \wedge \cdots \wedge \bar{\theta}^{i_q}$$

Then we have that $d\omega$ will, in each term, have either an extra θ or an extra $\bar{\theta}$. So $d\omega \in \Omega^{p+1,q}(M) \oplus \Omega^{p,q+1}(M)$, and this decomposition is unique, so

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[7] The book puts the indices $0, 1$ downstairs.

we may write $d\omega = d^{1,0}\omega + d^{0,1}\omega$. Thus, we have that $d = d^{1,0} + d^{0,1}$. We often write $d^{1,0} = \delta$ or ∂, and $d^{0,1} = \bar{\delta}$ or $\bar{\partial}$.

Lemma 8.1. *f is [pseudo-]holomorphic if and only if $\bar{\partial} f = 0$.*

We know that $d^2 = (\partial + \bar{\partial})^2 = \partial^2 + \partial\bar{\partial} + \bar{\partial}\partial + \bar{\partial}^2 = 0$.

Proposition 8.2. *$\partial^2 = 0$ if and only if J is integrable if and only if $\bar{\partial}^2 = 0$ if and only if $\bar{\partial}\partial = \partial\bar{\partial}$.*

In this case, we can just look at the complex

$$0 \to \Omega^{0,0} \xrightarrow{\bar{\partial}} \Omega^{0,1} \xrightarrow{\bar{\partial}} \Omega^{0,2} \xrightarrow{\bar{\partial}}$$

and the cohomology is called the Dolboex [[sp?]] cohomology.

Moving from linear algebra to geometry, the statement $(V_{\mathbb{C}})^* \simeq (V^*)_{\mathbb{C}}$ becomes the following. Think of $\mathbb{R}^m \subseteq \mathbb{C}^m$, and consider $C^\omega(\mathbb{R}^m)$, the real analytic functions on $\mathbb{R}^m$, and tensor with $\mathbb{C}$. We can also consider $C^\omega_{\mathbb{C}}(\mathbb{C}^m)$. If we restrict an analytic function to $\mathbb{R}^m$, you get a complex-valued real analytic function. If you start with a real analytic function on $\mathbb{R}^m$, you can extend it to a neighborhood U of $\mathbb{R}^m$.

$$\begin{tikzcd}
C^\omega(\mathbb{R}^m) \otimes \mathbb{C} \arrow[r, bend left, "\text{locally}"] & \mathscr{C}^\omega_{\mathbb{C}}(\mathbb{C}^m) \arrow[l]
\end{tikzcd}$$

figure 1

If you start with $(\mathbb{R}^{2n}, J)$, with coordinates $x^1, \dots, x^{2n}$, you get coordinates

$$\eta_j = \frac{\partial}{\partial x^j} + iJ^k_j(x)\frac{\partial}{\partial x^k}$$
$$\xi_j = \frac{\partial}{\partial x^j} - iJ^k_j(x)\frac{\partial}{\partial x^k}$$

where $J^k_j(x)$ is the matrix $J : T_x\mathbb{R}^{2n} \to T_x\mathbb{R}^{2n}$. Then the η_j (but only n of them, not $2n$) form a basis for the $0,1$ part and the ξ for the $1,0$ part.

hmmmm

$T_{x+i0}\mathbb{C}^{2n} \simeq T_x\mathbb{R}^{2n} \oplus iT_x\mathbb{R}^{2n} \simeq T_{\mathbb{C},x}\mathbb{R}^{2n}$. So we have these complexified vector fields on our real manifold.

If you assume that your almost complex structure is real analytic, you can analytically extend all the J^k_j to some neighborhood of $\mathbb{R}^{2n} \subseteq \mathbb{C}^{2n}$. So you get some sub-bundle $E \subseteq T\mathbb{C}^{2n}$ (the extension of the $0,1$ part) of dimension n, which is holomorphic (the η_j are the typical sections of E).

Now let's assume that J is integrable. This tells is that E is closed under bracket (it is an *involution*). Now we invoke the holomorphic Frobenius theorem, so we get a foliation of some neighborhood of $\mathbb{R}^{2n} \subseteq \mathbb{C}^{2n}$. So near any point, we can find $\phi^1, \dots, \phi^n$ independent functions which are constant on the leaves of the foliation. When restricted to $\mathbb{R}^{2n}$, they are still independent (the leaves intersection $\mathbb{R}^{2n}$ transversely) and J-holomorphic.

Since every $0,1$ leaf intersects (at least locally) each $1,0$ leaf exactly once, you get a map from the leaf to the original $\mathbb{R}^{2n}$. Since the leaf is complex, this induces a holomorphic complex chart on the $\mathbb{R}^{2n}$.

In fact, you can start of with something C^∞ and get complex coordinates (but this is hard).

Lecture 9 - Compatibility of Symplectic and Almost Complex Structures

Correction from last time: We said $\Omega^K = \bigoplus_{p+q=k} \Omega^{p,q}$, then $d\Omega^{p,q} \subseteq \Omega^{p+1,q} + \Omega^{p,q+1}$. This used the fact that $d\omega = \partial\omega + \bar\partial\omega$. It is only true when the complex structure is integrable. In local coordinates $\omega = \sum a_{IJ}\theta^I \wedge \theta^J$. When we took d of this thing, we forgot to take d of the θ^i's.

Let $\theta \in \Omega^{1,0}$, then to prove that $d\theta \in \Omega^{2,0} + \Omega^{1,1}$, it suffices to show that $(d\theta)(x,y) = 0$ if x and y have type $0,1$. We can calculate this using

$$d\theta(x,y) = \underbrace{x\theta(y)}_{0} - \underbrace{y\theta(x)}_{0} - \theta([x,y])$$

and the last term is zero if $T^{0,1}$ is involutive.

Once you have that, you can deduce that $\partial^2 = \bar\partial^2 = 0$.

There is a real analog of this. Suppose that $TM = E_1 \oplus E_2$, then we get that $T^*M \simeq E_1^* \oplus E_2^*$. Then we have

$$\Omega^k M = \bigoplus_{p+q=k} \Gamma(\wedge^p E_1^*) \otimes_{C^\infty M} \Gamma(\wedge^q E_2^*)$$

you get this structure on a "bifoliated" manifold (if both E_1 and E_2 are involutive).

$$\begin{tikzcd} V \arrow[r, "J"] \arrow[dr, "\tilde\omega"'] & V \arrow[d, "\tilde g"] \\ & V \end{tikzcd}$$

One way to state compatibility is to say that

$$(*) \qquad \omega(Jx, Jy) = \omega(x,y)$$

We define $g(x,y) = \tilde g(x)(y) = \tilde\omega(J^{-1}x, y) = -\omega(Jx, y)$. $(*)$ implies that g is symmetric such that $g(Jx, Jy) = g(x,y)$. Any two of these define the third. This triple is called a *pseudohermitian structure*. It is called *hermitian* if g is positive definite.

Consider $\mathbb{R}^2$, with coordinates (p,q), and $J(\frac{\partial}{\partial q}) = \frac{\partial}{\partial p}$. Let $z = q + ip$, and $\omega = dq \wedge dp$. Then we have that

$$g\left(\frac{\partial}{\partial q}, \frac{\partial}{\partial q}\right) = -\omega\left(J\frac{\partial}{\partial q}, \frac{\partial}{\partial q}\right) = 1$$

Good, so we have the correct sign.

If we are on a symplectic [almost complex] manifold, can we find compatible almost complex [almost symplectic[8]] structure. The answers are yes.

Consider the anulus in $\mathbb{C}$, with the inner and outer circles glued radially. This is a complex manifold, topologically $S^1 \times S^1$, which has a symplectic structure.

Figure 1

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[8] Not necessarily closed.

Now if we do the same thing in $\mathbb{C}^n$, we get a $S^{2n-1} \times S^1$, which cannot have a symplectic structure.

If J is integrable, we get a Kähler structure.

Locally, every complex structure looks like $\mathbb{C}^n$, and every symplectic structure looks like $\mathbb{R}^{2n}$. But when we put the two together, the g has weird twists and doesn't have to be equivalent to the usual metric.

What can we say anything about the geodesic flows of such things. I dunno. It can be tied to the behavior of the laplacian (on functions), but I've never seen anything about this.

If we think of almost complex structure as

Figure 2

then for $\omega \in \wedge^2 V^*$, we can complexify it to $\omega_{\mathbb{C}} \in \wedge^2 V_{\mathbb{C}}^*$ by requiring it to be complex bilinear.

If we apply $\omega_{\mathbb{C}}$ to $V^{0,1}$, since $V^{0,1}$ is the graph of J, we have

$$\omega_{\mathbb{C}}(x + iJx, v + iJv) = \underbrace{\omega(x,v) - \omega(Jx,Jv)}\,+ i(\underbrace{\omega(x,Jv) + \omega(Jx,v)})$$
$$= 0 \quad \text{iff } J, \omega \text{ compatible}$$

So another way to say J, ω compatible is to say that $V^{0,1}$ is lagrangian in $V_{\mathbb{C}}$. So looking for compatible almost complex structure on a symplectic manifold is the same as looking for lagrangian somethings, and on a Kähler manifold, it is the same as looking for lagrangian bi-foliations of the complexification.

What about the metric? We do a little computation. Remember that under the projection from $V^{1,0}$ to V, V gets the structure of a complex vector space.

$$\omega_{\mathbb{C}}(\underbrace{v - iJv}_{1,0}, \underbrace{w + iJw}_{0,1}) = \omega(v,w) + \omega(Jv,Jw) + i(\omega(v,Jw) - \omega(Jv,w))$$
$$= 2\omega(v,w) + 2ig(v,w)$$

So by taking the complexified form and restricting it to $V^{1,0}$, we get a hermitian form $\frac{1}{2i}\omega_{\mathbb{C}} = g - i\omega$. If we set $v = w$, we just get the g part.

On $V \oplus V^*$, we have two natural bilinear forms: $\omega_{\pm}((x,\alpha),(y,\beta)) = \alpha(y) \pm \beta(x)$. If we have a form $\omega \in \wedge^2 V$, then the graph of $\tilde{\omega}$ is Dirac (is killed by ω_+). On a manifold, we have $TM \oplus T^*M$. Suppose we are given $\omega \in \Omega^2 M$, then the graph of $\tilde{\omega}$ is a Dirac structure on the bundle TM (i.e., is lagrangian sub-bundle of $TM \oplus T^*M$).

T. Courant: Bracket on section of $TM \oplus T^*M$ defined by

$$[[(X,\alpha),(Y,\beta)]] = ([X,Y], \mathscr{L}_X\beta - \mathscr{L}_Y\alpha)$$

If we stop here, this is a semi-direct product of lie algebras (because $[\mathscr{L}_X, \mathscr{L}_Y] = \mathscr{L}_{[X,Y]}$). But it doesn't have the property that something is closed under it if and only if it is the graph of a closed form. So we change it to

$$[[(X,\alpha),(Y,\beta)]] = ([X,Y], \mathscr{L}_X\beta - \mathscr{L}_Y\alpha + \frac{1}{2}d(i_Y\alpha - i_X\beta))$$

This is no longer a lie algebra (doesn't satisfy Jacobi) and is not a semi-direct product. It has the property that the graph of $\tilde{\omega}$ is closed under this bracket if and only if $d\omega = 0$. A lagrangian sub-bundle of $(TM \oplus T^*M, \omega_+)$ is called an *almost dirac* structure, and an almost dirac structure whose sections are closed under bracket is called a *dirac structure*.

Theorem 9.1. *If E is a Dirac structure, then the restriction to E of the bracket satisfies the Jacobi identity. That is, E is a lie algebra under $[[\ ,\ ]]$.*

$$\begin{tikzcd}
E \arrow[r, hook] \arrow[dr] & TM \oplus T^*M \arrow[d, "\rho"] \\
 & TM
\end{tikzcd}$$

sends $[[\ ,\ ]]$ to $[\ ,\ ]$.

Suppose you have $\tilde{\pi} : T^*M \to TM$, which corresponds to $\pi \in \wedge^2 T^{**}M = \wedge^2 TM$. Given π, we get an almost Dirac structure, which is the graph of $\tilde{\pi}$. What is the condition on π that is equivalent to this being a Dirac structure? That is, when are things like $(\tilde{\pi}\alpha, \alpha)$ closed under the bracket?

Theorem 9.2. *Given $\pi \in \wedge^2 TM$, define $\{f, g\} = \pi(df, dg)$ on $C^\infty M$. Then the graph of $\tilde{\pi}$ is a Dirac structure if and only if $\{\ ,\ \}$ satisfies the Jacobi identity. In this case, we call π a* Poisson Structure*.*

If you have a symplectic structure $\tilde{\omega} : TM \to T^*M$, it is invertible, so we get a $\tilde{\pi}$. Define $\{f, g\} = \omega(\tilde{\omega}^{-1}df, \tilde{\omega}^{-1}dg)$

Lecture 10 - Hamiltonian Mechanics

Submit a proposal for a term paper by Tuesday via bspace.

If we start with a symplectic manifold (M, ω), we have $TM \xrightarrow{\tilde{\omega}} T^*M$. Given $H \in C^\infty(M)$, we can look at $X_H := \tilde{\omega}^{-1}(dH)$, which is called the *Hamiltonian vector field* generated by H, or the "symplectic gradient of H". This vector field has some nice properties:

$$\mathscr{L}_{X_H} H = dH(X_H) = \tilde{\omega}(X_H)(X_H) = \omega(X_H, X_H) = 0$$

This is conservation of energy (X_H is flow through time, and H is energy). We also have

$$\mathscr{L}_{X_H}\omega = i_{X_H}\underbrace{d\omega}_{0} + di_{X_H}\omega = d\tilde{\omega}(X_H) = ddH = 0$$

so ω is invariant in time. From this you can derive

$$\mathscr{L}_{X_H}\omega^k = 0$$

for any $k \geq 0$. This give us that $\mathscr{L}_{X_H}\omega^{\dim M/2} = 0$, which says that hamiltonian flows preserve volume of phase space.

$X_H = 0$ at critical points of H, which are called equilibria. You cannot have stable (or completely unstable) equilibria in hamiltonian mechanics because stable would give volume loss, and unstable would give volume gain.

Consider the case of a pendulum, with $\omega = dq^i \wedge dp_i$, then

$$dH = \frac{\partial H}{\partial q^i}dq^i + \frac{\partial H}{\partial p_i}dp^i$$

$$X_H = -\frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i} + \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i}$$

so

$$\begin{aligned}\frac{dq^i}{dt} &= \frac{\partial H}{\partial p_i}\\ \frac{dp_i}{dt} &= -\frac{\partial H}{\partial q^i}\end{aligned}$$

which are called Hamilton's equations. $H = \frac{1}{2}p^2 + V(q)$. In our case, q is the angle of the pendulum, $V(q)$ is height. Phase space looks like this, with level curves:

figure 1

Lagrange was studying celestial mechanics. He wrote down these equations for this. An elliptical orbit is given by six parameters. If there are multiple planets, then for each planet, the six parameters evolve in time. Later Hamilton did stuff.

Suppose we look at an arbitrary vector field X, then

$$\mathscr{L}_X\omega = d(\tilde{\omega}(X)) = 0 \Leftrightarrow X = \tilde{\omega}^{-1}(\alpha)$$

where $d\alpha = 0$, and α is determined by X. The flows preserving ω are those corresponding to closed 1-forms. The hamiltonian vector fields are those that correspond to exact 1-forms. These are the same locally, so such X are called "locally hamiltonian".

If we look at the cylinder, with coordinates (θ, z), with $\omega = d\theta \wedge dz$. Then $\frac{\partial}{\partial z}$ is closed, but it would have to be "$-d\theta$", so it is not globally hamiltonian.

Define $\{f, g\} = \omega(X_f, X_g)$. So $\{q^i, p_j\} = \omega(-\frac{\partial}{\partial p_i}, \frac{\partial}{\partial q_j}) = \delta^i_j$. In general, we have

$$\{f, g\} = \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i}$$

(note that we are summing over i). We can also write $\{f, g\} = \pi(df, dg)$, where $\pi = \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i}$ (again, we are summing over i). This is called a *bivector field*, and gives us a *Poisson structure*. We need to show that the Jacobi identity is satisfied. Note that $\{f, g\} = X_g f$.

The flow of X_g is hamiltonian, so the flow preserves symplectic structure, and therefore preserves poisson brackets. Let ϕ_t be the flow of X_h, then we have

$$\{\phi_t^* f, \phi_t^* g\} = \phi_t^*\{f, g\}$$

So you have a family of automorphisms. Suppose B is a bilinear operation, A_t is a family of automorphisms with $A_0 = \mathrm{Id}$, and you have $B(A_t f, A_t g) = A_t B(f, g)$. Then differentiating with respect to t (setting $X = \frac{\partial A_t}{\partial t}\big|_{t=0}$)

$$B(Xf, g) + B(f, Xg) = XB(f, g)$$

In our case, we have $B = \{\ ,\ \}$, and $A_t = \phi_t^*$, so we have

$$\{X_h f, g\} + \{f, X_h g\} = X_h\{f, g\}$$

, that is, X_h is a derivation of the bracket. This is the derivation property of hamiltonian vector fields. Jacobi first wrote this, so it is called the Jacobi identity. To see that this is the regular Jacobi identity, recall the definition of $X_h f = \{f, h\}$:

$$\{\{f, h\}, g\} + \{f, \{g, h\}\} = \{\{f, g\}, h\}$$

This is called the *right Liebniz property* (i.e. bracketing on the right is a derivation).

We also have that this is anti-symmetric, so we can put re-arrange everything to look like

$$\{g, \{h, f\}\} + \{f, \{g, h\}\} + \{h, \{f, g\}\} = 0$$

which is the usual Jacobi identity. (Note: there are some things that are Liebniz algebras, but are not anti-symmetric)

Some more manipulating yeilds

$$\begin{aligned}\{f, \{g, h\}\} &= \{\{f, g\}, h\} - \{\{f, h\}, g\} \\ X_{\{g,h\}} f &= X_h X_g f - X_g X_h f \\ &= -[X_g, X_h] f\end{aligned}$$

Some examples of Liebniz algebras:

(a) $M_n(\mathbb{R}) \oplus \mathbb{R}^n$, with the rule

$$[[(A, v), (B, w)]] = ([A, B], Aw)$$

which is called a *hemi-semidirect product* (the semidirect product would be $[[(A, v), (B, w)]] = ([A, B], Aw - Bv)$)

(b) What we did before was

$$[[(X, \alpha), (Y, \beta)]] = ([X, Y], \mathscr{L}_X \beta - \mathscr{L}_Y \alpha + \frac{1}{2}(d(X_2 \lrcorner \alpha_1) - d(X_1 \lrcorner \alpha_2)))$$

then something if we use $\mathscr{L}_X \beta - i_Y \alpha$ in the second part, we get another bracket $[[[\ ,\ ]]]$. You then have

$$\frac{1}{2}([[[e_1, e_2]]] - [[[e_2, e_1]]]) = [[e_1, e_2]]$$

So far we have two properties of the Poisson bracket:

$$\begin{aligned}X_h\{f, g\} &= \{X_h f, g\} + \{f, X_h g\} \\ X_h f g &= (X_h f) g + f(X_h g)\end{aligned}$$

we know that $\phi_t^*(fg) = (\phi_t^* f)(\phi_t^* g)$, and differentiating like before gives the second rule, which we can write as

$$\{fg, h\} = \{f, h\} g + f\{g, h\}.$$

What we have is two operations: pointwise product and bracket. One is commutative and the other is anti-commutative, and they are linked in this way. When we have all this, we say we have a *Poisson Algebra*.

Poisson showed that if $\{f, h\} = \{g, h\} = 0$, then $\{\{f, g\}, h\} = 0$. He knew that time evolution was given by bracketing, so it was important to look for invariants of motion, and this says that if you have two invariants, their bracket is another one (and it is sometimes something really new[9]).

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[9] For example, bracketing x and y angular momentum yields z angular momentum.

Jacobi first introduced the idea of vector fields as operators. Lie's contribution was the observation that you can study abstractly things that satisfy this identity.

If you have a Hamiltonian vector field X_h and a submanifold S, when is the vector field tangent to the submanifold?

(1) If S is a level of h, then yes (conservation of energy).
(2) If S is an orbit of X_h, then yes.
(3) If S is a level of f then if and only if $\{f, h\} = 0$ on S if and only if $\{h, f\} = 0$ on S, which is equivalent to saying that h is a constant of motion along the flow of f! This goes under the name of Noether's theorem.

    This says that looking for invariants of motion along the flow of h is equivalent to looking for functions whose hamiltonian flow preserve h. In other words, you are looking for symmetries of (M, ω, h). So looking for conserved quantities is equivalent to looking for symmetries.
(4) If S is lagrangian, then the condition is that $X_h \in TS = (TS)^\perp$. This is the same as saying that $dh = \tilde{\omega}(X_h) \in \tilde{\omega}(TS)^\perp = (TS)^0$ (the forms which annihilate TS). This is equivalent to saying that h is constant on S.
(5) In the special case where $M = T^*Q$, and $S =$ image of $d\phi$, where $\phi \in C^\infty(Q)$. The image of $d\phi$ is invariant under the flow of X_h if and only if $h \cdot d\phi$ is constant. That is, $h(q, \frac{\partial\phi}{\partial q})$ =constant ... this is the *Hamilton-Jacobi* equation.

LECTURE 11

The Lagrangian $L : TM \to \mathbb{R}$ is written as $L(x, v)$, where x are local coordinates on M and $v = \frac{dx}{dt}$. Then given a path $\gamma : [a, b] \to M$, we define

$$A(\gamma) = \int_a^b L(\dot{\gamma}(t))\, dt$$

Classically, we have $L = \frac{1}{2}\dot{x}^2 - V(x)$.

Consider paths from r to s. This forms a big space, and we look for paths which extremize (minimize or maximize) L. How do you find such a thing? Look at tangent vectors in path space. A path in path space is just a function of two variables.

$$A(\gamma_s) = \int_a^b L(\frac{\partial}{\partial t}\gamma_s(t))$$

$$\frac{d}{ds}A(\gamma_s) = \int_a^b (stuff)\frac{\partial\gamma_s}{\partial t}(t)\, dt$$

The second thing must be zero for all variations, which implies that $stuff$ must be zero. When you write it out,

$$stuff = \frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0$$

This gives a system of partial differential equations called the *Euler-Lagrange* equations. In the classical case, we get

$$0 = -\frac{\partial V}{\partial x} - \frac{d}{dt}(m\dot{x}) = -\frac{\partial V}{\partial x} - m\ddot{x}$$

What do these look like for more general Lagrangians? Well we could have written these last equations as

$$\begin{aligned}\frac{dx}{dt} &= \dot{x}\\ \frac{d\dot{x}}{dt} &= -\frac{\partial V}{\partial x}\end{aligned}$$

which is a vector field on TQ, given by

$$\frac{\partial}{\partial t} = \dot{x}\frac{\partial}{\partial x} - \frac{\partial V}{\partial x}\frac{\partial}{\partial \dot{x}}.$$

In general, we get

$$\frac{\partial L}{\partial x^i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} = \frac{\partial^2 L}{\partial x^j \partial \dot{x}^i}\dot{x}^j + \frac{\partial^2 L}{\partial \dot{x}^j \partial \dot{x}^i}\ddot{x}^j$$

It would be handy to be able to invert the matrix $\left(\frac{\partial^2 L}{\partial \dot{x}^j \partial \dot{x}^i}\right)$. This is the *Legendre condition.* If the condition holds, then (E-L) are 2nd order ODEs (i.e., a vector field on TQ).

If $L = \sqrt{\sum(\dot{x}^i)^2}$, you find that the E-L equations are degenerate, so the solution is not unique. This is because changing the speed of a length-minimizing path leaves it length-minimizing.

If you do the calculation for something of the form $L(x, \dot{x}) = g_{ij}(x)\dot{x}^i\dot{x}^j$, it turns out that the critical points of this are geodesics, parameterized by arc length.

If you let $p_i = \frac{\partial L}{\partial \dot{x}^i}$, then you can define $\mathscr{E} = p_i\dot{x}^i - L$, which we call "energy".

Theorem 11.1. *The Legendre condition holds if and only if* $\mathscr{L} : (x, \dot{x}) \mapsto (x, p)$ *is a local diffeomorphism if and only if* $\mathscr{L}^*(\underbrace{\sum dx^i \wedge dp_i}_{\omega})$ *is nondegenerate. Moreover, Lagrange's equations give hamiltonian vector field of* $\mathscr{E}$ *with respect to* $\mathscr{L}^*\omega$, *i.e.* $\widetilde{(\mathscr{L}^*\omega)}^{-1}(d\mathscr{E})$.

If the Legendre condition holds, then $\left(\frac{\partial p_i}{\partial \dot{x}^j}\right)$ is invertible, so $\dot{x} \to p$ locally a diffeomorphism, so $(x, \dot{x}) \mapsto (x, p)$ is a local diffeomorphism, given by

$$\begin{pmatrix} I & * \\ 0 & \frac{\partial^2 L}{\partial \dot{x}^j \partial \dot{x}^i} \end{pmatrix}$$

Figure 1

$dL_{(x,v)} : T_{(x,v)}TQ \to \mathbb{R}$, but we can restrict it to $T_{(x,v)}T_xQ$, so $dL_{(x,v)}|_{T_{(x,v)}T_xQ} \in T_{(x,v)}(T_xQ)^* \cong (T_xQ)^*$, which is T_x^*Q.

Thus, if $\pi : TQ \to Q$,

$$dL|_{\ker T\pi} : \qquad \begin{tikzcd} TQ \arrow[r, "\mathscr{L}"] \arrow[dr, "\pi"'] & T^*Q \arrow[d] \\ & Q \end{tikzcd}$$

The Legendre condition says that $T\mathscr{L}$ is invertible, which is equivalent to $\mathscr{L}$ being a local diffeomorphism, which is equivalent to $\mathscr{L}^*\omega$ being symplectic.

$\mathscr{E}(x,v) = \mathscr{L}(x,v)(v) - L(x,v)$, which is sometimes written $\mathscr{E} = \langle p, v\rangle - L$. Thus, from any Lagrangian, you get a pullback 2-form. When the Legendre condition fails, this 2-form is degenerate, so you cannot get a hamiltonian vector field from a function.

There is another way of looking at this. If we impose the *strong Legendre condition*, that $\mathscr{L}$ is globally a diffeomorphism, then we define $H = \mathscr{E} \circ \mathscr{L}^{-1} : T^*Q \to \mathbb{R}$. This is called the hamiltonian associated to this lagrangian. Then Lagrange's equations on TQ are transfromed by $\mathscr{L}$ into Hamilton's equations on T^*Q with canonical symplectic structure, and $H(x,p) = \langle p, \mathscr{L}^{-1}(x,p)\rangle - L(\mathscr{L}^{-1}(x,p))$.

In the case $L = \frac{1}{2}m(\dot{x}^i)^2 - V(x)$, with $i = 1,2,3$. Then $p_i = m\dot{x}^i$[10] Then we get

$$\begin{aligned} \mathscr{E} &= \sum m(\dot{x}^i)^2 - \frac{1}{2}\sum m(\dot{x}^i)^2 + V(x) \\ &= \frac{1}{2}\sum m(\dot{x}^i)^2 + V(x) \\ H &= \frac{1}{2}\sum \frac{p_i^2}{m} + V(x) \end{aligned}$$

If you strengthen the Legendre condition by requiring that the matrix is positive definite, then you know that you have an extreme point (not just a critical point)[is this right?]. If you follow the solution to the E-L equations too far, you may reach a point where the path is not even locally minimizing (e.g. consider shortest length paths on a sphere).

If we start with $H; T^*Q \to \mathbb{R}$, consider $dH|_{\text{fibres of } T^*Q} : T^*Q \xleftarrow{\mathcal{M}=weird} TQ$. So we're setting $v^i = \frac{\partial H}{\partial p^i}$, then define $L = (p^i \frac{\partial H}{\partial p^i} - H) \circ \mathcal{M}^{-1} : TQ \to \mathbb{R}$. If we take the Legendre transform of this L, we get H.

Hamilton's principle: Given some $H(x,p)$, define $L(x,p,\dot{x},\dot{p}) = p\dot{x} - H$. If we have a path in T^*Q, we can evaluate $L(\dot{\gamma}) = \alpha(\dot{\gamma}) - H \circ (projection)$. Then Lagrange's equations say

$$\begin{aligned} \frac{\partial L}{\partial x} &= \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} \\ \frac{\partial L}{\partial p} &= \frac{d}{dt}\frac{\partial L}{\partial \dot{p}} \end{aligned}$$

---

[10] If we write $\delta_{ij}\dot{x}^i\dot{x}^j$ instead of $(\dot{x}^i)^2$, we would get the right upper/lower indices.

so

$$-\frac{\partial H}{\partial x^i} = \frac{d}{dt}p_i$$
$$\dot{x}^i - \frac{\partial H}{\partial p^i} = \frac{d}{dt}(0) = 0.$$

Notice that the Legendre condition fails miserably, so Lagrange's equations are first order equations. The solutions to this set of equations are just the extrema for our variational problem. Consider things that solve the second equation, so $\dot{x}^i = \frac{dx^i}{dt} = \frac{\partial H}{\partial p^i}$. Then Hamilton's lagrangian is $p_i\dot{x}^i - H$. We identify solutions to this second half with paths in Q, not in T^*Q.

LECTURE 12 - MOMENTUM MAPS

Today's lecture was given by Christian.

Momentum maps are supposed to associate conserved quantities to symmetries. The symmetries are the basic assumptions on space-time and interactions, from which we get the conserved quantities of momentum and charge. If two quantities are conserved, so is their Poisson bracket, so the conserved quantities form a Lie algebra.

Lets start with an action, ρ, of a group G on a set S. An action is *transitive* if there is only one orbit. The action is *free* if $g \cdot s = s \Rightarrow g = e$. The action is *effective* if $g \cdot s = s$ for all s implies that $g = e$. If S_1 and S_2 are G-sets and $\phi : S_1 \to S_2$ respects the action, then it is called *G-equivariant.*

Now we can add a manifold structure.

Definition. A *Lie group* is a group-manifold such that multiplication and inverse are smooth maps. We also require it to be closed and connected.

This is equivalent to L (left multiplication) being a smooth action of G on itself (this implies that L_g is a diffeomorphism for each g, but the converse is not true). If $f \in C^\infty(G)$ is L-invariant, then we have that $f(g) = f(hg)$ for all g, h, so f is constant.

Invariant vector fields are more interesting. Say $L_g^* v = v$, that is, $(T_g L_h)v(g) = v(hg)$. Then we can define

$$\mathfrak{g} = \{v \in \chi^1(G) | v \text{ is } L\text{-invariant}\}$$

the lie algebra of G.

Note that we can get a left invariant vector field by simply taking some $v_e \in T_eG$ and defining $v(g) = (T_eL_g)v_e$. Thus, as a vector space, $\mathfrak{g} \cong T_eG$.

The vector field being invariant corresponds to the flow being L-equivariant. For any vector field, we have

$$\check{\phi}(\phi(g, t_1), t_2) = \check{\phi}(g, t_1 + t_2)$$

and L-equivariant means

$$\check{\phi}(\phi(g, t_1)h, t_2) = \check{\phi}(g, t_1)\check{\phi}(h, t_2)$$
$$\check{\phi}(gh, t) = g \cdot \check{\phi}(h, t)$$

So we get an exponential map:

$$\check{\phi}(e, t_1 + t_2) = \check{\phi}(\check{\phi}(e, t_1)e, t_2) = \check{\phi}(e, t_1) \cdot \check{\phi}(e, t_2)$$

So we can define exp : $T_eG \to \mathfrak{g} \to G$ given by $v_e \mapsto v \mapsto \check{\phi}(e,1)$. We have that $\mathbb{R} \to G$ given by $t \mapsto \exp(vt)$ is a group homomorphism. These things are called one-parameter subgroups. If they close up, they are isomorphic to S^1, and if they do not, they are isomorphic to $\mathbb{R}$.

$Ad : G \times G \to G$ is defined by $Ad_g h = ghg^{-1}$. You can look at the derivative in the second argument to get $Ad : G \times \mathfrak{g} \to \mathfrak{g}$. If you like, you can define $Ad_g \exp(tv) = g\exp(vt)g^{-1}$, in which case $Ad_g v := \frac{d}{dt} g\exp(vt)g^{-1}$. You can take the dual of the action to get $Ad^* : G \times \mathfrak{g}^* \to \mathfrak{g}^*$ given by $\langle (Ad^*)(g)\alpha, v\rangle = \langle \alpha, (Ad\, g^{-1})v\rangle$. Taking the derivative, we get

$$ad : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

given by $ad(v)w = [v,w]$.

Now we can add some symplectic structure. Say (M,ω) is symplectic. We require our actions to satisfy $\rho(g)^*\omega = \omega$. That is, $\rho : G \to Sym(M,\omega)$. Such actions are called symplectic. The infinitesimal version of this says that $(\exp vt)^*\omega = \omega$. The Lie derivative version is that $\mathscr{L}_{\tilde{\rho}(v)}\omega = 0$, where $\tilde{\rho} : \mathfrak{g} \to \chi^1_{sym}(M,\omega)$. If the vector fields $\tilde{\rho}(\mathfrak{g})$ are hamiltonian, then the action is called *hamiltonian.*

Figure 1

In this case, we can get a generating function for each element of the lie algebra. Let $\tilde{\mathfrak{J}} : \mathfrak{g} \to C^\infty(M)$ send a vector field v to its generating function f_v (this is not unique!). This map is called the (co)momentum map. We define the momentum map $\mathfrak{J} : M \to \mathfrak{g}^*$ by $\langle \mathfrak{J}(m), v\rangle = \tilde{\mathfrak{J}}(v)|_m$.

Proposition 12.1. *$\tilde{\mathfrak{J}}$ exists ($\tilde{\rho}$ is hamiltonian) if and only if $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}] \to H^1_{dR}(M)$, $[v] \to [\tilde{\omega}^{-1}\tilde{\rho}(v)]$, is the zero map. By the way, $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}] \cong H^1_{CE}(\mathfrak{g},\mathbb{R})$.*

Proof. We just have to show that the map is well defined (because hamiltonian vector fields correspond exactly to exact forms). Let $X, Y \in \chi_{sym}(M)$. Then we have to show that $i_{[X,Y]}\omega$ is the derivative of something.

$$\begin{aligned} i_{[X,Y]}\omega &= (\mathscr{L}_X i_Y - \overbrace{i_Y \mathscr{L}_X}^{0})\omega && (\mathscr{L}_X\omega = 0)\\ &= (i_x \circ d \circ i_y + d \circ i_x)\omega && \text{(by magic formula)}\\ &= (i_X(\underbrace{\mathscr{L}_Y - i_Y d}_{0}) + d i_X i_Y)\omega && \text{(magic formula)}\\ &= d\omega(Y,X) \end{aligned}$$

□

Corollary 12.2. (i) *If $H^1_{dR}(M) = 0$, then the action is hamiltonian.*

(ii) *If $\mathfrak{g}$ is semi-simple*[11]*, then the action is hamiltonian.*

An action on the hamiltonian system (M,ω,H) is called invariant if H is G-invariant, i.e. $H(g\cdot m) = H(m) \implies \tilde{\rho}(v)H = 0$. The punchline of the lecture is

Proposition 12.3. *If $\tilde{\rho}$ preserves dynamical system, then $\tilde{\mathfrak{J}}(v)$ is a conserved quantity for any v.*

---

[11] Which implies $\mathfrak{g} = [\mathfrak{g},\mathfrak{g}]$.

Proof. $X_H\tilde{\mathfrak{J}}(v) = \{\tilde{\mathfrak{J}}(v), H\} = -X_{\tilde{\mathfrak{J}}(v)}H = -\tilde{\rho}(v)H = 0.$ □

LECTURE 13 - MOMENTUM MAPS AND SYMPLECTIC REDUCTION

Next meeting is in 450C Moffit Library.

There are some nice ways to simplify your life. One way is to find symmetries, and the other is to find invariants of motion. If you can do one, you can do the other. This follows from

$$X_H F = \{F, H\} = -\{H, F\} = -X_F H$$

If one is zero, so is the other. In general, when you take a quotient or look at a subspace (on which something is constant), you lose the symplectic structure.

Lets take $\mathbb{R}^4$ with (q^1, q^2, p_1, p_2) and the standard form $dq^i \wedge dp_i$. We can take, for example

$$H(q^2, p_1, p_2) = \frac{1}{2}g^{ij}(q^2)p_i p_j + V(q^2)$$

We have that $\{H, p_1\} = 0$. Let's impose the condition $p_1 = c =$const. Now we are down to three dimensions. On $\{p_1 = c\}$, we have the coordinates (q^1, q^2, p_2), and the induced form, $dq^1 \wedge dq^2$, is degenerate.

On the other hand, $\frac{\partial H}{\partial q^1} = 0$, and $\mathscr{L}_{\frac{\partial}{\partial q^1}}\omega = 0$, so we can pass to the quotient by the flow of $\frac{\partial}{\partial q^1}$. Now we have coordinates (q^2, p_1, p_2). But we don't have a way of pushing forward the form (you cannot have a non-degenerate form be the pullback of a lower-dimensional form). But you can push the poisson structure down. $\pi = \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i}$ pushes down to $\frac{\partial}{\partial q^2} \wedge \frac{\partial}{\partial p_1}$. And $\{q^2, p_2\} = 0$, and $\{f, p_1\} = 0$ for all f (p_1 is called a Casimir function). This is a degenerate poisson structure.

In both cases, we get some degeneracy. How can we get back to symplectic geometry?

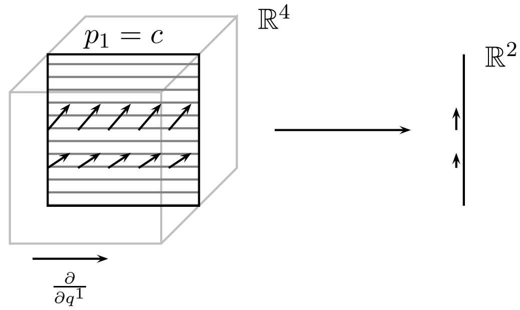
<u>Presymplectic</u>: (the first case) you can still find a hamiltonian vector field. So $X_H \lrcorner \omega = dH$, or $\tilde{\omega}(X_H) = dH$. We have that $\tilde{\omega}(\frac{\partial}{\partial q^1}) = dp^1 = 0$, $\tilde{\omega}(\frac{\partial}{\partial q^2}) = dp^2$, and $\tilde{\omega}(\frac{\partial}{\partial p_2}) = -dq^2$. $H = H(q^2, c, p_2)$, so it doesn't have any dq^1 in it, so $dH = \frac{\partial H}{\partial q^2}dq^2 + \frac{\partial H}{\partial p_2}dp_2$, so we get

$$X_H = -\frac{\partial H}{\partial q^2}\frac{\partial}{\partial p_2} + \frac{\partial H}{\partial p_2}\frac{\partial}{\partial q^2} + (anything)\frac{\partial}{\partial q^1}$$

We have an H which happens to be in the image of $\tilde{\omega}$, but the hamiltonian vector field is not well defined because of that *anything*. We can resolve this problem by passing to the quotient:

$$\{p_1 = c\}/\underbrace{(\text{flow of } X_{p_1} = \frac{\partial}{\partial q^1})}_{\text{ker of } \tilde{\omega}}$$

Now we have coordinates (q^2, p_2), with form $dq^2 \wedge dp_2$, and a "reduced hamiltonian", $H(q^2, c, p_2)$. We get dynamics on this $\mathbb{R}^2$, which is the projection of what we had before.



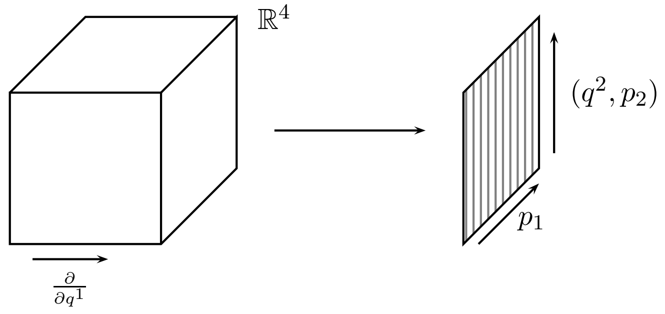
X_H tangent to the slice $p_1 = c$ and invariant along $\frac{\partial}{\partial q^1}$, then we look at the quotient.

For our hamiltonian, we end up with

$$H = \underline{\frac{1}{2}g^{22}(q^2)p_2p_2} + \overbrace{g^{12}(q^2)cp_2}^{\text{vector potential}} + \underbrace{\frac{1}{2}g^{11}(q^2)c^2 + V(q^2)}_{\text{new potential}}$$

This is pretty interesting, we again get something quadratic in the momentum plus an extra term (the *vector potential*), plus a new potential. So our reduces system behaves as though there was a magnetic field or something. For example, when you have a rotating system, you get a force perpendicular to the direction of motion (coriolis force).

On the other side, we have $\pi = \frac{\partial}{\partial q^2} \wedge \frac{\partial}{\partial p_2}$, so we get $\tilde{\pi}(dH) = X_H$, a well defined vector field. We have that $X_H F = \{F, H\}$. In particular, if F is Casimir function, it is a constant of motion for any hamiltonian flow. This picture is dual to the previous picture. Before, we could only solve for the hamiltonian if it was invariant along ***, and it wouldn't be well defined. Here, we also started with a four dimensional space, then we pass to the quotient. There is then a special direction (the p_1 direction). Each of the planes $p_1 = c$ are invariant under any hamiltonian flow, so these submanifolds have an induced symplectic structure (these are called the symplectic leaves).



We get exactly the same flow as before. So there are two ways to do this. Start with a big phase space, then pass to a presymplectic subspace, then take a quotient to get something symplectic. Or you can first take a quotient to get a poisson structure, then take a subspace to get something symplectic.

This is an example of symplectic reduction, but we've done the special case where the symmetry group is $\mathbb{R}$. This is more or less general. If $f \in C^\infty(M)$, with M symplectic, and if $(df)(x) \neq 0$, then near x, there are canonical coordinates (q^i, p_i), in which $f = p_1$. Thus, locally, this is the general case. Sometimes more interesting things happen. For example, the orbits of the flow might close up on themselves, which is exactly what happens in rotational motion.

Moment(um) Map(ping)s. We have a symplectic manifold (M, ω) (though there are versions for the presymplectic and the poisson cases (and dirac as well)), and we have a lie group G acting on M symplectically. So we have $g \mapsto g_M$ with $(gh)_M - g_m \circ h_M$. This induces a lie algebra action. Namely, if you take $v \in \mathfrak{g}$, you get a vector field $v_M \in \chi(M)$ given by

$$v_M = \frac{d}{dt}\Big|_{t=0} (\exp\ tv)_M.$$

It turns out that $(v + w)_M = v_M + w_M$, $(av)_M = av_M$, and $[v, w]_M = -[v_M, w_M]$. If we have

$$G \longrightarrow Diff(M) \qquad g \longmapsto g_M$$

then we get a map $G \to \operatorname{Aut}(C^\infty(M))$, $g \mapsto g_M^*$, then $\hat{v}_M = \frac{d}{dt}\Big|_{t=0}((exp\,tv)_M)^*$ as a derivation of functions, and the bracket of vector fields is just the bracket of operators. From this point of view, we have a map from one lie group to another, so it should induce a lie algebra homomorphism. But $g \mapsto g_M$ is a homomorphism and $g_M \mapsto g_M^*$ is an anti-homomorphism, which is why you get a minus sign.

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "antihom"] \arrow[d, dotted, "\mathfrak{J}"'] & \chi(M, \omega) \\
C^\infty(M) \arrow[ur, "antihom"'] &
\end{tikzcd}$$

If $H^2(\mathfrak{g}, \mathbb{R}) \simeq \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ vanishes, you get a lift $\mathfrak{J}$, the comomentum map. It is not unreasonable to require $\mathfrak{J}$ to be a lie algebra homomorphism. Alternatively, since G acts on $\mathfrak{g}$ by the adjoint representation and it acts on $C^\infty(M)$ by $g_{C^\infty(M)} = (g_M^{-1})^*$, we could require that $\mathfrak{J}$ G-equivariant.

Proposition 13.1. *$\mathfrak{J}$ is a lie algebra homomorphism if and only if $\mathfrak{J}$ is equivariant for the action of the identity component of G (if and only if J is Ad^*-equivariant for the action of the identity component of G).*

Proof. Look in the book. □

There is another nice way of understanding all this, which was introduced independently by several people. Kostant, Souriau, and Smale. Siriau is the most generat. Kostant only looked at transitive actions. Smale only looked at cotangent bundles. They all reformulated the comomentum map. To $\mathfrak{J} : \mathfrak{g} \to C^\infty(M)$ (any linear map, at this point), we associate $J : M \to \mathfrak{g}^*$ by $J(x)(v) = \mathfrak{J}(v)(x)$ for $x \in M, v \in \mathfrak{g}$. We are really looking at a function of two variables: $M \times \mathfrak{g} \to \mathbb{R}$. We can think of it as $M \to g^*$ or $\mathfrak{g} \to C^\infty(M)$.

$\mathfrak{J}$ is equivariant if and only if J is equivariant. What are the group actions in question? We have a group acting on M, and we have G acting on $\mathfrak{g}^*$ given by $g_{\mathfrak{g}^*} = (Ad_{g^{-1}})^*$ (this is called the coadjoint action). Some people will write Ad_g^*, so you have to watch out.

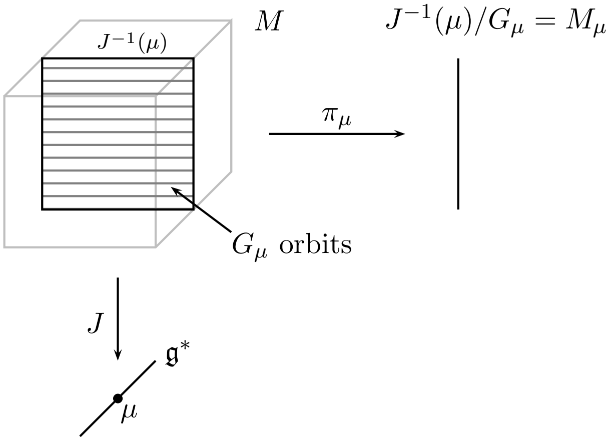
A *hamiltonian action* (sometimes called a *poisson action*) of G on (M, ω) is a symplectic action together with an equivariant momentum map $J : M \to \mathfrak{g}^*$. This isn't completely standard.

Where are we going? One place is a generalization of reduction from the case of one-parameter groups to more general groups

Theorem 13.2 (Symplectic Redcution Theorem, Marsden-Weinstien,Meyer)**.** *Say (M, ω, G, J) a hamiltonian action, and $\mu \in \mathfrak{g}^*$ quasi-regular [regular] value, then $J^{-1}(\mu)$ is invariant under the action of $G_\mu = \{g \in G | Ad^*_{g^{-1}}\mu = \mu\}$, the coadjoint isotropy, and the orbits of G_μ on $J^{-1}(\mu)$ form a regular foliation. Let $i_\mu^*\omega$ be the pullback of ω to $J^{-1}(\mu)$, where $J^{-1}(\mu) \overset{i}{\hookrightarrow} M$. At each $x \in J^{-1}(\mu)$, $\ker_x \widetilde{i_\mu^*\omega} = T_x(G_\mu x) = \{v_M(x) | v \in \mathfrak{g}_\mu\}$.*

Where if $M \xrightarrow{\phi} N$, $\mu \in N$ is *quasi-regular* if $\phi^{-1}(\mu) \subseteq M$ is a submanifold, and for any $x \in \phi^{-1}(\mu)$, the inclusion $T_x\phi^{-1}(\mu) \hookrightarrow \ker(T_x(\phi))$ is an equality. This simplest example of a non-quasi-regular value is when $\phi : x \to x^2$ on $\mathbb{R}$, $x = 0$ is not quasi-regular. The implicit function theorem tells you that regular values are quasi-regular.

This gives us the following picture:



If $J^{-1}(\mu)/G_\mu$ is a manifold, with π_μ a submersion, then there is a (unique) symplectic form ω_μ on $J^{-1}(\mu)/G_\mu$ such that $\pi_\mu^*(\omega_\mu) = i_\mu^*(\omega)$. Then (M_μ, ω_μ) is called the reduction of M at μ.

LECTURE 14 - LIBRARY MEETING

Today's lecture was given by Ann Jensen in Moffit. We went through a list of resources. Here is the information from the handout:

Name	Description	URL
	UCB Mathematics Library	`http://www.lib.berkeley.edu/math/`
Library Catalogs	MELVYL - Libraries of all the UC campuses GLADIS - Libraries of Berkeley only	
MathSciNet	Major index to mathematical literature 1940 - present Created by the AMS	`http://www.ams.org/mathscinet`
ArXiv	Timely preprints of articles, lecture notes, discussion notes mid-1990s - present	`http://front.math.ucdavis.edu/`
Web of Science Science Citation Index	Database of citations to research articles in all fields of science from 1945	
Proxy Server	How to access library resources off campus	`http://proxy.lib.berkeley.edu`

For assistance contact:

Ann Jensen
ajensen@library.berkeley.edu
Mathematics/Statistics Library
100 Evans Hall
642-5729

Lecture 15 - More Symplectic Reduction, Connections

Thursday is the final due date for the term paper topic. By Thursday, put your final proposal on bspace.

Recall the basic reduction theorem

Theorem 15.1. *Let (M, ω, G, J) as before, J Ad^*-equivariant. Then for $\mu \in \mathfrak{g}^*$, $M_\mu = J^{-1}(\mu)/G_\mu$. If $J^{-1}(\mu)$ is a manifold (i.e. μ is quasi-regular), then $T_x J^{-1}(\mu) = \ker(T_x J)$. This implies that $(T_x J^{-1}(\mu))^\perp \cap T_x J^{-1}(\mu) = T_x(G_\mu)$. We will say that μ is* immaculate *if the G_μ action on $J^{-1}(\mu)$ has M_μ as a "manifold quotient". Then M_μ has a symplectic structure ω_μ such that $\pi_\mu^* \omega_\mu = i_\mu^* \omega$. (M_μ, ω_μ) is called the* reduced symplectic manifold.

If G action is free and proper, then every value of J is immaculate (e.g. if G is compact). If the action is not free, then the quotient is called an "orbifold".

Example:Let $G = \mathbb{R}$, so $\mathfrak{g} \simeq \mathfrak{g}^* \simeq \mathbb{R}$. Then the J action is just flow.

Let $M = T^*X$, and ξ a vector field on X. X is configuration space. Define $J : T^*X \to \mathbb{R}$ by $J(\alpha) = \langle \alpha, \xi \rangle$, $J(\alpha) = \alpha(\xi(p(\alpha)))$. In coordinates $x_1, \ldots, x_n, p_1, \ldots, p_n$, then $\xi = \sum a^i(x) \frac{\partial}{\partial x_i}$ and $J = a^i(x) p_i$. If we look at Hamilton's equations, we get

$$\dot{x}^i = a^i(x)$$

$$\dot{p}_i = \frac{\partial a^j}{\partial x^i} p_j$$

The flow is linear on fibers, and is following the trajectory on the manifold downstairs. $\mu \in \mathbb{R}$.

If $\mu = 0$. $J^{-1}(0) \subseteq T^*X$ a codimension 1 sub-bundle. If we want the vector field to be non-zero so that these are well-defined.

We have a map $X \to X/\mathbb{R}$. Let's assume that the $\mathbb{R}$ action on X has a nice quotient $X/\mathbb{R}$. This happens in two cases: where the R action is free and proper, and where the flows are all circles. At any given point in $X/\mathbb{R}$, the cotangent space is isomorphic to the cotangent vectors which annihilate the direction of flow. $J^{-1}(0)$ are the annihilators of the orbits. So $J^{-1}(0)/\mathbb{R} \simeq T^*(X/\mathbb{R})$. It turns out that the symplectic structure is exactly the canonical symplectic structure.

A reference for these calculations: Abraham-Marsden(-Ratin)
Foundations of Mechanics anything after the 1st edition.

If $\mu \neq 0$, then what does $J^{-1}(\mu)$ look like? At any given point, we have the value of the vector field, and we want all 1-forms with a given value on this vector field. This means we are taking all the vectors with a given projection onto the vector field (this is a translation of $J^{-1}(0)$. If we don't have a metric, then there is no natural choice of one of these (we can't just take the shortest). So we can make a global choice of such vectors (a 1-form α), which identifies $J^{-1}(\mu)$ with $J^{-1}(0)$. This identification is compatible with the flow if α is compatible with the flow.

Note:

(1) The identification of $J^{-1}(\mu)$ with $J^{-1}(0)$ is non-canonical.
(2) The form ω_μ on $X/\mathbb{R}$ is in general not the canonical one.

In some non-canonical way, we've put a new symplectic structure on $T^*(X/\mathbb{R})$.

Figure 1

Let $G = \mathbb{R}$ or S^1, whichever makes the action free. Note $G = G_\mu$. This is a principal bundle. α is invariant under the flow and $\langle \alpha, \xi \rangle = \mu$. Assume that $\mu \neq 0$. Let $\beta = \frac{\alpha}{\mu}$, then

$$\mathscr{L}_\xi \beta \equiv 0$$
$$i_\xi \beta \equiv 1$$

Such a β is called a *connection form* on the principal bundle $X \to X/G$. You can look at $d\beta$, and notice that $\mathscr{L}_\xi d\beta = d\mathscr{L}_\xi \beta = 0$. So $d\beta$ is a two-form, which is invariant under the flow of ξ. Also, $i_\xi d\beta = \underbrace{\mathscr{L}_\xi \beta}_{0} - d(\underbrace{i_\xi \beta}_{1}) = 0$. So $d\beta$ kills vectors along the fibers. This exactly tells us that $d\beta$ comes from a two-form downstairs. Thus, there is a two-form F on X/G such that $\pi^* F = d\beta$. From this we have that $\pi^* dF = 0$, and since π^* is injective, we have that $dF = 0$. So F is a closed two-form on $X/\mathbb{R}$, and it is called the *curvature* of the connection β. Note that there is no reason for F to be exact, even though it becomes exact when you pull it up to X.

$J^{-1}(\mu)/G \overset{\alpha}{\simeq} T^*(X/G)$. On $J^{-1}(\mu)/G$, we have the reduced form on $(T^*X)_\mu$, and the corresponding thing on $T^*(X/G)$ is the canonical form plus $\mu(pr^* F_\beta)$, where μF_β is the 2-form on X/G which pulls back to $d\alpha$ on X. Here, $pr : T^*(X/G) \to X/G$. How does this depend on the choice of α. That is, how does the curvature depend on the choice of the connection? Say we replace α by $\alpha + \phi$, then we have $\mathscr{L}_\xi \phi = 0$ and $i_\xi \phi = 0$. $d(\alpha+\phi) = d\alpha + d\phi$, but these conditions tell us that ϕ is $\pi^*\psi$, where ψ is a 1-form on X/G. So we have that $d(\alpha + \phi) = d\alpha + d\phi = \pi^*(\mu F_\beta + d\psi)$.

Conclusion: If we replace α by $\alpha + \pi^*\psi$, the induced form on $T^*(X/G)$ is modified by the addition of $d\psi$.

This is consistent with something else we know about cotangent bundles.

Figure 2

If we compose the two identifications, we get a fibre-preserving map, which is translation by ψ. It is symplectic exactly when $\pi^* d\psi$ vanishes.

It also tells us when we can get rid of this F_α if and only if we can find ψ such that $F_\alpha + d\psi = 0$, which means that F_α is exact. The possible values of curvature therefore range over a cohomology class. That shows that if we do this cotangent reduction $(T^*X)_\mu \leftrightarrow T^*(X/G)$, then we can make the form on the left correspond to the canonical form on the right exactly when some cohomology class is zero. If there is a section of the projection, then the bundle is just a product. If the fibers are all copies for the real numbers, then you can always put together local sections by a partition of unity. So we need the fibers to be circles to get anything interesting.

Let $X = \mathbb{R}^2$ and let $J(x^1, x^2, p_1, p_2) = \frac{1}{2}({x^1}^2 + {x^2}^2 + p_1^2 + p_2^2)$. Hamilton's equations say

$$\begin{aligned} \dot{x}^j &= p_j \\ \dot{p}_j &= -x^j \\ \ddot{x}^j &= -\dot{x}^j \end{aligned}$$

Lets look at $J^{-1}(1/2)/S^1$. If we write $z^j = x^j + ip_j$, then we have that $\dot{z}^j = -iz^j$. The solutions are that $z^j = z_0^j e^{-it}$. So we have that $J^{-1}(1/2)/S^1 \simeq \mathbb{CP}^1 \simeq S^2$. This is not a cotangent reduction because the J is not linear. In any case, we get a symplectic structure on S^2. It isn't hard to see that the induced form on S^2 is the curvature.

We have a vector field which we can think of as $\eta = p_j \frac{\partial}{\partial x^j} - x^j \frac{\partial}{\partial p_j}$. Look at the 1-form $\alpha = (x^i dp_1 + x^2 dp_2 - p_1 dx^1 - p_2 dx^2)/2$, whose d is the canonical symplectic structure. If we take the inner product of our vector filed with α, what do we get? We get $-1/2$ ... we'll deal with it later. What we've show is that $i_\eta \alpha = -1/2$, and we also know that $\mathscr{L}_\eta \alpha = i_\eta d\alpha = i_\eta \omega = dJ = 0$ (because we are on a level set of J). So there is some 2-form on S^2 whose pullback is blah. That 2-form really is the curvature of the bundle.

Moral of the story: S^1 acts on M via a hamiltonian action, $J : M \to R$. $J^{-1}(\mu)/S^1$ is a symplectic manifold, and the symplectic structure is the curvature of some connection on the principal bundle $S^1 \to J^{-1}(\mu) \to M_\mu$. It is not quite the curvature because we don't have the right normalization, so we really have μ times the curvature of some connection.

Corollary 15.2. *If $J^{-1}(\mu)$ is compact, then the circle bundle $S^1 \to J^{-1}(\mu) \to M_\mu$ is* not *trivial.*

If it were, then the cohomology class (which is the first chern class) would be zero, so the symplectic structure would be exact, which it can never be.

Lecture 16 - About the Momentum Map

From last time:

General reduction: (M, ω, G, J), with $M_\mu = J^{-1}(\mu)/G_\mu$.

Cotangent reduction: $M = T^*X$, G acts on X. $J : M \to \mathfrak{g}^*$ linear on fibres. Then $J^{-1}(0)/G \simeq T^*(X/G)$, and $J^{-1}(a)/G$ could be identified with $T^*(X/G)$, but not canonically.

Then we did an example where $M = \mathbb{R}^4$, and we found that $J^{-1}(\mu) \simeq S^3$ and $M_\mu \simeq S^2$. We had the Hopf fibration.

Now for a real cotangent reduction example: Let

$$\begin{aligned} S^1 &= U(1) \\ \\ S^3 &\subseteq \mathbb{C}^2 \\ \\ S^2 &\simeq \mathbb{CP}^1 \end{aligned}$$

$T^*S^2 \xrightarrow{H} \mathbb{R}$ given by $H(x, \xi) = \frac{1}{2}||\xi||^2$. Hamilton's equations for such a hamiltonian are exactly the equations for geodesics. Then look at $H^{-1}(1/2)$.

[[If $M \xrightarrow{J} \mathfrak{g}^*$ and $H : M \to \mathbb{R}$ is G-invariant, then $H|_{J^{-1}(\mu)}$ induces H_μ on M_μ. $M \supseteq J^{-1}(\mu) \to M_\mu$.]]

We have a momentum map $T^*S^3 \xrightarrow{J} \mathbb{R}$ given by $J(v) = \langle v, X_H \rangle$. [[In the case of a surface of revolution, we have Clariout's Theorem, which tells us that the angle of a geodesic with the direction of rotation is fixed.]] $J^{-1}(0) =$ vectors perpendicular to the Hopf vector field.

Figure 1

Notice that the sub-bundle of spaces perpendicular to the hopf vector field is not integrable. What is the induced hamiltonian downstairs? $H_0 : T^*S^2 \to \mathbb{R}$ is also $\frac{1}{2}||\ ||^2$. What if we replace 0 by μ? Because we have a metric, we may as well lift a vector to μ times the hopf vector field plus stuff. Then we get the hamiltonian $H_\mu(v) = \frac{1}{2}||v||^2 + \mu^2$. What about the flow?

Consider the case where we have the hopf field itself, then it will project to the trivial flow on S^2. If it is close to the hopf field, then it must project to small circles for the trajectories. This suggests that if something is moving on the two sphere, it feels some force perpendicular to its direction of motion. How do we see that? The symplectic form on T^*S^2 has been modified ... it turns out to be modified by μ times the area form.

Say X is a Riemannian manifold with a free circle action, so $X \to X/S^1$. Suppose the action preserves the metric. If we pick a μ, we get induced motion on $T^*(X/S^1)$. We have $B \in \Omega^2(X/S^1)$ closed, which takes velocities to forces ... it behaves like a magnetic field. In many cases, you can turn this around. Say $X/S = Y$, and you want to describe motion in a magnetic field on Y. Then we can describe it as motion on something one dimension higher so long as we can realize B as the curvature of some connection. That is the case if and only if the cohomology class of B is "integral" (i.e., if you integrate B over any circle in Y, you get an integer). One can also lift the potential, if there is one. This is called Kaluza-Klien theory. This stuff came up when people were trying to unite magnetic and gravitational forces ...

the idea was to work in a 5-dimensional space rather than a 4-dimensional space-time. This was the beginning of gauge theory.

Now say we have (M, ω, G), and say we have a momentum map which is not equivariant. So

$$\begin{tikzcd}
v \in \mathfrak{g} \arrow[r] \arrow[d, "\mathfrak{J}"] & v_M \arrow[d, phantom, "\|"] \\
\mathfrak{J}(v) \arrow[r] & x_{\mathfrak{J}(v)}
\end{tikzcd}$$

We can ask that $\mathfrak{J} : \mathfrak{g} \to C^\infty(M)$ be G-equivariant, which would be a regular momentum map.

$$\begin{aligned}
[Ad_g v]_M &= \frac{d}{dt}\Big|_{t=0} (\exp(Ad_g v)) \\
&= \frac{d}{dt}\Big|_{t=0} (g_M \exp(tv)_M g_M^{-1}) \\
&= (Tg_M) v_M g_M^{-1}
\end{aligned}$$

$$\begin{aligned}
d\mathfrak{J}(Ad_g v) &= (Ad_g v)_M \lrcorner \omega \\
&= \underbrace{g_M^{-1*} v_M}\lrcorner \omega \\
&= (g_M^{-1})^*(v_M \lrcorner \omega) \\
&= (g_M^{-1})^*(d\mathfrak{J}(v))
\end{aligned}$$

$$d(d(Ad_g v) - g_M^{-1*}\mathfrak{J}(v)) = 0$$

Assume M connected, then $Ad_g v) - g_M^{-1*}\mathfrak{J}(v)$ is some constant $\langle c(g), v\rangle$. So $c(g) \in \mathfrak{g}^*$. So $c : G \to \mathfrak{g}^*$. This c is zero if and only if $\mathfrak{J}$ is equivariant if and only if $J : M \to \mathfrak{g}^*$ is equivariant.

This c is a 1-cocycle on G with values in $\mathfrak{g}^*$ with the Adjoint representation:

$$c(gh) \text{ “=” } c(g) + Ad^*_{g^{-1}} c(h)$$

Why are these called cocycles? Because we have a whole complex

$$C^0(G, \mathfrak{g}^*) \xrightarrow{\delta} C^1(G, \mathfrak{g}^*) \xrightarrow{\delta} C^2(G, \mathfrak{g}^*) \xrightarrow{\delta} \cdots$$

such that $\delta^2 = 0$, where $\delta(c)(g, h) = c(gh) - c(g) - Ad^*_{g^{-1}} c(h)$, and $C^i(G, \mathfrak{g}^*)$ are maps from G^i to $\mathfrak{g}^*$. And for $\alpha \in C^0$, we have $\delta(\alpha) = Ad^*_{g^{-1}}\alpha - \alpha$.

Recall that the condition on the momentum map is $d\mathfrak{J}(v) = v_M \lrcorner \omega$, so we can change $\mathfrak{J}$ by a constant, and when we do so, the c varies over the cohomology class.

If $p(g, h)$ is a 2-cochain, then we define

$$\delta p(g_0, g_1, g_2) = -p(g_0 g_1, g_2) + p(g_0, g_1 g_2) + Ad^*_{g_0^{-1}} p(g_1, g_2) - p(g_0, g_1)$$

There is another kind of equivariance. Given $c \in Z^1(G, \mathfrak{g}^*)$, we can define a new affine action of G on $\mathfrak{g}^*$ by the rule $g \cdot \alpha = Ad^{-1}_{g^*}\alpha + c(g)$. This it is easy to see that J is now equivariant with respect to this action.

Example: $M = \mathbb{R}^2$ with coordinates (q,p), $\omega = dq \wedge dp$. Let $G = \mathbb{R}^2$ with coordinates (a,b), then $\mathfrak{g} \cong \mathbb{R}^2$ with coordinates (x,y). Then we have the action $(a,b)_M(p,q) = (p+a, q+b)$. What is $(x,y)_M$? It is $x\frac{\partial}{\partial q} + y\frac{\partial}{\partial p}$. Then $(x,y)_M \lrcorner \omega = xdp - ydq = d(xp - yq)$ where d is the differential in (q,p) space. This is equal to $d\langle (p,-q),(x,y)\rangle$. So we can say $\mathfrak{J}(x,y) = xp = yq$. Then $J(q,p) = (p,-q)$. This is not equivariant because the coadjoint action is trivial. The cocycle is $c(a,b,q,p)(x,y) = \mathfrak{J}(x,y)(q,p) - \mathfrak{J}(x,y)(q+a,p+b) = xb - ya$. Then $c(a,b) = (b,-a)$.

Instead of just the lie algebra generators e_x and e_y, we add e_t such that $[e_x, e_y] = e_t$ and $[e_t, e_x] = [e_t, e_y] = 0$. The dual of this lie algebra has ξ, η dual to e_x, e_y, and τ dual to t. Define the lie algebra action by letting e_t act trivially: $(x,y,t)_M = x\frac{\partial}{\partial q} + y\frac{\partial}{\partial p}$. If we take $J(q,p) = (p,-q,1)$, then it turns out that this action is equivariant.

LECTURE 17

Did everybody get the email saying that you shouldn't use the bspace dropbox? Good.

We start with (G, ω, G, J), $J : M \to \mathfrak{g}^*$ with corresponding $\mathfrak{J} : \mathfrak{g} \to C^\infty(M)$. Now we define

$$\theta(v,w) := \{\mathfrak{J}(v), \mathfrak{J}(w)\} - \mathfrak{J}([v,w]) \in \mathbb{R} \quad \text{(generally, } H^0(M)).$$

This is a bilinear, skew-symmetric map, $\theta : \mathfrak{g} \wedge \mathfrak{g} \to H^0(M)$. It satisfies a kind of Jacobi identity:

$$\delta\theta(v,w,u) = \theta([v,w],u) + \text{(cyclic permutations)} = 0.$$

This is a piece of a complex which calculates the lie algebra cohomology.

$C^k(\mathfrak{g}) = \wedge^k \mathfrak{g}^* = \mathrm{Hom}(\wedge^k \mathfrak{g}, \mathbb{R})$, $\delta : C^k(\mathfrak{g}) \to C^{k+1}(\mathfrak{g})$ given by

$$(\delta\theta)(v_0, \dots, v_k) = -\sum_{i<j} (-1)^{s(i,j)} \theta([v_i, v_j], v_0, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_k).$$

This has a lot to do with the de Rham differential. [[

$$\begin{aligned}(\delta\theta)(v_0, \dots, v_k) = -\sum_{i<j} (-1)^{s(i,j)} \theta([v_i, v_j], v_0, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_k) \\ + v_i \cdot \theta(v_0, \dots, \hat{v}_i, \dots, v_k).\end{aligned}$$

]]

If G is a compact, connected lie group, then $H^k_{dR}(G, \mathbb{R}) \simeq H^k(\mathfrak{g}, \mathbb{R})$. If $G = \mathbb{T}^n$, then $\mathfrak{g} = \mathbb{R}^n$, and $C^k(\mathfrak{g}) = \wedge^k (\mathbb{R}^n)^* = H^k(\mathfrak{g})$, which has dimension $\binom{n}{k}$. If $G = \mathbb{R}^n$, then it has the same lie algebra, but the cohomology is different. So you definitely need compact and connected.

Recall that we can change $\mathfrak{J}$ by locally constant functions. Say we replace $\mathfrak{J} \rightsquigarrow \mathfrak{J} + b$, where $b : \mathfrak{g} \to H^0(M, \mathbb{R})$ (i.e. $b \in \mathfrak{g}^* \otimes H^0(M, \mathbb{R})$). Then

$$\begin{aligned}\theta_{\mathfrak{J}+b}(v,w) &= \{\mathfrak{J}(v) + b(w), \mathfrak{J}(w) + b(v)\} - \mathfrak{J}([v,w]) - b([v,w]) \\ &= \theta_{\mathfrak{J}}(v,w) + \underbrace{\{\mathfrak{J}(v), b(w)\} + \{b(v), \mathfrak{J}(w)\}}_{0+0} - \underbrace{b([v,w])}_{\pm\delta b(v,w)}.\end{aligned}$$

So when we change $\mathfrak{J}$ by a locally constant function, the θ is changed by a coboundary. Thus, given an action admitting a comomentum map, there is an associated element of $H^2(\mathfrak{g}, H^0(M))$ which vanishes if and only if there is a comomentum map which is a lie algebra homomorphism.

Last time we showed

$$\mathfrak{J}(Ad_g v) - {g_M^{-1}}^{*}\mathfrak{J}(v) = \langle \overbrace{c(g)}^{\in\mathfrak{g}^*}, v\rangle$$

for some $c : G \to \mathfrak{g}^*$. Now we're going to differentiate c at the origin. Let $g = \exp(tw)$ for $w \in \mathfrak{g}$, then take $\frac{d}{dt}$ at $t = 0$. Then we get

$$\begin{aligned}\mathfrak{J}([w,v]) + X_{\mathfrak{J}(w)} \cdot \mathfrak{J}(v) &= \mathfrak{J}([w,v]) + \{\mathfrak{J}(v), \mathfrak{J}(w)\}\\ &= \theta(v,w)\end{aligned}$$

on the LHS, and on the RHS, we have $\langle (T_e c)(w), v\rangle$. So if we think of c as a bilinear form on $\mathfrak{g}$, it is exactly θ. It turns out that if $\theta = 0$, you get that $c = 0$ on the connected component of the identity. So on a connected group, θ and c contain the same information. Not so on a non-connected group:

Consider $O(3)$ acting on $T^*\mathbb{R}^3$ with coordinates q^i, p_i $i = 1, 2, 3$. Take the one parameter group of rotations given by rotation of the 1-2 plane. $q^1\frac{\partial}{\partial p_1} - q^2\frac{\partial}{\partial p_2}$, with hamiltonian $q^1p_2 - q^2p_1$. Then $J = q \times p$. $L(q) \times L(p) = L(q \times p)$. The momentum map is SO_3 equivariant, but not O_3 equivariant because reflection introduces a minus sign. We've identified $\mathfrak{o}_3 \cong \mathfrak{so}_3 \simeq \mathbb{R}^3$.

Another example: let G be a lie group, then G acts on G by right translations. That is, $g \rightsquigarrow r_{g^{-1}}$. Then we can lift to T^*G: $g \cdot \alpha = r_g^*\alpha$. Since any lifted action is symplectic, we can expect a momentum map, and we can expect it to be linear on fibres. If you look at a one-parameter subgroup, the infinitesimal generator is a left-invariant vector field. $J : T^*G \to \mathfrak{g}^*$ given by left translation back to the identity. Then $J^{-1}(\mu)$ is all cotangent vectors, which when translated back to the identity become μ, so this is just the image of the left invariant 1-form μ, which is isomorphic to G as a manifold. So what is $J^{-1}(\mu)/G_\mu$? It is G/G_μ, which is the coadjoint orbit of μ. Thus, $G/G_\mu = G \cdot \mu$ inherits a reduced symplectic structure.

$$\begin{tikzcd}
T^*G \arrow[d] & J^{-1}(\mu) \arrow[l, hook'] \arrow[d] \\
T^*G/G & J^{-1}(\mu)/G_\mu \arrow[l, hook']
\end{tikzcd}$$

Here, T^*G/G is a poisson manifold, with the linear lie-poisson structure on $\mathfrak{g}^*$: $\{\mu_i, \mu_j\} = c_{ij}^k\mu_k$.

Convexity Theorem. Two big theorems in symplectic geometry, the Atiyah-Guilleman/Sternberg convexity theorem (1982), and Gromov pseudoholomorphic curve theorem (1985). We will talk about the first theorem

Theorem 17.1. *(M, ω) connected symplectic, $G = \mathbb{T}^m = (\mathbb{R}/\mathbb{Z})^m$ acting on M with equivariant $J : M \to \mathfrak{g}^* \cong \mathbb{R}^m$. [[If $v_1, \dots, v_m$ a basis for the lie algebra $\mathfrak{t}^m$ of $\mathbb{T}^m$, then $\{\mathfrak{J}(v_1), \mathfrak{J}(v_2)\} = 0$. Then the orbits of $v_{i,M}$ are* *isotropic.]] Then the J level submanifolds are connected, and $J(M) \subseteq \mathfrak{t}^{m*}$ is the convex hull of $J(M^G)$ (which is a finite set).*

If $m = 1$, we have a circle action, $J : M \to \mathbb{R}$ is just the hamiltonian for the circle action. Then the theorem says that the image is a closed interval whose endpoints are fixed points.

Local version: Assume we have a fixed point $x \in M$. Locally, we have a symplectic torus action on $\mathbb{R}^{2n}$ containing 0 as a fixed point.

Theorem 17.2 (Bachner)**.** *Any smooth action of a compact group may be linearized around any fixed point. i.e., given $x \in M$ fixed by a compact group action, there is some coordinate system around x such that the action on the coordinate system is linear.*

Choose a Riemannian metric on M, then use compactness of the group to replace the metric by an invariant metric under the action of the group. Then use the exponential map to give a coordinate chart. Then the action sends geodesics to geodesics, so on the tangent space, it is linear.

So we can say that around a fixed point, the torus action is linear. We would like to say that it is symplectic, but it is only symplectic with respect to $\exp^* \omega$, where ω is the symplectic form on M. We would like: if we choose something other than exp, ω will pull back to a constant symplectic form. What we really need is a Bochner-Darboux theorem, which says that if we have a compact group acting on a symplectic manifold near a fixed point, we can find a coordinate system where the group action is linear and the symplectic form is constant. That is not too hard to prove using the Moser method. First use the Bochner theorem. Then we have two symplectic structures on our T_xM. Then we need to take one to the other in such a way that it commutes with the action of the group. But if you look at the proof of the Darboux theorem, the construction commutes with any linear action. This gives us a linear symplectic action near a fixed point. Now we need to look at the momentum map.

Lecture 18

Theorem 18.1 (Schur-Horn)**.** *If you look at the action of $U(n)$ on $n \times n$ hermitian matrices, and let $\mathscr{O}_{(\lambda_1,\dots,\lambda_n)}$.*

$conv\{(/lambda_{\sigma(1)},\dots)\}\{diag(ADA^*)\} \in \mathbb{R}^n$*, where* $D = diag(\lambda_1,\dots,\lambda_n)$ *and σ a permutation.*

Then Kostant generalized, Heckman reinterpreted in terms of coadjoint orbits, Guilleman-Sternberg reinterpreted as tori acting on symplectic manifolds, Atiyah some other stuff. Then Kirwan generalized for non-tori ... N.T.Zung "the ultimate convexity theorem".

Ingredients of proof

1. Local normal form ⇒ local convexity
2. "Morse theory" → global result
3. Covering argument (C-D-M) (connected fiber

x fixed point for the torus action, action on T_xM linear, G-equivariant as in last lecture.

Then we find a G-invariant hermitian structure on T_xM.

G compact compact acting on vector space V. Then there is a G-invariant inner product on V. To see this, take any inner product B and define

$$\bar{B}(v,w) = \int_G B(gv, gw)\, dg$$

where dg is the haar measure.

This gives us a compatible complex structure. Now we are going to use that G is a torus $\mathbb{T}^m$, so we can diagonalize the action by choosing coordinates q_j, p_j. We get a map $\mathfrak{t}^m \to \mathbb{R}^n \simeq \mathfrak{t}^n$. Think of $G = \mathbb{R}^m/2\pi\mathbb{Z}^m$, then this map is an integer matrix.

$\langle J(z^1,\dots,z^n),(t_1,\dots,t_n)\rangle = \frac{1}{2}t_j r^j_k |z^k|^2 = \frac{1}{2}t_j r^j_k((q^k)^2+(p^k)^2)$, where r^j_k is an integer matrix. Hamilton's equations:

$$\dot{q}^k = \frac{\partial \mathfrak{J}}{\partial p^k} = t_j r^j_k p^k \qquad \text{(not sum over } k\text{)}$$

$$\dot{p}^k = -\frac{\partial \mathfrak{J}}{\partial q^k} = -t_j r^j_k q^k \qquad \text{(not sum over } k\text{)}$$

. So $J(z^1,\dots,z^n) = \{r^j_k \cdot \frac{1}{2}|z^k|^2\} = r^j_k \frac{1}{2}|z^k|^2 \tau_j = \underbrace{(r^j_k \tau_j)}_{e_k} \frac{1}{2}|z^k|^2$, where τ_j is the j-th basis vector of $\mathfrak{t}^*$, e_k a vector in $\mathfrak{t}^*$ with integer components. Thus, locally, the image of the momentum map are all the non-negative linear combinations of the e_k's. This gives us a polyhedral cone (if the e_k are pointing in sufficiently different directions that our cone is everything).

How do you know there are any fixed points?

Lemma 18.2. *For any torus action (in fact, any compact lie group), the fixed point set is a symplectic submanifold.*

This follows from our normal form near a fixed point (the fixed set must be a complex subspace, hence a symplectic submanifold). Then $\mathbb{T}^m = \mathbb{T}^1 \times \cdots$. The first has a fixed point because the action is hamiltonian, and the hamiltonian function must have a max and min ... then use induction.

We have that the image of J is locally in a bunch of these polyhedral cones. Now we use morse theory to show that the image is globally in the intersection of these cones.

Figure 1

look at the flow of $\exp(tv)$ on M. What is its hamiltonian? It is given by composing v with the momentum map. The hamiltonian has local maximum at x, and it must have some global maximum, but our local max isn't the global max because $J(y)$ is on the other side of the plane, contradicting

Lemma 18.3. *For any 1-parameter subgroup of the torus, its hamiltonian has a unique local maximum value.*

Proof. At each critical point, the function is quadratic with an even number of positive coefficients. The maxima are the points where that even number is zero. Connect two local maxima with a path. Along the path, there is a minimum value ... choose the path with the largest minimum. This must be a critical point, otherwise, you could find a better path. There cannot be two directions in which you increase, otherwise you could slide around the bowl and get a better path. Thus, there must be a point where there is only one positive direction (or none). This proves that the set of local maxima is connected. ☐

This tells us that the image is in the intersection of all the polyhedral cones. Suppose there were some inside point not in the image, then we may take the largest ball which is in the complement of the image. This touches the image somewhere. Look at the tangent hyperplane at the point of contact. We will show that the whole image is on the other side of the plane, which contradicts the assumption that we took the largest ball.

$x \in M$. $\mathbb{T}_x$ = stabilizer of x ... this is a closed subgroup of the torus. $M^{\mathbb{T}_x} \subseteq M$ the set of all points fixed by $\mathbb{T}_x$.

figure 2

The composite map is the momentum map for the action of $\mathbb{T}_x$. The image in $\mathfrak{t}_x^*$ is in a cone, so the image of J is in the "wedge" in $\mathfrak{t}^*$. Now we need to show that J fills up the wedge around $J(x)$. $M^{\mathbb{T}_x} \subseteq M$ symplectic submanifold. Lets look at $J(M^{\mathbb{T}_x})$ ... it must lie in the pre-image of the image of $J(x)$. It fills up that pre-image because we can find a complementary torus to $\mathbb{T}_x$, which must be free around x, so blah. Now the same morse theory argument shows that the image is globally in the wedge.

End of proof of convexity of image of J.

Now we need to show connectivity of the fibers. Define an equivalence relation on M. Equivalence classes are connected components of fibers of J. Let $\tilde{M} = M/\sim$.

$$\begin{tikzcd}
\tilde{M} \arrow[dr, "\tilde{J}"] & & & \\
M \arrow[u] \arrow[r, "J"] & J(M) & \subseteq & \mathfrak{g}^*
\end{tikzcd}$$

and we want to show that $\tilde{J}$ is bijective. Notice that $\tilde{M}$ is connected because it is a quotient of a connected set. $J(M)$ is simply connected because it is convex. so it is enough show that $\tilde{J}$ is a covering map.

1)$\tilde{J}$ is a local homeomorphism (from normal form).

2) if $f : X \to Y$ is local homeomorphism and f proper, then g is a covering map. To see this,

figure 3

since the map is proper, the inverse image of a point is finite. each point in the inverse image has a nbd which maps homeomorphically to a nbd of y ... intersect those. We just need that we have the complete inverse image of our nbd of y. Suppose not. Then there is a sequence $y_i \to y$ and $x_i \in f^{-1}(y_i)$ such that $x_i \notin$any U_j. $\{y_i, y\}$ compact, so by properness, the x_i have a convergent subseq, which must be in the inverse image of y. We win!

Lecture 19 - Torus Actions

Remark on the paper: the main idea of the paper is to cover a subject, not a particular article, or details of proofs.

There is a book, Toric actions on symplectic manifolds 2nd edition, by Michèle Audin, which is very good for the stuff we're doing.

The subject of torus actions fits into the more general area of completely integrable systems. The basic setup is this. You have (M, ω) $2n$-dimensional symplectic. Following a paper by Duistermaat (1980) in Comm. Pure Appl. Math., we have $M^{2n} \xrightarrow{J} W^n$ with rank n almost everywhere, with lagrangian fibres.

Lemma 19.1. *J has lagrangian fibres if and only if J is a Poisson map, where M has the symplectic poisson bracket, and W has the zero poisson bracket.*

That is, $\{f \circ J, g \circ J\} = 0$ for all $f, g \in C^\infty(W^n)$. If $W^n = \mathbb{R}^n$, and $J = (f_1, \ldots, f_n)$. Then this condition says that $\{f_i, f_j\} = 0$. We say that these functions are in involution.

Given $H : M \to \mathbb{R}$, J is a constant of motion for X_H if and only if H is constant on (connected components) of J-fibres. Roughly, H is a function of the f_is. Then we say that this structure forms a completely integrable system.

There are two steps to understanding these things: understanding J and understanding H dynamics.

Figure 1

Pick some $\mu \in W$ regular value for J, we get a map $T^*_\mu W \to \chi^1(J^{-1}(\mu))$. The images of different elements commute. So the image is a commuting subalgebra of vector fields spanning $T(J^{-1}(\mu))$. Take a function on W and pull it back. The pull back has a hamiltonian flow along the fibres. This map "integrates" to an action (perhaps partly defined) of the additive group $T^*_\mu W$ on $J^{-1}(\mu)$.

Let's assume completeness of the vector fields involved. e.g. this is implied by the assumption that $J^{-1}(\mu)$ is compact, which is implied by J proper. Let's assume J proper. Then we get an action of the cotangent space on the fiber (which is compact), and this action is transitive, locally free, so the fiber $J^{-1}(\mu)$ can be identified with $T^*_\mu W/\Lambda_\mu$, where Λ_μ is a lattice, so $J^{-1}(\mu)$ must be a torus.

Figure 2

Every fiber of T^*W is a group which acts on the $J^{-1}(\mu)$s, and the isotropy of all the fibers gives a lattice of 1-forms. $\bigcup_\mu \Lambda_\mu \subseteq T^*W$ is locally given by graphs of smooth 1-forms. Not globally! The fundamental group $\pi_1(W, \mu)$ acts on Λ_μ. This action is called "monodromy". Each fiber in M is a torus. There is an identification $\Lambda_\mu \simeq \pi_1(J^{-1}(\mu)) = H_1(J^{-1}(\mu), \mathbb{Z})$.

Note that you have to take out singular values, so you might get some non-simply connected W. Take the spherical pendulum. The configuration space is the two-sphere. There are two conserved quantities, so we can map $T^*S^2 \to \mathbb{R}^2$ by (E, L_z), where E is energy, and L_z is angular momentum around the z axis. The map is proper because the energy levels are compact. But there are singular values, and there is monodromy around the singular values. Cushman, Duistermaat, Vu have written some stuff.

Now let's go back and analyze integrable systems some more. If you think a little bit, any section of T^*W defines a map $M \to M$ by translating the torus fibres by it. This is a symplectic map if and only if the section is a closed 1-form. We showed that translation by a 1-form is symplectic if and only if it is closed. The elements of the lattice act by the identity, so they are smooth closed 1-forms. I claim this gives us a rigid structure on W. Let's look on W:

Figure 3

We have a lattice of closed 1-forms spanned by $\omega^1, \ldots, \omega^n$ (which are a local basis). $\omega^j = dx^j$, and since the ω^i are independent, the x^i form a coordinate system: $\mathbf{x} = (x^1, \ldots, x^n) : \mathcal{U} \to \mathbb{R}^n$. If we choose the ω^i, then the x^i are determined up to a constant. So we may change $\mathbf{x} \rightsquigarrow \mathbf{x} + \mathbf{b}$. We may also replace ω by ν by saying that $\nu^j = a^j_i \omega^i$, where $a^j_i \in GL(n, \mathbb{Z})$. If $\nu^j = dy^j$, then $y^j = a^j_i x^i + b^j$. We restrict ourselves to the case where the ω^i form a basis for the lattice. We get a "flat $GL(n, \mathbb{Z})$ structure". You can do on W any construction you can do on $\mathbb{R}^n$ so long as it is invariant under $GL(n, \mathbb{Z})$. You can talk about straight lines. You can proof that the torsion of the connection is zero. There is a distinguished family of bases of the tangent space up to $GL(n, \mathbb{Z})$. In each tangent space of W, there is a natural integer lattice, and there is a natural identification with nearby tangent spaces which identifies these lattices. If the base is compact, it eliminates certain cases. If the base is compact, it must be parallelizable, so it cannot be the 2-sphere, for example (in fact, it must be the torus or the klein bottle). A.T. Fomenko has written a bunch of stuff about related things.

Locally, $\Lambda \simeq W \times \mathbb{Z}^n$, so $M \simeq W \times \mathbb{T}^n$ locally (in W). The local coordinate function on W obtained from one of these bases generate flows which rotate the torus. This is a symplectic isomorphism ... you get coordinates $(I^1, \ldots, I^n, \theta_1, \ldots, \theta_n)$, and $\omega = dI^j \wedge d\theta_j$. Then H is a function of the I^js, so hamilton's equations give

$$\frac{dI^j}{dt} = 0 \qquad , \qquad \frac{d\theta_j}{dt} = -\underbrace{\frac{\partial H}{\partial I_j}(I_1, \ldots, I_n)}_{\phi_j}.$$

If H is linear in the Is, then it is a very degenerate system ... you get the same flow on each torus.

The non-degenerate case: If $\left(\frac{\partial^2 H}{\partial I_k \partial I_j}\right)$ is invertible. The determinant is $\det(\frac{\partial \phi_j}{\partial I_k})$. Torus actions are the ones for which these lattices are trivial bundles.

Delzant's Theorem. M is compact, and $M \xrightarrow{J} \mathfrak{t}^{n*}$ the momentum map for an effective torus action. We're assuming there is no monodromy around the singular points. Then we know that $J(M) = conv(J(M^{\mathbb{T}}))$. Note that $\mathfrak{t}^{n*}$ has a natural $GL(n, \mathbb{Z})$ structure. The kernel of the exponential map is a natural lattice in $\mathfrak{t}^n$. Delzant's theorem says that $J(M)$ is a Delzant polytope, i.e. has the following properties:

- From each vertex, there emanate exactly n edges, containing vectors which form a basis for the natural lattice Λ, the dual lattice to the kernel of the exponential map.

Theorem 19.2. *Every Delzant polytope arises as the $J(M)$ for a unique (up to equivariant symplectomorphism) hamiltonian torus action on a compact symplectic manifold.*

Notice that the Delzant structure on a polytope doesn't depend on the geometry (you can slide the faces of the polytope parallel to themselves). Consider the one-dimensional case. A delzant polytope is an interval:

Figure 4

If we slide the polytope up and down, it doesn't change anything. If we just move one end, it changes the size of the sphere.

How does the proof go? To prove that $J(M)$ has these properties is not too bad. The hard part is showing that it arises and then showing uniqueness. The way we do that is by exhibiting the manifold M for a polytope δ as a reduced manifold of $\mathbb{C}^d_\lambda$ for a linear action of $\mathbb{T}^r$ on $\mathbb{C}^d$. How do we get such a linear action out of a polytope? The simplest example is the one-dimensional case. The 2-sphere is $\mathbb{CP}^1$, which is $\mathbb{C}^2_\lambda$ for the Hopf circle action. In general, we have a polytope, so it is the intersection of half-spaces. Δ can be defined as $\{\mu \in \mathfrak{t}^{n*} | \langle v_j, \mu \rangle \leq \lambda_j\}$, where $v_1, \dots, v_d \in \mathfrak{t}^n$, and $\lambda_j \in \mathbb{R}$. The $\{v_j\}$ define a map $\mathbb{R}^d \to \mathfrak{t}^n$, which is an integer map (we can choose all the v_i integer). $v_j = (v_j^k)$, for $v_j^k \in \mathbb{Z}$. so we get an $d \times n$ integer matrix. How do we build an action of some torus on $\mathbb{C}^d$ so that the reduced manifold will have the right dimension? We can think of this as a map $\mathfrak{t}^n \to \mathbb{R}^d \simeq \mathfrak{t}^d$, so we get a $\mathfrak{t}^n$ action on $\mathbb{C}^d$. We'll do this next time.

LECTURE 20

Our basic symplectic manifold is $\mathbb{C}^d$. The idea of the Delzant construction is that we start with a delzant polytope Δ in $\mathfrak{t}^{n*}$, given by $\langle x, v_i \rangle \leq \lambda_i$ for some numbers λ_i, v_i. We use these v_is to construct $a : \mathfrak{t}^* \to \mathbb{R}^d$, letting the i-th component be $\langle x, v_i \rangle$. Then $\Delta = a^{-1}(\lambda + \mathscr{O}_-)$, where $\mathscr{O}_-$ is the negative orthant.

Figure 1

$\mathfrak{t}^d$ acts naturally on $\mathbb{C}^d$, and the momentum map is $J(z_1, \dots, z_n) = -\pi(|z_1|^2, \cdots, |z_n|^n) + \lambda$. Then the negative orthant is the image of the momentum map. Let $\mathfrak{n} \subseteq \mathfrak{t}^d$ be the elements of $\mathfrak{t}^d$ which annihilate $\mathfrak{n}^\perp$. Then $\mathfrak{n}^* \simeq \mathfrak{t}^{d*}/\mathfrak{n}^\perp$. This corresponds to some $N \subseteq \mathbb{T}^d$ (this follows from the Delzant condition on the polytope). Now we restrict from the $\mathbb{T}^d$ action on $\mathbb{C}^d$ to an N action. The momentum map $J_N \mathbb{C}^d \to \mathfrak{n}^* = \mathfrak{t}^{d*}/\mathfrak{n}^\perp$ is composition of J with the projection onto $\mathfrak{n}^*$. Next we form the reduced manifold $\mathbb{C}^d_0$, reduced by the N action. That is, we just take the part of $\mathbb{C}^d$ which maps to $\mathfrak{n}^\perp$ (this is compact because J is proper). This is $J_N^{-1}(0)$. The reduced manifold is $J_N^{-1}(0)/N$. It turns out that $J_N^{-1}(0)$ is a manifold, and that N acts freely on it, so our reduction is a smooth manifold. The entire torus still acts on this manifold (because the torus action commutes with the N action). Choose a complement $\mathfrak{m}^\perp \subseteq \mathfrak{t}^{d*}$ to $\mathfrak{n}^\perp$, to which there corresponds some $M \subseteq \mathbb{T}^d$. Thus, $\mathbb{T}^d = M^n \times N^{d-n}$. Now we look at the action of M on the reduced manifold. Notice that M is an n-dimensional torus, so we can identify $\mathfrak{m}^* \simeq \mathfrak{t}^{d*}/\mathfrak{m}^\perp$, which we may identify with $\mathfrak{n}^\perp$, which is identified with $\mathfrak{t}^{n*}$. Thus, we identify M with our original $\mathbb{T}^n$. What is J_M? There is a quotient map from $J_N^{-1}(0)$ to $J_N^{-1}(0)/N$. The image of the momentum map is our original polytope Δ.

You can often answer questions about polyhedra by realizing them as the images of momentum mappings.

Geometric Quantization. Souriau introduced this term. The idea was developed over some time by VanHove (who directly influenced Souriau), Segal, Kirillov, Kostant. There is a dictionary between classical and quantum mechanics

Classical	Quantum	What it is
symplectic manifold	hilbert space	"phase space"
symplectic mappings	unitary tranformations	time evolution
locally hamiltonian vector fields (functions modulo constants)	i·hermitian operators (i·hermitian modulo scalar)	infinitesimal time evolution
phase space T^*X	hilbert space $L^2(X)^{12}$	
M	$Quan(M)$	say you have such a procedure
$M \times N$	$Quan(M) \otimes Quan(N)$	
$T^*X \times T^*Y$ $= T^*(X \times Y)$	$L^2(X) \widehat{\otimes} L^2(Y)$ $= L^2(X \times Y)$	
$C^\infty(T^*X)$	Operators on $L^2(X)$	

Logically, you should be able to go from quantum to classical, and illogically, you should be able to go the other way. There are several ways to do this.

If $X = \mathbb{R}$, and T^*X has coordinates q, p, then $q \mapsto m_x$, multiplication by x (we will say $m_x = Quan(q)$), and $p \mapsto i\hbar\frac{\partial}{\partial x}$. What should qp go to? It should go to $i\hbar m_x D_x$, but pq "should" go to $i\hbar D_x m_x$. These are classically equal, but quantumly not (unless $\hbar \to 0$). But since we were thinking of these operators as the lie algebra of unitary operations, so multiplication is not the big operation ... the lie bracket is! So let's look at $m_x D_x - D_x m_x = -m_1$. Thus, we have

$$[Quan(q), Quan(p)] = -i\hbar m_1 = -i\hbar Quan(1) = -i\hbar Quan(\{q, p\})$$

which is great. It turns out that this is not exact either (this = poisson bracket yields lie bracket), but you can get it to come out right modulo higher order terms.

This is an infinitesimal version of saying that we'd like the group of symplectic mappings to go to the group of unitary transformations (as groups). But this doesn't work in general. You can make it work modulo higher order terms, or you can ask it to work for some subset of symplectic mappings.

There are some nice papers written in the last few years by N. Landsman (what should go on the RHS if you put the category where the morphisms are lagrangian submanifolds of products on the LHS?).

If M is symplectic. How do we assign a vector space to M in such a way that symplectic transformations correspond to unitary operators?

Try $C^\infty(M)$, or $L^2(M)$ with respect to the symplectic measure $\frac{\omega^n}{n!}$, where $\dim M = 2n$. These don't give you the right answer for $M = T^*\mathbb{R}^n$ ... it's too big! Another problem is when you try to assign operators to functions. Say $f \in C^\infty(M)$ gives you $i \cdot X_f$. Then if you take $f = 1$, you get the operator 0, not multiplication by 1, so this is no good. You might think you should just take $iX_f + m_f$, but this doesn't work either.

<u>Polarization</u>: takes care of the "too big" problem. Choose a Lagrangian foliation of M ... this is called a real polarization. Now instead of looking at all the functions on M, look only at functions constant on the leaves of the polarization. If you take T^*X with the polarization by fibers, then the space of leaves is X, so we are looking at functions on X. How should symplectic transformations act on this space? Symplectic transformation do not preserve the polarization in general. So lets only look at functions preserving the fibration. There are two sources of such things: diffeomorphisms of X, and 1-forms. $p \rightsquigarrow i\hbar X_p = i\hbar\frac{\partial}{\partial q}$, $q \rightsquigarrow i\hbar X_q = -i\hbar\frac{\partial}{\partial p} = 0$ since we are acting on things independent of p. phooey. We will use something like $i\hbar X_f + m_g$, but g cannot just be f, it has to depend on f in some more complicated way.

<u>Prequantization</u>: takes care of the problem with the constants. Think of q, p space. The problem with using just the hamiltonian vector fields was that the vector fields commute, and $\{q, p\} \neq 0$. So we add another dimension, which we call θ.

FIgure 2

We need a distribution in q, p, θ space. Consider the 1-form $i\hbar d\theta - pdq$. Now consider the horizontal lifts with respect to this form. $q \rightsquigarrow -\frac{\partial}{\partial p} \rightsquigarrow -\frac{\partial}{\partial p}$, but $p \rightsquigarrow \frac{\partial}{\partial q} \rightsquigarrow \frac{\partial}{\partial q} + \frac{}{i\hbar}p\frac{\partial}{\partial \theta}$. Now if we take the bracket of these two vector fields, we don't get zero any more. Let's look at functions of the form $f(q,p)\theta$. And let's introduce a polarization by requiring that f depends only on p, then q goes to $\frac{\partial}{\partial p}$, and p goes to something which gets rid of θ. So let's use $e^{i\theta}$ instead of θ. Now p gives us something closer to m_p.

LECTURE 21 - PREQUANTIZATION

$C^\infty(M)$. We are looking for a principal bundle $\mathbb{R}/2\pi\mathbb{Z} \cong U(1) \to Q \xrightarrow{p} M$. Let $\xi \in \chi(Q)$ generate the $U(1)$ action. Then the connection is a form $\phi \in \Omega^1(Q)$ such that $\phi(\xi) = 1$, $\mathscr{L}_\xi \phi = \xi \lrcorner d\phi = 0$. Then $\ker\phi$ is a horizontal distribution. $d\phi = p^*\omega$. That is, we want ω to be the curvature of this connection. Every $X \in \chi(M)$ has a unique horizontal lift $\hat{X} \in \Gamma(\ker\phi)$. This lifting is not a lie algebra homomorphism. In fact, $[\hat{X}, \hat{Y}] = \widehat{[X, Y]} + \phi([\hat{X}, \hat{Y}])\xi$. Recall that

$$d\phi(\hat{X}, \hat{Y}) = \hat{X}(\phi(\hat{Y})) - \hat{Y}(\phi(\hat{X})) - \phi([\hat{X}, \hat{Y}]).$$

Since these are horizontal, the first two terms are 0. so $\phi([\hat{X}, \hat{Y}]) = -d\phi(X, Y) = -p^*\omega([X, Y])$. This gives you the vertical component of the bracket of two horizontal vectors.

Given $f \in C^\infty(M)$, we attach to it X_f (an anti-homomorphism), then we take $Y_f := -\hat{X}_f + f\xi \in \chi(Q)$. This vector field has the property that it is the unique vector field whose horizontal component is $-\hat{X}_f$, and is a contact vector field. Let's check that

$$\begin{aligned}\mathscr{L}(-\hat{X}_f + g\xi)\phi &= d(-X_f + g\xi)\lrcorner\phi + (-\hat{X}_f + g\xi)\lrcorner d\phi \\ &= dg - \hat{X}_f \lrcorner d\phi \\ &= dg - p^*(X_f \lrcorner \omega) \\ &= dg - p^* df\end{aligned}$$

which implies that $g = p^* f + const$. Let's choose the constant to be zero.

Now we compute

$$\begin{aligned}[Y_f, y_g] &= [-\hat{X}_f + f\xi, -\hat{X}_g + g\xi] \\ &= [\hat{X}_f, \hat{X}_g] - [\hat{X}_f, g\xi] + [\hat{X}_g, f\xi] \\ &= [\widehat{X_f, X_g}] - \omega(X_f, X_g)\xi - (X_f \cdot g)\xi + (X_g \cdot f)\xi \\ &= -\widehat{X_{\{f,g\}}} + (-\{f,g\} - \{g,f\} + \{f,g\})\xi \\ &= Y_{\{f,g\}}.\end{aligned}$$

This correspondence ($f \rightsquigarrow Y_f$) is faithful ($Y_f = 0 \Rightarrow f = 0$). If we think of f as a function of q, p, θ. We can write $\phi = d\theta - p dq$ in local coordinates. Then $X_f = f_q \frac{\partial}{\partial p} - f_p \frac{\partial}{\partial q}$, so $\hat{X}_f = f_q \frac{\partial}{\partial p} - f_p \frac{\partial}{\partial q} + p f_p \frac{\partial}{\partial \theta}$, so

$$Y_f = -f_q \frac{\partial}{\partial p} + f_p \frac{\partial}{\partial q} - p f_p \frac{\partial}{\partial \theta} + f\xi.$$

So in particular, $Y_q = -\frac{\partial}{\partial p} + q\frac{\partial}{\partial \theta}$ and $Y_p = \frac{\partial}{\partial q}$. There is a geometric interpretation for this coordinate stuff. Consider the case where $M = T^*X$, and then $Q = T^*X \times U(1)$ with $\phi = d\theta - \alpha$. In this case, we find that for any $f \in C^\infty(X)$ (not M), $Y_f = -vert(df) + f\frac{\partial}{\partial \theta}$. On the other hand, if we have a vector field, ζ, which is linear on the fibers, then we can talk about $Y_\zeta = \frac{\partial}{\partial q}$, which is the cotangent lift of ζ.

Remember we are acting on functions of q, p, θ, which is too many variables. If we look at function independent of p, then Y_p acts by differentiation, which is good, and Y_q acts trivially. If we add a θ dependence, then we get multiplication by q, which is what we want. What is the right kind of θ dependence?

$U(1)$ acts on $C^\infty(Q)$. When you have a group acting on a vector space, you can break it up into irreducible representations. The representations of $U(1)$ are classified by the eigenvalues of the generator ξ. Typically, we have $f(e^{i\theta})$, and $\xi = \frac{\partial}{\partial \theta}$. The eigenvalues are $n \in \mathbb{Z}$. We can decompose $C^\infty(Q)$ into eigenspaces $\mathscr{H}_n$ of ξ. In particular, $\mathscr{H}_0 = p^* C^\infty(M)$. On $\mathscr{H}_n$, we find that "$\frac{\partial}{\partial \theta} = in$". Let $n = -1$. Then $Y_q = -\frac{\partial}{\partial p} - iq$, $Y_p = \frac{\partial}{\partial q}$. These are vector fields on a manifold, which have unitary flows, whose derivatives are skew. To get hermitian operators, look at $iY_q = -\frac{\partial}{\partial p} + q$, $iY_p = i\frac{\partial}{\partial q}$. If we now let these act on functions independent of p, then the operator corresponding to q is multiplication by q and that of p is $i\frac{\partial}{\partial q}$, just like we wanted. This works whenever we have a circle bundle.

In the special case where $Q = M \times U(1)$, then $u \in C^\infty(Q)$ can be written as $\sum a_n(x) e^{in\theta}$, where the a_n are the fourier coefficients. Then $\xi(u) =$

$\sum ina_n(x)e^{in\theta}$. For $u \in \mathscr{H}_n$, we have $u = a_n(x)e^{in\theta}$. For $n = -1$, $u = a_{-1}e^{-i\theta}$, which is just a function of x. This is in the case of the trivial bundle. In general, we don't have a trivial bundle. $\mathscr{H}_n$ is some vector space, but there is more structure. Locally, an element is just a complex function on the base. If we use pointwise multiplication, $\mathscr{H}_n\mathscr{H}_m \subseteq \mathscr{H}_{n+m}$ because ξ is a derivation. In particular, $C^\infty(M) \cong \mathscr{H}_0$, and $\mathscr{H}_n$ is a module over it ... it is a locally free module of rank 1. So $\mathscr{H}_n$ is a line bundle. We can identify the fibres. We have $Q \xrightarrow{p} M$. For $x \in M$, we define $\mathscr{H}_x^n$[13] to be $\{u \in C^\infty(p^{-1}\{x\}) | \xi u = inu\}$. This is a complex line, so $E^n := \bigcup H_x^n$ is a complex line bundle, so $\mathscr{H}_n = \Gamma(E^n)$. We can also say that $E^n \cong (E^1)^{\otimes n}$, where a negative tensor power is a tensor power of the dual bundle. The conclusion is that if we have a circle bundle over Q whose curvature is ω, we can construct an action of $C^\infty(M)$ such that poisson brackets go to commutator brackets. If we think of them as sections of a complex line bundle, what are these operators? We have vector fields on the base, and a connection gives us a covariant derivative. You can read Kostant's article in Lec. Notes in Math. 170.

How do we know there is such a Q? We know that we can take the trivial bundle in the case of a cotangent bundle (or whenever ω exact). What if ω is not d of some 1-form, as in the case of a compact symplectic manifold.

Theorem 21.1 (A. Weil)**.** *$\omega \in \Omega^2(M)$ is the curvature of a $U(1)$-bundle over M if and only if $[\omega] \in Im(H^2(M,\mathbb{Z}) \to H^2(M,\mathbb{R}))$ (where this is the map corresponding to coefficient homomorphism $\mathbb{Z} \xrightarrow{2\pi} \mathbb{R}$) if and only if $\int_\sigma \omega \in 2\pi\mathbb{Z}$ for every 2-cycle σ on M.*

This is often called the integrality condition. In this context, it is also called the prequantization condition. If we take the standard 2-sphere in $\mathbb{R}^3$, with the usual area element, it is prequantizable. In $\mathfrak{g}^* = \mathfrak{su}(2)^*$, we have $\{x,y\} = z, \{y,z\} = x, \{z,x\} = y$, and the levels of $x^2+y^2+z^2$ are the symplectic leaves ... coadjoint orbits. Then the symplectic area of the sphere is equal to $4\pi r$. The quantizable ones are those for which $r \in \frac{1}{2}\mathbb{Z}$ (including $r = 0$). This corresponds to spin. The representations of $SU(2)$ are indexed by this spin. But $\mathfrak{su}(2) = \mathfrak{so}(3)$, but the reps of $SO(3)$ require spin to be a whole integer. In general, for complact simply-connected groups, there is one to one correspondence between irreducible representations and quantizable coadjoint orbits (Borel-Weil Theorem). This is also true for nilpotent groups (Kirillov).

$0 \to \mathbb{Z} \xrightarrow{2\pi[i]} [i]\mathbb{R} \to [i]\mathbb{R}/2\pi[i]\mathbb{Z} \to 0$. Some people put the i because then the quotient is more directly isomorphic to $U(1)$. There is a corresponding long exact sequence in cohomology (in the same direction (these are the coefficients))

$$\cdots \to H^1(M,U(1)) \to \underbrace{H^2(M,\mathbb{Z}) \xrightarrow{2\pi} H^2(M,\mathbb{R})} \to H^2(M,U(1)) \to H^3(M,\mathbb{Z}) \to \cdots$$

---

[13] $\mathscr{H}^n = \mathscr{H}_n$.

The condition for prequantizability is that $[\omega]$ lies in the image of $H^2(M,\mathbb{Z})$. It turns out that $H^2(M,\mathbb{Z})$ classifies [hermitian, unitary] complex line bundles up to isomorphism. The element of $H^2(M,\mathbb{Z})$ corresponding to the bundle is called the chern class c_1. The kernel of the underbraced map are the complex line bundles whose churn classes go to the symplectic class. There is a lack of uniqueness if the previous map is non-trivial. If there is a connection whose cohomology is zero, then there is a connection which is flat (curvature is identically zero). Remember that there is a group structure on complex line bundles (given by tensor product), which makes the map to $H^2(M,\mathbb{Z})$ a group homomorphism. The point is that if you have two line bundles with connections $\mathit{curv}((E_1,\ \mathit{phi}_1)\otimes(E_2,\ \mathit{phi}_2)) = \mathit{curv}(E_1,\phi_1)+\mathit{curv}(E_2,\phi_2)$. So there is another group here, the one of hermitian line bundles with connection.

$$\begin{tikzcd}
 & & \text{bundles with connection} \arrow[dl, "\text{forget connection}"'] \arrow[dr, "\text{curvature}"] & & \\
 & \text{line bundles} \arrow[d, "\text{chern class}"'] & & Z^2(M) \arrow[d, "{[\ ]}"] & \\
\cdots \arrow[r] & H^2(M,\mathbb{Z}) \arrow[rr, "2\pi"] & & H^2(M,\mathbb{R}) \arrow[r] & \cdots
\end{tikzcd}$$

Lecture 22 - Geometric Quantization

The first draft of the paper is due next Tuesday!

We have (M,ω) a symplectic manifold, and a prequantization is a circle bundle (Q,ϕ) over M whose curvature is ω. A necessary condition is that $[\omega]$"$\in$"$H^2(M,2\pi\mathbb{Z})$. If it exists, it is unique up to isomorphism and tensor product with a flat $U(1)$ bundle-with-connection. These correspond to elements of$H^1(M,U(1)) = \mathrm{Hom}(H_1(M,\mathbb{Z}),U(1)) = \mathrm{Hom}(\pi_1(M),u(1))$. If you look at the set of all such pre-quantization bundle, it is a torus (at least if $H_1(M,\mathbb{Z})$ is free, otherwise a torus times finite cyclic group).

Once we choose such a thing, we get $\mathscr{H}^n \subseteq C^\infty(M)$ which is the $\sqrt{-1}n$ eigenspace of ξ (the generator of the $U(1)$ action). There is a sense in which $n\to\infty$ is like $\hbar\to 0$. If we take $\mathscr{H}^n_{(M,\omega)} \simeq \mathscr{H}^1_{(M,n\omega)}$. Recall that $\mathscr{H}^1_{(M,\omega)} \simeq \Gamma(E)$, where E is a complex line bundle. $\mathscr{H}^n_{(M,\omega)} \simeq \Gamma(E^{\otimes n})$. We know that the chern class of $E^{\otimes n}$ is n times that of E, so it is like taking $n\omega$ in place of ω. ω has units of action, so you have to divide by the unit of action, which is $\hbar$. Working in Q, we can understand what happens as $n\to\infty$.

Recall also that given $f\in C^\infty(M)$, we get $Y^n_f = -\hat{X}_f + inf$ (or $-\hat{X}_f + f\frac{\partial}{\partial\theta}$ as a vector field). Notice that in the classical limit ($n\to\infty$), it looks more and more like pointwise multiplication (Y^n_f looks like f).

Quantomorphisms: A quantomorphism is an automorphism of (Q,ϕ). Note that anything preserving ϕ preserves ξ. $\mathit{Quant}(Q,\phi)\to \mathit{Symp}(M,\omega)$ because ω can be gotten from ϕ. What are the quantomorphisms which give you the identity? It is just the $U(1)$ action (all the fibres have to rotate with the same speed). You can check that the Y_f are all the quantomorphisms. You can check this by breaking it up into vertical and horizontal parts.

$$1 \to U(1) \to Quant(Q, \phi) \to Symp(M, \omega) \xrightarrow{?}$$

On the lie algebra level, $Quant(Q, \phi)$ is $C^\infty(M)$. In $Symp(M, \omega)$, we have $\chi(M, \omega)$, the symplectic vector fields. In $U(1)$, we have $H^0(M, \mathbb{R})$ $(\simeq \mathbb{R})$.

$$0 \to H^0(M, \mathbb{R}) \to C^\infty(M) \to \chi(M, \omega) \to H^1(M, \mathbb{R}) \to 0.$$

If we have a group $G \hookrightarrow Symp(M, \omega)$, and we want to lift it to a group of quantomorphism, this is related to lifting the map $\mathfrak{g} \to \chi(M, \omega)$ to $\mathfrak{g} \to C^\infty(M)$, which is the story of momentum maps. If we can lift G, then $\mathfrak{g}$ has to lift, so we have to have a hamiltonian action. If we have a hamiltonian action, then we can lift $\mathfrak{g}$, then does it follow that G lifts? If we have a homomorphism of lie algebras and we want to lift it to a homomorphism of lie groups, we can do it locally. We can only do it globally if G is simply connected. The problem is that if you have a closed 1-parameter subgroup, when you lift it, it might not close up.

Example: $M = S^2$, with the circle action (by rotation about the vertical axis). We use coordinates (I, ψ) on the sphere (I is height). Then $\omega = d\psi \wedge dI$. If $-\sigma \le I \le \sigma$, then the total (symplectic) area is $4\pi\sigma$. So the is prequantizable if and only if $\sigma \in \frac{1}{2}\mathbb{Z}$ (because the cohomology class has to be in $2\pi\mathbb{Z}$). The quantization is unique because it is simply connected (the quantization is the hopf fibration). $SU(2)$ and $SO(3)$ act on S^2 is the usual way. Does the action lift to the quantization? Let's look at rotations around the z-axis. This rotation is generated by the function I and has period 2π. What happens when we try to lift the action? We have to look at the vector field $Y_I = -\hat{X}_I + I\frac{\partial}{\partial\theta}$. Above a critical point of I, Y_I is just $I\frac{\partial}{\partial\theta}$. Let N and S be the north and south poles. Over N, we have that $Y_I = \sigma\frac{\partial}{\partial\theta}$ and over S, $Y_I = -\sigma\frac{\partial}{\partial\theta}$. So the period of the action is $2\pi/\sigma$ upstairs. Remember that the action on the two-sphere has period 2π, so we must require that 2π is an integer multiple of $2\pi/\sigma$, so only the integer σs allow the action of $SO(3)$ to lift. This exactly corresponds to the fact that the irreducible representations of $SU(2)$ are parameterized by spin, which can be half integer, but only the integer spins give irreducible representations of $SO(3)$. If $\sigma = 1/2 + k$, then when we go half way around $SU(2)$, we go all the way around S^2, but only half way around upstairs. If you let it act on the space $\mathscr{H}^{\pm 1}$, you get multiplication by -1. We have $SU(2)$ acting on its coadjoint orbits, we have an infinite dimensional representation, which is not irreducible. We cut it down to size by polarization.

Polarization. Integrable Lagrangian sub-bundle $F \subseteq TM$. The leaves gives us a foliation, which we sometimes call the polarization. The example above is not quite a foliation because we have singularities at the two poles. Let's say we work away from this badness. We have (Q, ϕ) sitting over M (via $p : Q \to M$). If we restrict Q to one of the leaves $\mathscr{L}$ of our foliation, we have $p^{-1}\mathscr{L}$. The curvature is given by the pullback of ω, but since $\mathscr{L}$ is lagrangian, this is zero. So Q is flat on the leaves of the foliation. $\ker\phi$ is a contacts structure, and when you restrict to the leaves, you have a closed 1-form. Over each $\mathscr{L}$, there is the holonomy map $\pi_1\mathscr{L} \to U(1)$, which is in $H^1(\mathscr{L}, U(1))$.

Let's look at $\mathscr{H}^{-1}_{(M,\omega)}$ "$\supseteq$" $\mathscr{H}^{-1}_{(M,\omega,F)}$ = functions constant along the leaves of the lifted foliation given by $\hat{F}$. When do we have such a function? When we move along a fibre, it is just some value times $e^{i\theta}$. If the holonomy is non-trivial, something would have to be zero. So such functions are supported on leaves with trivial holonomy. This set of leaves is called the "Bohr-Sommerfeld set" $\mathcal{BS}$. If the leaves are simply connected, then great ... we don't have to worry about this condition.

Notice that if you fix a leaf which is in the Bohr-Sommerfeld set, and you look at the functions supported on that leaf which are in $\mathscr{H}^{-1}$, and which are constant along leaves, then the function is determined by one value in the leaf. We have that $\mathcal{BS}(F)/F \subseteq M/F$ = the leaf space of F. Upstairs, we have $p^{-1}(\mathcal{BS}(F))/\hat{F}$, with kernel $U(1)$ (we can identify the different circles by parallel translation). We've reduced ... instead of looking at all functions on Q, we restrict to $\mathcal{BS}$, and we are constant on the leaves.

In our example, what is $\mathcal{BS}$? We want to know the holonomy of a leaf. If $\omega = -d\alpha$ in some area of your manifold, and assume M is simply connected (for simplicity), then we can take Q to be $M \times U(1)$ with $\phi = d\theta - \alpha$. If we take a loop in M, we can define the holonomy, but it wont be homotopy invaiant. The holonomy of some loop γ is just $\int_\gamma \alpha$. When we lift a loop, $\phi = 0$, so $d\theta = \alpha$. Then to find the holonomy, we integrate $d\theta = \alpha$. If γ bounds a surface Σ, then this is $\int_\Sigma \omega$ by Stokes theorem. Notice that this is constant with other things. If we have two different surfaces bounded by γ, then the difference is required to be a multiply of 2π. We should really take the holonomy to be $e^{\int_\gamma \alpha}$. The $\mathcal{BS}$ set is the set where the holonomy is trivial (a multiple of 2π). So $\mathcal{BS}$ consists of the leaves at height $\sigma, \sigma - 1, \ldots, 1 - \sigma, -\sigma$. So there are $2\sigma + 1$ leaves (if you allow us to count the poles as leaves. Over each of these $2\sigma + 1$ points, we have a complex line. Something is sections of this bundle. So we get a $2\sigma + 1$ dimensional space. How does the circle group act on this space? Take the generator $\frac{\partial}{\partial\psi}$. Take an element which is 1 on the leaf corresponding to s for $-\sigma \leq s \leq \sigma$ an integer. Something in this space is killed by the horizontal distribution, so ξ acts by $-\sqrt{-1}s$. This operator of lifting is called spin along the z-axis. In the language of representation theory, the circles are maximal tori, and the $-\sqrt{-1}s$ are the weights.

What if we change the foliation ... do we get the same representation or not? Let's use the foliation with respect to the east pole and the west pole. What can we say about spin around the z-axis? It doesn't leave the polarization invariant. Geometric quantization works very nicely to build representations of groups which leave a polarization invariant. In this case, the action descends to an action on $\mathcal{BS}$. But in most examples, the groups don't leave the polarization invariant. Two ways out. First way is to deal with non-invariant polarizations ... this is called the Blattner-Kostant-Sternberg pairing or the projection method or Toeplitz quantization. The other approach is to widen your notion of what a polarization is. There is no polarization invariant under the action of $SO(3)$, so we allow complex polarizations (a complex structure compatible with the symplectic structure ... a Kähler structure).

Example: $M = T^*X$ with $Q = T^*X \times U(1)$, $\phi = -\alpha + d\theta$, where α is the liouville form, F is the foliation by the fibres. In this case, everything is trivial, and you find that the fibres are simply connected so $\mathcal{BS}$ can be identified with the base manifold, so we just have functions on the base X. The functions preserving the fibration are the ones which are affine on the fibres.

LECTURE 23 - COMPACT POLARIZATIONS

(M, ω, Q, ϕ) as usual ... Q with connection ϕ is a prequantization. $\mathscr{H}_{-1} =$ "antiequivariant functions from Q to $\mathbb{C}$" are the (-1)-eigenvalues. Think of Q as a bundle of frames for a complex line bundle $E \to M$. $Q \times \mathbb{C} \ni (q, z) \mapsto qz \in E$. Note that $(qe^{i\theta}, e^{-i\theta}z)$ goes to the same element, so we can identify E with $Q \times \mathbb{C}/U(1)$, where $U(1)$ acts on Q by the opposite of the given action and on $\mathbb{C}$ by the standard representation. So given Q, we can recover E. The fibre of E over x is the set of anti-equivariant functions of Q over x to $\mathbb{C}$. So sections of E are just elements of $\mathscr{H}_{-1}$.

A connection ϕ gives a "horizontal distribution" $H = \ker \phi$ on Q. A polarization is an integrable lagrangian distribution F on M. Its horizontal lift $F \subseteq TQ$ is integrable because F is lagrangian (the curvature is the symplectic structure, which is zero on a lagrangian ... vertical component of bracket is curvature). We cut down $\mathscr{H}_{-1}$ by considering function constant on leaves, supported on the $\mathcal{BS}$ set.

Complex Polarizations: $F \subseteq TM \rightsquigarrow F_{\mathbb{C}} \subseteq T_{\mathbb{C}}M$. A complex polarization on (M, ω) is an integrable lagrangian sub-bundle G of $T_{\mathbb{C}}M$ of constant dimension such that $G \oplus \bar{G}$ is also integrable.

There are two extreme cases. First is where $G = \bar{G}$, in which case $G = F_{\mathbb{C}}$ for some real polarization F (we call G a real polarization). The other is where $G + \bar{G} = T_{\mathbb{C}}M$, in which case G is a complex structure. We call this a totally complex polarization. Together with ω, this gives a pseudo-Kähler structure (the inner product may not be positive definite). In the general case, we have that $G \cap \bar{G}$ has constant dimension and is equal to its own conjugate, so $G \cap \bar{G} = \mathscr{F}_0^{\mathbb{C}}$ isotropic. So you get an isotropic foliation and a complex structure on the normal spaces to the foliation. So there is an induced complex structure on a submanifold transverse to the foliation in such a way that sliding along the leaves is a complex map. We call this a "transversely complex isotropic foliation".

We can cut down $\mathscr{H}_{-1}$ using a complex polarization. Given a complex polarization G. We have $Q \to M$ which induces $H \subseteq TQ \to TM$, so $H_{\mathbb{C}} \subseteq T_{\mathbb{C}}Q \to T_{\mathbb{C}}M \supseteq G$. G lifts to $\tilde{G} \subseteq H_{\mathbb{C}}$, which is integrable by the lagrangian condition. If G is real, then $\tilde{G}$ is the complexification of $\tilde{F}$. In the case where G is a totally complex polarization ... . Reduce $\mathscr{H}_n$ to $\mathscr{H}_{n,G} = \{$function in $\mathscr{H}_n$ annihilated by sections of $\tilde{G}\}$. We have that $\mathscr{H}_{n,G} \times \mathscr{H}_{m,G} \to \mathscr{H}_{n+m,G}$. $\mathscr{H}_{0,G} =$ functions on M annihilated by G. In the totally complex case, $\mathscr{H}_{0,G} =$ holomorphic functions on M. $\mathscr{H}_{n,G}$ is a module over $\mathscr{H}_{0,G}$, the holomorphic functions on M. If M is compact, there are no non-constant holomorphic functions, so instead of looking at global sections, we look at the sheaf of local section. So think of all of these $\mathscr{H}_{n,G}$s as sheaves. These are locally free modules. In particular, $\mathscr{H}_{-1,G}$ is a module over $\mathscr{H}_{0,G}$.

We don't know that the $\mathscr{H}_{n,G}$ are non-empty, so you prove

Lemma 23.1. *$\mathscr{H}_{n,G}$ has local sections.*

The proof involves solving a non-homogeneous $\bar{\partial}$ problem: $\bar{\partial} f = \alpha$, where α is a 1-form with $\bar{\partial}\alpha = 0$. You can solve this by the Dolbeault lemma.

So the $\mathscr{H}_{n,G}$ define holomorphic vector bundles. So attached to any complex polarization, we get a complex structure on the line bundle E and all its tensor powers.

The idea of sheaves is useful in the real case when then leaves might be dense or because the $\mathcal{BS}$ set is discrete. The idea here is "cohomological quantization": $\tilde{\mathscr{H}}_{n,G}$ be the sheaf (over M) of local elements of $\mathscr{H}_{n,G}$. Then $\mathscr{H}_{n,G} = H^0(\tilde{\mathscr{H}}_{n,G})$. There is also higher cohomology ... $H^k(\tilde{\mathscr{H}}_{n,G})$, which is also invariant under the action of the group. In the pseudo-Kähler case, H^0 may be zero, but higher cohomology is interesting. If you take the 2-sphere with the poles removed. The global sections are only supported on the $\mathcal{BS}$ set, but there are local sections elsewhere because the leaves are locally simply connected.

Let's look at the 2-sphere again, with area 2π. The circle bundle is $S^3 \subseteq C^2$. The circle action is the opposite of the usual action (i.e. $\theta \cdot (z_1, z_2) = (e^{-i\theta} z_1, e^{-i\theta} z_2)$). The corresponding line bundle E is the dual of the $\mathscr{H}_{-1}$ tautological bundle (over a point in $S^2 = \mathbb{CP}^1$, you put the line associated to that point). A section of E corresponds to a function on S^3 which is anti-equivariant with respect to the reversed circle action, so it is $U(1)$-equivariant. This correspond to functions $\mathbb{C}^2 \to \mathbb{C}$ which are linear. In general, sections of $E^{\otimes k}$, $\mathscr{H}_{-k}$ are functions $\mathbb{C}^2 \to \mathbb{C}$, homogeneous of degree k.

We haven't used the polarization yet. The polarized sections correspond to holomorphic functions, which must then be polynomials, homogeneous of degree k on $\mathbb{C}^2$ (note that k must be positive). On S^2, $\mathscr{H}_{-n,G}$ is spanned by $z_1^n, z_1^{n-1} z_2, \dots, z_2^n$, of which there are $n+1$. This is the representation spin$(\frac{1}{2}n)$. For $n = 0$, you get constants; for $n = 1$, you get a 2-dimensional space of linear functions, which is the usual representation of $SU(2)$ on $\mathbb{C}^2$.

Suppose we look at rotations around an axis. Do we get the correct eigenvalues? Take the maximal torus in $SU(2)$, given by things like $\left(\begin{smallmatrix} e^{i\psi} & 0 \\ 0 & e^{-i\psi} \end{smallmatrix}\right)$, which acts on $z_1^k z_2^{n-k}$ by sending it to $e^{ik\psi} e^{-i(n-k)\psi} z_1^k z_2^{n-k} = e^{i(2k-n)\psi} z_1^k z_2^{n-k}$. The eigenvalue is $e^{i(2k-n)\psi}$, so $2k - n$ goes from $-n$ to n in steps of 2, just as they should.

Where is the square norm of the section $z_1^k z_2^{n-k}$ going to be largest on the 2-sphere? Use lagrange multipliers: $I_1 = |z_1|^2$, $I_2 = |z_2|^2$, $I_1 + I_2 = 1$, and we want to maximize $I_1^k I_2^{n-k}$. You get $kI_2 = (n-k)I_1$, so $I_1 = \frac{n-k}{n}$, $I_2 = \frac{k}{n}$. So the maxima are equally spaced along the I axis (which is the vertical axis). This is the set of integer points on the moment polytope of the action of $SU(2)$. They correspond exactly to representations.

Guillemin-Sternberg: something with "geometric quantization" in the title.

LECTURE 24

Papers distributed for refereeing.

N. Woodhouse, *Geometric Quantization* (2nd edition), Oxford University Press.

We have (M, ω, Q, ϕ), from which we get $\mathscr{H}_{-1}$, and we choose a polarization $F \subseteq T_{\mathbb{C}}M$. Then we get $\mathscr{H}_{n,F}$, which are F-parallel sections of $E^{\otimes -n} = (E^*)^{\otimes n}$.

Example: F is a Kähler structure on (M, ω), then the $\mathscr{H}_{n,F}$ are the holomorphic sections of $E^{\otimes -n}$.

Theorem 24.1 (Riemann-Roch)**.** *If (M, ω) is compact Kähler, then*

$$\sum (-1)^k \dim H^k(E^{\otimes n}) = \chi_n$$

is a polynomial in n of degree $\frac{1}{2}\dim M$, with leading term $n^{\frac{1}{2}\dim M} \cdot \int_M \frac{\omega^{\dim M/2}}{n!} = \int_M \frac{(n\omega)^{\dim M/2}}{n!}$.

Theorem 24.2 (Kodaira Vanishing Theorem)**.** *For $n \gg 0$, $H^{>0}(E^{\otimes n}) = 0$.*

Let $M = T^*X$, with $F =$ foliation by fibres, $Q = M \times U(1)$, $\phi = d\theta - \alpha$. Then $\mathscr{H}_{-1,F}$ is the set of functions constant along fibres, so it is just functions on X. Diffeomorphisms act on $\mathscr{H}_{-1}$ by pullback. If $S \in C^\infty(X)$, then dS acts on $\mathscr{H}_{-1}$ by multiplication by e^{iS}.

If V a finite dimensional vector space over $\mathbb{R}$, then $\wedge^{top}V^*$ is the set of maps (frames of V) $\xrightarrow{\sigma} \mathbb{R}$ such that $\sigma((e_1, \dots, e_n) \cdot A) = \sigma(e_1, \dots, e_n) \cdot \det A$. Look instead at things which transforms as $\sigma((e_1, \dots, e_n) \cdot A) = \sigma(e_1, \dots, e_n) \cdot |\det A|$, which you can integrate without an orientation ... call these things *densities*, denoted $|V|$. Instead of $|\det A|$, you can use $|\det A|^\alpha$ ... such things are α-densities. Then $|V|^\alpha \otimes |V|^\beta \xrightarrow{\sim} |V|^{\alpha+\beta}$. In particular, half-densities are the things we should take to form an L^2 space. On a manifold M, the compactly supported sections of $|TM|^{\frac{1}{2}}$ form a natural inner product space, where the inner product of r and s is $\langle r, s\rangle = \int_M rs$. If instead of maps to $\mathbb{R}$, you take maps to $\mathbb{C}$, they form a pre-hilbert space, not just an inner product space, with $\langle r, s\rangle = \int_M r \cdot \bar{s}$.

In local coordinates, $dx^1 \wedge \cdots \wedge dx^n$ basis for $\wedge^{top}TX$, so $|dx^1 \wedge \cdots \wedge dx^n|^\alpha$ basis for $|TX|^\alpha$. So a typical α-density is $a(x)|dx^1 \cdots dx^n|^\alpha$. If you like, you can complete our pre-hilbert space to a hilbert space.

We can relate the spaces attached to different polarizations. Take $M = \mathbb{R}^2$ with coordinates q, p, $\alpha = pdq$, $\omega = dq \wedge dp$. $M \times U(1)$, with connection $\phi = d\theta - pdq$. $F_q = \langle \frac{\partial}{\partial p}\rangle$. $\tilde{F}_q = \langle \frac{\partial}{\partial p}\rangle$. $a(q,p)e^{-i\theta}$ is an element of $\mathscr{H}_{-1}$. If it is in $\mathscr{H}_{-1,F_q}$, it must be of the form $a(q)e^{-i\theta}$. To make them half-densities, we take things of the form $a(q)e^{-i\theta}\sqrt{|dq|}$. Now take another polarization

$F_p = \langle \frac{\partial}{\partial q} \rangle$, so $\tilde{F}_p = \langle \frac{\partial}{\partial q} + p\frac{\partial}{\partial \theta} \rangle$. In order for $b(q,p)e^{-i\theta} \in \mathscr{H}_{-1,F_p}$, we must have that

$$0 = b_q(q,p)e^{-i\theta} - ipb(q,p)e^{-i\theta}.$$

So $b_q(q,p) = ipb(q,p)$, so $b = b(p)e^{ipq}$. Thus, $\mathscr{H}_{-1,F_q} = \{b(p)e^{ipq}e^{-i\theta}\sqrt{|dp|}\}$.

There is a pairing, the Blattner-Kostant-Sternberg (BKS) pairing, which multiplies things in these two spaces: $\mathscr{H}_{F_q} \otimes \mathscr{H}_{F_p} \to \mathbb{C}$, given by $\langle A, B \rangle = \int_M A\bar{B}\sqrt{\mathit{Liouville}} = \int_{\mathbb{R}^2} a(q)\bar{b}(p)e^{-ipq}\underbrace{\sqrt{dq}\sqrt{dp}\sqrt{dqdp}}_{dq\,dp}$. Here the pairing is conjugate linear. We can rewrite the inner product as $\langle \beta(A), B \rangle_{\mathscr{H}_{F_p}}$ since conjugate linear functionals are just inner product with something. It is clear that $\beta(A) = (\int_{\mathbb{R}^2} a(q)e^{-ipq}dq)\sqrt{|dp|}$. This is the Fourier transform (there should probably be a 2π somewhere in there).

So we have that the Fourier transform comes from this symplectic construction. It also points out that if you take functions on V, you get measures on V^*, and vice versa. If you want to get the same kind of object, you should take half-densities!

$2\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial q} + i\frac{\partial}{\partial p}$, which we will call F_z. $\phi = d\theta - \frac{1}{2}(pdq - qdp)$. If you look at a general function $C(q,p)e^{-i\theta}$, it belongs to $\mathscr{H}_{-1,F_z}$ if and only if $C = e^{-\frac{1}{4}z\bar{z}}c(z)\sqrt{|dz|}$, where $c(z)$ is holomorphic. Then the inner product is $\int_{\mathbb{C}=\mathbb{R}^2} c_1(z)e^{-\frac{1}{4}z\bar{z}} \cdot \bar{c}(z)e^{-\frac{1}{4}z\bar{z}}(\mathit{Liouville\ measure}) = \int_{\mathbb{R}^2} c_1\bar{c}_2e^{-\frac{1}{2}z\bar{z}}|dq\,dp|$. This is called the Bargmann space or the Fock space.

LECTURE 25 - GEOMETRIC QUANTIZATION TO DEFORMATION QUANTIZATION

Geometric quantization: $(M,\omega) \rightsquigarrow \mathscr{H}$ so that $C^\infty(M) \rightsquigarrow$ operators on $\mathscr{H}$. Then $\{,\} \rightsquigarrow [,]$, and hamilton's equations $\frac{df}{dt} = \{f, H\}$ become $\frac{dA}{dt} = i\hbar[A, \hat{H}]$ for some operator A. If you can get at this algebra, then you're good ... the $\mathscr{H}$ is a representation of the algebra.

Deformation quantization: forget about $\mathscr{H}$ and just try to get the algebra. Start with $C^\infty(M[,\mathbb{C}]) = A_0$, and embed it into a family of algebras $A_\hbar$. Let's assume that the $A_\hbar$ are all the same (as vector spaces). So on some vector space, we have a family $*_\hbar$ of associative products. We will assume that $*_0$ is commutative. Then

$$0 = \frac{d}{d\hbar}\Big|_0 [(f *_\hbar g) *_\hbar h - f *_\hbar (g *_\hbar h)]$$

which gives you a condition on $\frac{d}{d\hbar}\big|_0 (f *_\hbar g) = B_1(f,g)$.

$$f *_\hbar g = f *_0 g + \hbar B_1(f,g) + O(\hbar^2)$$

we will write fg for $B_0(f,g)$, and assume it is commutative. We have

$$\delta B_1(f,g,h) = B_1(f,g)h + B_1(fg,h) - B_1(f,gh) - fB_1(g,h) = 0$$

which is part of the Hochschild complex. Look at $B_1(f,g) - B_1(g,f)$ ... it turns out to be a bi-derivation, which we will call $\{,\}$. If

$$f *_\hbar g = f *_0 g + \hbar B_1(f,g) + \hbar^2 B_2(f,g) + O(\hbar^3)$$

for some B_2, then $\{,\}$ satisfies Jacobi identity. $\{f,g\}$ is the first order part of the commutator of f and g, and gives a Poisson structure.

If you start with a Poisson manifold, does there exist a deformation such that the above construction gives the given poisson bracket? It is hard to define these $*_\hbar$ for a given $\hbar$. Given $B_0, B_1, \cdots : A \times A \to A$, where B_0 is multiplication, they define a bilinear product on $A[[\hbar]]$ given by

$$a *_\hbar b = \sum_{j=0}^{\infty} B_j(a,b)$$

where the B_j are extended to $A[[\hbar]]$ by $\mathbb{C}[[\hbar]]$-linearity.

Today we'll talk about how on a Kähler manifold, you can go from a geometric quantization to a deformation quantization; this is sometimes called Berezin-Toeplitz quantization, but the people who did the first work were Boutet de Monvel and Guillemin.

(M, ω, Q, ϕ, F) with M compact, from which we get a vector bundle E, with $\Gamma(E) \simeq \mathscr{H}_{-1}$, and $\Gamma_{hol}(E) \simeq \mathscr{H}_{-1,F} \subseteq \mathscr{H}_{-1}$, and $\mathscr{E}^n = \Gamma_{hol}(E^{\otimes n}) \simeq \mathscr{H}_{-n,F} \subseteq \mathscr{H}_{-n}$. The Riemann-Roch theorem tells us about what happens as n gets big. Let $\mathscr{H}_{-n} \xrightarrow{\pi_n} \mathscr{H}_{-n,F}$ be the projection. For $f \in C^\infty(M)$, define the nth Toeplitz operator $T_f^{(n)} = \pi_n M_f \pi_n : \mathscr{E}^n \to \mathscr{E}^n$.

Theorem 25.1 (Boutet de Monvel-Guillemin, Bordemann-Meinrenken-Schlichenmaier,...)**.** *There are (unique!) bidifferential operators* $B_j : C^\infty M \times C^\infty M \to C^\infty M$ *such that, if we set* $f *_{[I]} g = \sum_{i=0}^{I} \frac{1}{n^i} B_i(f,g)$, *then* $||T^{(n)}_{f*_{[I]}g} - T_f^{(n)} T_g^{(n)}|| \leq C_I \frac{1}{n^{I+1}}$. *So as* $I \to \infty$, $T^{(n)}$ *gets closer and closer to a homomorphism. In particular,* $B_0(f,g) = fg$, $B_1(f,g) - B_1(g,f) = i\{f,g\}$.

$\{f,g\} = \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}$, we may write $P = \frac{\partial}{\partial q^i}\frac{\partial}{\partial p_i} - \frac{\partial}{\partial p_i}\frac{\partial}{\partial q^i}$, with the operators acting in the right direction, then we have fPg.

It follows from the theorem that the B_i form a deformation quantization. The theorem tells you about the existence of a deformation quantization, but it also gives you information about the holomorphic sections. All the proofs use the picture of a tower of line bundles. The $T_f^{(n)}$ act on different spaces, so that's annoying, but all the $\mathscr{H}_{-n} \subseteq L^2(Q)$, so you can look at the direct sum of all the spaces in $L^2(Q)$.

In the simplest example, $M = \mathbb{P}^1 = S^2$, then $Q = S^3$. Then the $\mathscr{H}_{-n}$ are degree n homogeneous polynomials, so when you take the direct sum and take the appropriate closure, you get holomorphic functions on the 4-disk. You can look at those functions which are in $L^2[?]$ on S^3.

What do the operators B_i look like? In local coordinates, it's too complicated to write the formula. Let's assume that there is a piece of the manifold which is flat. Then in local coordinates in a flat place,

$$B_1(f,g) = \frac{1}{i}(\underbrace{\frac{\partial}{\partial z^j}_{\leftarrow}\frac{\partial}{\partial \bar{z}^j}_{\rightarrow}}_{\Phi})g$$

$$\begin{aligned}\frac{\partial f}{\partial z}\frac{\partial g}{\partial \bar{z}} &= \frac{1}{2}(f_q - if_p)(g_q + ig_p)\\ &= \frac{1}{2}(f_q g_q + f_p g_p + i(\underbrace{f_q g_p + f_p g_q}_{\{f,g\}}))\end{aligned}$$

We have that

$$f *_\hbar g = f(e^{\hbar\Phi})g.$$

where $\Phi^n = \partial_z^n \partial_{\bar{z}}^n$, or whatever. This is also called the [Anti-?]Wick product. If g is holomorphic or f is anti-holomorphic, then $f *_\hbar g = fg$.

This is a local model for compact manifolds, but the product also works globally on $\mathbb{C}^n \simeq \mathbb{R}^{2n}$. Since tangent spaces of Kähler manifolds look like $\mathbb{C}^n$, this is the local picture in general. There is a real theory too, which was used in the more algebraic proofs of deformation quantization. In that case, the local model is the Moyal Product on $\mathbb{R}^{2n}$ (Von Neumann, Weyl):

$$\begin{aligned}f *_\hbar g &= f(e^{\frac{i\hbar}{2}P})g\\ &= fg + \frac{i\hbar}{2}\{f,g\} + (\frac{i\hbar}{2})^2\frac{1}{2!}f(P)^2 g + \cdots\end{aligned}$$

this is a standard star product, and it is unique if you assume that it is invariant under the action of the symplectic group $Sp(2n;\mathbb{R})$, which acts on functions by pulling back. On quadratic functions, $\mathfrak{sp}(2n;\mathbb{R})$, $f *_\hbar g - g *_\hbar f = i\hbar\{f,g\}$ (it is clear that the third and higher order terms vanish, but so do the second order terms!). In fact, you only need one of f, g to be quadratic. This thing lives naturally on any symplectic vector space, so you have one on each tangent space of a symplectic manifold.

Another possible product:

$$f *'_\hbar g = f(e^{i\hbar\frac{\partial}{\partial q}\leftarrow\frac{\partial}{\partial p}\rightarrow})g$$

which is much simpler.

You want to get from Polynomials on $\mathbb{R}^{2n}$ to operators on $L^2(\mathbb{R}^n)$. We take $q \mapsto M_z$, $p \mapsto i\hbar\frac{\partial}{\partial x}$. Then where does qp go? It could be $i\hbar x\frac{\partial}{\partial x}$ or $i\hbar\frac{\partial}{\partial x}x$, or if you can't decide, you can take $\frac{i\hbar}{2}(x\frac{\partial}{\partial x} + \frac{\partial}{\partial x}x)$. The first rule corresponds to the star product $*'_\hbar$ (that's where composition goes), whereas the last one leads to the Weyl product.

In general, the Weyl product doesn't converge ... it is just formal. But there is an integral formula. Let x, y, z be general points in $\mathbb{R}^{2n}$.

$$(f *_\hbar g)(z) = (\mathit{something})\int K(x,y,z)f(x)g(y)\,dx\,dy$$

To get the Weyl product, take $K = e^{\frac{i}{\hbar}4Area(\Delta_{xyz})}$ where Δ_{xyz} is the triangle with vertices x, y, z. This is how Von Neumann comes into the story.

What does this have to do with the Weyl product formula we had? Principal of Stationary Phase: if S a morse function, $\int e^{\frac{i}{\hbar}S(x)}a(x)\,dx \sim_{\hbar\to 0} \sum_{p\in crit\,S}\hbar a(p)\frac{1}{\sqrt{\det\frac{\partial^2 S}{\partial x^i\partial x^j}}} + \cdots$

Lecture 26 - Poisson Geometry

A poisson structure is$\pi \in \wedge^2 TM$, we we get $\tilde{\pi} : T^*M \to TM$, then define $\{f, g\} := \pi(df, dg)$. $X_f = \{\cdot, f\} = \tilde{\pi}(df)$. To figure out the right signs, we want

$$\omega = dq \wedge dp \tilde{\omega}(\frac{\partial}{\partial q} = dp$$

$$\pi = \frac{\partial}{\partial q} \wedge \frac{\partial}{\partial p} \tilde{\pi}^{-1} = \tilde{\omega} \qquad\qquad \tilde{\pi}(dp) = \frac{\partial}{\partial q}$$

so $\alpha(\tilde{\pi}(\beta)) = \pi(\alpha, \beta)$, $\tilde{\omega}(x)(y) = \omega(x, y)$. Then we also have that $[X_f, X_g] = X_{\{f,g\}}$, so hamiltonian vector fields are closed under $[,]$.

In the symplectic case, if X, Y are symplectic vector fields (i.e. $\mathscr{L}_X \omega = \mathscr{L}_Y \omega = 0$), then the bracket, $[X, Y]$, is hamiltonian. However, this is not true in the Poisson case. Note, by the way, that the zero poisson structure is a poisson structure.

Examples

(a) $\pi = 0$.
(b) $\pi = \phi(x, y) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$ for any ϕ in $\mathbb{R}^2$.
(c) $\pi = \frac{1}{2} c_k^{ij} x^k \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}$ on $\mathfrak{g}^*$.

What do poisson structures look like? Locally, symplectic structures all look the same (Darboux). $\tilde{\pi}(T^*M) \subseteq TM$ is not a sub-bundle in general, it is a distribution. In case b, the dimension goes to zero when ϕ vanishes. The natural sections of $\tilde{\pi}(T^*M)$ are $\tilde{\pi}$(a 1-form).

Theorem 26.1. *$\tilde{\pi}(T^*M)$ is integrable in the sense that M is a disjoint union of connected integral manifolds.*

a: integral manifolds are points
b: integral manifolds are components of the open set where $\phi \neq 0$.

A point $m \in M$ is *regular* if rank$(\tilde{\pi})$ is constant on a neighborhood of m. π induces on each leaf of $\tilde{\pi}(T^*M)$ a symplectic structure because $T_m^*M / \ker \tilde{\pi} \simeq \tilde{\pi}(T_m X)$. The leaves are the *symplectic leaves* of (M, π).

[e.g. let $\pi = 0$ for $y \leq 0$ and non-zero elsewhere]

On a surface, the generic situation is this: you have curves where $\pi = 0$, and a bunch of open symplectic leaves. The thesis of O. Radko classifies all of these structures.

The f such that $X_f = 0$ are *Casimir functions*. These are exactly the functions constant along leaves.

Local Structure:

Theorem 26.2 (Splitting theorem)**.** *There are local coordinates q^i, p_i, y^j $1 \leq i \leq k$, $1 \leq j \leq r$ such that*

$$\pi = \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i} - \pi^{rs}(y) \frac{\partial}{\partial y^r} \frac{\partial}{\partial y^s}$$

where $\pi^{rs}(0) = 0$.

So you have a symplectic manifold, and a bunch of transverse guys with poisson structures. You can flow along a hamiltonian vector filed, preserving poisson structure, to get between transverse sections. So near a leaf, there is a product structure, but these things can glue together funny. See Vorobev, Davis-Wade, and somebody in Belgium.

Going back to the local question. It is enough to classify the π^{rs}. Consider a tweak of example c:

$$(*) \qquad \pi = \frac{1}{2} c^{ij}_k y^k \frac{\partial}{\partial y^i} \wedge \frac{\partial}{\partial y^j} + O(y^2)$$

is the most general version.

If you just write $\pi = \pi^{ij} \frac{\partial}{\partial y^i} \wedge \frac{\partial}{\partial y^j}$, you don"t get a poisson structure in general, you need $[\pi, pi] = 0$ for the Schouten-Nijenhuis bracket. The c term something lie algebra.

Something ... the transverse leaves are lie algebras. At a regular point, the rank can't change when you move around, so the transverse structure (the π^{rs}s) is zero, and the y's are local Casimirs. This was done by Lie.

How general are the linear structures? Given a π like $*$, do there esist new coordinates $z^i = y^i + O(y^2)$ such that $\pi = \frac{1}{2} c^i_k j \frac{\partial}{\partial z^i} \wedge \frac{\partial}{\partial z^j}$. Sometimes you do. But there are some theorems that say that if the lie algebra is complicated enough, you can linearize like this.

Linearization theorems: We say a lie algebra $\mathfrak{g}$ is [formally, smoothly, analytically] non-degenerate if any [formal,C^∞,C^ω] poisson structure whose linearization is isomorphic to $\mathfrak{g}^*$ is locally isomorphic to $\mathfrak{g}^*$.

Theorem 26.3 (Arnol'd)**.** *If* $\mathfrak{g} = \{[x, y = y]\}$*, i.e.* $\{x, y\} = y + O(x, y)^2$*, then you can linearize in "any category".*

Theorem 26.4 (Weinstein)**.** *If* $\mathfrak{g}$ *is semi-simple, you can formally linearize.* $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$*, then not* C^∞*.*

Theorem 26.5 (Conn)**.** $\mathfrak{g}$ *semi-simple, analytic works, and* C^∞ *only if* $\mathfrak{g}$ *is also of compact type (Killing form definite, note just non-degenerate).*

. . . , most due to Dufour or students of his, or people who work with him, like Wade, Nguyen Tien Zung, Monnier, Stolovich (holomorphic poisson), etc. There are some other lie algebras (non-semi-simple) which still have this stability.

Consider the structure

$$\begin{aligned} \{x, y\} &= 0 \\ \{x, z\} &= ax + by \\ \{y, z\} &= cx + dy \end{aligned}$$

the hamiltonian flow of z is given by $\dot{x} = ax + by, \dot{y} = cx + dy$, which is an arbitrary linear differential equation. The flow of y is $\dot{x} = \dot{y} = 0, \dot{z} = -cx + dy$, so you can move up and down (almost always). So the leaves will be cylinders on the trajectory on the plane.