Contents

How these notes are coming to exist

It is Fall 2007. QFT classes are being taught at UC Berkeley by Richard Borcherds (RB), Nicolai Reshetikhin (NR), and Peter Teichner (PT). Anton LaTeXs these notes in class and edits them later.[1] The version you're currently reading was compiled October 4, 2014. They should be available at
http://math.berkeley.edu/~anton/index.php?m1=writings.

– When something doesn't make sense to me, I mark it with three big, eye-catching stars [[★★★ like this]]. If you can clear any of these up for me, let me know.

– If you have notes that I'm missing or if you have a correct/clear explanation for something which is incorrect/unclear, let me know (either tell me what you'd like to modify, give me some notes to go on, or update the tex yourself and send me a copy). Real (mathematical) errors should be fixed because it would be immoral to let them propagate (er . . . that is, sit there), and typographical errors hardly take any time to fix, so you shouldn't be shy about telling me about them.

[1] With the exception of NR22, which was done by Chris Schommer-Pries.

1 NR 08-27

Quantum field theory is a very big subject (in both physics and math), even though it is relatively new (late 50s and 60s). It was designed to describe the interactions of particles and the structure of the micro-world. From the beginning, there were some formidable mathematical (and intrinsic) problems:

1. renormalization problem
2. perturbation theory

On Tuesdays, Richard Borcherds will have a seminar which will be focused on these problems, so these things won't be in this course.

By the 60s and 70s, there were well-developed ways to get around these things, but there were more and more particles showing up, and they needed explanation. The main outcome of this was the Standard Model and Gauge theory. This stuff is very interesting, but we won't talk about it.

The goals of this course: Give a mathematical summary of the basic ideas in classical and quantum field theory. You can't really talk about these theories unless you start from classical and quantum mechanics, so this will be the subject of the first few lectures. Newton did stuff ... 2nd order ODE; variational principals and Lagrangians; Hamiltonian mechanics (reduce to first order ODEs); Symplectic geometry (M, ω), $C^\infty(M)$. The symplectic form induce a Poisson structure $\{\cdot, \cdot\}$. So you get $C^\infty(M)$, which is (1) a commutative algebra, (2) a Lie algebra with $\{f, g\} = (df \wedge dg, \omega^{-1})$, where $\omega \in \bigwedge_x^2 T^*M$ and $\omega^{-1} \in \bigwedge_x^2 TM$, and we have that

$$\{f, gh\} = \{f, g\}h + \{f, h\}g.$$

Such an object is called a Poisson algebra.

How to go from 2nd order to first order, well we have $\ddot{q} = F(q)$, which can be turned into the system $p = \dot{q}$ and $\dot{p} = F(q)$, so we have twice as many variables.

Physicists would not approve of this approach because it isn't very physical.

We have $C^\infty(M)$ with $\{\cdot, \cdot\}$. We want to deform this algebra in the category of associative algebras. That is, we want to find some (associative) multiplication

$$f *_h g = fg + \frac{h}{2}\{f,g\} + O(h^2)$$

This is called deformation quantization. There is a powerful technique called geometric quantization which Peter will talk about.

This does the classical mechanics to quantum mechanics. Hopefully we'll do this in the first two weeks.

What is a classical field theory? It is the same sort of thing, but where the n-dimensional manifold N ($M = T^*N$) is replaced by $\Gamma(E \to M)$ for some bundle E (which we can't even say is a manifold).

The real goal of this course is to explain invariants of 3-manifolds and corresponding conformal field theories (we'll just focus on Chern-Simons theory). (Something about affine lie algebras. Lately, there is the theory of SLE processes which relates this stuff to probability theory. If you have Brownian motion, you get a random curve you'd like to describe. There is an analytic technique)

When I started thinking about this course, I realized that Peter is teaching a seminar course, so we'll try to coordinate (for the first half of the semester). There will be a certain division of labor. I'll focus on bosonic field theories, and Peter will be focusing (at least in the beginning) on fermionic field theories, which are really important for bosonic field theories (for Chern-Simons theory). There is an infinite-dimensional group which makes the Lagrangian something. The way people deal with this in QFT and perturbation theory is known as the theory of Faddeev-Popov ghosts. These are fields which you don't see, but they play a certain role. It is important to have these objects because they give you finite type invariants of 3-manifolds. One of the goals of this course is to explain that there are different points of view on the same thing, and they produce different kinds of results. Fermionic fields are essential for this part. For this reason, it is a good idea to go to both classes (at least for the first half of the semester).

There is a syllabus on my website.

Q: What are the prerequisites? NR: I assume you know differential geometry and some symplectic geometry. If you don't know this stuff, you can look in a textbook.

We'll start next lecture with Lagrangian mechanics and then Hamiltonian mechanics (that will be this week). Next week, we'll spend some time on Hamilton-Jacobi theory. After that, we'll move on to quantization and semi-classical analysis. Then we'll go to classical field theories. Then we'll talk about quantum mechanics and quantum field theory. Then symmetries (action of the gauge group) with some examples. In about a month (or a month and a half), we'll start focusing on Chern-Simons theory.

PT: if you decide to take this class, then register so that we can get a bigger room.

There will be homework, but it won't be graded. The office hours are on Tuesdays by appointment. At the end of the class, we'll have a mini-conference. Everybody registered will give a short presentation, and this will be instead of the final exam.

1 PT 08-28

There will be notes online. Office hours are Th 2-3:30 in 703. You can find the website from my site. There are also notes from a previous class there which are relevant.

Thursday, we'll be in 939. After that, we'll see.

This semester seems to be a QFT semester.

1. Kolya's class MWF 1-2
2. This class TT 11-12:30
3. Richard's class Tu 1-2
4. Hot topics course on topological conformal field theory. Tu 2-3:30. This one will be lectures by students.
5. Student seminar W 2-3
6. Topology seminar W 4-5
7. Th 3-4 QFTea

In this course, there will be homework every 2 weeks. This is to keep you honest. I think I convinced Kolya to do the same. Submit it in groups of 2-4 students. Find a group of people you like, get together with them to do the problems, and split up the writing. The homework is optional. The mini-conference will be the thing that counts for the grade. We've decided that if you give one talk, it can count for both this class and Kolya's class. The first homework will be this Thursday.

This course will have three parts.

1. Super mathematics. First super algebra, then super differential topology, then super geometry. This part should only last two weeks (this will only be a survey). There is a very good reference (Deligne-Morgan [DEF+99, Vol. 1, Notes on Supersymmetry]).
2. Fermionic field theory. Kolya is starting with bosonic field theory. We'll follow everything he does bosonically and do it fermionically. This should take four to six weeks.
3. 6-8 weeks left. Kolya will go into Chern-Simons theory etc. and we'll go into the relation to algebraic topology.

In case you don't make it all the way to act 3, I'll tell you about it today.

Let Man be the category of smooth manifolds with smooth maps. Let GRing be the category of $\mathbb{Z}$-graded commutative rings. $\mathbb{Z}$-graded means that $R = \bigoplus_{i\in\mathbb{Z}} R^i$ as an abelian group, and that $R^i \cdot R^j \subseteq R^{i+j}$. A graded ring is said to be commutative if for homogeneous elements $a, b \in R$, we have

$$b \cdot a = (-1)^{|a|\cdot|b|} a \cdot b.$$

From now on, when we say ring, we mean $\mathbb{Z}$-graded commutative ring.

Note that this kind of graded commutativity makes sense for $\mathbb{Z}/2$-graded rings. A $\mathbb{Z}/2$-graded ring with this flavor of commutativity is called a *commutative super algebra*.

Definition 1.1. A *(generalized) multiplicative cohomology theory*[1] is a homotopy functor $h^* \colon \mathsf{Man}^\circ \to \mathsf{GRing}$ with the Mayer-Vietoris property: for open subsets $U, V \subseteq M$, there is a long exact sequence

$$\cdots \xrightarrow{\delta} h^i(U \cup V) \to h^i(U) \oplus h^i(V) \to h^i(U \cap V) \xrightarrow{\delta} h^{i+1}(U \cup V) \to \cdots$$

[[★★★ probably with some naturality condition]] ◇

This connecting homomorphism δ is part of the data of a cohomology theory. The Mayer-Vietoris axiom tells you that the ring $h^*(M)$ is computable (modulo knowing the cohomology of a point) since you can break a manifold up into contractible open sets with contractible intersections [[★★★ It looks like you don't even have to know what δ is! So $h^*(pt)$ really completely determines h^*?]]. $h^*(pt)$ is called the *coefficient ring* of the cohomology theory.

Example 1.2. Singular cohomology $H^*(X) = \bigoplus_{i\in\mathbb{Z}} H^i(X)$ with cup product. This H is the letter for ordinary cohomology. The coefficient ring is $\mathbb{Z}$ concentrated in degree zero. ◇

[1] These days, you just say "cohomology theory", leaving off the word "generalized".

Remark 1.3. Why restrict to manifolds? Don't you usually define cohomology on arbitrary topological spaces? Somehow, the definition of a topological space is way too general; it was really defined for analysis. In topology, we usually restrict to CW-complexes by imposing the weak homotopy axiom: if $f\colon X \to Y$ induces isomorphisms on all homotopy groups with all base points (i.e. is a weak homotopy equivalence), we require that $h^*(f)\colon h^*(X) \to h^*(Y)$ be an isomorphism. For any topological space X, there is a CW-complex $X' = |S_\bullet X|$ with a map to X which is a weak homotopy equivalence. This tells you that you can't see anything with homotopy groups which doesn't show up in CW-complexes. Note, by the way, that π_* is not a cohomology theory because there is no Mayer-Vietoris sequence, so homotopy groups are not (yet) easily computable. By the weak homotopy axiom, $h^i f\colon h^i X' \xrightarrow{\sim} h^i X$ is an isomorphism, so you may as well study cohomology of CW-complexes.

There is a beautiful fact that any smooth manifold has the structure of a CW-complex. Finally, if you have a CW-complex which is finite-dimensional and countable, then you can thicken it and get a manifold. So in this class, we'll only talk about manifolds. ⋄

Okay, now we have one of the definitions. The next one will take the rest of the class. But first, some examples.

1900s, Poicaré, Lefschetz. The usual H^*. I hope you've all learned how wonderful the usual cohomology is; its is a wonderful tool.

1950s. Grothendieck, Atiyah-Hirzebruch. K^*. This was the first theory which didn't satisfy the dimension axiom: $K^*(pt) = \mathbb{Z}[u, u^{-1}]$ with u of degree 2. This is very geometric: you start with vector bundles modulo stable isomorphisms. Maybe the surprising thing is that this satisfies MV. You can prove more things with this, like find the number of independent vector fields on a sphere. Division algebras over $\mathbb{R}$ have dimension 1, 2, 4, or 8. Atiyah-Singer index theorem. This got people excited about other cohomology theories.

1990s. Hopkins-Miller, Lurie. TMF^* (topological modular forms) is the universal elliptic cohomology theory. Hopkins-Miller proved that TMF^* exists and Lurie constructed it. None of it is published, but Lurie's stuff is coming out slowly. Why "elliptic"? There is a relationship between cohomology theories and formal groups (which we won't explain). Let's say R is your favorite ring (to be the coefficient ring), then if you have a formal group law on R, then you can construct a cohomology theory so that the formal group has to do with $\mathbb{CP}^\infty$. If you take the additive group law $(a + b)$, you get the usual thing; if you take ab, you get K-theory; if you take the group law of an elliptic curve, then you get some "elliptic cohomology theory". Since there are many elliptic curves, there are many elliptic cohomology theories. TMF is tricky to construct because there is no universal elliptic curve (so you can't just use the construction); you have to deal with stacky stuff.

What are the applications of TMF? There is a beautiful open question. If you have a manifold, say $\mathbb{RP}^n$, then we know that we can embed it into $\mathbb{R}^{2n}$, but what is the minimal k so that there is an embedding $\mathbb{RP}^n \hookrightarrow \mathbb{R}^k$. (Ralph Cohen did the case of an immersion instead of an embedding). For many n it is known using H^*, K^* and TMF^*. The other thing that TMF gives us is that it allows us to understand the homotopy groups of spheres up to 60. In particular, it allows you to detect Lie groups. The real exciting application is not yet worked out: it has to do with the index theorem on loop space. Witten has a Dirac operator on loops space, and it's index is an element in $TMF^*(pt)$. The index theorem for loop space would be nice, but we're missing the analytic side, so the theorem cannot yet be formulated.

By the way $TMF^*(pt)$ is completely known (unlike the stable homotopy groups of a point).

If the picture relating this stuff to QFTs is right, then you can use all this machinery to describe QFTs.

Definition 1.4. Let X be a manifold, $d = 0, 1, 2$ and $n \in \mathbb{Z}$. Then $QFT^n_{d|1}(X)$ are supersymmetric QFTs of dimension $d|1$ and degree n over X. ⋄

We don't know how to do this for $d > 2$. Here, $d|1$ is the super-dimension of the world-sheet $\Sigma^{d|1}$.

One reason that this class will be so different from physics classes is that we'll actually give a definition, but it will be done by sucking as much intuition from the physicists and turning it into a definition.

Physicists would never look at $d = 0$, $d = 1$ is quantum mechanics, $d = 2$ is the first interesting case, and they really want to study $4|16$-dimensional world-sheets. Even through we're learning a lot from physics, we're focusing on different things (we're not trying to understand the real world, like they are).

For $d = 0$, $QFT^n_{d|1}(X)$ is a set, for $d = 1$, it's a category, and for $d = 2$, its a 2-category. $QFT^n_{d|1}$ is actually a (contravariant) functor.

Theorem 1.5 (Conjecture?)**.**

d	$QFT^n_{d\|1}(X)$	$QFT^n_{d\|1}[X]$
0	$\Omega^n_{closed}(X)$	$H^n_{dR}(X)$
1	*super vector bundles on* πTX *with Quillen connection?*	$K^n(X)$
2	*something new*	$TMF^*(X)$*?*

You might know that $[Y, X]$ are homotopy classes of maps from Y to X, so it is $Map(Y, X)/\simeq$. If you have any contravariant functor, you can make the same definition. $QFT^n_{d|1}[X]$ is $QFT^n_{d|1}(X)$/concordance. It will be an exercise that two closed forms differ by an exact form if and only if there is a closed form on $X \times I$ which restricts to the two forms at the ends. By definition of concordance, $QFT^n_{d|1}[X]$ is automatically a homotopy functor.

If you leave out supersymmetry (the $|1$), then you still get a beautiful definition, but the third column is all zeros.

The whole third act of the class is a joint project with Stephan Stolz (at Notre Dame, Indiana).

1 RB 08-28

This won't really be a seminar, it will be a short course on quantum field theory for mathematicians. Aim of the course: give mathematical answers to the following questions

1. What is a QFT? There are many incompatible answers.
2. How do you construct them? Nobody knows how to construct them non-perturbatively, so we'll do everything perturbatively. What is a Feynman measure? It is easy to prove they don't exist.
3. What is renormalization and reregularization?
4. What is gauge-invariance? Anomalies.

In the first seminar, we'll try to give a quick survey without proofs.

Recall what a classical field theory is. There are two basic ingredients for classical field theory.

1. Fiber bundle.
2. a Lagrangian.

A fiber bundle is locally (on M) something of the form $F \times M \to M$. We call M the base space and F the fiber., so we have a copy of F sitting over every point in M.

A *classical field* is a section of the fiber bundle. In the case of a product, this is just a function $M \to F$.

Example 1.1. Classical mechanics is a (trivial) field theory. Take $M = \mathbb{R}$ (thought of as time) and F is configuration space, which is typically a finite-dimensional manifold (the possible positions of some mechanical system). The fields are maps from $\mathbb{R}$ (time) to the configuration space F. ◇

Example 1.2. Statistical field theory. Take M = space (say $\mathbb{R}^3$) and $F = \mathbb{R}$ (say). Then a field is a real function on $\mathbb{R}^3$ ◇

Example 1.3. Quantum field theory. Take M = spacetime (some Lorentzian manifold, usually flat Minkowski space $\mathbb{R}^{1,3}$). Unfortunately, one of the major unsolved problems in physics is whether it is $\mathbb{R}^{3,1}$ or $\mathbb{R}^{1,3}$. Take $F = \mathbb{R}$ for a Hermitian scalar field. In this case, a classical field is just a real function on $\mathbb{R}^{1,3}$. $\diamond$

In more complicated classical field theories, you could take $F = SU(3)$ (which underlies quantum chromodynamics). Most of the problems of field theory show up in the simple case where you take $F = \mathbb{R}$.

Remark 1.4. More generally, instead of taking total space to be $F \times M \to M$, you take some twisted version of it (a more general fiber bundle $E \to M$, which locally looks like $F \times M \to M$). Physicists almost always take the fiber bundle to be a product. $\diamond$

The next ingredient is a Lagrangian.

Example 1.5. Take statistical field theory. $L(\phi) = \sum_i \left(\frac{\partial \phi}{\partial x_i}\right)^2 + m^2\phi^2 + \lambda\phi^4$. Here ϕ is a field (e.g. a real-valued function). This Lagrangian is a function from $L\colon \text{Fields}\times\text{Base space}\to \mathbb{R}$, but it isn't any old function; it is sort of local. L depends only on ϕ and its derivatives at the point $x \in M$. This means that L is a function on the *Jet space* of the fiber bundle. $\diamond$

A jet space is just the set of pairs $(\phi, x) \in (\text{functions}\times M)$, where we identify (ϕ_1, x) with (ϕ_2, x) if ϕ_1 and ϕ_2 agree to all orders at the point x (i.e. all their derivatives agree). There are also finite order jet spaces where you only look at a finite number of derivatives (identify (ϕ_1, x) with (ϕ_2, x) if all derivatives to third order are the same). Finite jet spaces aren't as nice as infinite order jet spaces.

There are some variations on Lagrangians. We have Lagrangians, Lagrangian densities, and actions. The purpose of a Lagrangian is to produce an action, which is what gets you into business. A Lagrangian is a function on jet space. You want to integrate the Lagrangian over spacetime (the base space) to get an action. So the action of a classical field is "$\int_M L(\phi)$". However, you can't integrate functions: you often explain to undergraduates that things like $\int x^2$ don't make sense; you need to multiply a function by a form before you can integrate it. A Lagrangian density is a form on the manifold times a Lagrangian. This is a bit untidy. We have Jet space$\to M$, so if we have a form on M, we can pull it back to Jet space, so you can think of a Lagrangian density as a special kind of form on Jet space.

Often, M has a canonical n-form, in which case you can identify Lagrangians with Lagrangian densities. For example, if M has a metric and an orientation, this gives a volume form. You do need to worry about the difference if you're studying gravity, because then M has no canonical metric (the point being that gravity varies with the metric on M). In that case, Lagrangians and Lagrangian densities are not the same thing, and the right thing to use is a Lagrangian density. But usually, you don't need to worry too much about the difference.

Finally, what is an action? If you have a Lagrangian density, you can integrate it over spacetime to get an action $\int_M L(\phi)$, except you can't because M is usually non-compact, so there is no reason for the integral to converge. Most physics books ignore the problem by pretending the integral converges even when it is clear it doesn't. What seems to be going on is that even though you can't define the action of a field, you can define the DIFFERENCE of the actions of two fields ϕ_1 and ϕ_2 provided that they differ on a compact set.

So if we've written down a Lagrangian density, we can define variation. Now you're in business. You can define the classical equations of motion (Euler-Lagrange equations): $\int L(\phi)$ is STATIONARY under variations of ϕ on COMPACT SETS.

For classical mechanics, you need a fiber bundle and a Lagrangian density. Once you have these things, you hand them to somebody who knows classical mechanics and they'll get excited and solve the equations of motion for you.

Now let's talk about what a quantum field theory is (let's do the case of a Hermitian scalar field $\phi : \mathbb{R} \times M \to M$). The basic idea: we should have (1) a Hilbert space H and (2) an operator $\phi(x)$ on H for each $x \in M$ satisfying various axioms. The problem is that it is impossible to make sense of the operator $\phi(x)$. The problem is that $\phi(x)\phi(y)$ has really dreadful singularities as x and y get close together. All of quantum field theory is in some sense trying to get around the problem of how to define $\phi(x)\phi(x)$. You get around this in two steps. The first thing you do whenever you have singularities is to smooth them out by convolving with smooth functions with compact support. Instead of using operators $\phi(x)$ for $x \in M$, we use $\phi(f) =$"$\int \phi(x) f(x)\, d^n x$" where f is smooth with compact support. There is another problem: even smoothed operators $\phi(f)$ are not defined on H. This turns out to be a fairly minor problem. They are defined on a dense subset $D \subseteq H$ and map D to D (they are "unbounded operators").

So what you end up with is an unbounded operator on H for each smooth compactly supported classical field f.

Now we'll give a minimal definition of a quantum field theory. We need (1) a module D over some $*$-commutative ring (in practice, you do perturbative stuff, so you use formal power series) (a $*$-ring is a ring with involution $*$ so that $(ab)^* = b^* a^*$), (2) a hermitian inner product $(\cdot,\cdot)\colon D \times D \to \mathbb{R}$, and (3) a $*$-algebra A of operators acting on D (generated by $\phi(f)$). Furthermore, $(Ax, y) = (x, A^* y)$.

This is the minimal amount of stuff you need to reasonably say you have a quantum field theory.

Example 1.6. QFT satisfying the Wightman axioms (we'll discuss these later). In this case, D is a dense subset of a Hilbert space and A is the algebra generated by the field operators. $\diamond$

Example 1.7. Let L be a Lie algebra acting on a vector space D with an invariant symmetric inner product. Then you can construct an algebra by taking the universal enveloping algebra $U(L)$ with $a^* = a$ for all $a \in L$. $\diamond$

Example 1.8. If G is a group and D is an orthogonal representation of G, then we can take A to be the group-ring of G with $g^* = g^{-1}$ for all $g \in G$. $\diamond$

Example 1.9 (generalizing the last two examples)**.** A a Hopf algebra and D to be an orthogonal representation of A. $\diamond$

Example 1.10. Take A to be any C^*-algebra or von Neumann algebra and D to be any Hilbert space that is a $*$-representation of A. $\diamond$

Warning 1.11. People often define quantum field theories in terms of C^*-algebras. You have to watch out, because in our cases, the operators will usually be unbounded, and the C^*-algebra examples have bounded operators. $\lrcorner$

To construct a QFT, we need to give (1) a $*$-algebra A, and (2) a module D, and these things should satisfy some extra axioms.

How to construct examples. The algebra A is easy to construct. You could just take it to be the universal $*$-algebra generated by classical fields. The hard part is to construct the right representation. Start with a state w (a linear map $w\colon A \to R$ such that $w^* = w$) and define $(\cdot,\cdot)$ on A, given by $(a, b) = w(ab^*)$ and take $D = A/\ker(,)$ (this is basically the GNS construction). ω is constructed using Feynman integral, which is formally given by $\int e^{i\int L(\phi)} D\phi$ which can be expanded as a series of Feynman diagrams.

2 NR 08-29 Lagrangian Mechanics

Today we'll start with classical mechanics. Recall some basic facts.

Recall that Newton's equations say that a trajectory γ in $\mathbb{R}^n$ should satisfy the following equation.

$$m\ddot{\gamma}(t) = -\frac{\partial U}{\partial q}\big(\gamma(t)\big)$$

People often write $F := -\frac{\partial U}{\partial q}$ and $a := \ddot{\gamma}$, in which case this is written as $F = ma$. Thinking of a solid as a collection of constrained points, you can understand the motion of solids. (Euler)

Let's reformulate these differential equations as a variational problem (Lagrange) on $T\mathbb{R}^n$, or more generally, on TN for a smooth manifold N (which we'll refer to as *configuration space*). Choose a *Lagrangian* $\mathcal{L} \in C^\infty(TN)$. If N is equipped with a Riemannian metric (as in our example $N = \mathbb{R}^n$), then we take the Lagrangian

$$\mathcal{L}(\xi, q) = \frac{(\xi,\xi)}{2} - U(q) \tag{2.1}$$

for $(\xi, q) \in TN$, where $(\cdot,\cdot)$ is the Riemannian metric and U is some potential function. For a parameterized path $\gamma = \{\gamma(\tau)\}_{\tau=0}^t$ in N, define the *action functional*

$$\mathcal{A}[\gamma] = \int_0^t \mathcal{L}\big(\dot{\gamma}(\tau), \gamma(\tau)\big)\, d\tau.$$

As we'll see later, solutions to Newton's equations are parameterized paths γ_{cl} in N on which $\mathcal{A}[\gamma_{cl}]$ is extremal (i.e. the variation vanishes).

Heuristically the (first) variation is the infinitesimal change in action when γ is changed infinitesimally.

$$\delta\mathcal{A}[\gamma] = \text{“}\mathcal{A}[\gamma + \delta\gamma] - \mathcal{A}[\gamma]\text{”}$$

More precisely, if f is a function on paths, let γ_s be a family of parametrized paths such that $\gamma_0 = \gamma$ and define $\delta f(\gamma) = \frac{df(\gamma_s)}{ds}\big|_{s=0}$. Note that this depends on the choice of the family $\{\gamma_s\}$. In particular, $\delta\gamma = \frac{d\gamma_s}{ds}|_{s=0} = \big\{\big(\delta\dot{\gamma}(\tau), \delta\gamma(\tau)\big) \in T_{\dot{\gamma}(\tau)}(T_{\gamma(\tau)}N)\big\}_{\tau=0}^t$ is a vector field along the path $\{\dot{\gamma}(\tau), \gamma(\tau)\}_{\tau=0}^t$ which descibes how we are wiggling γ [[★★★ It looks like $\delta\gamma$ is more naturally a vector field along γ, not along $(\dot{\gamma}, \gamma)$, but this does induce a vector field along $(\dot{\gamma}, \gamma)$.]].

The variation of $\mathcal{A}$ is

$$\begin{aligned}
\delta\mathcal{A}[\gamma] &= \frac{d\mathcal{A}[\gamma_s]}{ds}\Big|_{s=0} \\
&= \sum_i \int_0^t \Big(\frac{\partial\mathcal{L}}{\partial\xi^i}(\dot{\gamma},\gamma)\delta\dot{\gamma}^i(\tau) + \frac{\partial\mathcal{L}}{\partial q^i}(\dot{\gamma},\gamma)\delta\gamma^i(\tau)\Big)\, d\tau \\
&= \underbrace{\sum_i \int_0^t \Big(-\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\xi^i}(\dot{\gamma},\gamma) + \frac{\partial\mathcal{L}}{\partial q^i}(\dot{\gamma},\gamma)\Big)\delta\gamma^i(\tau)\, d\tau}_{\text{bulk term}} \qquad \text{(integrating by parts)} \\
&\quad + \underbrace{\sum_i \Big(\frac{\partial\mathcal{L}}{\partial\xi^i}\big(\dot{\gamma}(t),\gamma(t)\big)\delta\gamma^i(t) - \frac{\partial\mathcal{L}}{\partial\xi^i}\big(\dot{\gamma}(0),\gamma(0)\big)\delta\gamma^i(0)\Big)}_{\text{boundary terms } = \frac{\partial\mathcal{L}}{\partial\xi}(\delta\gamma(t)) - \frac{\partial\mathcal{L}}{\partial\xi}(\delta\gamma(0))}
\end{aligned}$$

We call the first term a “bulk term” and the last terms “boundary terms”.

If γ is a solution to the Euler-Lagrange equations

$$-\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\xi^i}(\dot{\gamma},\gamma) + \frac{\partial\mathcal{L}}{\partial q^i}(\dot{\gamma},\gamma) = 0 \qquad \text{(Euler-Lagrange equations)}$$

then the bulk term vanishes, so $\delta\mathcal{A}[\gamma]$ is given by boundary terms. Note that this is the *only* way to get the bulk term to vanish because if this factor doesn't vanish, we can change our choice of the family $\{\gamma_s\}$ (thus changing $\delta\gamma$) to get the integral to not vanish. The Euler-Lagrange equations for (2.1) are Newton's equations in $\mathbb{R}^n$. Classical trajectories are those that satisfy the Euler-Lagrange equations.

Boundary Problems

A *boundary problem* is a submanifold $B \subseteq TN \times TN$. A solution to the boundary problem is a classical trajectory γ_{cl} (one that solves the Euler-Lagrange equations) such that $\big(\dot{\gamma}(t), \gamma(t), \dot{\gamma}(0), \gamma(0)\big) \in B$. Here are three examples of boundary problems (conditions we can impose on our paths).

(i) Boundary value problems: $\gamma(0) = Q$ and $\gamma(t) = Q'$. In this case, $B = T_{Q'}N \times T_Q N$.

(ii) $\gamma(0) = Q$ and $\dot\gamma(t) = V$. In this case, $B = V(N)\times T_Q N$, where $V(N)$ is the image of the vector field $V\colon N \to TN$ in TN.

(iii) Initial value problems: $\gamma(0) = Q$ and $\dot\gamma(0) = V$. In this case, $B = TN \times \{(V,Q)\}$. This means we fixed a point on TN at $\tau = 0$.

Consider the first problem. If we restrict to families $\{\gamma_s\}$ for which $\big(\dot\gamma_s(t), \gamma_s(t), \dot\gamma_s(0), \gamma_s(0)\big) \in B$, then we see that $\delta\gamma(t)$ and $\delta\gamma(0)$ are zero, so the boundary terms in $\delta\mathcal{A}[\gamma]$ vanish. Thus, the extrema of $\mathcal{A}[\gamma]$ (subject to the boundary conditions) are precisely solutions to this boundary problem.

In general, if we restrict to families $\{\gamma_s\}$ satisfying the boundary conditions it is not true that a solution to the Euler-Lagrange equations is an extremum of the action (because the boundary terms of $\delta\mathcal{A}[\gamma]$ may not vanish). To remedy this, we can try to change the action functional so that solutions to the boundary problem are extrema of the new action functional.

Consider the second boundary problem. We have that $\delta\gamma(0) = 0$ (because we fixed $\gamma(0)$), so the remaining boundary terms are (in the Riemannian case, using the Lagrangian in (2.1))

$$\begin{aligned}
d\mathcal{L}\big(\delta\gamma(t)\big) &= \sum_i \frac{\partial\mathcal{L}}{\partial\xi^i}\big(\dot\gamma(t),\gamma(t)\big)\delta\gamma^i(t) && \text{(in coordinates)}\\
&= \sum_i \dot\gamma^i(t)\delta\gamma^i(t) = \sum_i V^i_{\gamma(t)}\delta\gamma^i(t) && \Big(\mathcal{L}(\xi,q) = \tfrac{(\xi,\xi)}{2} - U(q)\Big)\\
&= \big(V_{\gamma(t)}, \delta\gamma(t)\big) && \text{(the pairing from the metric)}
\end{aligned}$$

So extrema of our action functional are not solutions to the boundary problem. If $N = \mathbb{R}^n$, consider the modified action functional

$$\mathcal{A}_V[\gamma] = \mathcal{A}[\gamma] - \sum_i V^i_{\gamma(t)}\gamma^i(t).$$

Then it is easy to see that extrema of $\mathcal{A}_V$ are exactly the solutions to the boundary problem. [[★★★ For $N \neq \mathbb{R}^n$, what is the analogue of this weird term $\sum V^i_{\gamma(t)}\gamma^i(t)$?]]

General strategy: If we want solutions to a boundary problem $B \subseteq TN \times TN$ to be extrema of an action functional, we should consider the modified action functional

$$\mathcal{A}_{F,B}[\gamma] = \mathcal{A}[\gamma] + F$$

for some function F on B. We want solutions to the equations of motion to be exactly the extrema of the action $\mathcal{A}_{F,B}$, so we want

$$\delta\mathcal{A}_{F,B}[\gamma_{cl}] = \text{boundary terms}|_B + \delta F = 0$$

where γ_{cl} is a solution to the Euler-Lagrange equations and the family $\{\gamma_s\}$ is contstrained by the boundary conditions (in particular, the vector $\big(\delta\dot\gamma(t), \delta\gamma(t), \delta\dot\gamma(0), \delta\gamma(0)\big)$ is tangent to B). If we use coordinates (V,Q,v,q) on $TN\times TN$,[1] this is equivalent to F satisfying the differential equations

$$dF = -d\mathcal{L}\circ dQ + d\mathcal{L}\circ dq.$$

If we choose local coordinates $\{x^a\}$ on B, these become

$$\frac{\partial F}{\partial x^a} = -\sum_i \frac{\partial\mathcal{L}}{\partial\xi^i}\big(V(x),Q(x)\big)\frac{\partial Q^i(x)}{\partial x^a} + \sum_i \frac{\partial\mathcal{L}}{\partial\xi^i}\big(v(x),q(x)\big)\frac{\partial q^i(0)}{\partial x^a}.$$

The question is whether we can always find such an F. If you try to do this for the third boundary problem (iii), you'll discover it's impossible. It is clear that a necessary condition is that the form $-d\mathcal{L}\circ dQ + d\mathcal{L}\circ dq$ is closed. In local coordinates, this means that

$$\Big[\Big(\frac{\partial}{\partial x^b}\sum_i \frac{\partial\mathcal{L}}{\partial\xi^i}\big(V(x),Q(x)\big)\frac{\partial Q^i(t)}{\partial x^a}\Big) - \big((V,Q)\leftrightarrow(v,q)\big)\Big] - \big[a\leftrightarrow b\big] = 0$$

Once this condition is satisfied, the obstruction to finding such an F is an element of $H^1_{dR}(B)$. [[★★★ that is, $\frac{\partial^2 F}{\partial x^a\partial x^b} = \frac{\partial^2 F}{\partial x^b \partial x^a}$]]

Example 2.2. If N is a Riemannian manifold with Lagrangian $\mathcal{L}(\xi,q) = \frac{1}{2}(\xi,\xi)$, then the classical trajectories are geodesics in N. [[★★★ HW1]]

If $N = \mathbb{R}^n$ with the usual metric, the Euler-Lagrange equations say that $0 = -\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\xi}(\dot\gamma,\gamma) + \frac{\partial\mathcal{L}}{\partial q}(\dot\gamma,\gamma) = \ddot\gamma$, so the trajectories are $\gamma(t) = A + Bt$.

[1] [[★★★ Q and q are well-defined globally (and these are the only coordinates we care about), but V and v don't make sense. I'd like to fix this without making the meaning unclear.]]

If $N = S^2$, with the metric induced by the standard embedding in $\mathbb{R}^3$, then [[★★★]]. On S^2, there are generically two geodesics connecting a pair of points, demonstrating that there can be more than one trajectory connecting two points. $\diamond$

The physical systems we've descibed so far are called *conservative*, meaning that $\mathcal{L}$ is independent of time. More generally, we could take $\mathcal{L}$ to be a smooth function on $TN \times \mathbb{R}$. In this case, the action functional is

$$\mathcal{A}[\gamma] = \int_{t_1}^{t_2} \mathcal{L}\big(\dot\gamma(\tau), \gamma(\tau), \tau\big)\, d\tau.$$

[[★★★ how much of the previous analysis works?]]

2 PT 08-30

We'll be in 87 Evans starting Tuesday. Today we'll start super mathematics. There are three books I've been reading. Dan Freed: Five lectures on supersymmetry. Darajan: Supersymmetry for mathematicians. The best notes are in "quantum fields and strings: a course for mathematicians" volume 1; this is online (in the IAS website), but it isn't well organized. The one you want is: Deligne and Morgan, Notes on Super Symmetry. `math.ias.edu/qft`

Super stuff started in physics and there are competing schools of mathematicians trying to clean it up. This sheaf-theoretic approach seems to be dominating for now.

(Physical) Motivation: you may already know that one quantum particle is represented by a vector $v \in H$ in a Hilbert space (up to phase). A 2-particle system is represented by $H \otimes H'$. If these two particles are indistinguishable, we have $H = H'$ and $v \otimes v' = \lambda v' \otimes v$ as a physical state (you can pick up some phase $\lambda \in S^1 \subseteq \mathbb{C}$). Doing this twice, we should pick up the same phase (for some reason, which doesn't work in dimension 2 for example), so we get $\lambda^2 = 1$, so $\lambda = \pm 1$. If you keep track of the world lines, you see that two switches is not the same as doing nothing. If $\lambda = 1$, you get a *boson*; if $\lambda = -1$, you get a *fermion*; if $\lambda \in S^1$, you have an *anyion* (for any phase).

Bosons live in $Sym^2 H \subseteq H \otimes H$ and fermions live in $\bigwedge^2 H \subseteq H \otimes H$. As a consequence, you get Pauli's exclusion principle (two fermions can't be in the same state because $v \otimes v = 0$. Physicists decided to write (this is a new H) $H^{new} = H^b \oplus H^f = H^e \oplus H^o$ (even and odd parts). The usual symmetries you have preserve the even and odd parts. Physicists think there should be symmetries which switch the two subspaces, called *supersymmetries* (these haven't been observed). Mathematically, this will be an odd operator on H^{new}. We'll see that $Sym^2 H^{new} = Sym^2 H^b \oplus (H^b \otimes H^f) \oplus Sym^2 H^f$ (and $Sym^2 H^f = \bigwedge^2 H^f$ because H^f is odd).

Definition 2.1. A *super vector space* (over $\mathbb{C}$ or $\mathbb{R}$) is a ($\mathbb{Z}/2$-)graded[1] vector space $V = V^e \oplus V^o$. $\diamond$

[1]From now on, "graded" will mean $\mathbb{Z}/2$-graded

Noah: I don't want to call this a super vector space until you have a tensor product. PT: you're right, we'll get to this later today.

If V and W are super vector spaces, then $\mathrm{Hom}(V,W)$ is a super vector space, where the even homomorphisms are the ones which preserve the grading (sending V^e and V^o to W^e and W^o respectively), and the odd ones reverse the grading. So you can think of a homomorphism as [[★★★]].

If $V = W$ is finite-dimensional and $f = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, then we have $\mathrm{str}(f) = \mathrm{tr}(A) - \mathrm{tr}(D)$ and (Berezinian superdeterminant) $\mathrm{Ber}(f) = \det(A - BD^{-1}C) \cdot \det D^{-1}$ if $\det D \neq 0$ (Lemma: this matrix is invertible if and only if A and D are invertible [[★★★ well, we'll fix this to something correct later]]. Determinant is only defined for invertible matrices?) The property you want is $\mathrm{Ber}(e^X) = e^{\mathrm{str}(X)}$ [[★★★ Homework 1]], [[★★★ Homework 2: $\mathrm{Ber}(X \cdot Y) = \mathrm{Ber}(X) \cdot \mathrm{Ber}(Y)$]]

Definition 2.2. A *superalgebra* (over $\mathbb{C}$) is a super vector space A with an even algebra structure. That is, $\mu\colon A \times A \to A$ is even in the sense that $\mu(A^i \times A^j) \subseteq A^{i+j}$. ⋄

Example 2.3. If V is an ordinary vector space, then $\bigwedge^* V$ is an example of a superalgebra. It is $\mathbb{Z}$-graded a priori, so in particular it is $\mathbb{Z}/2$-graded. This is a quotient of the tensor algebra $T^*V \twoheadrightarrow \bigwedge^* V$. The tensor algebra is still graded, but it is not supercommutative (or finite-dimensional). ⋄

Example 2.4. $H^*(X;\mathbb{C})$ is a superalgebra (with the cup product), so is $\Omega^*(M)$ for M a manifold. So even if you don't care about physics, super stuff shows up. ⋄

These two examples are supercommutative. That is, $a \cdot b = (-1)^{|a|\cdot|b|} b \cdot a$ for homogeneous elements a and b.

Example 2.5. If V is a space with symmetric bilinear form b, then $Cl(V,b) = T(V)/(v \otimes v' + v' \otimes v - b(v,v') \cdot 1)$. If b is identically zero, you get $\bigwedge^* V$. If b is non-zero, this algebra is not supercommutative. Note that the Clifford algebra is not $\mathbb{Z}$-graded, it is only $\mathbb{Z}/2$-graded. ⋄

NR: what's the difference between superalgebras and a $\mathbb{Z}/2$-graded algebra? PT: no difference yet. Noah: Well, there is a small difference: the algebra structure has to be even.

Definition 2.6. If V and W are super vector spaces, then define $(V \otimes W)^e = V^e \otimes W^e \oplus V^o \otimes W^o$ and $(V \otimes W)^o = V^e \otimes W^o \oplus V^o \otimes W^e$. This makes $V \otimes W$ into a super vector space. ⋄

This is good. For example, $\mu\colon A \otimes A \to A$ is even means exactly that it is an even map in $\mathrm{Hom}(A \otimes A, A)$. NR: still unhappy; it's still a $\mathbb{Z}/2$-graded algebra. Noah: Aren't there two homs flying around? PT: We haven't defined the category yet. I hope you'll soon understand why the experts are confused.

Remark 2.7. Associativity of $\mu\colon A \otimes A \to A$ can be written as the following commutative diagram.

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "\mu\otimes \mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\mu"'] & A \otimes A \arrow[d, "\mu"] \\ A \otimes A \arrow[r, "\mu"] & A \end{tikzcd}$$

In this class, we'll assume the tensor product of vector spaces is associative. That is, we'll pretend that $(V \otimes W) \otimes X = V \otimes (W \otimes X)$. These aren't really equal, but there is a totally canonical isomorphism. I'm not sure if Kolya will need the associator. NR: at the very end.

Similarly, for sets, we'll pretend $(S_1 \sqcup S_2) \sqcup S_3 = S_1 \sqcup (S_2 \sqcup S_3)$. ⋄

Definition 2.8. A *(strict) monoidal category* is (1) a category $\mathcal{C}$ (think Set, **Vect** or **GVect** (with grading-preserving maps)), (2) an associative product functor $\mathcal{C} \times \mathcal{C} \xrightarrow{\otimes} \mathcal{C}$ ($\otimes$ or $\sqcup$), and (3) a unit object $\mathbb{1}$ such that $\mathbb{1} \otimes X = X = X \otimes \mathbb{1}$. The diagrams above define an algebra in $\mathcal{C}$. ⋄

An algebra in **GVect** is a superalgebra.

To define *commutative algebras* (A,μ) in $\mathcal{C}$, we want to say that

$$\begin{tikzcd} A \otimes A \arrow[r, "\mu"] \arrow[d, "c"'] & A \\ A \otimes A \arrow[ur, "\mu"'] & \end{tikzcd}$$

commutes, where c is some kind of "flip map". If $\mathcal{C} = \mathsf{Vect}$, then the usual flip map is ok, but if $\mathcal{C} = \mathsf{GVect}$, then you need to involve the sign rule somehow.

Definition 2.9. A *symmetric* monidal category $(C, \otimes)$ is a monoidal category with a *braiding* natural isomorphism $c : V \otimes W \xrightarrow{\sim} W \otimes V$ satisfying

– (Yang-Baxter equation)

$$\begin{tikzcd} U \otimes V \otimes W \arrow[r, "c_{U\otimes V,W}"] \arrow[d, "\mathrm{id} \otimes c_{V,W}"'] & W \otimes U \otimes V \\ U \otimes W \otimes V \arrow[ur, "c_{U,W} \otimes \mathrm{id}"'] & \end{tikzcd}$$

And the obvious symmetric diagram [[★★★]]. So far, we've defined a *braided monoidal category*.

– $c_{V,W} \circ c_{W,V} = \mathrm{id}_{V \otimes W}$ ◇

Now we have defined a commutative algebra in a symmetric monoidal category. Note that in a braided monoidal category, you have to make a choice between $c_{A,A}$ and $c^{-1}_{A,A}$.

Definition 2.10. The symmetric monoidal category SVect of super vector spaces has objects graded vector spaces, morphisms are even morphisms, monoidal structure defined as before, and braiding $V \otimes W \xrightarrow{c_{V,W}} W \otimes V$ is defined on homogeneous elements as $v \otimes w \mapsto (-1)^{|v|\cdot|w|} w \otimes v$. ◇

I invite you to check the little diagrams.

Lemma 2.11. *A commutative superalgebra is the same as a commutative algebra object in* SVect.

This categorical point of view has huge advantages because you can define all the usual objects you define in linear algebra. Once you define something categorically, you know how to "superize it". Noah: you may have seen this sign rule before with tensor products of Clifford algebras. PT: let me expand on that. If A, B are algebra objects in $(\mathcal{C}, \otimes)$, then is $A \otimes B$ an algebra object? Well, we have to find a map $A \otimes B \otimes A \otimes B \xrightarrow{\mu_{A\otimes B}} A \otimes B$

$$\begin{tikzcd} A \otimes B \otimes A \otimes B \arrow[r, "\mu_{A\otimes B}"] \arrow[d, "\mathrm{id} \otimes c \otimes \mathrm{id}"'] & A \otimes B \\ A \otimes A \otimes B \otimes B \arrow[ur, "\mu_A \otimes \mu_B"'] & \end{tikzcd}$$

You need the braiding to do this, and you can check that this gives you an algebra structure on $A \otimes B$. What Noah was saying is that with this tensor product, we have $Cl(V_1, b_1) \otimes Cl(V_2, b_2) \cong Cl(V_1 \oplus V_2, b_1 \perp b_2)$. Note that $T(V_1 \oplus V_2) \cong TV_1 \otimes TV_2$.

Definition 2.12. If $(\mathcal{C}, \otimes, c)$ is symmetric monoidal, a *Lie algebra* in $\mathcal{C}$ is an object L together with a bracket $[,] : L \otimes L \to L$ such that you have (diagrammatically) (1) skew-symmetric and (2) Jacobi. ◇

Antisymmetry:

$$\begin{tikzcd} L \otimes L \arrow[r, "{[\cdot,\cdot]}"] \arrow[d, "c_{L,L}"'] & L \\ L \otimes L \arrow[r, "{[\cdot,\cdot]}"] & L \arrow[u, "-\mathrm{id}"'] \end{tikzcd}$$

I think we need $\mathcal{C}$ to be an additive category so that $-\mathrm{id}$ makes sense. You'll need additive for Jacobi as well.

3 NR 08-31 Legendre transform, Hamiltonian formulation

The Legendre transform

Definition 3.1. A smooth function $\mathcal{L} \in C^\infty(TN)$ is *non-degenerate* if $\det\left(\frac{\partial^2 \mathcal{L}}{\partial \xi_i \partial \xi_j}\right) \neq 0$ for all (ξ, q). ⋄

Definition 3.2. $\mathcal{L}$ is *strongly non-degenerate* if $\left(\frac{\partial^2 \mathcal{L}}{\partial \xi_i \partial \xi_j}\right)$ is positive definite for all (ξ, q). ⋄

For a function $\mathcal{L} \in C^\infty(TN)$, define $\mathcal{H} \in C^\infty(T^*N)$ by

$$\mathcal{H}(p, q) = \max_{\bar\xi \in T_qN} \big(p(\bar\xi) - \mathcal{L}(\bar\xi, q)\big).$$

To find the $\bar\xi$ for which $p(\bar\xi) - \mathcal{L}(\bar\xi, q)$ is maximum, we differentiate with respec to $\bar\xi$, and we see that we are trying to solve for $\bar\xi$ in

$$p(\bar\xi) = \frac{\partial \mathcal{L}}{\partial \xi}(\bar\xi, q). \tag{3.3}$$

Definition 3.4. The *Legendre transfrom* is the map $TN \to T^*N$ given by $(\xi, q) \mapsto \left(\frac{\partial \mathcal{L}}{\partial \xi}(\xi, q), q\right)$. [[★★★ notation . . . two different ξ's.]] ⋄

This definition of the Legendre transform also works for $\mathcal{L}$ a non-smooth convex function (some higher than first derivatives may be discontinuous).

Proposition 3.5. *If $\mathcal{L}$ is strongly non-degenerate (resp. non-degenerate), then the Legendre transform is an isomorphisms (resp. local isomorphism). In particular, equation (3.3) has a unique solution $\bar\xi$ for a given (p, q).*

Thus, if $\mathcal{L}$ is strongly non-degenerate, there is a unique solution to (3.3), so we can define $\mathcal{H}$.

The "inverse transformation" is

$$\tilde{\mathcal{L}}(\xi, q) = \max_{\bar p \in T_q^*N} \big(\bar p(\xi) - \mathcal{H}(\bar p, q)\big).$$

This $\tilde{\mathcal{L}}$ is the convex hull of $\mathcal{L}$ (see, e.g. [CdS03]).

Theorem 3.6. *If $\mathcal{L}$ is fiberwise convex (for all ξ, each q), then $\tilde{\mathcal{L}} = \mathcal{L}$.*

[[★★★ how is convexity used . . . wikipedia picture]] If we assume $\mathcal{L}$ is smooth, convexity is equivalent being strongly non-degenerate.

In our applications, $\mathcal{L}$ will either be non-degenerate, which insures fiberwise convex, or the Hessian $\left(\frac{\partial^2 \mathcal{L}}{\partial \xi_i \partial \xi_j}\right)$ will be identitcally zero.

Theorem 3.7. *The image of classical trajectories on TN in T^*N (with respect to the Legendre transform) are solutions to the following first order system.*

$$\dot p_i = \frac{\partial \mathcal{H}}{\partial q^i} \qquad \dot q^i = -\frac{\partial \mathcal{H}}{\partial p^i}$$

Where $\mathcal{H}$ is the Hamiltonian of the system (given by the Legendre transform of the Lagrangian).

Proof. [[★★★ HW4. It's easy from the ingredients.]] □

We'll see a coordinate-free formulation of this theorem later (Corollary 3.11).

Elements of symplectic geometry 1

Recall that a symplectic manifold is a pair (M, ω) where ω is a closed nondegenerate 2-form on M. Nondegeneracy means that when you think of ω_x as a map $T_xM \to T_x^*M$, it is an isomorphism. Thus, we have an inverse map ω_x^{-1}, and we can think of ω^{-1} as a section of $\bigwedge^2 TM$ (i.e. as a bivector field). If we choose local coordinates x^i on M, then we can write ω_x as

$$\omega_x = \sum_{i,j} (\omega_x)_{ij}\, dx^i \wedge dx^j$$

$$\omega_x^{-1} = \sum_{i,j} (\omega_x)^{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}$$

Example 3.8 ($M = T^*N$)**.** For a smooth manifold N, the projection $\pi\colon T^*N \to N$ induces a 1-form $T^*T^*N \overset{\alpha}{\leftarrow} T^*N$ on M. This 1-form

α is called the *canincal 1-form* on T^*N. Explicitly, if $x \in T_{(p,q)}T^*N$ is a tangent vector,

$$\alpha_{(p,q)}(x) = p\big(d\pi(x)\big)$$

where $p(\beta)$ is the natural pairing of $p \in T_q^*N$ with $\beta \in T_qN$. and $d\pi\colon TT^*N \to TN$ is the differential of the canonical projection $\pi\colon T^*N \to N$.

In local coordinates $\{q^i\}$ and corresponding coordinates $\{p_i\}$ on T_q^*N, one can check that

$$\alpha = \sum_i p_i dq^i.$$

The 2-form $\omega = d\alpha$ is a sypmlectic form on $M = T^*N$. In local coordinates, $\omega = \sum_i dp_i \wedge dq^i$. ◇

Definition 3.9. *A a commutative algebra over $\mathbb{C}$ is a Poisson algebra if it is a Lie algebra with some bracket $\{\,,\}$ and the Lie algebra structure acts by derivations on the commutative algebra structure:$\{ab, c\} = a\{b, c\} + b\{a, c\}$.* ◇

[[★★★ HW2 formulate the notion of super Poisson algebras. c.f. PT's lectures for commutative super algebras and Lie super algebras]]

Theorem 3.10. *For a syplectic manifold (M, ω), $C^\infty(M)$ with point-wise multiplication and the Poisson bracket given by*

$$\{f, g\} = \omega^{-1}(df \wedge dg) = \sum_{ij} (\omega^{-1})^{ij} \Big(\frac{\partial f}{\partial x^i}\frac{\partial g}{\partial x^j} - \frac{\partial g}{\partial x^i}\frac{\partial f}{\partial x^j}\Big)$$

is a Poisson algebra.

Proof. It is clear that it is a commutative associative algebra. The operation $\{\,,\}$ is a first order bidifferential operator, it will satisfy the Liebniz rule, so we need only check the Jacobi identity, which follows from $d\omega = 0$ [[★★★ HW3, it's very easy]]. □

A function $f \in C^\infty(M)$ induces a vector field $v_f := \omega^{-1}(df)$ on M.

Corollary 3.11 (to Theorem 3.7)**.** *The image $x(\tau)$ in T^*N of a classical trajectory $\big(\dot\gamma(\tau), \gamma(\tau)\big)$ on TN (with respect to the Legendre transform) is a flow line of the Hamiltonian vector field $v_{\mathcal{H}} = \omega^{-1}(d\mathcal{H})$. In other words, $\dot x(\tau) = \omega^{-1}(d\mathcal{H})(x(\tau))$ [[★★★ can this be simplified]].*

The main moral of this transformation is that under the Legendre transform, the trajectories become flow lines of the Hamiltonian vector field.

Variational principle in Hamiltonian mechanics

Assume $\mathcal{L}$ is strongly non-degenerate, so it gives an isomorphism between tangent and cotangent bundles. Define $p(\tau) := \frac{\partial\mathcal{L}}{\partial\xi}\big(\dot\gamma(\tau), \gamma(\tau)\big)$. Recall that the Hamiltonian is a function on T^*N given by $\mathcal{H}\big(p(\tau), \gamma(\tau)\big) = p(\bar\xi) - \mathcal{L}\big(\bar\xi, \gamma(\tau)\big)$, where $\bar\xi$ satisfies $p = \frac{\partial\mathcal{L}}{\partial\xi}(\bar\xi, q)$ (that is, $\bar\xi = \dot\gamma(\tau)$). So we have that

$$\mathcal{L}\big(\dot\gamma(\tau), \gamma(\tau)\big) = p\big(\dot\gamma(\tau)\big) - \mathcal{H}\big(p(\tau), \gamma(\tau)\big).$$

We can write the action as

$$\begin{aligned}\mathcal{A}[\gamma_*] := \mathcal{A}[\gamma] &= \int_0^t \Big(p(\tau)\dot\gamma(\tau) - \mathcal{H}\big(p(\tau), q(\tau)\big)\Big)\, d\tau \\ &= \int_{\gamma^*} \alpha - \int_0^t \mathcal{H}\big(p(\tau), q(\tau)\big)\, d\tau\end{aligned}$$

where $\gamma_* = \{p(\tau), \gamma(\tau)\}_{\tau=0}^t$ is the image of $\gamma = \{\dot\gamma(t), \gamma(t)\}$ in T^*N under the Legendre transformation. The first term universal (in the sense that it only depends on γ_* and the manifold N, but not on the Lagrangian), and the other term really describes the dynamics.

The variation of the action on a classical trajectory can now be written in terms of γ_{cl}^*.

$$\delta\mathcal{A}[\gamma_{cl}^*] = \sum_i \frac{\partial\mathcal{L}}{\partial\xi^i}\big(\dot q(t), q(t)\big)\delta q^i(t) - (t \leftrightarrow 0) = \alpha\big(di(\delta q(t))\big) - \alpha\big(di(\delta q(0))\big)$$

where $\alpha(\delta q)$ is the value of the form α on the vector field $di(\delta q)$ and $i\colon N \hookrightarrow T^*N$ is the zero section.

The goal now is to show that for $B \subseteq TN \times TN$, the image $B_* \subseteq T^*N \times T^*N$ is Lagrangian if we take the difference of the form on the two cotangent bundles. $\Omega = \omega_1 - \omega_2$. There is some L_t some family of Lagrangians with the following property. Solutions to the E-L equations with given boundary conditions (L_*) are $L_* \cap L_t$.

Let γ_{cl}^* be a classical trajectory in the phase space originating at $q \in N$ at time t_1 and ending at time t_2 at $Q \in N$. We can evaluate the action on such a trajectory, giving a function on $N \times N \times \mathbb{R} \times \mathbb{R}$

$$\mathcal{A}[\gamma_{cl}^*] = \mathcal{A}(q, Q, t_1, t_2).$$

Theorem 3.12 (Hamilton-Jacobi)**.**

$$\frac{\partial \mathcal{A}}{\partial t_2} + H(d_Q\mathcal{A}, Q) = 0 \qquad \frac{\partial \mathcal{A}}{\partial t_1} - H(-d_q\mathcal{A}, q) = 0.$$

3 PT 09-04-2007

Your feedback:

- Problem 2 is wrong in the original problem set: the formula for the Berezinian is not multiplicative. The new version has the right assumptions for the formula to be true.
- However, we're still missing some assumptions for the exponential map.
- the Stokes' theorem hint is bad, you just need Stokes' theorem on the interval.

Let's extend the submission date to next week (Sept. 11).

If A is a commutative super algebra, a *free (right) module* of super dimension $(m|n)$ is a free (right) module with m (free) even generators and n (free) odd generators. Recall that $F, A \in \mathsf{SVect}$, so we have $\mu_F \colon F \otimes A \to F$ in SVect (in particular, it is an even map) with the usual commutative diagrams. There is a nice adjunction formula.

$$\mathrm{Hom}_{A\text{-}\mathsf{mod}}(F, V) \cong \mathrm{Hom}_{\mathsf{SVect}}(\mathbb{C}^{m|n}, V_{\text{forget}}).$$

[[★★★ check that super dimension is unique given the module structure. No, over $\mathbb{C}$, declare odd or even. PT: no, that's not a counterexample because F should be in SVect to begin with, so you have a fixed super dimension over $\mathbb{C}$.]] In problem 2, replace the exponential formula by the following property of the Berezinian. $\mathbb{A}[e]/e^2$ with e even, and let $f \colon F \to F$ be an even morphism, then $\mathrm{Ber}(1+ef) = 1 + e\,\mathrm{str}(f)$. Replace the exponential property with this.

Feedback from Barbara: I said that 87 is still too small for the class. She said that those that aren't registered shouldn't get a chair. So register, and then drop whenever you like (you can drop until the very last day of class). The only disadvantage of registering is that I might learn your name.

If you try to say something precisely, you should try to do it categorically because it will keep you honest. $X \in \mathcal{C}$ means that X is an object in $\mathcal{C}$. If $X, Y \in \mathcal{C}$, then we'll write $\mathcal{C}(X, Y)$ for $\mathrm{Hom}_{\mathcal{C}}(X, Y)$. We talked about the notion of a monoidal category. If $\mathcal{C}$ has products[1] (this is a property of $\mathcal{C}$, not extra structure).

Example 3.1. $\mathcal{C} = \mathsf{Set}, \mathsf{Top}, \mathsf{Man}$. ⋄

We could define $X \otimes Y := X \times_{\mathcal{C}} Y$. We have to use choice to pick a representative of $X \times_{\mathcal{C}} Y$, which is only defined up to unique isomorphism. This $\otimes$ is only as associative as the product of sets (there is a canonical associator, which we're ignoring).

Now note that we have two different monoidal structures on Vect. In general, you have to decide which monoidal structure you use. For manifolds or topological spaces, we'll use this monoidal structure (in bordism categories, we'll use $X \otimes Y = X \sqcup Y$ instead), but for vector spaces, we'll use tensor product.

Definition 3.2. Let $(\mathcal{C}, \otimes)$ be a monoidal category. A *group object* G in $\mathcal{C}$ is an object $G \in \mathcal{C}$ together with morphisms $\mu\colon G \otimes G \to G$, $e\colon \mathbb{1} \to G$,, and $\nu\colon G \to G$ satisfying the usual axioms. ⋄

Shenghao: you need a map $G \to G \times G$ for the inverse axiom. PT:

$$\begin{tikzcd} G \times G \arrow[r, "\mathrm{id} \times \nu"] & & G \times G \arrow[d, "\mu"] \\ G \arrow[u, "\Delta"] \arrow[r, dashed, "\varepsilon"] & \mathbb{1} \arrow[r, "e"] & G \end{tikzcd}$$

We need $\mathbb{1}$ to be a zero object to get the dashed arrow (Noah: or just pick the map $G \to \mathbb{1}$, and then you get a Hopf algebra object. PT: yeah, let's do that, so add Δ and ε to the definition, but then you have to say what diagrams they satisfy).

If we take the monoidal structure given by product, then Δ and ε exist canonically.

Remark 3.3. For the notion of a commutative group in $\mathcal{C}$, you need a symmetric monoidal category $(\mathcal{C}, \otimes, c)$. Just as you get different notions of groups by using different monoidal structures, you get different notions of groups by using different braidings. ⋄

[1] If $X_1, X_2 \in \mathcal{C}$, then there is an object $X_1 \times_{\mathcal{C}} X_2 \in \mathcal{C}$ with two projection maps $p_i\colon X_1 \times_{\mathcal{C}} X_2 \to X_i$ such that the map $\mathcal{C}(Y, X_1 \times_{\mathcal{C}} X_2) \xrightarrow{(p_1\circ -)\times(p_2 \circ -)} \mathcal{C}(Y, X_1) \times_{\mathsf{Set}} \mathcal{C}(Y, X_2)$ is a bijection.

Last time, we defined a Lie algebra object in a symmetric monoidal $\mathbb{C}$-linear category $\mathcal{C}$. Noah: I think additive is really the right thing. PT: okay, let's try to always get the right axioms, so let's just say additive instead of $\mathbb{C}$-linear. We'll also say that $\mathbb{C} \hookrightarrow \mathcal{C}(\mathbb{1}, \mathbb{1})$. In this case, we'll have $\mathcal{C}(X, Y) \otimes_{\mathbb{C}} \mathcal{C}(Y, Z)$ instead of product in Set and require composition the map to $\mathcal{C}(X, Z)$ to be $\mathbb{C}$-bilinear?[[★★★]]

Definition 3.4. We'll call this kind of thing (enriched over $\mathbb{C}$) a (symmetric) *tensor category*. ⋄

Definition 3.5. A monoidal category $(\mathcal{C}, \otimes)$ is *closed* if there exist "inner Homs" $\underline{\mathcal{C}}(X, Y) \in \mathcal{C}$ for all $X, Y \in \mathcal{C}$ with natural isomorphisms $\mathcal{C}(W, \underline{\mathcal{C}}(X, Y)) \xrightarrow{\sim} \mathcal{C}(W \otimes X, Y)$. [[★★★ this is weaker than rigid because you don't get a coevaluation map]] ⋄

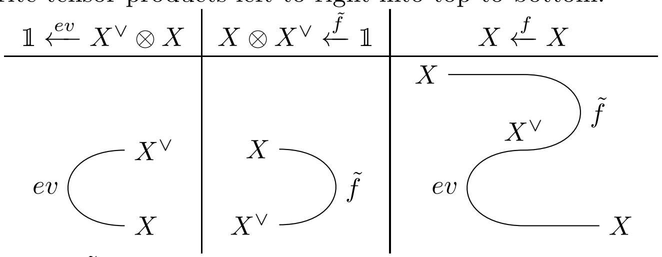
Example 3.6. For vector spaces, you can think of the hom sets as vector spaces, so there is an inner hom which is the same as the usual hom (as a set). What is $\underline{\mathrm{Hom}}_{\mathsf{SVect}}$? It is *all* homomorphisms, not just even morphisms. This is $\mathbb{Z}/2$-graded as before. ⋄

Example 3.7. $(\mathcal{C}, \otimes) = (\mathsf{Top}, \times)$. Is this closed? No; if you want to use the compact-open topology on the hom sets, you need to take the compactly generated product $\times_c$ to get the right adjunction. If you start with Top_c, the category of compactly generated topological spaces, with it's product $\times_c$, then you get a category which *is* closed monoidal. If you take $(\mathsf{Man}, \times)$, this subtlety is gone (but you have to allow infinite-dimensional manifolds for the hom sets to be manifolds). Interesting thing: $\underline{\mathrm{Hom}}_{\mathsf{SMan}}(\mathbb{R}^{0|n}, M)$ is a (finite-dimensional!) supermanifold. ⋄

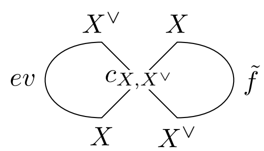
Note that $\mathcal{C}(\mathbb{1}, \underline{\mathcal{C}}(X, Y)) \cong \mathcal{C}(X, Y)$ canonically. For SVect, since a map from $\mathbb{1}$ to $\underline{\mathrm{Hom}}_{\mathsf{SVect}}(X, Y)$ must be even, so it is picking out an even map from X to Y.

We can define the dual to X by $X^\vee := \underline{\mathcal{C}}(X, \mathbb{1})$. Deligne and Morgan get the order in this next thing wrong. There is a canonical evaluation map $X^\vee \otimes X \xrightarrow{ev} \mathbb{1}$ by taking $W = X^\vee$ and $Y = \mathbb{1}$ and taking $\mathrm{id}_{X^\vee}$ on the left hand side. You also get a canonical map $X \otimes Y^\vee \to \underline{\mathcal{C}}(Y, X)$ (this order is important!).

Now we can define a trace. Taking $X = Y$, we have $\mathbb{1} \xrightarrow{\tilde{f}} X \otimes X^\vee \to \underline{\mathcal{C}}(X, X)$ for any $f \in \mathcal{C}(X, X)$. Note that for an infinite-dimensional vector space X, there is no trace $\mathcal{C}(X, X) \to \mathcal{C}(\mathbb{1}, \mathbb{1})$. You can only define trace for $\tilde{f}$. Let's use the convention that we draw pictures with the maps going from right to left, so that it is easy to translate into symbols. We also write tensor products left-to-right into top-to-bottom.



Given $\tilde{f}$, you'd like to stick it onto the evaluation map, but you have to throw in a switch. The picture below is by definition the trace $\mathbb{1} \xleftarrow{\mathrm{tr}(\tilde{f})} \mathbb{1}$ or $\mathrm{tr}_{\mathcal{C}}(\tilde{f}) \in \mathcal{C}(\mathbb{1}, \mathbb{1})$.



Lemma 3.8. $\mathrm{tr}_{\mathsf{SVect}}$ *is* str.

2 RB 09-04-2007

Today we'll continue with what should have been the second half of the first lecture.

Recall that last time we said that a classical field theory consists of (1) a fiber bundle over some manifold M (which might be space-time) and (2) a Lagrangian density, which is a special kind of form on Jet space.

For a quantum (or statistical) field theory, you have to specify a $*$-algebra A acting on some module with a sesquilinear form $(,)$ satisfying some axioms which we'll talk about next week. This algebra will usually be generated by classical fields.

Roughly, to get a form, you take a state $\omega\colon A \to R$ (with $R = \mathbb{R}$ or $\mathbb{R}[[\lambda]]$) on A such that $\omega^* = \omega$ ($\omega^*(a) = \omega(a)$). Then you can define $(a, b) = \omega(ab^*)$. So the main problem is to construct a state ω on the algebra A. The state ω is constructed (at least formally) in terms of Feynman integrals, which look something like

$$\int (\textstyle\int \phi(x)^* f(x) dx) e^{i \int L(\phi) d^4x} D\phi$$

You integrate over the space of all fields. A problem: there is no invariant measure on infinite-dimensional spaces. To get around this, think about what a measure on a space X really is. It is a map from some subsets of X to $\mathbb{R}$ which is countably additive, etc. This approach to thinking about a measure doesn't work very often when you work on an infinite-dimensional space. A "Radon measure" is a linear map from continuous functions with compact support to $\mathbb{R}$, given by taking f to a real number which you think of as $\int f d\mu$. If you have a usual measure, then usual measure theory tells you how to construct the integral of a reasonable function. However, it turns out that you can't even define a Radon measure on an infinite-dimensional space (there probably aren't any continuous functions with compact support except for the zero function).

Instead: define a measure to be a linear map from *some space* of functions to $\mathbb{R}$. We only define the integral of functions we are interested in. But which functions are you interested in? Thinking about it, you see that most things you're interested in are of the form

$$\int (\textstyle\int \phi^4(x) f(x) dx)(\int \phi^2(x) \partial\phi(x) dx) \cdots e^{i \int \text{quadratic}\, dx} D\phi$$

Where the first functions are given by integrating a form on Jet space. So we want to integrate polynomials in (certain forms on Jet space)$\times e^{\text{quadratic}}$. A *Feynman measure* is a linear functional on this space with certain properties (which we'll talk about later). This is a perfectly rigorous concept. You have to remember that physicists are doing perfectly reasonable mathematics, but they lie and say that they're doing something else. If you examine it carefully, a physicist means by "measure" a linear function on this space. PT: what if the fiber is not linear? RB: you want the fiber to at least be an affine space and you choose a vacuum to turn it into a vector space.

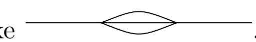
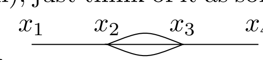
Next problem: is the Feynman measure unique? This is one of the most confusing things about quantum field theory. (1) Most physics books imply that it is, but they are well aware it is NOT. We need to explain why it behaves as if it is unique. Why can we get away with pretending it is unique? The key point is the following: There is an (infinite-dimensional) group of *renormalizations* which acts simply transitively on the Feynman measures. So there is a unique well-defined *orbit* of Feynman measures.

If you try to read a physics book, it gives the following misleading picture. If you start with a Lagrangian, you get a QFT (this is wrong). The correct picture: If you start with a Lagrangian and a Feynman measure, you get a QFT. Furthermore, there is a group acting on Lagrangians and Feynman measure which preserves the QFT, so you should take (Lagrangians)$\times$ (Feynman measures)/(Group of renormalizations). The group of renormalizations is the same size as the renormalization group, so you can often get away with just picking a Lagrangian. AJ: what you're calling "Feynman measure" is what is usually called a regularized path integral? RB: yes, a choice of Feynman measure is equivalent to a choice of regularization and renormalization scheme.

Toy example. Can we find an invariant measure on a 1-dimensional space? Yes, but there is no canonical way to do it if you want a translation-invariant measure: if $d\mu$ is a translation-invariant measure, so is $sd\mu$. Rescaling acts on the space of all translation-invariant measures. Furthermore, the group of rescalings acts simply transitively on the invariant measures on $\mathbb{R}^1$, so it doesn't really matter which measure you take most of the time. The same sort of thing happens in QFT, but the group of renormalizations is infinite-dimensional and non-abelian.

Consequences of non-uniqueness of Feynman measures are things called ANOMALIES. An anomaly is given by the following. Suppose the Lagrangian L is invariant under some group G (say G is the group of gauge transformations). We would expect G to act on the QFT of L. There is a problem because the QFT depends on the Feynman measure as well as L. Can we find a Feynman measure which is also invariant under G (and hence get a G-invariant QFT)? Sometimes you can and sometimes you can't. What is the obstruction to doing this?

We have a group H which acts simply transitively on a space X. In this case, by choosing a point of X, you can identify X with H. Now suppose another group G acts on H and X. Can we identify X with H in a G-invariant way? The answer is that there is an obstruction given by an element of the non-abelian cohomology group $H^1(G, H)$ (so you can do it when this element is zero in the cohomology group). This group H will be the group of renormalizations and G will be the group of symmetries. The problem of whether you can get G acting on the QFT amounts to working out an element of $H^1(G, H)$ and checking if it is zero. Thus, anomalies are elements of $H^1(G, H)$.

Now I'll try to explain quickly what a Feynman diagram is. A Feynman integral can be formally expanded as an infinite series of Feynman diagrams. A Feynman diagram looks like . Each Feynman diagram is an abbreviation for a finite-dimensional integral. Each point represents a point of spacetime and each line represents a propagator $\Delta(x, y)$ (which is a Green's function for the quadratic part of the Lagrangian); just think of it as some function of x and y. Then the Feynman diagram 

represents $\Delta(x_1, x_2)\Delta(x_2, x_3)^3\Delta(x_3, x_4)$ integrated over some of the x_i's. You usually know that this integral does NOT converge.

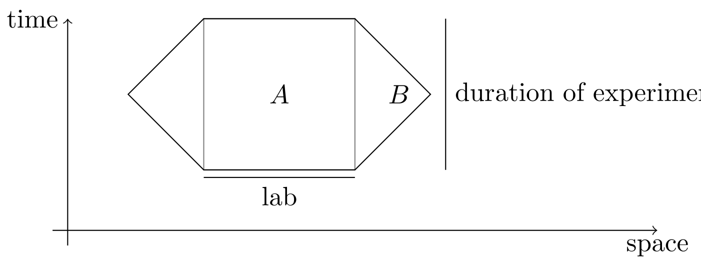
Three problems:

1. infrared divergences.
2. ultraviolet divergences.
3. divergent series (even if each integral converges, the series of them may not).

The obvious thing to do is give up, which is what people historically did at first.

Why does the integral $\int_{-\infty}^{\infty} f(x)dx$ not converge? There are two basic reasons. (1) f may be locally bad (e.g. $f(x) = x^{-10}$, which blows up at zero), which is called an *ultraviolet divergence* (it is short distance singularity). The other reason an integral might not converge is (2) f is globally too large (e.g. $f(x) = x^2$), which is called an *infrared divergence*. In general, these integrals will have both kinds of divergences.

(1) Dealing with infrared divergences. The solution is to ignore them. The key point is that individual Feynman diagrams have infrared divergence, but if you sum over all Feynman diagrams of a given order, the infrared divergences automatically cancel out. There is a simple physical reason you'd expect this to happen. Suppose we're looking at the Lagrangian $m^2\phi^2 + (\partial\phi)^2 + \lambda(x)\phi^4$ for some function λ on spacetime.



A=lab during experiment. It doesn't matter what happens what happens outside of the box A. Well it does matter because something can leave the lab and then come back. Let B=all points where you can send a signal and get it back. We don't care what goes on outside the region B. So there is some compact region which is all we care about. If we want, we can just replace the coupling coefficient λ by a function vanishing outside of B. If we do that, then we find that all the integrals are over a compact set, so there are no large-distance singularities.

Q: what do you mean by "ignoring them"? Do you pretend they are zero, or keep them around until they cancel? RB: The infrared divergence will be given by some parameter going to zero. The integral will diverge if you let this parameter go to zero. However, if you add all the integrals and then let the parameter go to zero, you get a finite number. Q: does it all go away, or do you get a non-zero finite term? RB: there is a non-zero finite term.

(2) Ultraviolet divergences. These are much trickier to deal with and they definitely don't cancel out. Distributions t have no ultraviolet divergences (almost by definition, $\int t(x)f(x)dx$ is defined for all f smooth of compact support). So we could get rid of ultraviolet divergences if we could replace $\Delta(x_1,x_2)\Delta(x_2,x_3)^3\Delta(x_3,x_4)$ by a distribution, we could eliminate ultraviolet divergences. Each of the factors is a distribution, but the product of distributions need not be a distribution (e.g. if you square the Dirac delta, you get nonsense).

Regularize the propagator $\Delta(x_1,x_2)$. This means you add an extra variable $\Delta(x_1,x_2,\varepsilon)$. For example, $\Delta(x_1,x_2)$ might be a Fourier transform of $(p^2+m^2)^{-1}$. If you integrate over large values of p, it will be infinite. What you can do is replace $(p^2+m^2)^{-1}$ by $(p^2+m^2)^{-1-\varepsilon}$. For ε large and positive, everything converges nicely and we can define all Feynman integrals. Then we want to take $\varepsilon = 0$. What we do is we get an analytic function of ε and take the analytic continuation to $\varepsilon = 0$ and look at the value there. It turns out you can't do this because there is a pole at $\varepsilon = 0$ in general. You deal with this pole by renormalization.

4 NR 09-05

Recall from lecture 2 that given a boundary problem $B \subseteq TN \times TN$, we try to find a function F on B satisfying the condition

$$dF(V,Q,v,q) = \frac{\partial \mathcal{L}}{\partial \xi}(v,q)dq - \frac{\partial \mathcal{L}}{\partial \xi}(V,Q)dQ = p\,dq - P\,dQ.$$

where $p = \frac{\partial \mathcal{L}}{\partial \xi}(v,q)$ and $P = \frac{\partial \mathcal{L}}{\partial \xi}(V,Q)$. If the form on the right is closed (i.e. if $dp \wedge dq - dP \wedge dQ = 0$), then such an F exists locally. [[★★★ I'm not sure this coordinate-free description is quite right.]] In coordinates $\{x^a\}$ on B, the closedness condition is

$$\sum_i \Big(\frac{\partial P_i}{\partial x^a}\frac{\partial Q^i}{\partial x^b} - \frac{\partial q_i}{\partial x^a}\frac{\partial q^i}{\partial x^b}\Big) - [a \leftrightarrow b] = 0$$

In other words, a necessary condition for existence of such an F is that

$$\Omega|_{B_*} = 0$$

where B_* is the Legendre transform of B and $\Omega = \pi_1^*\omega - \pi_2^*\omega_2$ is the symplectic form on $T^*N \times \overline{T^*N}$.[1] Recall that such a submanifold is called *isotropic*.

Elements of symplectic geometry 2

Definition 4.1. B_* is an *isotropic submanifold* of a symplectic manifold if the restriction of the symplectic form to B_* is zero. ⋄

Definition 4.2. An isotropic submanifold of maximal dimension is called a Lagrangian submanifold. ⋄

It is clear that if the dimension of the symplectic manifold is $2n$, then the maximal dimension of an isotropic submanifold is n (this follows from non-degeneracy of the symplectic form).

Now let's look at the E-L solutions. From the theory of ODEs, we know that we generically need to fix $2n$ coordinates to get finitely many solutions. Thus, if you want extrema, this boundary manifold B_* should be Lagrangian.

Thus, a boundary condition B is variational and the number of solutions is generically finite if and only if B_* is a Lagrangian submanifold in $T^*N \times T^*N$ with the symplectic form Ω.

For each extremum of $\mathcal{A}$ with fixed end points (Q,q), we have the function $\mathcal{A}[\gamma^*_{cl}]$ on a neighborhood of $(Q,q) \in N \times N$. Assume that each of these functions extends to a function on $N \times N$. In free motion on a Riemannian manifold N these will be the geodesics connecting two generic points. Now we can take the differential of this function with respect to the first or second argument gives the submanifold

$$L_t^{(\gamma_{cl})} := \{(P,Q,p,q) \in T^*N \times T^*N | P = d_Q\mathcal{A}[\gamma_{cl}], p = -d_q\mathcal{A}[\gamma_{cl}]\}$$

[[★★★ the condition that $(P,Q,p,q) \in L_t$ is exactly "if you start at (p,q) and flow for time t along the Hamiltonian vector field v_H, you will end up at the point (P,Q)]]

Proposition 4.3. *L_t is a Lagrangian submanifold.*

Proof. [[★★★ HW]] [[★★★ This follows from the fact that flowing for time t, $F_t\colon T^*N \to T^*N$ is a symplectomorphism, and L_t is the graph of this symplectomorphism. In general, the graph of a symplectomorphism $M \to M$ is a Lagrangian in $M \times \overline{M}$.]] □

Now solutions to the E-L equations with boundary conditions B can be identified with points in $L_t \cap B_*$. Note that solutions to the E-L equations can be regarded as points on L_t, and imposing the boundary conditions on them intersects with B_*.

[1] Note that $\lambda_1\pi_1^*\omega + \lambda_2\pi_2^*\omega$ gives a symplectic form on $T^*N \times T^*N$ for any (non-zero) λ_1 and λ_2.

4 PT 09-06

One bit of cleanup from Tuesday. I tried to define a category enriched over abelian groups and there was a question of whether to use $\otimes$ or $\times$ for composition. This is kind of an advertisement for monidal categories. Fix a monoidal category $(\mathcal{A}, \otimes_{\mathcal{A}})$ (think $\mathcal{A} = \mathsf{Ab}$, with monoidal structure given by either $\times$ or by $\otimes_{\mathbb{Z}}$). An $(\mathcal{A}, \otimes_{\mathcal{A}})$-enriched category $\mathcal{C}$ is a class of objects, and for any pair of objects $X, Y \in \mathcal{C}$, a hom object $\mathcal{C}(X, Y) \in \mathcal{A}$ with identity morphisms $\mathrm{id}_X \in \mathcal{A}(\mathbb{1}_{\mathcal{A}}, \mathcal{C}(X, X))$ and with associative composition morphisms $\mathcal{C}(X, Y) \otimes_{\mathcal{A}} \mathcal{C}(Y, Z) \to \mathcal{C}(X, Z)$ in $\mathcal{A}$ (we're shutting associators out of the discussion, but you could throw them in). Q: so this isn't a category? PT: no, it isn't a category. If there exists a functor $(\mathcal{A}, \otimes_{\mathcal{A}}) \to (\mathsf{Set}, \times)$, then you get a category structure on $\mathcal{C}$.

Example 4.1. $\mathcal{C} = \mathsf{Vect}$ (over $\mathbb{C}$) is enriched over $(\mathsf{Vect}, \otimes)$. $f \circ (g + h) = fg + fh$, so the composition is actually bilinear. A category enriched over Vect is called a *linear category.* ◇

Today I want to get to supermanifolds, so I need to take some shortcuts. Let A be a commutative super algebra (i.e. a commutative algebra object in $(\mathsf{SVect}, \otimes, c)$; in particular, the multiplication is even and commutativity means that $a \cdot b = (-1)^{|a|\cdot|b|} b \cdot a$).

Example 4.2. $A = \bigwedge^* \mathbb{C}^n$. Remember that the Clifford algebra is not super commutative. ◇

The category cSAlg of commutative super algebra is itself a monoidal category via

$$\begin{tikzcd} (A \otimes B) \otimes (A \otimes B) \arrow[r, "\mu_{A\otimes B}"] \arrow[d, "\mathrm{id}_A \otimes c_{B,A} \otimes \mathrm{id}_B"'] & A \otimes B \\ (A \otimes A) \otimes (B \otimes B) \arrow[ur, "\mu_A \otimes \mu_B"'] & \end{tikzcd}$$

Fix such an A. Then the category $\mathsf{mod}\text{-}A$ is the category of right A-modules with even A-module homomorphisms.

Lemma 4.3. $\mathsf{mod}\text{-}A$ *is a closed monoidal category.*

Next we will introduce supermanifolds, which can be thought of as a commutative super algebra, and modules will be like sheaves of modules on the supermanifold.

"Proof". We have to define the tensor product and inner hom and verify the adjunction formula. In general, $M \otimes_A N$ makes sense if M is a right module and N is a left module (which it isn't). To define the monoidal structure, we'll turn N into a left module via $\mu_N^\ell := \mu_N \circ c_{A,N}$.

$$\begin{tikzcd} A \otimes N \arrow[r, "c_{A,N}"'] \arrow[rr, bend left, "\mu_N^\ell"] & N \otimes A \arrow[r, "\mu_N"'] & N \end{tikzcd}$$

[[★★★ To check this, you must use that A is commutative]] Now we have two maps $\mathrm{id}_M \otimes \mu_N^\ell, \mu_M \otimes \mathrm{id}_N \colon M \otimes A \otimes N \to M \otimes N$. We define the monoidal structure $M \otimes_A N$ to be the coequalizer in SVect of these two: $M \otimes A \otimes N \rightrightarrows M \otimes N \to M \otimes_A N$.

Now let's define the inner hom. Recall that it should come with isomorphisms $\mathcal{C}(X, \underline{\mathcal{C}}(Y, Z)) \cong \mathcal{C}(X \otimes Y, Z)$.

Lemma 4.4. *If* $M, N \in \mathsf{mod}\text{-}A$*, the A-module* $\mathrm{Hom}_A(M, N)$ *(this includes even and odd morphisms) is an inner hom.*

I have to give you a right A-action on $\mathrm{Hom}_A(M, N)$. $\mathrm{Hom}_A(M, N)$ is a left module via $(a \cdot \phi)(m) = a \cdot \phi(m) = \mu_N^\ell(a \otimes \phi(m))$. Now make this into a right A-module as above. □

Remark 4.5. We used that the two actions (left and right action on N) commute, which follows from commutativity of A. ◇

Q: does Yoneda's lemma work for enriched categories? PT: let's check during the break.

Office hours are moving today; this week, they'll be Friday at 2.

Super-manifolds

These are defined via sheaves.

Definition 4.6. A *super-manifold* $M = (|M|, \mathcal{O}_M)$ *of dimension* $(m|n)$ is a sheaf $\mathcal{O}_M$ of commutative super algebras over a (Hausdorff, second countable) topological space $|M|$ which is locally isomorphic (as a ringed space) to $(U, C^\infty(U) \otimes \bigwedge^* \mathbb{R}^n)$ where $U \subseteq \mathbb{R}^m$ is an open subset. $\diamond$

Remark 4.7. We'll see that $|M|$ will be come a smooth manifold of dimension n. $\diamond$

Definition 4.8. If X is a topological space and $\mathcal{C}$ is a category. A ($\mathcal{C}$-valued) presheaf on X is a functor $\mathcal{F}\colon \mathit{Open}(X)^\circ \to \mathcal{C}$, where $\mathit{Open}(X)$ has objects open subsets of X and morphisms inclusions (i.e. $\mathrm{Hom}_{\mathit{Open}(X)}(U,V) = *$ if $U \subseteq V$ and $\varnothing$ otherwise). $\diamond$

Example 4.9. $\mathcal{F}(U) = C^0(U)$ (we could take $\mathcal{C}$ to be commutative algebras). If X is a smooth manifold, we could define $\mathcal{F}(U) = C^\infty(U)$. $\diamond$

Definition 4.10. Assume $\mathcal{C}$ has all products. A presheaf $\mathcal{F}$ is a *sheaf* if the gluing property is satisfied: for any open covering $U = \bigcup_i U_i$, the sequence

$$\begin{tikzcd} F \arrow[r, "\prod F(\iota_i)"] & \prod_i \mathcal{F}(U_i) \arrow[r, shift left, "\prod(\mathcal{F}\iota_{ij} \circ p_i)"] \arrow[r, shift right, "\prod(p_j \circ \mathcal{F}\iota_{ij})"'] & \prod_{i,j} \mathcal{F}(U_i \cap U_j) \end{tikzcd}$$

is an equalizer in $\mathcal{C}$. $\diamond$

Remark 4.11. If follows (by taking the empty cover of $\varnothing$) that $\mathcal{F}(\varnothing)$ is the terminal object in $\mathcal{C}$. $\diamond$

Definition 4.12. A smooth structure on a topological manifold X is a sheaf $C^\infty(X) \subseteq C^0(X)$ so that $(X, C^\infty(X))$ is locally isomorphic to $(U, C^\infty(U))$ for some $U \subseteq \mathbb{R}^n$ open. $\diamond$

Morphisms of sheaves. (a) Say $\mathcal{F}, \mathcal{G}\colon \mathit{Open}(X)^\circ \to \mathcal{C}$ are two sheaves on the same space X. Then a morphisms of sheaves ($\mathcal{F} \to \mathcal{G}$ is a natural transformation between them. That is, it is a T such that the following diagram commutes for every inclusion $\iota\colon V \hookrightarrow U$.

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\mathcal{F}\iota"] \arrow[d, "T(U)"'] & \mathcal{F}(V) \arrow[d, "T(V)"] \\ \mathcal{G}(U) \arrow[r, "\mathcal{G}\iota"] & \mathcal{G}(V) \end{tikzcd}$$

(b) Let $\mathcal{F}$ be a sheaf on X and $\mathcal{G}$ a sheaf on Y (both valued in $\mathcal{C}$. A morphism from $(X, \mathcal{F})$ to $(Y, \mathcal{G})$ is a continuous map $f\colon X \to Y$ and a natural transformation from $\mathcal{G}$ to $\mathcal{F} \circ f^{-1}$ [[★★★ The other way?!? Ok, this is supposed to be a morphisms of "$\mathcal{C}$-sheaved spaces". Clean up.]].

$$\begin{tikzcd} \mathit{Open}(X) \arrow[r, "\mathcal{F}"] \arrow[dr, phantom, "T\Downarrow"] & \mathcal{C} \\ \mathit{Open}(Y) \arrow[u, "f^{-1}"] \arrow[ur, "\mathcal{G}"'] & \end{tikzcd}$$

Another way to think of it:

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d] & f_*\mathcal{F} \arrow[d] & \mathcal{G} \arrow[l] \arrow[dl] \\ X \arrow[r, "f"] & Y & \end{tikzcd}$$

5 NR 09-07

Last time I gave the Hamiltonian interpretation of these variational problems. Let's move forward now, and we'll come back to it when it becomes more relevant to the goals of the class. I want to talk about Chern-Simons theory, which is an infinite-dimensional version of something. How you should look at the previous sections: $S(A) = \int_M \operatorname{tr}(A \wedge dA + \frac{2}{3} \wedge^3 A)$ is the Chern-Simons action, and $\delta S(A) = 0$. I gave the impression that there is a unique trajectory which connects two points, which was an (untrue) assumption. The condition that $\delta S(A) = 0$ says that $F(A) = 0$ (A is flat connection). $\mathfrak{g} = Lie(G)$ (compact Lie group). Gauge classes of flat connections correspond to classical trajectories. The counterpart to these classical trajectories are γ_{cl}, solutions of the E-L equations. There could be several such γ_{cl}, so all sentences where I said "the classical solution" should be replaced with "a classical solution". For example, if we fix $\gamma_{cl}(0) = q$, $\gamma_{cl}(t) = Q$, then we get $\mathcal{A}[\gamma_{cl}] = \mathcal{A}(q, Q, t)$. $\mathcal{A}_{\gamma_{cl}}$ is then a function on $N \times N \times \mathbb{R}$, but there is a value for each solution γ_{cl}.

I explained that boundary conditions B that have a variational interpretation are, by the Legendre transform, the Lagrangian submanifolds $B_* \subseteq T^*N \times T^*N$ (under the form $\Omega = \omega_1 - \omega_2$). I also described for each γ_{cl} a Lagrangian submanifold $L_t = \{(P, Q, p, q) \in T^*N \times T^*N | P = d_Q \mathcal{A}_{\gamma_{cl}}, p = -d_q \mathcal{A}_{cl}\}$.

Proposition 5.1. *For every γ_{cl}, L_t is a Lagrangian submanifold (I'm a little loose with the term "submanifold"; it's generically smooth).*

Then we have the following intrepretations for solutions to the E-L equations. A solution is a system of Lagrangian submanifolds in $T^*N \times \overline{T^*N}$. Solutions to E-L equations with given boundary conditions B_* are intersection points in $B_* \cap L_t^{\gamma_{cl}}$. If there is only one trajectory connecting two points, then there is only one such L_t, and if B_* and L_t are in generic position, they'll only intersect once, so you'll have a unique trajectory. You can already see the benifit of the Hamiltonian point of view: it gives this geometric interpretation of solutions to the E-L equations.

Now let's move on.

Hamiltonian dynamics on a symplectic manifold

So far, we've had $M = T^*N$. Now let (M, ω) be any symplectic manifold. Fix the Hamiltonian function $H \in C^\infty(M)$. It defines a vector field $v_H = \omega^{-1}(dH)$. The Hamiltonian dynamical system generated by H has trajectories which are flow lines of this vector field v_H (you can take this as the definition of a Hamiltonian dynamical system); i.e. they are solution to the (generally nonlinear) sytem of differential equations

$$\dot{x}(t) = v_H(x(t)). \tag{$*$}$$

We can think of these flow lines as the motion of points. Define $F_t(x) = x(t)$ (assuming the existence and uniqueness of solutions t0 $(*)$, at least for some small interval), where $x(t)$ is the flow line passing through x at $t = 0$. You can think of this as the local action of $\mathbb{R}$ on M, given by shifting points along flow lines. Suppose $f \in C^\infty(M)$, then define $f_t = F_t^*(f)$ (i.e. $f_t(x) = f\big(x(t)\big)$). This gives an action of $\mathbb{R}$ on $C^\infty(M)$.

Theorem 5.2.

1. *$\frac{df_t}{dt} = \{H, f_t\}$ for $f_0 = f$. Now even though the space is infinite-dimensional, the equations of motion are linear.*

2. *$F_t^*(\{f, g\}) = \{F_t^*(f), F_t^*(g)\}$ (it is clear that $F_t^*(fg) = F_t^*(f)F_t^*(g)$, so F_t^* is a Poisson algebra homomorphism).*

[3.] *An infinitesimal version: $\partial_H f \overset{def}{=} \{H, f\}$ is a derivation of the Poisson algebra $(C^\infty(M), \{,\})$.*

Proof. (1)

$$\begin{aligned}
\frac{df_t}{dt} &= \frac{d}{dt} f(x(t)) \\
&= \big(\dot{x}(t), df(x(t))\big) \\
&= \big(v_H(x(t)), df(x(t))\big) \\
&= \big(\omega^{-1}(dH), df(x(t))\big) \\
&= \omega^{-1}(dH \wedge df_t) = \{H, f_t\}
\end{aligned}$$

(3), then we can exponentiate it to get (2).

$$\partial_H(\{f,g\}) = \{\partial_H f, g\} + \{f, \partial_H g\}$$
$$\{H,\{f,g\}\} = \{\{H,f\},g\} + \{f,\{H,g\}\}$$

which is just the Jacobi identity for $\{\,,\}$. □

Proposition 5.3. *v_H is tangent to the level surfaces of H.*

In other words, the dynamics is parallel to the level surfaces of H; if a trajectory originates on a particular level surface, it will remain there. This is good news because instead of solving differential equations on a $2n$-dimensional manifold, we can solve them on a $(2n-1)$-dimensional manifold. The level surfaces of H are physically surfaces of constant energy. Notice that this proposition is true only for conservative systems, when the Lagrangian (and thus the Hamiltonian) do not depend on time. This is also known as conservation of energy.

This property of the Hamiltonian function inspires the following definition.

Definition 5.4. A function $G \in C^\infty(M^{2n})$ is an *integral* of the Hamiltonian dynamics generated by H if

– $F_t^*(G) = G$

– equivalently (because of the Theorem 5.2), $\{H,G\} = 0$. That is, G is in involution with H. ⋄

When we have such a function, it is clear that the level surfaces $G^{(c)} = \{x \in M | G(x) = c\}$ are also preserved by evolution (i.e. v_H is tangent to $G^{(c)}$).

If the system is conservative, we are guarenteed at least one integral (namely H). If another integral G exists, then we know that if a trajectory starts at the intersection $H^{(E)} \cap G^{(c)}$, it will remain there. So we will have reduced our dynamics to a $(2n-2)$-dimensional space. In general, if we have lots of integrals, we can significantly reduce the dimension of our space.

Elements of symplectic geometry 3

What is the maximal number of integrals we can have? What are the extra properties of these integrals we should require?

First let's prepare the ingredients. Let's return to Lagrangian submanifolds. Remember that if we have $L \subseteq M$ such that $\omega|_L = 0$ (i.e. that $T_xL \subseteq T_xM$ is an isotropic subspace for the symplectic form ω_x for all $x \in L$), we call L *isotropic*, and a maximal-dimension isotropic submanifold is called *Lagrangian*, and this maximal dimension is n when M is $2n$-dimensional. Recall that $L \subseteq M$ is called *coisotropic* if $T_xL \subseteq T_xM$ is a coisotropic subspace for the symplectic form ω_x, which means that $(T_xL)^\perp \subseteq T_xL$. Thus, Lagrangian submanifolds are both isotropic and coisotropic. Let's translate this notion to the algebra of functions. What does it mean for the Poisson algebra that a submanifold is Lagrangian?

$C^\infty(M)$ has (i) pointwise commutative multiplication and (ii) a Lie bracket $\{,\}$. From the point of view of the first structure, a submanifold $L \subseteq M$ corresponds to the ideal $I_L \subseteq C^\infty(M)$ of functions which vanish on L. What happens with this ideal if we take into account the second structure. Suppose $f, g \in I_L$ (so f and g vanish on L), then what can we say about $\{f,g\}$? This I_L could be nothing special, a Lie subalgebra, or a Lie ideal. We have

$$\{f,g\}|_L = \omega^{-1}(df \wedge dg)|_L = 0$$

[[★★★ HW; prove this; it's true for L coisotropic]] This means that for an isotropic submanifold L, the vanishing ideal I_L is a Lie-subalgebra. It is *not* a Lie ideal, so $\{I_L, C^\infty(M)\} \not\subseteq I_L$.

Definition 5.5. A *Lagrangian fibration* $M_{2n} \xrightarrow{\pi} B_n$ is a fibration over a base of dimension n, where the generic fibers are Lagrangian submanifolds. ⋄

When we have such a map, we get $\pi^* \colon C^\infty(B_n) \hookrightarrow C^\infty(M_{2n})$, so we have a subalgebra. What special property does this subalgebra have if the fibers are Lagrangian?

Claim. *If $\pi \colon M_{2n} \to B_n$ is a Lagrangian fibration, then $C^\infty(B_n) \hookrightarrow C^\infty(M_{2n})$ is a maximal commutative Lie subalgebra.*

So far I introduced Hamiltonian dynamics on a symplectic manifold, and said that you can think of it as flow lines or as linear dynamics on the space of functions. An isotropic submanifold is an ideal which is a Lie subalgebra.

More generally, if $M_{2n} \to B_k$ such that generic fibers are coisotropic, then $\pi^* : C^\infty(B_k) \hookrightarrow C^\infty(M_{2n})$ is a commutative Lie subalgebra.

A completely integrable system is: if you have a Lagrangian fibration so that the flow lines are parallel to the fibers. Hamiltonian dynamics on symplectic manifold with Lagrangian fibration such that blah is a completely integrable system.

6 NR 09-10

I want to try to finish Hamiltonian dynamics this lecture. Recall that last time, we introduced Hamiltonian dynamics on a symplectic manifold (M, ω), and we defined Lagrangian fibrations. Geometrically, we fix a function $H \in C^\infty(M)$, then we get $v_H = \omega^{-1}(dH)$, and trajectories of our system are flow lines of this vector field: $\dot{x}(t) = v_H(x(t))$. Algebraically, $f_t(x) = f(x(t))$, $\frac{df_t}{dt} = \{H, f\}$.

An integral of motion is a function whose values along these flow lines are constant () $F(x(t))$ =const, () $\{F, H\} = 0$, or () flow lines are paralel to level surfaces of F.

Isotropic subspaces. We have $\omega \in \wedge^2 T^*M$. $L \subseteq M$ is *isotropic* if $\omega(\xi \wedge \eta) = 0$ for all $\xi, \eta \in TL \subseteq TM$. That is, $TL \subseteq (TL)^\perp$ (where the $\perp$ is with respect to the pairing ω). We say L is *coisotropic* if $(TL)^\perp \subseteq TL$. We say L is *Lagrangian* if $(TL)^\perp = TL$ (i.e. if L is both isotropic and coisotropic).

Proposition 6.1. *If L is isotropic $I_L \subseteq C^\infty(M)$ is a Lie subalgebra of the poisson algebra of functions on M.*

Last time I said this for isotropic L, but I was corrected that this should be true for L coisotropic. Let's do an experiment.

Example 6.2. $M = \mathbb{R}^{2n}$, and take $L \subseteq \mathbb{R}^{2n}$ to be the submanifold given by $\{q_1 = 0\}$ (this is $(2n-1)$-dimensional). This is clearly coisotropic. We have that $I_L = \{q_1 f(p, q)\}$, and $\{q_1 f, q_1 g\}$ will be proportional to q_1, so the proposition is true for coisotropic; we have $\{I_L, I_L\} = 0$. ⋄

Proof. [[★★★ HW. At least half of the class should do this.]] □

At the end of this week, we'll start talking about term paper stuff.

Suppose F_1 and F_2 are two integrals of motion. If a point belongs to some level surface $F_1^{(c)}$, it will remain there. The same is true of F_2. If $x \in F_1^{(c_1)} \cap F_2^{(c_2)}$, then it will stay there, but what is the dimension of the submanifold spanned by these trajectories? There are two possibilities. If F_1 and F_2 are integrals, then $\{F_1, F_2\}$ is also an integral. So you can keep taking brackets to get more and more integrals. We want to know how this impacts the dynamics. I don't want to get into this. It is called Hamiltonian dynamics with constraints, and it is important.[[★★★ In the lecture notes, NR will write what this is and give some references]] There is one particular case which is imporatant to us, which is when $\{F_1, F_2\} = 0$. This is the beginning of the notion of integrable systems.

Completely integrable systems

These are the systems where you have the maximal number of Poisson commuting integrals. $F_1, \dots, F_n \in C^\infty(M)$ integrals, with $\{F_i, F_j\} = 0$, such that $dF_1 \wedge \cdots \wedge dF_n \neq 0$ (the F_i are *independent*.

Definition 6.3. The Hamiltonian system generated by $H \in C^\infty(M)$ is *(completely) integrable* if there exists such integrals $F_1, \dots, F_n$. $\diamond$

Geometrically, this means that we have a Lagrangian fibration $F_1 \times \cdots \times F_n \colon M_{2n} \to \mathbb{R}^n$ (i.e. the generic fiber is a Lagrangian submanifold in M). Why is this definition important?

Theorem 6.4. *For an integrable sustem, we have the following.*

1. *Level surfaces of this fibration map (i.e. fibers) are invariant with respect to flow.*
2. *Each generic fiber has an affine structure (i.e. each generic fiber we can cover by an atlas with transition functions which are rotations and translations).*
3. *Compact fibers are n-dimensional tori.*

Let $(\phi_1, \dots, \phi_n)$ be local coordinates in this affine coordinate system. These are called *angle coordinates.* Coordinates on $\mathbb{R}^n$ given by the values of these functions, $c_1, \dots, c_n$ are called *action coordinates.* The flow lines of any H which Poisson commutes with these F_i are straight lines in these coordinates:

$$\phi_i(t) = \omega_i(H, c_1, \dots, c_n)t + \phi_i$$

These ω_i are called *frequencies* and in the case of a compact fiber, these really are the frequencies of the trajectories around the torus; if we fix a coordinate, the flow is given by the action of $\mathbb{R}$ on the torus. If these frequencies are rational, then you have periodic trajectories around the torus. If only some of them are rational, in which case we get a dense path in the torus. If all of them are irrational, you also get a dense covering. You can also imaging the situation that for every generic fiber, these frequencies are rational. Then the flow lines span one dimensional manifolds (or maybe 2-dimensional).

This is called *superintegrability (or degenerate integrability).* In this case, the invariant submanifolds have dimension smaller than n. In a normal integrable system, the dimension of invariant submanifolds is at most n. If you relax the condition of commutativity of the integrals, then you can get such a system. The F_i generate a subalgebra of $C^\infty(M)$. Normally, this is a commutative subalgebra, and if we want it to be commutative, it is not possible to get more than n such functions. Instead, we can require (i) the commutative (multiplication) subalgebra F in $C^\infty(M)$ generated by $F_1, \dots, F_k$ (where $k \geq n$) is a Poisson subalgebra (i.e. $\{F_i, F_j\} = G_{ij}(F_i \dots F_k)$, (ii) $\{F_i, H\} = 0$, (iii) the center $Z(F)$ has rank $2n-k$. Note that if $k = n$, condition (iii) says that the whole algebra is commutative. In such a system, the Liouville theorem holds, but level surfaces of the fibration should be replaced by level surfaces of the center $Z(F)$. Then the dimension of invariant tori will be $2n - k \leq n$.

Integrable systems are the ones where one can say something really explicit about the dynamics. When working with non-integrable systems, you try to find an integrable system which is close by, and then use some perturbation theory. I'll put up references for Hamiltonian dynamics and integrable systems.

PT: can you say something about this affine structure. NR: roughly, $\phi_1, \dots, \phi_n$ are given by flow lines of $F_1, \dots, F_n$ (you can think of each of the integrals as a Hamiltonian). $(F_1, \dots, F_n) \colon M_{2n} \to \mathbb{R}^n$. PT: And the coordinate changes are affine linear because of the Poisson bracket? NR: I think that's probably right. If I try to reconstruct the proof now, I'll waste the rest of the time, so I'll talk about these coordinates next time. It follows from the involuativity of the integrals. These are the same reasons you use the center $Z(F)$ for the more general Liouville theorem. PT: and you get disjoint unions of tori? NR: yes. I'll return to this question next time.

5 PT 09-11

Definition 5.1. A *supermanifold* $M = (|M|, \mathcal{O}_M)$ of dimension $(m|n)$ is a (second countable, Hausdorff) topological space $|M|$, together with a sheaf $\mathcal{O}_M$ of commutative super algebras, which is locally isomorphic to $\mathbb{R}^{m|n} := (\mathbb{R}^m, \mathcal{O}_{\mathbb{R}^{m|n}})$, where $\mathcal{O}_{\mathbb{R}^{m|n}} = C^\infty(\mathbb{R}^m) \otimes \bigwedge^* \mathbb{R}^n$. ("$m$ even variables, n odd variables") ◇

If you like sheaves, you should use them, but we'll see that you can avoid sheaves if you really hate them.

Remark 5.2. Depending on the ground field ($\mathbb{R}$ or $\mathbb{C}$), you get a different notion because $C^\infty\mathbb{R}^m$ could be smooth $\mathbb{R}$-valued or $\mathbb{C}$-valued functions. This gives us real or complex super manifolds. The usual convention is that real super manifolds are called *super manifolds* and complex super manifolds are called *CS manifolds*. You don't say "complex supermanifolds" because it gets confused with "super complex manifolds". You know that complex and real manifolds are very different, but this distinction between supermanifolds and CS manifolds is new ... we didn't have two different notions before. In the super world, these are very different beasts. ◇

Definition 5.3. A *super complex manifold* is something locally isomorphic to $\mathcal{O}^{an}_{\mathbb{C}^{m|n}} := C^{an}(\mathbb{C}^m) \otimes \bigwedge^* \mathbb{C}^n$. ◇

NR: you can draw an analogy with representation theory; real and complex representations are very different. PT: yes, we'll see that it is basically the same reason that supermanifolds and CS manifolds are different.

Let $J(U)$ (for $U \subseteq |M|$ open) be the ideal in $\mathcal{O}_M(U)$ generated by the odd elements. Note that any section of J is nilpotent because a high enough power of a (finite) linear combination of odd elements eventually has a square of an odd element in each term, and odd elements square to zero.

Claim. *$M_{red} := (|M|, \mathcal{O}_M/J)$ is a smooth structure on $|M|$.*

Locally, we have $\left(C^\infty(\mathbb{R}^m) \otimes \bigwedge^* \mathbb{R}^n\right)/J \cong C^\infty(\mathbb{R}^m)$. I won't explain why this quotient is actually a sheaf (you don't need to sheafify); it will be in the homework. A smooth structure is actually an embedding of the sheaf of smooth functions into the sheaf of continuous functions. So given $\phi \in \mathcal{O}_M/J$, we construct the function $(x \mapsto \lambda(x))$ where λ is uniquely determined by the condition that $\phi - \lambda(x) \cdot 1 \in \mathcal{O}_M/J$ is *non*-invertible in any neighborhood of x.

The procedure above gives you a way to take any section of $\mathcal{O}_M$ and produce an honest function on $|M|$. The projection $\mathcal{O}_M \to \mathcal{O}_M/J$ induces an embedding of supermanifolds $M^{m|0}_{red} \hookrightarrow M^{m|n}$ (interpreting a manifold as a supermanifold). The map on underlying spaces is the identity, and the projection $\mathcal{O}_M \to \mathcal{O}_M/J$ is the morphism of sheaves. Sometimes we'll confuse $M^{m|0}_{red}$ with $|M|$, just like we do all the time (even though the first one has an extra structure: a sheaf of smooth functions).

Now let's look at the main examples. We've already seen one example: the local model $(\mathbb{R}^m, \mathcal{O}_{\mathbb{R}^{m|n}})$. We can take any smooth manifold and tensor with $\bigwedge^* \mathbb{R}^n$, throwing in a constant odd fiber. But you can also twist the odd fiber around.

Example 5.4. Let $E^n \to X^m$ (n is the dimension of the fiber, so E is $(m+n)$-dimensional) be a smooth vector bundle over a manifold X. Define the super manifold πE as the pair $(X, \mathcal{O}_{\pi E})$ via $\mathcal{O}_{\pi E}(U) = C^\infty(U, \bigwedge^* E) = \{\text{smooth sections of the bundle } \bigwedge^* E \to X\}$. This is a super manifold of dimension $m|n$. ◇

Let's discuss the different notions from the remark [[★★★]]. We have two different notions of vector bundle: real and complex. Depending on which one we use, you get a different notion. Take your favorite example of a complex vector bundle which is not a real vector bundle tensor $\mathbb{C}$ and you get the following corollary.

Corollary 5.5. *There are "more" CS manifolds than (real) supermanifolds.*

Take the canonical line bundle on a complex manifold. Say $\mathcal{O}(-1)$ on $\mathbb{CP}^1 = S^2$. This is a complex bundle, but every real line bundle is trivial on S^2 is trivial because S^2 is simply connected. But $\mathcal{O}(-1)$ has non-trivial Chern class, so it is non-trivial. There could still be some weird real manifold so that when you tensor up with $\mathbb{C}$ you get this $\pi\mathcal{O}(-1)$. To rule that out, you need the following theorem.

Theorem 5.6 (Batchelor)**.** *Any ($\mathbb{R}$ or $\mathbb{C}$) super manifold M is isomorphic to πE for some smooth vector bundle E over M_{red}.*

Noah: do these isomorphisms play well with the morphisms? PT: No. if the answer were yes, then we wouldn't have introduced super manifolds. Santiago: you want to think of differential forms as sections of the odd tangent bundle, so shouldn't you be taking sections of the dual of $\bigwedge^* E$? PT: that's true if you're thinking of a quotient of a tensor power, but I'm thinking of multilinear forms (which maybe I should denote $\bigwedge^*(E^*)$).

Proof. I'll write $\mathcal{O}_M \to |M|$ to mean a sheaf over $|M|$. We can form $\mathcal{O}_M/J \to |M|$, which is the smooth manifold M_{red}. We have $J/J^2 \to |M|$, a sheaf of $\mathcal{O}_M/J$-modules over M_{red}. Locally, $J/J^2 \cong C^\infty(\mathbb{R}^m)\otimes\bigwedge^1\mathbb{R}^n \cong C^\infty(\mathbb{R}^m)\otimes\mathbb{R}^n$ (since $J = C^\infty(\mathbb{R}^m)\otimes\bigwedge^{\geq 1}\mathbb{R}^n$ and $J^2 = C^\infty(\mathbb{R}^m)\otimes\bigwedge^{\geq 2}\mathbb{R}^n$. This is a vector bundle over M_{red}.[1]

[[★★★ HW to finish the proof. In particular, prove that all these things are sheaves. Show that if E is the vector bundle J/J^2, $\pi E \cong M^{m|n}$. The dimensions are right and the underlying spaces are the same. In the proof, you have to use partitions of unity (we're in the smooth category); in the analytic context, this is not true. It may look obvious, but you have to check. If you take a filtered vector space, take the associated graded, and think of it as a filtered vector space, you don't get the original thing back. You really have to check that the two things are the same.]] □

Thanks for the homework. I'm saying the homework in class today, and waiting for feedback before I put it on the website.

I put the first project up: Let $(\mathcal{A},\otimes)$ be a closed monoidal category. Formulate a proof of the Yoneda lemma for $\mathcal{A}$-enriched categories.

Kolya and I discussed this business. We agreed that you can do the projects in groups of one, two, or three people. If there is just one of you, you can write a paper or give a talk. If there are three of you, you should do both. If there are two, it will be something in between. If you want this problem, you should take it, but if you think there will be more interesting things, then you should wait. It will be first come first served.

Next we'll do a theorem which will get rid of sheaves. We've already gotten rid of sheaves on objects (the theorem that specifying a vector bundle is enough). Now let's get rid of sheaves on morphisms.

We'll define $C^\infty(M) := \mathcal{O}_M(|M|)$ to be the commutative super algebra of functions. Note that this is not $C^\infty(M_{red})$.

Theorem 5.7. *The map $\mathsf{SMan}(M,N) \to \mathsf{SAlg}\big(C^\infty(N), C^\infty(M)\big)$ is bijective. That is, a morphism of supermanifolds is completely determined by the induced morphism on global sections of the structure sheaves.*

I'm going to skip the theorem. We could make it HW, but let's not. Again, you need partitions of unity and it fails in the analytic case.

Corollary 5.8. *There is an equivalence of categories $\bigwedge$-$\mathsf{Vect} \to \mathsf{SMan}$, where the objects of $\bigwedge$-Vect are smooth vector bundles and the morphisms are given by the theorem: a morphism $(E \to X) \to (E' \to X')$ are pairs (f,ϕ), where $f\colon X \to X'$ and a $\phi \in \mathsf{SAlg}(C^\infty(X', \bigwedge^*(f^*E')), C^\infty(X, \bigwedge^* E))$ (these are very different from usual bundle maps; this is a morphism from $C^\infty(\pi f^*E') \to C^\infty(\pi E)$).*

Example 5.9. $\mathbb{R}^{0|2} \xrightarrow{\alpha} \mathbb{R}^{1|0}$ given by "$(\theta_1,\theta_2) \mapsto \theta_1\cdot\theta_2$", is described via the algebra homomorphism $\Lambda^*[\theta_1,\theta_2] = C^\infty(\mathbb{R}^{0|2}) \leftarrow C^\infty(\mathbb{R}^1) \ni x$ given by $x \mapsto \lambda\theta_1\theta_2$ for some $\lambda \in \mathbb{R}$. Why is this well-defined? It looks like it's only defined on the polynomial algebra $\mathbb{R}[x]$. If this were well-defined, we could take any map $f\colon \mathbb{R} \to \mathbb{R}$ and get $x \mapsto f(\theta_1\theta_2)$. The point is that we have Taylor series, and $\theta_1\theta_2$ is nilpotent, so the Taylor series automatically converges (normally, it only converges for analytic functions).

A more interesting example might have been $\mathbb{R}^{1|2} \to \mathbb{R}$, given by $(y,\theta_1,\theta_2) \mapsto y + \theta_1\theta_2$. Here you'd need to find

$$f(y+\theta_1\theta_2) = \sum_{n=0}^{\infty} \frac{f^{(n)}(y)}{n!}(\theta_1\theta_2)^n$$

which will converge. ⋄

Remark 5.10. There is no (good) theory of C^k super manifolds because you don't get Taylor series. ⋄

[1] By the way, a vector bundle can be though of as a sheaf. If you have a vector bundle $E \to X$, then the sheaf of smooth sections of $E \to X$ is a sheaf of modules over $\mathcal{O}_X = C^\infty(X)$. Locally, this is just smooth fiber-valued (so $\mathbb{R}^n$-valued) functions on X, i.e. $C^\infty(\mathbb{R}^m)\otimes\mathbb{R}^n$. This means that the sheaf is locally free of rank n. Now the statement is that if you start with a sheaf of modules which is locally free of rank n, then you can construct a vector bundle. I invite you to check this [[★★★]]

[[★★★ HW 2 (this even holds in the analytic setting, no partition of unity needed): $\mathsf{SMan}(M, \mathbb{R}^{p|q}) \cong (C^\infty(M)^{ev})^p \times (C^\infty(M)^{odd})^q$. That is, specifying a map to $\mathbb{R}^{m|n}$ is just specifying m even functions and n odd functions. If x_i, θ_j are coordinates on $\mathbb{R}^{m|n}$, the even functions are F^*x_i and the odd functions are $F^*\theta_j$, where $F \in \mathsf{SMan}(M, \mathbb{R}^{m|n})$. If you like, you can think about $\mathsf{SAlg}(C^\infty(\mathbb{R}^{p|q}), C^\infty(M))$ instead of $\mathsf{SMan}(M, \mathbb{R}^{m|n})$]]

3 RB 09-11

Today we'll try to finish off the first lecture. We said last time the infrared divergences all cancel out. You deal with ultraviolet divergences by regularization and renormalization. Regularization is where you raplace the propagator (say $\frac{1}{p^2+q^2}$) by, say, $\frac{1}{(p^2+q^2)^{1+\varepsilon}}$. Everything is a function of ε with poles at ∞. Renormalization is a cunning way to choose Lagrangian as a function of ε.

The third problem we had was divergent series. The problem is that we only know how to define Feynman integrals of the form $\int\int(\text{poly in } \phi)e^{\text{quadratic}}D\phi$, but we want to be able to define things like $e^{\text{non-quadratic}}$. The non-quadratic Lagrangian will be something like $(\partial\phi)^2 + m^2\phi^2 + \lambda\phi^4$. You can expand this as a formal power series in λ to get $\int e^{\text{quadratic}}D\phi + \int\int\lambda\phi^4 e^{\text{quadratic}}D\phi + \int\int\lambda\phi^4\int\lambda\phi^4 e^{\text{quadratic}}D\phi + \cdots$. Each of these integrals can be defined useing regularization and renormalization. So you get a well-defined formal power series in λ, which (probably) doesn't converge (unless $\lambda = 0$). Nobody knows for certain that it doesn't converge in four dimensions, but it is ridiculously improbably. It doesn't converge because the coefficient of λ^n is the sum on Feynman diagrams on n points. The number of Feynman diagrams on n points grows like $n!$ or something. So we get series like $\sum\lambda^n n!$ (this shouldn't really be taken seriously), which doesn't converge for any non-zero λ. For λ small, the first few terms decrease, so you get a very accurate approximation to something (but nobody knows what). In QED, λ is something like 10^{-2}, so the first hundred terms decrease and then the terms are incredibly tiny, so you have a very good estimate of something, and these agree with experiment to great accuracy. This final problem nobody knows what to do about.

There is no caninical way to regularize and renormalize Feynman integrals. I vauguely indicated that you can insert ε in a particular way, but there are loads of other way to do it. There are something like a dozed different ways it's done in the litiriture. This corresponds to the fact that Feynman measure is not unique.

Suppose we have two different regularization/renormalization methods A and B, and suppose we have a Lagrangian L. This means we get two different theories for the Lagrangian L, which don't give the same answer. In other words, the QFT we get depends on L AND on the choice of renormalization method. The space of Lagrangians is well-behaved (it's finite-dimensional), but the space of renormalization schemes is not. If your theory depends on an unknown point in an infinite-dimensional space, it's really bad beacause there is no way to identify this point with a finite number of measurements.

Fortunately, there is a group of (finite) renormalizations which acts simply transitively on the space of renormalizations (i.e. Feynman measures), and it also acts on the space of Lagrangians. The theory of (L, M) (Feynman measure M) is the same as the theory of $(g(L), g(M))$ where g is a renormalization. So physicist A may be using a Lagrangian L, but phisicist B has to use a slightly different Lagrangian to get the same results. Suppose A uses (L, A) and B uses measure B. Find a renormalization g taking A to B and then the second physicist has to use Lagrangian $g(L)$. So there is no such thing as a theory associated to a Lagrangian, you need a Lagrangian together with a Feynman measure.

Warning 3.1. The group of renormalizations is not the same as the renormalization group, though they are closely related. The first is infinite-dimensional and acts on the space of Lagrangians and measures. The second is one-dimensional and is the group of "rescalings" of space-time. Rather confusingly, the second group also acts on measures and Lagrangians. Suppose you choose a measure A. By rescaling space-time, you get a new measure A'. You might think that A' is obtained by scaling A by a constant, but renormalization and regularization are not scale-invariant, so the relationship is more complicated than you'd think. These two measures are related by a renormalization. In other words, if you take a measure A and an element of the renormalization group, this gives you a renormalization. This isn't actually a homomorphism, but the way. ⌟

That's it for introduction. Let's get to producing examples and seeing what axioms they satisfy.

First recall representation theory of a Heisenberg algebra. It turns out that the theory of free quantum field theory is equivalent to the representation theory of Heisenberg algebras. The Heisenberg algebra is a Lie algebra of dimension 3 with $[X, Y] = Z$ and Z in the center. A typical example is $X = \frac{d}{dx}$ and $Y = x$ and $Z = 1$ acting on $\mathbb{C}[x]$. We want to find all "lowest weight" representations of the Heisenberg algebra generated by "vacuum vector" v with $Y(v) = 0$ and $Z(v) = v$. If you've done quantum mechanics, this is the harmonic oscillator. Y is the annihilation operator and X is the creation operator. It is easy to work out what the representation looks like. The only interesting thing you can do is act on v by X.

$$\begin{tikzcd} v \arrow[r, bend left, "X"] \arrow[loop below, "Z"] & Xv \arrow[l, bend left, "Y"] \arrow[r, bend left, "X"] \arrow[loop below, "Z"] & X^2v \arrow[l, bend left, "Y"] \arrow[r, bend left, "X"] \arrow[loop below, "Z"] & \cdots \arrow[l, bend left, "Y"] \end{tikzcd}$$

So the representation we get is the symmetric algebra $H = \mathbb{C}[x]$.

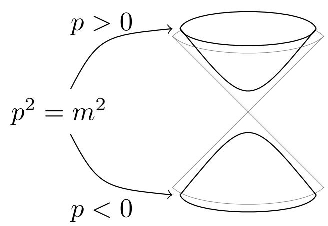
We want to put an inner product on H so that $X^* = -Y$ and $(v, v) = 1$. From this you can work out everything else. For example, $(Xv, Xv) = (v, -YXv) = (v, v) = 1$. It turns out that

$$(X^n v, X^n v) = n!(v, v).$$

PT: do you care if it's positive definite? RB: No, I don't care. The point is that the representation theory of Heisenberg algebras is easy (at least if you only care about highest weight representations). The same thing happens if instead of X and Y you take an arbitrary vector space.

Let V be a vector space with some inner product $(,)$. For a Heisenberg algebra $V^- \oplus \mathbb{R} \oplus V^+$, where $V^\pm$ are copies of V, with $[V^+, V^+] = 0 = [V^-, V^-]$ and $[u^+, v^-] = (u, v)$. Highest weight rep: acting on $S(V^+)$ with $V^-(1) = 0$ and $\mathbb{R}$ acts as multiplication by the identity. So the free quantum field theory we're going to construct is essentially a highest weight rep of a Heisenberg algebra.

Basic example: Free Hermitian scalar field. To construct a QFT, we need to give (1) a $*$-algebra A and (2) a $*$-representation H. A is generated by operators $\phi(f)$ where f is a classical field with compact support. "classical field" just means a smooth real function on space-time. $\phi(f) = \phi^+(f) + \phi^-(f)$ for certain operators $\phi^+(f)$ and $\phi^-(f)$ which are part of a Heisenberg algebra. The commutation relations are $[\phi^+(f), \phi^+(g)] = 0 = [\phi^-(f), \phi^-(g)]$ and $[\phi^+(f), \phi^-(g)] = i\int_{\mathbb{R}^n} \tilde{f}(p)\tilde{g}(-p)m(p)d^n p$, where the $\tilde{f}$ and $\tilde{g}$ are Fourier transforms of f and g, and $m(p)$ is the measure with support $p^2 = m^2$



$m(p)$ supported on top sheet and is rotationally invariant. Take H to be a lowest weight representation of a Heisenberg algebra. there is a (very degenerate) inner product, whose kernel we quotient out by to get H. Then A is the algebra of operators generated by $\phi(f) = \phi^+(f) + \phi^-(f)$. $\phi(f)^* = \phi(f)$. Now we have all the basic data.

PT: why don't you use the whole Heisenberg algebra, just these combinations? RB: because the Wightman axioms don't care about them; you can keep them if you like.

Any unitary representation of a group is going to satisfy these conditions, so we need to narrow our definition of a QFT. What extra conditions do A and H need to satisfy to be a QFT? Answer: Wightman axioms.

1. (minor) The algebra A is generated by $\phi(f)$ where f is a classical field on space-time with compact support.

2. The inner product on H is positive definite. If you're an analyst you'll probably want to complete to get a Hilbert space, but there is no need for that.

3. Lorentz invariance: $O^+_{1,3}(\mathbb{R})$ (the $+$ means preserving the time direction)[1] acts on $\mathbb{R}^{1,3}$, and A, H should be invariant under this action. In fact, this is Poincaré invariant because you also require translation invariance.

4. (Positive Energy condition) E =translation (forward) in time is a positive operator (i.e. $(Ea, a) \geq 0$ for any $a \in H$).

5. (Locality) $[\phi(f), \phi(g)] = 0$ if the supports of f and g are spacelike separated (i.e. if $f(x) \neq 0$ and $g(x) \neq 0$, then $x - y$ is spacelike). This is non-trivial to check in our example.

6. (Vacuum vector) There is a vector fixed by the Lorentz group $\mathbb{R}^{1,3}O^+_{1,3}(\mathbb{R})$.

7. (minor) the vacuum is essentially unique

8. (minor) Tempered.

[1] Switching the time direction would switch which sheet of the hyperboloid you're on.

7 NR 09-12

From last lecture there was a question about where this affine structure came from on level sets of integrals. $F\colon M_{2n} \to \mathbb{R}^n$. When we have $\{F_i, F_j\} = 0$ with $dF_1 \wedge \cdots \wedge dF_n \neq =$, the corresponding vector fields v_{F_i} commute. On corresponding components of $M^{(c)} = \{x \in M | F_i(x) = c_i\}$, we have an action of $\mathbb{R}^n$ which is locally free and transitive, giving you an affine structure. If there is a subgroup $\Gamma \subseteq \mathbb{R}^n$ that acts trivially (Γ is the stabilizer), then this connected component is isomorphic to $\mathbb{R}^n/\Gamma$. Any such Γ must be isomorphic to $\mathbb{Z}^k$, so any connected component is isomorphic to $\mathbb{R}^{n-k} \times \mathbb{T}^k$.

[[★★★ Project 1: "noncommutative integrability", which we called superintegrablity last time.]] There is a tool which allows you to construct basically all known integrable systems. These are Poisson-Lie groups. [[★★★ This can either be in Project 1, or it can be Project 2]] You probably know about Lie groups. I'll talk about the Poisson part when we talke about Hamiltonian reduction.

States in classical mechanics

Definition 7.1. A *state* on manifold M (doesn't have to be symplectic) is a probability measure on M (i.e. a measure μ so that $\mu(M) = 1$). ⋄

Last time, we assigned to a function $f \in C^\infty(M)$ a measure μ_f on $\mathbb{R}$. If $E \subseteq \mathbb{R}$, then $\mu_f(E)$ is supposed to be the probability that f has value in E. In the view of our new definition, $\mu_f(E) = \mu\big(f^{-1}(E)\big)$, or $\mu_f(E) = \int_{M_f(E)} \mu$, where $M_f(E) = f^{-1}(E) = \{x \in M | f(x) \in E\}$.

Remark 7.2. If μ_1 and μ_2 are such measures, then any convex combination $\alpha\mu_1 + (1-\alpha)\mu_2$ is also a probability measure. ⋄

Definition 7.3. *Pure states* are states μ that are supported at points. That is, they are measures $\mu_x(U) = \begin{cases} 1 & x \in U \\ 0 & x \notin U \end{cases}$. ⋄

$C^\infty(M)$ is the algebra of classical observables. M can be identified with the set of pure states. We get an expectation value $E_\mu(f) = \int_M f\mu =: \langle f \rangle_\mu$ for an observable f. Another important value is the disperssion around $E_\mu(f)$

$$\sigma_\mu(f)^2 = \big\langle (f - \langle f \rangle_\mu) \big\rangle_\mu = \langle f^2 \rangle_\mu - \langle f \rangle_\mu^2 \geq 0.$$

Say we have a (Hamiltonian) dynamics[1] on M is $f \mapsto f_t$, with $\frac{df_t}{dt} = \{H, f_t\}$. The dual dynamics on states is $\mu \mapsto \mu_t$, where $E_{\mu(t)}(f) := E_\mu(f_t)$. This evolution is called *Liouville evolution* of states. If M is symplectic with some volume measure ω^n and if our state is continuous with respect to this measure (i.e. if $\mu = \rho\omega^n$ for some continuous function ρ), then $\frac{d\rho_t}{dt} = -\{H, \rho_t\}$.

This is the picture of states in classical mechanics. Q: what does a state mean? do you have a set, and the measure is the number of particles in that set? NR: it depends what measure you take. If you have a pure state, then $E_{\mu_x}(f) = f(x)$. Another example is the Gibbs state. If you have an energy functino (usually the Hamiltonian) $E(x) > 0$, then the Gibbs state is $\mu(x) = \exp\big(-E(x)/T\big)\omega^n$, where T is a parameter analogous to temperature. If you let T go to zero (if you cool the system), then the measure will become concentrated at the points which have minimal energy.

Q: how do we know there is a μ_t like that? NR: This is something like the Rees-Nikodim theorem. It says that if you have a functional on the space of functions, then there is measure giving it. PT: aren't you just pushing forward the measure along the flow? NR: yes, that's a better way to say it: $\mu_t(E) = \mu\big(g_t(E)\big)$, where g_t is the evolution map.

Hamilton-Jacobi equation

Now we return to the situation where our symplectic manifold is T^*N for some smooth manifold N, Hamiltonian $H \in C^\infty(T^*N)$ with corresponding Lagrangian $\mathcal{L}$ (assume strongest possible non-degeneracy). Suppose $\gamma(q, Q)$ is a solution to the Euler-Lagrange equations with fixed end points so that $\gamma_0 = q$ and $\gamma_t = Q$. Let's assume that this solution is unique for every pair (q, Q). Then we have

$$\mathcal{A}[\gamma(q,Q)]_{t_1,t_2} = \int_{t_1}^{t_2} \mathcal{L}\big(\dot\gamma(\tau), \gamma(\tau)\big)\, d\tau =: \mathcal{A}_{t_1,t_2}(q,Q)$$

[1] By "dynamics" I mean a local action of $\mathbb{R}$ telling you how points evolve in time.

PT: actually, in geometry, you just say $\dot\gamma$ because it has a well-defined basepoint. NR: Let's stick to this mixed version.

So we have a family of functions $\mathcal{A}_{t_1,t_2}$ on $N\times N$. These functions are quite remarkable. It defines a Lagrangian submanifold in $T^*N\times\overline{T^*N}$, and this Lagrangian submanifold is $L_t=\{(P,Q),(p,q)|P=dQ\mathcal{A}_{t_1,t_2},p=-dq\mathcal{A}_{t_1,t_2}\}$. The second wonderful property is that if we want to compute it we don't have to use the Euler-Lagrange equations.

Theorem 7.4 (Hamilton-Jacobi[2])**.** —

$$\frac{\partial\mathcal{A}_{t_1,t_2}}{\partial t_2}+H(d_Q\mathcal{A}_{t_1,t_2},Q)=0 \qquad \frac{\partial\mathcal{A}_{t_1,t_2}}{\partial t_1}-H(-d_q(\mathcal{A}_{t_1,t_2}),q)=0$$

Proof. Use the Legendre transformation.

$$\mathcal{A}_{t_1,t_2}=\int_{t_1}^{t_2}(p\dot q-\underbrace{H(\gamma_*(\tau))})\,d\tau$$

the second term is the image of $(\dot\gamma,\gamma)$ under the Legendre transform, so

$$\mathcal{A}_{t_1,t_2}=\int_{\gamma^*}\alpha-\int_{t_1}^{t_2}H(p(\tau),q(\tau))\,d\tau$$

[[★★★ NR: it is better to use t_1 and t_2 so that we can apply this stuff to non-conservative systems.]] So we have

$$\frac{\partial A_{t_1,t_2}}{\partial t_2}=-H(p(t_2),q(t_2))=-H(P,Q).$$

Q: the path γ depends on the t_1 and t_2? NR: yes, but the integral $\int_{\gamma^*}\alpha$ doesn't depend. Q:but we only fixed the q's, not the p's. NR: maybe [[★★★ HW1]] □

We'll see how these equations will appear as a kind of justification of the path integral in quantum mechanics.

Another way to think about it: these equations give you the generating function for the Lagrangian submanifolds L_t. There are many ways to get the same information about classical evolution.

[2] Maybe this was proven by somebody else, but these are the names everybody attaches.

Hamiltonian reduction

So far, we've been using symplectic manifolds of the form T^*N with the symplectic form $\omega=d\alpha$, but there is a general source of examples not of this form

Definition 7.5. A *Poisson manifold* is a pair $(M,p\in\wedge^2TM)$ with M a smooth manifold and p a bivector field such that the bracket

$$\{f,g\}=\langle p,df\wedge dg\rangle$$

induces a Poisson algebra structure on $C^\infty(M)$. ◇

Since this bracket is a bidifferential operator of first order, it acts by derivations on $C^\infty(M)$ (with pointwise multiplication). The Jacobi identity gives a bilinear differential identity for p. In local coordinates it can be written as

[[★★★]]

The coordinate-free approach using the Schouten bracket can be found in [[★★★]].

[[★★★ Project 3: Poisson geometry and the Schouton bracket.]]

Example 7.6. Suppose p is non-degenerate. Then $p\colon T^*M\to TM$ has an inverse, which can be regarded as a 2-form $p^{-1}\in\wedge^2T^*M$. [[★★★ HW2: show that the corresponding 2-form p^{-1} is a symplectic structure on M. You need to prove that it is closed.]] ◇

Example 7.7 (Lie-Kirillov-Kostant)**.** Let $\mathfrak{g}$ be the Lie algebra of a (finite-dimensional) Lie group G. Let $\mathfrak{g}^*$ be the dual space. Then on $C^\infty(\mathfrak{g}^*)$, we can define the operation

$$\{f,g\}(x)=\big\langle x,[df(x),dg(x)]\big\rangle$$

(note that $x\in\mathfrak{g}^*$ and $df(x)\in\mathfrak{g}$, $dg(x)\in\mathfrak{g}$). This defines a bivector field p on $\mathfrak{g}^*$. If $\{e_i\}$ is a basis for $\mathfrak{g}$, $\{x^i\}$ are coordinate functions on $\mathfrak{g}$, and $\{\frac{\partial}{\partial x^i}\}$ is a basis for the tangent space, then

$$p=\sum_{i,j,k}c^i_{jk}x_i\frac{\partial}{\partial x_j}\wedge\frac{\partial}{\partial x_k}$$

where $[e_i,e_j]=\sum_k c^k_{ij}e_k$ ◇

Equivalently, the Poisson bracket between coordinate functions has the form

$$\{x_i, x_j\} = \sum_k c_{ij}^k x_k.$$

[[★★★ inconclusive … maybe this section should be moved to the next lecture.]]

6 PT 09-13

Three main theorems on super manifolds.

(a) $M \cong \pi E$ for some smooth vector bundle $E \to M_{red}$. Recall that $C^\infty(\pi E) = C^\infty(M_{red}, \bigwedge^* E^*)$. [[★★★ HW1]]

(b) $\mathsf{SMan}(M, N) \cong \mathsf{SAlg}(C^\infty N, C^\infty M)$. Super manifolds are in some sense “affine”.

(c) If $U \subseteq \mathbb{R}^{p|q}$ is a *domain* (the restriction of the super manifold structure on $\mathbb{R}^{p|q}$ to an open subset U), then $\mathsf{SMan}(M, U) \cong \{f_1, \dots, f_p \in C^\infty(M)^{ev}, \phi_1, \dots, \phi_q \in C^\infty(M)^{odd} | (f_1(x), \dots, f_p(x)) \in |U| \text{ for all } x \in M\}$.[[★★★ HW2]]

The first two use partitions of unity, so you must be in the smooth category, not the analytic category.

Combining, $C^\infty(U) = C^\infty(U_{red})[\xi_1, \dots, \xi_q] \supseteq \mathbb{R}[x_1, \dots, x_p] \otimes \bigwedge^*[\xi_1, \dots, \xi_q]$, which is a free commutative super algebra on x_i and ξ_j, so it is really easy to specify a morphism from this subalgebra (it is determined by choosing an even guy for each x and an odd guy for each ξ). To extend this to the whole algebra, you use Taylor expansion.

Last time we saw that there are maps

$$\mathsf{SMan}(M, U) \to \mathsf{Man}(M_{red}, U_{red}) \to C^0(|M|, |U|)$$

[[★★★]].

Reimundo: do you have a map $M \to M_{red}$. PT: there is a map which comes from the zero section using $M \cong \pi E$, but since this isomorphism is not canonical, they map is not canonical.

Remark 6.1 (Historical digression?)**.** Once you have this third result, you can throw the other stuff away and define super manifolds locally.

If $S, U \subseteq \mathbb{R}^{p|q}$ are domains, then $\mathsf{SMan}(S, U) \cong \{f_i, \phi_j \text{ as above}\}$, with $f_i \in C^\infty(S)^{ev} = \left(C^\infty(S_{red}) \otimes \bigwedge^*[\xi_i]\right)^{ev}$, so

$$f_i = \sum_{I \text{ even}} f_I^i(x) \cdot \xi_I = \underbrace{f_\varnothing^i(x)}_{\text{body}} + \underbrace{\quad\cdots\quad}_{\text{soul}}$$

I varies over index sets (sequences of 0's and 1's) $(i_1, \ldots, i_r)$ where $1 \leq i_a \leq q$ and $f_I \in C^\infty(S_{red})$. Similarly, the ϕ_j have the same, but for odd. The soul is nilpotent. The ϕ_j don't have a body, just a soul.

Historically, people defined morphisms of super manifolds this way (before defining objects). Then you have to figure out how to compose these guys. I'm going to skip this. You have to use the Taylor expansion on all of these f_i's around the body. If you do the second homework, you have to deal with these Taylor expansions.

Then you know how to deal with maps between domains, so you can talk about isomorphisms. Now you can glue domains together just like you'd glue ordinary manifolds, using these isomorphisms to glue.

What we've done is equivalent, but is more elegant in my opinion. To do computations, you should use the local language, but to formulate precise statements, use the global language. $\diamond$

Example 6.2. $f\colon U \to \mathbb{R}^{1|1}$ given by $\sum_I f_I(x) \cdot \xi_I$. The body is $f_\varnothing(x)$ and the rest, $f_s(x)$, is the soul. The soul has some even stuff in it, but it is nilpotent because it has odd factors. The body is (like Reimundo was saying) an ordinary function in $C^\infty(M_{red})$. You cannot write is as this body/soul as a direct sum if U is not a domain. You can always pick out the body, but the soul is not canonical.[1]. Q: isn't the soul just f minus the body? PT: no, the body is a function on the reduced guy.

[[From (c), by the way, you get $C^\infty(M) \cong \mathsf{SMan}(M, \mathbb{R}^{1|1}) \cong \mathsf{SMan}(M, \mathbb{R}^{1|0}) \times \mathsf{SMan}(M, \mathbb{R}^{0|1})$. This gets into this inner Hom stuff. This is saying that any function breaks up uniquely as an even and an odd function.]]

Now we have $g\colon \mathbb{R}^{1|1} \to \mathbb{R}^{1|0}$. Ah, there is trouble, you can think of g as a function (not necessarily even) on $\mathbb{R}^{1|1}$ or as a morphism in the category, so even. I mean that $g(y, \eta) = g_0(y)$, so I mean a morphism in the category. That is, g is an even function. Reimundo: write $\mathbb{R}^{1|0}$ on the target; then there is no confusion. PT: yes, that's a good idea.

Now let's try to calculate $g \circ f(x, \xi)$. I'll write in terms of body and soul because I have to Taylor expand around the body. g is equal to its body.

$$\begin{aligned} g\big(f_0(x) + f_s(x,\xi)\big) &= \sum_{n=0}^\infty \frac{g^{(n)}(f_0(x))}{n!}(f_x(x,\xi))^n \\ &= g f_0(x) + g'(f_0(x)) f_s(x,\xi) + g''(f_0(x))??? + \cdots \end{aligned}$$

And we know that this will be a finite sum because $f_s(x,\xi)$ is nilpotent. $\diamond$

Notation: Let $A \in \mathsf{SAlg}$ and $M \in \mathsf{mod}\text{-}A$. Then we get the *parity reversed module* πM, given by defining $(\pi M)^{ev} = M^{odd}$ and $(\pi M)^{odd} = M^{ev}$. For example, A is a free module on one even generator, and πA is a free module of dimension $(0|1)$ with free generator $1_A \in A^{ev} = (\pi A)^{odd}$.

Thus, a free module of dimension $(p|q)$ is (by definition, if you want) isomorphic to $A^{\oplus p} \oplus (\pi A)^{\oplus q} =: A^{p|q}$. If the algebra is commutative (super), then p and q are determined by the module, otherwise they are not.

Super objects in differential geometry.

Philosophy: Anything one can formulate in terms of functions (more generally, sections of bundles) has a super analogue.

Definition 6.3. M a super manifold. Then a *vector bundle* (of dimension $(p|q)$) E over M is a locally free sheaf of $\mathcal{O}_M$-modules over $|M|$ (of rank $(p|q)$). $\diamond$

Q: Are these bundles completely determined by their global sections. PT: that is an excellent question. I think they are, but I haven't checked it. It should follow from the same techniques we used to prove result (b).

Definition 6.4. If A is a commutative super algebra, $Der(A) = \{D\colon A \to A | D(ab) = D(a)b + (-1)^{|D||a|} a D(b)\}$ is an A-module. $\diamond$

Example 6.5. $TM \to M^{m|n}$ is a vector bundle of dimension $(m|n)$ given by $\mathcal{O}_{TM}(U) = Der(\mathcal{O}_M(U))$.

"Proof". in a coordinate chart (x_i, ξ_j), basis is given by ∂_{x_i} and ∂_{ξ_j}, which act like you'd think on the coordinate functions. $\diamond$

[1] "The body is canonical. Everything else you have to search for."

[[break]]

Consider $U^{p|q} \xrightarrow{f} \mathbb{R}^{1|1} \xrightarrow{g} \mathbb{R}^{1|1}$ as before. Body and soul decomposition only makes sense because we're working in $\mathbb{R}^{p|q}$. The even part of f is $f_0(x) + f_x^{ev}(x,\xi)$, which is like your y and the odd part is $f_x^{odd}(x)$, which is like your η. So we have

$$g_0(f_0(x) + f_x^{ev}(x,\xi)) + g_1(f_0(x) + f_s^{ev}(x,\xi)) \cdot f_x^{odd}(x,\xi)$$

You do the Taylor expansion around the body, with variable the *even part* of the soul.

Q: in the $\mathbb{R}^{1|1}$ case, the soul only has an odd part. PT: yes, this is a bad example.

Let's get back to these vector bundles. Jonah asked a good question over the break: we defined $TM \to M$ as a sheaf, but is TM a super manifold? The answer is yes. there is an alternative way to define vector bundles where you have a total space and you say it's locally trivial.

Example 6.6. $M = \mathbb{R}^{1|1}$ with coordinates (t,θ), and consider the odd vector field $D = \partial_\theta + \theta\partial_t$. What is $D^2 = \frac{1}{2}[D,D]$? It is not a vector field in general, but remember that $\frac{1}{2}[D,D]$ is not zero because D is odd. Given $f \in C^\infty(\mathbb{R}^{1|1})$, we can write it as $f_0(t) + f_1(t)\theta$. So we have

$$\begin{aligned} D(f) &= (\partial_\theta + \theta\partial_t)(f_0(t) + f_1(t)\theta) \\ &= f_1(t)\underbrace{\partial_\theta(\theta)}_{1} + \theta(f_0'(t) + f_1'(t)\theta) \\ &= f_1(t) + f_0'(t)\theta \\ D^2(f) &= f_0'(t) + f_1'(t)\theta \\ &= \partial_t(f) \end{aligned}$$

So $D^2 = \partial_t$. ◇

The notation is bad. I should have denoted the tangent bundle as $\chi(M)$ (vector fields). $\Omega^1 M$ are sections of the "cotangent bundle", given by $\Omega^1 M(U) = \mathrm{Hom}_{\mathcal{O}_M(U)}(\chi M(U), \mathcal{O}_M(U))$.

Then there is a beautiful operator $d\colon \mathcal{O}_M \to \Omega^1 M$, the de Rham differential. It is determined by $\langle D, df\rangle = D(f)$ for $D \in \chi M$. This is an even map. d extends to a graded derivation of square zero $\Omega^\bullet M = \bigwedge^\bullet_{\mathcal{O}_M}(\Omega^1 M)$, which is $\mathbb{Z}$-graded as usual; this is what I mean by *graded derivation* …there are no other signs because d is even. Reimundo: you're making a choice of what is the even and odd parts. PT: there is a whole chapter in Deligne and Morgan about two very natural choices. Reimundo: I think you're making a choice by saying that d is even. PT: I don't know about that.

Theorem: $H^*(\Omega^* M, d) \cong H^*_{dR}(M_{red})$.

You should think of the reduced guy as having all the real features ... the odd part is like a nilpotent cloud.

8 NR 09-14

Definition 8.1. Let $x \in M$. The *symplectic leaf* of M through x is the subset $S_x = \{\text{points connected to } x \text{ by piecewise Hamiltonian paths}\}$, where a Hamiltonian path is a flow line of a Hamiltonian vector field $v_H = p(dH)$ for a smooth function $H \in C^\infty(M)$. $\diamond$

Theorem 8.2. *$S_x \subseteq M$ is a submanifold. Furthermore, S_x is a symplectic manifold with symplectic structure given by restricting p. Two symplectic leaves either coincide or do not intersect.*

One of the main questions in Poisson geometry is: given a Poisson manifold (M, p), find the symplectic leaves. This is the geometric analogue of classifying isomorphism classes of irreducible representations of a given associative algebra. Deformation quantization deforms an associative algebra in such a way that the first order deformation is given by the Poisson bracket. The symplectic leaves then correspond to ideals in the algebra so that quotienting by them gives irreps. [[★★★ part of Project 3: symplectic leaves in Poisson-Lie groups. These examples can be very involved and complecated, but are very interesting.]]

Example 8.3. If M is symplectic, there is only one symplectic leaf, namely M. $\diamond$

Example 8.4. $\mathfrak{g}^*$ is a Poisson manifold.

Theorem 8.5 (Lie-Kostant-Kirillov+others)**.** *Symplectic leaves in $\mathfrak{g}^*$ are coadjoint G-orbits.*

G naturally acts on $\mathfrak{g}$ by the adjoint action. The dual action of G on $\mathfrak{g}^*$ is the *coadjoint action.* $\diamond$

Summary: we extended the notion of symplectic manifolds to Poisson manifolds. Poisson manifolds in many ways behave like associative algebras in the sense that symplectic leaves correspond to irreducible representations.

The moment map and Hamiltonian reduction

Recall that a vector field v on a symplectic manifold (M, ω) is *Hamiltonian* if there exists a function $H \in C^\infty M$ such that $v = \omega^{-1}(dH)$. Since $[v_{H_1}, v_{H_2}] = v_{\{H_1,H_2\}}$, Hamiltonian vector fields form a Lie subalgebra $HVect(M)$ of the Lie algebra $Vect(M)$ of all vector fields on M. Assume a Lie group G is acting on M. This induces a Lie algebra homomorphism $\phi \colon \mathfrak{g} \to Vect(M)$.

Definition 8.6. The action of G on M is *Hamiltonian* if the image of $\mathfrak{g}$ in $Vect(M)$ lies in $HVect(M)$. $\diamond$

In other words, if $x \in \mathfrak{g}$, then $e^{tx} \in G_e \subseteq G$ (neighborhood of the indentity), and we have

$$\frac{d}{dt} f(e^{tx} m)\Big|_{t=0} = \langle \phi(x), df \rangle(m) = x \cdot f(m)$$

So Hamiltonian action means that for each $x \in \mathfrak{g}$, there exists a function $h_x \in C^\infty(M)$ such that

$$x \cdot f(m) = \{h_x, f\}(m).$$

From the definition, we can see that h_x is linear in x (modulo a constant, so let's require that $h_0 = 0$), so $h_x(m) = \langle \mu(m), x \rangle$ where $\mu \colon M \to \mathfrak{g}^*$. [[★★★ for NR: what is the standard notation here?]]

PT: do you assume $h_{[x,y]} = \{h_x, h_y\}$? NR: it follows. PT: then you need a stronger assumption.

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "Lie"] \arrow[dr, dashed, "Lie"'] & HVect(M) & \subseteq & Vect(M) \\
& C^\infty(M) \arrow[u] & & \\
& consts \arrow[u] & &
\end{tikzcd}$$

Theorem 8.7. *(1) $\mu \colon M \to \mathfrak{g}^*$ is a Poisson map (i.e. $h_{[x,y]} = \{h_x, h_y\}$) and (2) μ is G-equivariant.*

[[★★★ for NR: clarify the following disscussion.]]

Natural question: what can we say about M/G? This is already a quite complicated question. There are various ways to make it into a manifold if there are geometric problems. Let's assume there are no such geometric problems.

Theorem 8.8. *M/G is a Poisson manifold.*

If we want to study this manifold, one way to do it (corresponding to the notion of a categorical quotient) is to consider $C^\infty(M)^G$ (G-invariant functions). For this, we should be in the algebraic category. The subalgebra $C^\infty(M)^G \subseteq C^\infty(M)$ is a Poisson subalgebra, which means that $M \to M/G$ is Poisson (assuming M/G makes sense as a manifold).

We produced a Poisson manifold out of a symplectic manifold. What are the symplectic leaves of M/G? [[★★★ Project 3 is not for one person. You can include the precise statement of the following and a precise discussion of what kind of quotients we can have when we have a Lie group action on a manifold. One notion that was used effectively is the theory of GIT quotients.]]

Theorem 8.9. *Symplectic leaves of M/G are $\mu^{-1}(\mathcal{O})/G$ (where $\mathcal{O}$ is a coadjoint orbit).*

Remember that symplectic leaves in $\mathfrak{g}^*$ are coadjoint orbits $\mathcal{O} \subseteq \mathfrak{g}^*$.

In particular, we always have the distinguished orbit zero. So in particular, $\mu^{-1}(0)/G$ will be a symplectic leaf. This symplectic leaf is called the *Marsdon-Weinstein reduction* of M by G. The origin of this theory is in physics and angular momentum, when $G = SO(3)$

[[★★★ Refferences]]

Classical field theory

[[★★★ for NR: is there a page of corrections to this section somewhere?]]

If you're a (classical) physicist and all you care about are coordinate functions $q_i(t)$ $1 \le i \le n$. If you have infinitely many particles in a box, then you have infinitely many degrees of freedom. In this case, does it make sense to ask how many particles are in a given region? NO. You should ask what is the density of particles in the region. So we pass from finitely many degrees of freedom to densities of particles. In field theory, you have no individual particles, just fields.

<u>The idea</u>: replace the time interval $[t_1, t_2]$ by some (possibly complicated) manifold M with possibly non-empty boundary ∂M. Classically, $M = I$, and $\partial M = pt \sqcup \overline{pt}$ (the initial point should come with a minus sign). It makes sense to assume M is oriented, so ∂M is oriented.

These densities evolve in time. There is some distinction between the time direction and the other directions. We can either choose to take this into account or to ignore it. We can also choose to think of M as a Riemannian manifold.

Recall that the action can be written in terms of the Hamiltonian

$$\int_{\gamma^*} \alpha - \int_{t_1}^{t_2} H(\gamma_*(t))\, dt$$

The first term is independent of the parameterization. If we take $H = 0$, then a phisicist would say this is an empty system, but the evolution is non-trivial. This is called *topological mechanics*.

At Berekeley, Michael Green (not Brian Greene) gave some lectures. The title of the colloquim talk was "string theory = theory of nothing". His last slide was that in some cases the action is non-zero even though the hamiltonian is zero. Thus, TQFT is "the theory of nothing", which we'll study a lot.

9 NR 09-17

Two things we'll have to come back to: (1) constrained mechanical systems, Lagrangian and Hamiltonian, and (2) systems with degenerate Lagrangian (gauge systems)[[★★★ Project]].

Back to classical field theory. Last time I explained that the main idea is to replace the interval $[t_1, t_2]$ with an oriented manifold M. We will assume that the boundary of M is naturally polarized; that is, it has two connected components $\Sigma_1 \sqcup \overline{\Sigma}_2$. We will replace paths γ by section of some fiber bundle. This is the first approximation of what we want. Let's do the second approximation (still not the final version)

Space-time categories

A *d-dimensional spacetime category* is a category in which objects are oriented closed $(d-1)$-dimensional manifolds Σ (possibly with extra structure: e.g. Riemannian metric or symplectic form) and morphisms $\Sigma_1 \to \Sigma_2$ are d-dimensional oriented compact manifolds M (usually with extra structure) together with an isomorphism $\partial M \cong \overline{\Sigma}_1 \sqcup \Sigma_2$ (respecting any extra structure on M, Σ_1, and Σ_2). The composition is given by gluing.

Example 9.1 (Riemannian category)**.** Objects are $(d-1)$-dimensional Riemannian manifolds. The morphisms are (equivalence classes) of d-dimensional Riemannian manifolds (in the true version, there will be some extra information, "collars" on objects which tell you how to glue an object to a morphism) M with $\partial M \cong \overline{\Sigma}_1 \sqcup \Sigma_2$. Gluing is more involved. ◇

Example 9.2 (Minkowski category)**.** Here there will be similar subtleties. The objects are $(d-1)$-dimensional manifolds (most likely with collars). Morphisms are d-dimensional Minkowski manifolds M (the metric is not unique, just the signature should be $(d-1,1)$, so one minus sign). ◇

Example 9.3 (Topological category)**.** Objects are compact oriented $(d-1)$-dimensional smooth manifolds Σ and morphisms $M\colon \Sigma_1 \to \Sigma_2$ are homotopy classes of d-dimensional manifolds M with smooth $(d-1)$-dimensional boundary together with isomorphisms $\partial M \cong \overline{\Sigma}_1 \sqcup \Sigma_2$. This is the category of d-cobordisms. ◇

Example 9.4 (Cell decomposition)**.** Objects are $(d-1)$-dimensional manifolds with a cell decomposition and morphisms are d-dimensional manifolds with a cell decomposition $M\colon \Sigma_1 \to \Sigma_2$ together with $\partial M \cong \overline{\Sigma}_1 \sqcup \Sigma_2$ so that the boundary is a subcomplex. The cell decomposition is part of the extra structure. ◇

Example 9.5 (Metrized cell decomposition)**.** A metrized cell decomposition is a cell decomposition where you assign volumes to all cells. Take this as the extra structure. This is an intermediate case between the topological and Riemannian categories. By taking finer and finer approximation, we can obtain the Riemannian category. One of the ideas of dealing with infinite-dimensional field theory is to approximate it by combinatorial approximations on a cell complex. ◇

Example 9.6. One can weaken the previous example by assigning volumes only to d-dimensional cells. So objects are (non-metrized) cell decompositions of $(d-1)$-dimensional manifolds, but morphisms are d-dimensional cell complexes of manifolds together with volumes of the d-cells. ◇

Example 9.7 (Classical mechanics)**.** Objects are (oriented) points, and morphisms are (oriented) intervales connecting points. ◇

Objects in spacetime categories are analogs of the endpoints of time intervals, and morphisms are the analogs of of time intervals.

Now we need the notion of fields to talk about classical field theory. The space of fields is the space of smooth sections $C^\infty(F \to M)$ of a fiber bundle $F \to M$ with fiber X. In mechanics, we take $M = [t_1, t_2]$, and F is the trivial bundle with fiber $X = N$.

In classical mechanics a Lagrangian function $\mathcal{L}$ is a function on TN. We will only take first order Lagrangians (this is a fundamental principal that to fix a trajectory you only need to fix a position and velocity. In general, the Lagrangian could be some function on Jet space). In mechanics, we had $L(\xi, q)$. In field theory, we will have $\mathcal{L}\big(\phi(x), d\phi(x)\big) \in \bigwedge^n T_x^* M$. A classical field ϕ is a section of $\pi\colon F \to M$, so over each point $x \in M$ of space time we have a fiber $F_x \cong_{i_x} X$, which contains the point $\phi(x)$. Then $d\phi(x) \in T_x^* M \otimes_{\mathbb{R}} T_{\phi(x)} F_x \cong_{\mathbb{R}} T_x^* M \otimes T_{i_x(\phi(x))} X$. Q: are you assuming a connection? How did you get into the tangent space to the fiber? AJ: are we secretly using the Jet space. NR: yes, we are secretly using the Jet space. Maybe I should just pick a connection right now. What I mean is that

$$\langle d\phi(x), \xi \rangle = \frac{d}{dt}\phi(\gamma_t)$$

where γ_t has $\dot{\gamma}_t(0) = \xi$ (in particular $\gamma_t(0) = x$).

The Lagrangian defines the classical action functional as

$$\mathcal{A}[\phi] = \int_M \mathcal{L}(x).$$

It is a functional on the space of fields.

Example 9.8 (Scalar $\mathbb{R}$-valued field in the Riemannian category)**.** $X = \mathbb{R}$. $\mathcal{L}\big(\phi(x), d\phi(x)\big) = \frac{1}{2}\langle d\phi(x), d\phi(x)\rangle - \frac{m^2}{2}\phi(x)^2 - V(\phi(x))$ where V is a finite polynomial $\frac{1}{3!}\lambda_3\phi(x)^3 + \cdots \frac{\lambda_n}{n!}\phi(x)^n$ (this is n-th order). These are the "kinetic term", "massive term", "self-interaction term". $\diamond$

7 PT 09-18

Ways of defining:

	super manifolds	vector bundles
1	sheaf $\mathcal{O}_M$ in SAlg	locally free $\mathcal{O}_M$-modules
2	gluing domains $U \subseteq \mathbb{R}^{m\|n}$	gluing $U \times \mathbb{R}^{m\|n}$, fiberwise linear
3	$C^\infty(M) \in \mathsf{SAlg}$	projective modules[1] over $C^\infty(M)$
4	functor of points (Yoneda)	

[[★★★ Project 2: characterize $C^\infty(M)$ for a supermanifold M, *algebraically*, among all commutative super algebras. It turns out there is a beautiful characterization: if you give me a commutative super algebra, there is a beautiful way to decide if it is the functions on a supermanifold. This is known for ordinary manifolds: you prove that points in the manifold correspond to maximal ideals with real residue field, then use the Zariski topology, then work some more to get the sheaf of functions. This is all explained in some book; I'll put a pdf on my web site.]]

Once you have various points of view, you jump around and always use the most convenient interpretation.

Q: does 3 work for non-compact things? PT: yes, this is because manifolds are paracompact, so any vector bundle lies in a trivial bundle.

<u>Pullbacks of vector bundles</u>. You know the construction of pullbacks for ordinary manifolds.

$$\begin{tikzcd} f^*E \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & E \arrow[d] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

Point of view 3 is most convenient here. Given $f\colon M \to N$, we have $f^*\colon C^\infty(N) \to C^\infty(M)$. If P_E is a projective module, we can define $f^*(P_E) := P_E \otimes_{C^\infty(N)} C^\infty(M)$.

Remark 7.1. If P_E is a $C^\infty(M)$-module, it can also be viewed (via our algebra map f^*) as a $C^\infty(N)$-module. We'll call this push-forward. Note that this operation doesn't preserve "projective". $\diamond$

To check that we preserve projective, it is enough to observe that we preserve direct sum and freeness. It is clear that the first construction (pull-back) does this, but the second (push-forward) does not.

Example 7.2. Let's pull back to a point. What is a map from a point to N? If X is an ordinary manifold, then $\mathsf{SMan}(X, N) \cong \mathsf{SAlg}(C^\infty N, C^\infty X)$. I claim this is the same as $\mathsf{SAlg}(C^\infty N_{red}, C^\infty X)$; this is because $C^\infty X$ has no odd part, so the ideal in $C^\infty N$ generated by the odd functions must be killed. But this is the same as $\mathsf{Man}(X, N_{red})$. [[★★★ This is just saying that $_{red}$ is adjoint to the forgetful functor from manifolds to super manifolds.]]

When you pull back,

$$\begin{tikzcd} f^*E \arrow[d] & & E^{p|q} \arrow[d] \\ pt \arrow[r, "f"] & N_{red} \subseteq & N \end{tikzcd}$$

This fiber f^*E is the fiber, but remembering that it is $(p|q)$-dimensional. ◇

Products in SMan. I want to define these so I can talk about super Lie groups.

Definition 7.3. A super Lie group is a super manifold with $G \times G \xrightarrow{\mu} G$, $G \xrightarrow{\nu}$, and $\mathbb{R}^0 = \mathbb{R}^{0|0} = pt \xrightarrow{e} G$. ◇

In an arbitrary monoidal category, you need to define a Hopf algebra object, but if you have products, you don't need the diagonal and counit maps.

The easiest way to define products is using description 2 (there was a homework which told you how to get maps to $\mathbb{R}^{p|q}$, from which you can prove that products are what you think), but let's use 3. Question: what is the coproduct in Alg. Answer: it is tensor product (over whatever the base is). Two maps $A_1 \to A$ and $A_2 \to A$ gives you a map $A_1 \otimes A_2 \to A$ (for this, you need to be in commutative rings), and you can go the other way by restriction. Question: is this ok for supermanifold (or just regular manifolds)? For ordinary manifolds, is $C^\infty(X_1 \times X_2) \cong C^\infty X_1 \otimes_{alg} C^\infty X_2$? No! There is a map $C^\infty X_1 \otimes_{alg} C^\infty X_2 \to C^\infty(X_1 \times X_2)$, but this map is only onto if one of them is discrete. The point is that any function which depends interestingly on both variables isn't a finite linear combination of simple tensors.

Theorem 7.4. *If one completes $\otimes_{alg}$ to the projective tensor product $\otimes$ of Frechét spaces, the above map becomes an isomorphism.*

This Frechét spaces stuff will be relevent later when we do quantum field theories, so let's explain a little of this now.

We'll have to prove that $C^\infty M$ is a Frechét space. It turns out that there is a unique structure of a Frechét algebra on $C^\infty(M)$ (check the reference for Project 2). So $C^\infty M \in \mathsf{FSAlg}$.

To define a super Lie group, you turn around all the arrows to see that $C^\infty G$ has a Hopf algebra structure. The only subtlety is that this is a Hopf algebra in the category FSAlg (commutative as an algebra, but possibly non-cocommutative).

[[break]]

Let me explain how these projects should work. It's first come first served, and Matthias grabbed this one and somebody is already interested in Project 1. But since there can be more than one person per project, it's ok to say that you'd also like to work on that project.

Frechét spaces

We'll call them F-spaces. Banach spaces are B-spaces. The difference is like compact and non-compact spaces, or C^k versus C^∞. For supermanifolds, even if we restrict to compact super manifolds, you still have this trouble with C^k. (Andy: if you limit the odd dimensions, you can do C^k.)

Start with a topological space X. Then what structure do we have on $C^0(X)$? We have the compact-open topology. We only have a norm if X is compact (the supp-norm). Matt: you could take functions vanishing at infinity if it isn't compact. PT: yes, but I don't want to; this amounts to looking at functions on the one point compactification. If X is not compact, we might still want to give a good description than the compact-open topology. This is done by the notion of uniform convergence on compact sets. If X is any topological space, you get a semi-norm for each compact $K \subseteq X$, $\rho_K(f) := \max_{x \in K} |f(x)|$. This family of semi-norms leads to the topology of uniform convergence on compact sets. The nice thing about manifolds is that they are second countable, so we just need a countable sequence of these semi-norms.

On a (topological) manifold, we may pick a countable sequence $K_1 \subseteq K_2 \subseteq \cdots$ such that $X = \bigcup K_i$. Then we get $\rho_{K_1} \leq \rho_{K_2} \leq \cdots$ defining the topology on $C^0(X)$. If you want, you can even define a metric now by some formula. This implies that $C^0(X)$ is metrizable.

Theorem 7.5. *$C^0(X)$ is complete with respect to this topology.*

Completeness of a topological vector space means that Cauchy sequences converge (this is ok so long as the space has a countable basis; otherwise you need to use nets). [[★★★ btw, how does this work?]]

Definition 7.6. A *Frechét space* (or *F-space*) is a complete topological vector space whose topology is given by an increasing countable sequence of semi-norms. ⋄

You can get around picking the norms; here is an equivalent definition.

Definition 7.7. A *Frechét space* (or *F-space*) is a complete locally convex[2] topological vector space that is metrizable. ⋄

Remark 7.8. Locally convex is important because the Hahn-Banach theorem tells you that linear continuous maps V' can detect points (there will always be a continuous linear map which doesn't vanish at a given point). ⋄

Example 7.9. $L^p[0,1]$ $(p > 0)$ is locally convex if and only if $p \geq 1$ (in which case they are actually Banach spaces). ⋄

Example 7.10. Any Banach space. In this case, you have just one norm which defines the topology. As we saw (by taking X non-compact), there are other F-spaces. ⋄

Example 7.11. Let X be a compact smooth manifold. Then $C^k(X)$ is a B-space for all $0 \leq k < \infty$ using the C^k semi-norms [[★★★ a sequence converges if derivatives up to order k converge pointwise?]]. The norm will depend on some choices (like a choice of some charts or a choice of a Riemannian metric), but the induced topology does not depend on these choices. An F-space only has the topological structure, not on the sequence of semi-norms. Similarly, C^k has lots of norms, but they all define the same topology, so it is more naturally an F-space than a B-space.

If X is not compact, then $C^k(X)$ is only an F-space.

Finally, if X is not compact, you can make $C^\infty(X)$ into an F-space via controlling more and more derivatives on larger and larger compact sets. It is enough to check completeness. [[★★★ We're taking an inverse limit of Frechét spaces $\cdots \to C^{k+1}(X) \hookrightarrow C^k(X)$. Shouldn't completeness follow immediately from the fact that all the C^kX are complete and $C^\infty X = \bigcap C^k X$?]] ⋄

If V and W are topological vector spaces, then $V \otimes_{alg} W$ has lots of possible topologies. This is one of the problems with topological vector spaces. We'll use the projective topology, characterized by: given a continuous bilinear $V \times W \to Z$, it factors uniquely through $V \otimes_{proj} W$ by a continuous linear map (through the usual bilinear map $\varepsilon\colon V \times W \to V \otimes_{proj} W$).

Definition 7.12. If V and W are F-spaces, then define $V \otimes W$ to be the completion of $V \otimes_{proj} W$. ⋄

The result is that if we take complete topological vector spaces Z, the property given above characterizes $\otimes$ (because a map to a complete thing extends uniquely to the completion).

Lemma 7.13. *If X_1 and X_2 are smooth manifolds, then $C^\infty(X_1) \times C^\infty(X_2) \to C^\infty(X_1 \times X_2)$ given by $(f_1, f_2) \mapsto \big((x_1, x_2) \mapsto f_1(x_1) \cdot f_2(x_2)\big)$ is bilinear* and continuous.

Theorem 7.14. *The induced map $C^\infty X_1 \otimes C^\infty X_2 \to C^\infty(X_1 \times X_2)$ is an isomorphism.*

For any super manifold, $C^\infty(M)$ is a Frechét algebra. In fact, the structure sheaf $\mathcal{O}_M$ is a sheaf of Frechét algebras. Locally, that sheaf is $C^\infty(\mathbb{R}^m) \otimes \bigwedge^* \mathbb{R}^n$, and this exterior algebra is finite dimensional. We know that $C^\infty(\mathbb{R}^m)$ has a unique Frechét structure. $\otimes$ defines a product on super manifolds which is compatible with the product on manifolds.

[2] A subset is convex if the line between any two points in the set is in the set. Locally convex means that any neighborhood of 0 contains a convex neighborhood.

4 RB 09-18

In the last lecture, we were discussing the free Hermitian scalar quantum field theory. Recall that, as for all quantum field theories, we need to give a $*$-algebra A acting on H. For a free quantum field theory, A is generated by operators $\phi(f)$ where f is a smooth compactly supported function on $\mathbb{R}^{1,3}$. You define $\phi(f) = \phi^+(f) + \phi^-(f)$, creation and annihilation operators. All the creation operators commute, and all annihilation operators commute, and $[\phi^-(f), \phi^+(g)]$ is some scalar ($\int_{p^2=m^2,p_0>0} \hat{f}(p)\hat{g}(-p)$). And $\phi^-(f)^* = \phi^+(f)$. This more or less gives you a Heisenberg algebra, and these algebras have really easy representation theory. H is the (essentially unique) representation generated by the vacuum vector which is killed by all $\phi^-(f)$.

I was commenting on various properties that this has: positivity, Lorentz invariance, positive energy. These are easy to check. A harder property to check is locality: $[\phi(f), \phi(g)] = 0$ if f and g are spacelike separated (i.e. $\mathrm{Supp}(f)$ and $\mathrm{Supp}(g)$ are spacelike separated). This has something to do with the fact that you can't send signals faster than light.

We have

$$\begin{aligned}
[\phi(f), \phi(g)] &= [\phi^-(f), \phi^+(g)] - [\phi^-(g), \phi^+(f)] \\
&= \int_{p^2=m^2,p_0>0} \hat{f}(p)\hat{g}(-p) - \int_{p^2=m^2,p_0<0} \hat{f}(p)\hat{g}(-p) \\
&= \int f(x)g(y)m(x-y)\, d^4x\, d^4y
\end{aligned}$$

[[★★★ picture of cone and two sheet hyperboloid; first integral over top sheet and second integral over bottom sheet.]] This is the same as integrating over the whole two sheet hyperboloid with a measure M of coefficient $+1$ on the top sheet and a coefficient of -1 on the bottom sheet. Let m be the Fourier transform of this measure. So to prove locality, we have to show that m vanishes on spacelike vectors. You could explicitly compute m, but let's not. M has the following properties:

1. It is invariant under rotations preserving the time direction.

2. It changes sign under the reflection $t \mapsto -t$.

It follows that m has the same symmetries. Any function m with these symmetries is zero on spacelike vectors x because there is a time-reversing rotation σ fixing x. So $m(x) = m(\sigma x) = -m(x)$ as m changes sign under σ. With a little more work, we see that any measure m must also vanish on spacelike vectors.

So locality can be traced back to the fact that the measure we integrate over has an antisymmetry property under reversing the time direction.

Why do we define $[\phi^-(f), \phi^+(g)] = \int_{m^2=p^2,p_0>0} \hat{f}(p)\hat{g}(-p)d\cdots$? (1) We want translation invariance. Any translation invariant distribution like this can be written as $\int \hat{f}(p)\hat{g}(-p)d(\text{measure})$ by taking Fourier transforms and fiddling with them. Conversely, you can take any measure you like and this will be translation invariance. (2) We want rotation invariance. This forces the measure to be rotationally invariant. (3) Positive energy condition needs the condition that the support of the measure has "positive energy". This means that the support is in the positive cone of momentum space ($p^2 > 0$). [[★★★ only the top part of the cone for some reason]] (4) Why do we need the condition $p^2 = m^2$? We don't. For other measures, we get things called "generalized free field theories" which are pretty similar. The only use I know of for these things is as a source of counterexamples. Q: the ones for fixed m are irreducible and the others aren't? RB: depends on what you mean by that. It's true that if you want the state space (?) to be irreducible, you have to use the free field theory. The generalized guys still have H irreducible as an A-representation.

What is a free field (theory)? That is, how do you recognize a free field theory when you see it? There doesn't seem to be any definition, but everybody can recognize one when it comes up.[1] A free field theory should be roughly equivalent to a representation of some Heisenberg algebra. The $*$-algebra should be generated by a set of "annihilation operators" which commute (this isn't too important) and the commutator of an annihilation operator with an adjoint of one should be some scalar (this is the key property).

What does a free field theory actually look like? That is, what is the Hilbert space H? $H = \mathbb{C} \oplus Sym^*(H_1)$ (H_1 is "one particle states"). H_1 is the space of (well-behaved)[2] functions on the manifold $p^2 = m^2, p_0 > 0$.

[1] A dog can't define a rat, but knows one when it smells it.
[2] L^2 or rapidly decreasing, or whatever.

Classically, the one particle states with $p^2 = m^2$ correspond to the points of this manifold. In quantum mechanics, the space of one particle states is functions on this manifold. The interpretation is that $Sym^n(H_1)$ is the space of n non-interacting particles ("sort of like mathematicians at a party I guess"). It would be good to have a field theory where you have the particles interacting. Nobody has found such a field theory which satisfies the Wightman axioms. The problem with the Wightman axioms is that they don't allow perturbative QFTs, where instead of working over $\mathbb{R}$, you work over a formal power series ring $\mathbb{R}[[\lambda]]$ (λ is called a coupling constant). Solution: extend the Wightman axioms to work over $\mathbb{R}[[\lambda]]$; make A an algebra over the power series ring instead of over $\mathbb{R}$. Let's check if there are any problems.

- Lorentz invariance works over $\mathbb{R}[[\lambda]]$ with no changes.
- Locality works over $\mathbb{R}[[\lambda]]$ with no changes (so long as you don't do anything stupid).
- Positive energy and positivity gives a problem. $(a, a) \geq 0$ and $(a, Ea) \geq 0$. You have to decide what it means for a formal power series to be positive. There are several different ways to define positivity. We need to explicitly choose a set of positive elements of $\mathbb{R}[[\lambda]]$. It doesn't seem to matter very much which method you choose. For example, you could say $x \geq 0$ if x is a square of a formal power series (any sum of squares is already a square). AJ: does this admit $\Phi3$(?) theory in 6-dimensions, where you don't have a positive definite Hamiltonian by perturbatively it doesn't matter? RB: I'm not sure. The problem is that the potential is not positive, but perturbatively it is (because the kinetic term is positive). I would guess this works. This illustrates that working perturbatively, you don't see a lot of important things.

To describe a free field theory, we'll describe its n-point "functions" (distributions). To describe a representation H from A using a "state" ω, we do $(a, b) = \omega(ab^*)$. What is the state ω for a free field theory? A typical element of A looks like $\phi(f_1)\cdots\phi(f_n)$. The corresponding element ω is given by $\langle vac, \phi(f_1)\cdots\phi(f_n) vac\rangle$. This is sometimes written as $\langle vac|\phi(f_1)\cdots\phi(f_n)|vac\rangle$ or as $\langle\phi(f_1)\cdots\phi(f_n)\rangle$ [[★★★ I think]]. This is really a distribution on the product of n copies of spacetime (sometimes called a Wightman distribution). Knowing this ω is the same as knowing these Wightman distributions explicitly.

The idea is that to compute $\langle vac, \big(\phi^+(f_1) + \phi^-(f_1)\big)\cdots\big(\phi^+(f_n) + \phi^-(f_n)\big) vac\rangle$, you push the $\phi^-(f_i)$ to the right, as these kill the vacuum. Let's do the case of a two-point function first.

$$\begin{aligned}\langle(\phi^+(f_1) + \phi^-(f_1))(\phi^+(f_2) + \phi^-(f_2))\rangle &= \langle\phi^-(f_1)\phi^+(f_2)\rangle \\ &= \langle[\phi^-(f_1), \phi^+(f_2)]\rangle\end{aligned}$$

which is a scalar distribution given by $\Delta^+(x_1, x_2)$, Fourier transform of $p^2 = m^2$, $p_0 > 0$. This is some kind of Bessel function. [[★★★ rewrite this. since the commutator $[\phi^-(f_1), \phi^+(f_2)] =: \Delta^+(x_1, x_2)$ is a scalar operator, $\langle vac, [\phi^-(f_1), \phi^+(f_2)] vac\rangle = [\phi^-(f_1), \phi^+(f_2)]\langle vac, vac\rangle = [\phi^-(f_1), \phi^+(f_2)]$]]

Lemma 4.1. *The "n-point function" $\langle\phi(f_1)\cdots\phi(f_n)\rangle$ is given by $\sum_{\sigma \text{ perfect pairing}} \prod_{i<\sigma(i)} \Delta^+(x_i, x_{\sigma(i)})$, where a* perfect pairing *is an order 2 permutation with no fixed points.*

Proof. It suffices to prove that

$$\langle\phi(f_1)\cdots\phi(f_n)\rangle = \sum_{k=2}^{n} \Delta^+(x_1, x_i) \cdot \langle\widehat{\phi(f_1)}\phi(f_2)\cdots\widehat{\phi(f_k)}\cdots\phi(f_n)\rangle$$

(where the hats indicate omission) and to observe that the 1-point function $< vac, (\phi^+(f_1) + \phi^-(f_1)) vac\rangle$ is zero because the ϕ^- kills the vacuum on the right and the ϕ^+ kills the vacuum on the left. This formula follows almost immediately from the observation

$$\begin{aligned}\phi^-(f_1)\phi(f_k) &= [\phi^-(f_1), \phi^+(f_k)] + \phi(f_k)\phi^-(f_1) \\ &= \Delta^+(x_1, x_k) + \phi(f_k)\phi^-(f_1).\end{aligned}$$

[[★★★ it wouldn't hurt to write this out more explicitly]] □

Example 4.2. By the lemma, we can compute the 4-point function $\langle\phi(f_1)\phi(f_2)\phi(f_3)\phi(f_4)\rangle$ to be

$$\begin{aligned}&\Delta^+(\,1 - 2 \quad 3 - 4\,) + \Delta^+(\,1 \quad 2 \quad 3 \quad 4\,) + \Delta^+(\,1 \quad 2 - 3 \quad 4\,) \\ =&\Delta^+(x_1, x_2)\Delta^+(x_2, x_4) + \Delta^+(x_1, x_3)\Delta^+(x_2, x_4) + \Delta^+(x_1, x_4)\Delta^+(x_2, x_4) \quad \diamond\end{aligned}$$

So in free field theories, everything can be determined in terms of the two-point functions. This two-point functions is called the Wightman 2-point function or the *cut propagator*.

Next we'll discuss propagators, which are confusing because there are six different sorts of propagators for a QFT. We'll talk about them and the relations between them.

10 NR 09-19 Pure Yang-Mills theory

Fix a principal G-bundle $E \to M$. Assume M is Riemannian and $\mathfrak{g} = Lie(G)$ has a non-degenerate bilinear form $\langle , \rangle$. The space of fields in pure Yang-Mills theory is the space of connections on E (see Appendix 1 [[★★★ does not yet exist]]). The Lagrangian is

$$\mathcal{L}(A) = \frac{1}{2}(F(A), F(A))$$

where $F(A) = dA + A \wedge A \in \Omega^2(M, E^{ad})$ is the curvature of the connection A. If $\{x^i\}$ are local coordinates on M and $\{e_a\}$ is a basis for $\mathfrak{g}$, then

$$F(A) = \sum_{i,j,a} F^a_{ij} dx^i \wedge dx^j e_a \quad \text{where} \quad F^a_{ij}(A) = \partial_i A^a_j - \partial_j A^a_i + c^a_{bc} A^b_i A^c_j$$

There are several names for this: Yang-Mills, gauge theory, chromodynamics. There are observables which are guage invariant when you cannot express in terms of F, like $W(A) = tr(P\exp(\int_\gamma A))$ for $\gamma \subseteq M$. [[★★★ for NR: edit this an write Appendix 1]]

If we assume $G = GL_N$, then $A^g = gAg^{-1}$ and $\omega^R_g = dg \cdot g^{-1}$. This is an infinite dimensional group. It acts on connections: $F(g(A)) = gF(A)g^{-1}$. You can see that the Lagrangian $\mathcal{L}$ is invariant with respect to this action. So we have a Lie group acting on the space of fields and the Lagrangian is invariant. This is very bad news because it means that the Lagrangian is invariant. The Legendre transform assumes that $\det(\frac{\partial^2 \mathcal{L}}{\partial \xi_i \partial \xi_j}) \neq 0$, but we have that $\mathcal{L}(\xi^g, q^g) = \mathcal{L}(\xi, q)$ which implies that this determinant is identically zero. I think this was the second project, and it is still open.

Why chromodynamics? According to the accepted theory of strong and weak interactions, these fields are supposed to describe the particles which glue the nucleus together. If you consider the case where $G = U(1)$, then

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}(F(A), F(A))_{\text{metric on } M} \\ &= \frac{1}{2} F_{ij}(A)^2 \end{aligned}$$

This is called Pure electrodynamics. [[★★★]] Something about $d = 4$. As an exercise, you can derive the Maxwell equations as the Euler-Lagrange equations for this action. In this case, A is called the vector potential. The electric field induced by this vector postential $A = (A_0, \vec{A})$ (in coordinates x_0 (time), x_1, x_2, x_3) is

$$\vec{E} = \partial_0 \vec{A} \qquad \vec{B} = \vec{\nabla} \times \vec{A}$$

From these you can derive the Maxwell equations. Q: are you assuming Minkowski metric? NR: strictly speaking, I'm assuming we're in dimension $d = 3 + 1$.

The moral is that there are many systems with degenerate Lagrangians. We should be able to somehow reduce the manifold since the Lagrangian is invariant with respect to the guage group, and formulate the mechanics on orbits of the action. Then we can hope the Lagrangian will be non-degenerate so that we can do the Hamiltonian formulation.

Example (3); Chern-Simons (CS) theory. M is a 3-dimensional smooth manifold, $F = \Omega^1(M, E^{ad})$, $E \to M$ is a principal G-bundle, and $\mathfrak{g}$ has $\langle a, b\rangle$ non-degenerate. Let's assume $\mathfrak{g}$ is simple, so this is the killing form $\operatorname{tr}(ab)$.

Remark 10.1. The Lagrangian should always be a top form so that we can integrate it, so you should have a volume form whenever you describe it as a function. ◇

Take the Lagrangian $\mathcal{L} = \operatorname{tr}(A \wedge dA) + \frac{1}{3}\operatorname{tr}(\wedge^3 A)$ (this is a 3-form on M. PT: isn't it a 3-form on the total space E?), then

$$\mathcal{A} = \int_{M_3} \mathcal{L}(x)$$

Is this guage invariant? What happens if $A \mapsto g(A)$.

PT: isn't A a 1-form on E with values in the Lie algebra?

8 PT 09-20 Super Lie algebras (over $\mathbb{C}$)

Classical super Lie algebras (see Kac):

$$\begin{array}{ccc} \mathfrak{gl}(p|q) & \supseteq & \mathfrak{sl}(p|q) \\ \cup\!| & & \\ \mathfrak{osp}(p|q) & & \end{array}$$

$\mathfrak{osp}$ has several versions over $\mathbb{R}$.

$$P(n) = \left\{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \,\middle|\, \operatorname{tr}(A) = 0, B \text{ symm}, C \text{ skew symm} \right\}$$

$$Q(n) = \left\{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} \,\middle|\, \operatorname{tr}(B) = 0 \right\}$$

$\mathfrak{g}$	F_4	G_3	$D_{2\|1}(\alpha)$
sdim	$24\|16$	$17\|14$	$9\|8$
$(\mathfrak{g}_0, \mathfrak{g}_1)$	$\cdots$	$(G_2 \times \mathfrak{sl}_2, \mathbb{C}^7 \otimes \mathbb{C}^2)$	$((\mathfrak{sl}_2)^3, \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2)$

Note that $\mathfrak{g}_1$ is a representation of $\mathfrak{g}_0$. The table doesn't give you the information of what brackets of $\mathfrak{g}_1$ with $\mathfrak{g}_1$ are.

A super Lie algebra is a super vector space $\mathfrak{g}$ together with a Lie bracket $[,]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that is skew symmetric and satisfies the Jacobi identity (in SVect, so $[b, a] = -(-1)^{|a|\cdot|b|}[a, b]$).

Example 8.1. If $\operatorname{sdim}(V) = (p|q)$, then $\mathfrak{gl}(p|q) = \operatorname{Hom}(V, V) = \mathsf{SVect}(V, V)$. This contains $\mathfrak{sl}(p|q) := \{\alpha \in \operatorname{Hom}(V, V) | \operatorname{str}(\alpha) = 0\}$. Note that if $p = q$, then the identity has supertrace zero. If you want a simple Lie algebra, you'd have to quotient out by the multiples of the identity. The bracket here is $[\alpha, \beta] := \alpha\beta - (-1)^{|\alpha|\cdot|\beta|}\beta\alpha$. $\mathfrak{sl}(p|q)$ inherits this bracket. ◇

Example 8.2. If ϕ is a non-degenerate symmetric bilinear form on $V = V^e \oplus V^o$ (i.e. it is symmetric on V^e, skew on V^o, and vanishes on pairing even with odd), we have $\mathfrak{osp}(p|q) = \{\alpha \in \operatorname{Hom}(V, V) | \phi(\alpha v, w) +$

$(-1)^{|\alpha|\cdot|v|}\phi(v,\alpha w)=0\}$. Over $\mathbb{R}$ there are many non-isomorphic forms like this, so you should really write $\mathfrak{osp}(p_1,p_2|q)$, where (p_1,p_2) is the signature. Over $\mathbb{C}$ there is only one such form up to isomorphism. The Lie bracket is inherited from $\mathfrak{gl}(p|q)$. ⋄

Example 8.3. $P(n)$ and $Q(n)$ ⋄

Example 8.4. Finally, there are a few exceptional super Lie algebra. The last one has a continuous parameter α. This completes the list of *simple* super Lie algebras (actually, not all of these examples are simple, because you have to quotient $\mathfrak{sl}(p|p)$ by identities to get simple, and erase $\mathfrak{gl}(p|q)$. Then you have all the simples. This is a theorem of Kac [[★★★ ref]]). You have to form extensions between these things ⋄

Andy: why don't you also have one for skew-symmetric bilinear forms? PT: It turns out to be isomorphic to $\mathfrak{osp}$ (as a super Lie algebras), so it is already on the list. Reimundo: It's not the same vector space though; $\mathfrak{spo}(p|q)$ should be isomorphic to $\mathfrak{osp}(q|p)$. What happens if you take a form which pairs the even and the odd part non-trivially? [[★★★ something I didn't catch]]

Super Lie groups

Definition 8.5. A super Lie group is a super manifold G with $\mu\colon G\times G\to G$, $\nu\colon G\to G$, and $e\colon pt\to G$ satisfying .... Here we had to use existence of products in SMan. ⋄

We were asking if SMan has products. We thought of it as some category of sheaves over ordinary manifolds, together with some Frechét structure. Then you can take products by taking these completed tensor products. Another way of proving this was that we had an equivalence of categories $\bigwedge$-Vect to SMan, given by taking $(E^q\to X^p)\mapsto C^\infty(\pi E)=C^\infty(\bigwedge^* E^*)$, where the morphisms in $\bigwedge$-Vect are induced by the morphisms in SMan. Now it suffices to show that $\bigwedge$-Vect has products. Define $(E_1\to X_1)\times(E_2\to X_2)=(E_1\oplus E_2\to X_1\times X_2)$ (since $\bigwedge^*(E_1^*\oplus E_2^*)\cong\bigwedge^* E_1^*\otimes\bigwedge^* E_2^*$). You can use whatever model for the product you like.

Definition 8.6. A *(left) G-action* on a super manifold M is a morphism $\ell\colon G\times M\to M$ satisfying the associativity and unit diagrams. ⋄

Example 8.7. $G=\mathbb{R}^{1|1}$, with $\mathbb{R}^{1|1}\times\mathbb{R}^{1|1}\xrightarrow{+}\mathbb{R}^{1|1}$ given by $\big((t_1,\theta_1),(t_2,\theta_2)\big)\mapsto(t_1+t_2,\theta_1+\theta_2)$ (this actually works on any $\mathbb{R}^{p|q}$). ⋄

Example 8.8 (Super Heisenberg group)**.** $\mathbb{R}^{1|1}\times\mathbb{R}^{1|1}\xrightarrow{\mu}\mathbb{R}^{1|1}$ given by $\big((t_1,\theta_1),(t_2,\theta_2)\big)\mapsto(t_1+t_2+\theta_1\theta_2,\theta_1+\theta_2)$. Note that this group is not commutative (not even super commutative), but you can prove that this is associative. You can check that $\nu(t,\theta)=(-t,-\theta)$ and $e=(0,0)$ (this is just the point 0 in the underlying $\mathbb{R}^1$. You don't actually have a choice of θ because choosing e amounts to choosing an even and an odd function on $\mathbb{R}^{0|0}$, and there aren't any odd functions. ⋄

We want to go from Lie groups to Lie algebra, so I have to define left invariant vector fields. What are left invariant vector fields on a G-manifold M? The usual formula: $\xi\in Vect(M)$ is left invariant if $\ell_g^*\xi=\xi\ell_g^*$ for all $g\in G$ ($\ell_g\colon M\to M$ is multiplication by g). This is problematic in our case because this is defined pointwise; you have to use arrows. Let's check if the formula itself is a problem. Given $\xi\in Vect(M)=Der(C^\infty M)$, we have $C^\infty M\xrightarrow{\xi}C^\infty M\xrightarrow{\ell_g^*}C^\infty M$ is a derivation and $C^\infty M\xrightarrow{\ell_g^*}C^\infty M\xrightarrow{\xi}C^\infty M$ is also, so the equation makes sense.

Define $L:=(\pi_1\times\ell)\colon G\times M\to G\times M$. Note that $C^\infty(G\times M)=C^\infty G\otimes C^\infty M$ (using our definition of $\otimes$). $\mathrm{id}\otimes\xi$ is a derivation on this thing. You have to check that this is continuous on the usual tensor (then there is a unique continuous extension to the completion). You would call this "the vertical vector field on $G\times M$ corresponding to ξ".

Definition 8.9. $\xi\in Vect(M)$ is *G-invariant* if $L^*(\mathrm{id}\otimes\xi)=(\mathrm{id}\otimes\xi)L^*$. ⋄

If you think about it just a bit, you'll see that this L has all group elements built in, and this is equivalent to the usual definition.

The left invariant vector fields form a Lie subalgebra of all vector fields (the derivations form a Lie algebra, but it is infinite-dimensional): $Vect(M)$ is a super Lie algebra under the bracket and $Vect(M)^G$ forms a Lie subalgebra.

Theorem 8.10. *Let G be a super Lie group and $\mathfrak{g}$ be the super Lie algebra of left invariant vector fields on G (so $M = G$ and the action ℓ is the multiplication μ). Then the restriction map $\mathrm{res}_e \colon \mathfrak{g} \to T_eG$ is an isomorphism of super vector spaces.*

If our group G had dimension $(m|n)$, then T_eG has dimension $(m|n)$.
[[break]]
There is a little confusion about what this restriction map. Let's start with a vector bundle "$E \to M$", which is by definition a locally free $\mathcal{O}_M$-module $C^\infty E$ (this is the "sheaf of sections"). Then we have

$$\text{“}\begin{tikzcd} f^*E \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & E \arrow[d] \\ N \arrow[r, "f"] & M \end{tikzcd}\text{”}$$

We specify $C^\infty(f^*E) := C^\infty N \otimes_{(C^\infty M, f^*)} C^\infty E$. Now if $N = \mathbb{R}^{0|0}$ is a point, then $C^\infty N = \mathbb{R}$ or $\mathbb{C}$, so the module $C^\infty(f^*E)$ is just a finite-dimensional super vector space, and this is what I meant by $T_eG = C^\infty(e^*TG)$ (as a $C^\infty(pt)$-module). What is this restriction? Remember that $\mathfrak{g} \subseteq Vect(G) := Der(C^\infty(G)) =: C^\infty(TG)$. I claim there is a map $C^\infty(TG) \to C^\infty(e^*TG)$. This is just saying that there is a map $C^\infty(TM) \to C^\infty(pt) \otimes C^\infty(TM)$ given by $1 \mapsto 1 \otimes s$ for $s \in C^\infty(TM)$.

$$\text{“}\begin{tikzcd} f^*E \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & E \arrow[d] \\ N \arrow[r, "f"] \arrow[u, dashed, bend left, "f|"] & M \arrow[u, bend right, "s"'] \end{tikzcd}\text{”}$$

Remark 8.11. By the way, the notation is $C^\infty(E) = \Gamma_{C^\infty}(X; E)$, which is confusing because this is not the same as functions on E (thought of as a manifold). ◇

Proof of 8.10. We construct the inverse map $T_eG \to \mathfrak{g}$. What is T_eG? An element $v \in T_eG$ is a derivation $v \colon C^\infty G \to \mathbb{R}$ in the sense that $v(f \cdot g) = v(f) \cdot g(e) \pm f(e) \cdot v(g)$. Remember that we can evaluate functions at points by composition: $pt \xrightarrow{e} G \xrightarrow{f} \mathbb{R}$. If you had a global vector field (derivation) $V \colon C^\infty G \to C^\infty G$, then you could compose with evaluation at the point e. So whenever you have a global vector field, you can evaluate at any point you want to get the value of the vector field at the point.

Now we construct $T_eG \to \mathfrak{g}$, given by $v \mapsto \xi_v$. We define $\xi_v \colon C^\infty G \xrightarrow{\mu^*} C^\infty(G \times G) = C^\infty G \otimes C^\infty G \xrightarrow{\mathrm{id} \otimes v} C^\infty G \otimes \mathbb{R} = C^\infty G$ [[★★★]]. Symbolically, $f(x) \mapsto f(x \cdot y) \mapsto v\big(y \mapsto f(xy)\big)|_{y=e}$.

Claim. *ξ_v is left invariant and $v \mapsto \xi_v$ is the inverse to restriction to the identity.*

□

Example 8.12. $(\mathbb{R}^{1|1}, \mu)$ and $v = \partial_{\theta_2}$. Then $\xi_v f = \partial_{\theta_2}\big(f(t_1 + t_2 + \theta_1\theta_2, \theta_1 + \theta_2)\big)|_{(t_2,\theta_2)=(0,0)}$

$$\begin{aligned} f(t,\theta) &= f_0(t) + f_1(t)\theta \\ \xi_v f &= \partial_{\theta_2}\Big(f_0(t_1+t_2) + f_0'(t_1+t_2)\theta_1\theta_2 + f_1(t_1+t_2)(\theta_1+\theta_2)\Big) \\ &= -f_0'(t_1)\theta_1 + f_1(t_1) \end{aligned}$$

So $\xi_v = \partial_\theta - \theta\partial_t$ on $\mathbb{R}^{1|1}$ which is that vector field for which $\frac{1}{2}[\xi,\xi] = -\partial_t$. Then since the dimension of the Lie algebra is $1|1$, we have that it is generated by ξ and $[\xi,\xi]$. ◇

11 NR 09-21

Functionally, classical Hamiltonian field theory can be regarded as a functor from a spacetime category to the category of symplectic manifolds.

Definition 11.1 (First approximation)**.** A *Hamiltonian classical field theory* in a d-dimensional spacetime category is an assignment of a symplectic manifold $S(N_{d-1})$ to each object N_{d-1} in the spacetime category and an assignment of a lagrangian submanifold $L(M_d) \subseteq S(\partial M_d)$ for each morphism M_d. Axioms:

1. $S(\varnothing) = pt$.
2. $S(\overline{N}_{d-1}) = \overline{S(N_{d-1})}$.
3. $S(N_1 \sqcup N_2) = S(N_1) \times S(N_2)$.

   $L_{M_1 \sqcup M_2} = L_{M_1} \times L_{M_2} \subseteq S(\partial M_1) \times S(\partial M_2)$.
4. If $\partial M_d = N \sqcup \overline{N} \sqcup N'$ and if $M' = M/\langle N \sim \overline{N}\rangle$ (so $\partial M' = N'$), then $L_{M'} = \{\ell \in S(N') |$ there is $m \in S(N)$ with $(m, m, \ell) \in L_M \subseteq S(N) \times \overline{S(N)} \times S(N')\}$. [[★★★ for NR: there needs to be some transversality assumption to get a Lagrangian submanifold]] ◇

If you ignore the transversality problem, this gives a functor from a spacetime category to the category of symplectic manifolds (where morphisms in $\mathrm{Hom}(M_1, M_2)$ are lagrangian submanifolds of $M_1 \times \overline{M}_2$).

Let $\mathbb{R} \to E \to M_d$ with M Riemannian, and let $\mathcal{L}$ be a first order Lagrangian (written $\mathcal{L}(\phi, d\phi)$). Then we get an action functional. Let's compute it's variation. We can write $\mathcal{L} = \mathcal{L}_0(\phi) + \mathcal{L}_1(\phi, d\phi) + \mathcal{L}_2(\phi, d\phi) + \cdots$

$$\begin{aligned}\delta\mathcal{A}[\phi] &= \int_M \Big(\frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial d\phi} \wedge \delta d\phi\Big) \\ &= \int_M \Big(\frac{\partial \mathcal{L}}{\partial \phi} - d\big(\frac{\partial \mathcal{L}}{\partial d\phi}\big)\Big) d\phi + \int_{\partial M} \big(\frac{\partial \mathcal{L}}{\partial d\phi}\big)\delta\phi = \alpha(\delta\phi).\end{aligned}$$

$\delta\phi$ is a vector field on the space of fields and α is a 1-form on the space of fields. But at the same time this is a $(d-1)$-form on ∂M_d.

The Euler-Lagrange equations are the condition that the bulk term of the variation vanishes.

$$\frac{\partial \mathcal{L}}{\partial \phi} - d\Big(\frac{\partial \mathcal{L}}{\partial d\phi}\Big) = 0.$$

In this case $S(\partial M) = \Omega^0(\partial M) \oplus \Omega^{d-1}(\partial M)$. The symplectic structure is

$$\omega = \int_{\partial M} D\varphi \wedge D\pi$$

where $\varphi \in \Omega^0(\partial M)$ and $\pi \in \Omega^{d-1}(\partial M)$. [[★★★ what are these Ds? NR: the symplectic form is a pairing on the cotangent space of the space $S(\partial M)$]] It's the same formula as $\sum dp_i \wedge dq^i$ in local coordinates.

$$\omega((\delta\phi_1, \delta\pi_1), (\delta\phi_2, \delta\pi_2)) = \int_{\partial M} (\delta\phi_1 \wedge \delta\pi_2 - \delta\phi_2 \wedge \delta\pi_1)$$

Now I have to describe $L_M \subseteq S(\partial M)$.

$$L_M = \Big\{(\varphi, \pi) \;\Big|\; \varphi = \phi|_{\partial M}, \pi = \tfrac{\partial \mathcal{L}_{(\phi, d\phi)}}{\partial d\phi}\Big|_{\partial M}, \text{ where } \phi \text{ solves E-L eqns}\Big\}$$

[[★★★ HW: prove this is isotropic. to prove it is maximal isotropic, you might need some extra assumption.]]

In the case of classical mechanics, $M = [t_1, t_2], \partial M = p \sqcup \overline{p}$, $S(\partial M) = S \times \overline{S}$, where $S = T^*N$. Solution to the E-L equations

$$L^{(\gamma)}_{t_1,t_2} = \{(P, Q), (p, q) | P = d_Q\mathcal{A}[\gamma], p = -d_q\mathcal{A}[\gamma]\}$$

In principal you can have several solutions to the Euler Lagrange equations, so you might have many Lagrangian submanifolds.

The primal idea is the variational principle. You can then impose some constraints and the result will be a more complicated dynamics. Hamiltonian dynamics gives a very natural framework for such constraints. For field theories, it is a similar story. You start with a variational field theory, but then you can pass to the Hamiltonian description. If $\mathcal{L}$ is invariant under some group action, one should reduce the symplectic manifold to get a Hamiltonian description.

12 NR 09-24

Last time I gave an outline of classical field theory. Our main goal is Chern-Simons theory, where the fields are connections on a principal G-bundle on a [[★★★ Riemannian?]] manifold. The other example is Yang-Mills theory. So let's talk about connections on principal bundles today.

Connections

Definition 12.1. Let $\pi \colon P \to M$ be a fiber bundle over M. A *connection* on P is a distribution on P which projects isomorphically to TM. ⋄

A distribution is a subbundle of the tangent bundle, so for each $p \in P$, we select a linear subspace $A(T_pP) \subseteq T_pP$. We have $d\pi \colon TP \to TM$, which has a kernel. So we get the sequence

$$0 \to \ker(d\pi) \xrightarrow{i} T_pP \to T_{\pi(p)}M \to 0$$

A connection is a choice of a splitting $T_pP = \ker(d\pi)) \oplus A(T_pP)$ with $A(T_pP) \cong T_{\pi(p)}M$ under $d\pi$. Equivalently, a connection is a section $A \colon T_pP \to \ker(d\pi))$ (so $i \circ A = \mathrm{id}_{\ker(d\pi)}$.

An important notion is *parallel transport*. If we have two points $x, y \in M$, and a path γ connecting them, then we can lift γ to an isomorphisms $h(\gamma) \colon P_x \to P_y$. Because $T_pP \supseteq A(T_pP) \cong T_{\pi(p)}M$, we can lift a tangent vector from the base to P. Integrating these lifted tangent vectors gives a lifted path [[★★★ how does this give the isomorphism?]].

Alternatively, you could define $\Gamma_1^{sm}(M)$ to be the category whose objects are points of M with $[-\varepsilon, \varepsilon) \subseteq M$ so that 0 maps to the point (call such a thing c_x) [[★★★ germs of paths]] and $\mathrm{Hom}(x, y)$ consists of smooth paths from x to y with $c_x, \bar{c}_y \subseteq \gamma$ (i.e. with the given germ). Gluing gives the composition. [[★★★ will this be a category ...any trouble with identity morphisms?]] [[PT: you could take the objects to be $[0, \varepsilon)$ collars and the morphisms look like [[★★★ picture with "outside collar"]] so morphisms are also collars, so the identity is just the object itself. Unfortunately, when you go to infinite-dimensional something, it turns out that the natural direction is different from what you think. This gives you a nice category; it works well. It probably isn't the only choice.]] [[★★★ you have to vary parameterizations to get associativity of composition]]

If P is a fiber bundle over M, a representation of $\Gamma_1^{sm} \to P$ is an assignment $c_x \mapsto P_x$ and $(c_x, c_y \subseteq \gamma) \mapsto (h(\gamma) \colon P_x \xrightarrow{\sim} P_y)$. In otherwords, it is a functor from $\Gamma_1^{sm}(M)$ to the category where objects are fibers of P and morphisms are linear morphisms between the fibers [[★★★ what if we don't require isomorphisms?]]. This gives an equivalent description of a connection. PT: you have to put some subtle conditions to give you smoothness of the distribution. NR: I always ignore these smoothness conditions and we'll see why in a minute. You're right that there should be some conditions on smoothness.

Consider M_T a cell decomposition of M. We can try to modify this definition to define a connection on a fiber bundle over a cell decomposition.

Definition 12.2. A fiber bundle over M_T is an assignment to each vertex (0-cell) $x \in V(M_T)$ a fiber P_x such that $P_x \cong P_y$ (non-canonically) for all $x, y \in V(M_T)$. ⋄

Clearly a fiber bundle over M induces a fiber bundle over M_T by restriction. It is not true that any fiber bundle over the decomposition extends to a fiber bundle over M. If you can construct a fiber bundle for any decomposition M_T, then you can expect that it will give you a fiber bundle on M. Q: isn't any fiber bundle over M_T trivial?

We define $\Gamma_1(M_T)$ to have objects vertices of M_T and morphisms are edge paths (in the 1-skeleton) from x to y. A connection is a choice of isomorphisms $P_{e_+} \xrightarrow{\sim} P_{e_-}$ for every path from e_+ to e_- (in a compatible way). A connection is thus a representation of $\Gamma_1(M_T)$.

Connections on principal G-bundles

Let G be a Lie group and $P \to M$ a principal G-bundle (i.e. G acts simply transitively on the fibers of the bundle).

Definition 12.3. A *connection* on a principal G-bundle $P \to M$ is a G-invariant distribution on P that projects isomorphically to TM. ⋄

Again we have

$$0 \longrightarrow \ker(d\pi) \underset{A}{\overset{i}{\rightleftarrows}} TP \xrightarrow{d\pi} TM \longrightarrow 0$$

so that $i \circ A = \mathrm{id}$ and G-invariance.

I claim that a connection can be viewed as an element $A \in \Omega^1(P, \mathfrak{g})^G$, where G acts on $\mathfrak{g}$ by the adjoint action. For this we need the following lemma.

Lemma 12.4. $\ker(d\pi) \cong P \times \mathfrak{g}$ *G-equivariantly (where there is the diagonal action on the product $P \times \mathfrak{g}$).*

Proof. [[★★★ HW]] □

Now it is completely clear; this is exactly what 1-forms do. A element of $\Omega^1(P, \mathfrak{g})^G$ is a G-equivariant morphism $TP \to \mathfrak{g}$. Fiberwise, this means we have a canonical isomorphism $T_pP \cong \mathfrak{g} \times P_p$. [[★★★]]

Suppose A_1 and A_2 are two connections. Will a linear combination be a connection? No, because of the condition $i \circ A = \mathrm{id}$. However, the difference satisfies $i \circ (A_1 - A_2) = 0$, so $A_1 - A_2 \in \mathrm{Hom}_{\mathsf{Vect}}(T_pP/\ker(d\pi), \mathfrak{g}) =: V$. In other words, the space of connections is not a vector space, but it is an affine space over V.

Definition 12.5. An affine space L over a vector space V is a triple $(L, V, \theta\colon L \times L \to V)$ (we usually denote $\theta(a, b)$ by "$a - b$") so that for any $b \in L$, $a \mapsto a - b$ is a bijection $L \cong V$ and $(a - b) + (b - c) = a - c$. In otherwords, it is a principal V-set. ◇

Claim. *Connections on a principal G-bundle P form an affine space over* $V = \mathrm{Hom}_{\mathsf{Vect}}(T_pP/\ker(d\pi), \mathfrak{g})$.

Claim. $V \cong \Omega^1(M, \mathfrak{g}^{ad})$.

This was my initial definition of the space of connections. This is wrong because the space of connections is an *affine space* over this space. The easiest way to see this is through the transformation properties with respect to the action of the guage group.

If you have a connection A, you get an action of $g \in G$ given by $A^g = gAg^{-1} + dgg^{-1}$ (this is the action on connections), so $(A_1 - A_2)^g = g(A_1 - A_2)g^{-1}$ (which is how 1-forms transform). Next time we'll go into details about what this means.

Let P be a principal G-bundle over M_T (a cell decomposition of M), so over each $x \in V(M_T)$, we have $P_x \cong G$ (non-canonically).

Example 12.6. Given $P \to M$, it induces $P \to M_T$. ◇

If A is a connection on P and γ is a path connecting x and y, we have a G-equivariant isomorphism $h(\gamma)\colon P_x \xrightarrow{\sim} P_y$. If we fix a trivialization $P_x \cong G$, then we have the action of G on P by right multipication by the inverse $(g\colon h \mapsto hg^{-1})$. The map $h(\gamma)\colon G \to G$ is multiplication by some element $g(\gamma)$.

Moral: a connection on $P \to M_T$ with fixed trivialization is an assignment to each edge e a group element $g(e) \in G$, so a connection is a mapping $E(M_T) \to G$. If you change the trivialization, this mapping changes.

9 PT 09-25

About the first homework: nobody got the characterization of the Berezinian because I didn't put enough conditions. You need that the Berezinian is natural. If F is free over A of finite dimension, then $\mathrm{Ber}\colon \mathrm{Hom}_A(F,F)^\times \to GL_1(A^e)$ is multiplicative. If $\phi\colon A \to B$ is a morphism of commutative super algebras, then the diagram

$$\begin{tikzcd} \mathrm{Hom}_A(F,F) \arrow[r, "\mathrm{Ber}"] \arrow[d, "\alpha\otimes\mathrm{id}_B"'] & GL_1(A) \arrow[d, "GL_1(\phi)"] \\ \mathrm{Hom}_B(F\otimes_A B, F\otimes_A B) \arrow[r, "\mathrm{Ber}"] & GL_1(B) \end{tikzcd}$$

Recall that we proved the following theorem.

Theorem 9.1. *If G is a super Lie group with $\mathfrak{g} = Lie(G)$ the left invariant vector fields on G (a super Lie algebra), then $res_e\colon \mathfrak{g} \to T_eG$ is an isomorphism in* SVect.

We proved this by constructing an explicit inverse.

If M is a super manifold and $m \in M$, what is T_mM? It is a super vector space (it is a vector bundle over a point). I want to say that $T_mM \cong Der(C^\infty M, \mathbb{R}_m)$ (as a super vector space) [[★★★ HW1. this is an easy one]]. Where is m? $\mathbb{R}$ is a bimodule over $C^\infty M$ via evaluation at m. By the way, a points of M are $\mathsf{SMan}(\mathbb{R}^{0|0}, M) \cong \mathsf{Man}(\mathbb{R}^0, M_{red})$, so a point really is a point of the underlying manifold. You get evaluation at a point by $C^\infty M \to C^\infty M_{red} \xrightarrow{ev_m} \mathbb{R}$.

Ok, so what was this explicit inverse? Remember that $Vect(M) = Der(C^\infty M, C^\infty M)$ is a $C^\infty M$-module. Given $v \in T_eM$, $v\colon C^\infty G \to \mathbb{R}$ a derivation,we produced $\xi_v\colon C^\infty G \xrightarrow{\mu^*} C^\infty(G\times G) \cong C^\infty G \otimes C^\infty G \xrightarrow{\mathrm{id}\otimes v} C^\infty G \otimes \mathbb{R} = C^\infty G$. [[★★★ HW2. fill in the gaps. In particular, check that ξ_v is left invariant]] Explicitly, this ξ_v is given by $f(x) \mapsto f(x\cdot y) \mapsto v(y \mapsto f(x\cdot y))$. This will be left invariant because we are using right multiplication by y. If you wanted right invariant, you'd use $v\otimes\mathrm{id}$ instead of $\mathrm{id}\otimes v$. Part of this homework is to prove the following lemma.

Lemma 9.2 (Inverse Function Theorem [[★★★ this is the right name?]])**.** *If $f\colon M \to N$ induces an isomorphism $df_m\colon T_mM \to T_{f(m)}N$, then f is an isomorphism in some neighborhood of $m \in M$.*

What is this df_m? It is given by $T_mM \cong Der(C^\infty M, \mathbb{R}_m) \xrightarrow{-\circ f} Der(C^\infty N, \mathbb{R}_{f(m)}) \cong T_{f(m)}N$.

Warning 9.3. We do not have $df\colon Vect(M) \to Vect(N)$. ⌟

Examples of super Lie groups and their Lie algebras

Example 9.4. Let $V = V_0 \times V_1$ be a super vector space. Then V is a super Lie group under addition. We have to produce a super manifold and a multiplication, so we need to define $C^\infty(V)$. It is $C^\infty V := C^\infty(V_0) \otimes \bigwedge^*(V_1^*) \supseteq V^*$ (the linear functions on V_0 are smooth and $\bigwedge^1 V_1^* = V_1^*$).

Now we need to describe the super Lie group structure. We have morphisms of super manifolds $V\times V \xrightarrow{+} V$, $V \xrightarrow{-} V$, and $pt \xrightarrow{0} V$. I hope it is clear what exactly these maps are. Note that this is a commutative super Lie group. ◇

Super Heisenberg groups are really key.

Example 9.5 (Super Heisenberg groups)**.** Let $V \in \mathsf{SVect}$, and let b an even skew form on V. So $b\colon V\otimes V \to \mathbb{R}$ is a skew symmetric morphism in SVect, so it is skew on the even part, symmetric on the odd part, and the cross terms are zero because $\mathbb{R}$ is even. Then we can construct $H(V,b)$. As a super manifold, it is $\mathbb{R}\times V$ (or $\mathbb{C}\times V$) (the $\mathbb{R}$ will be the center). But as a group, it is not the product (I hope it is clear that there are products of super Lie groups; you have to be a little careful because you get a sign). The group structure is $(\mathbb{R}\times V)\times(\mathbb{R}\times V) \to \mathbb{R}\times V$, given by $\big((t_1,v_1),(t_2,v_2)\big) \mapsto \big(t_1+t_2+b(v_1,v_2), v_1+v_2\big)$. This is an extension of V by $\mathbb{R}$:

$$1 \to \mathbb{R} \to H(V,b) \to (V,+) \to 1$$

Taking $v_1 = 0$, we see that this $\mathbb{R}$ is central. This is not a semi-direct product because there is no splitting (as groups) if $b \neq 0$. Explicitly, in terms of functions, we have $C^\infty(\mathbb{R}\times V\times\mathbb{R}\times V) \leftarrow C^\infty(\mathbb{R}\times V)$, given by sending $t \mapsto t_1+t_2+b$, where $b\colon V\times V \to \mathbb{R}$ pulled back to $\mathbb{R}\times V\times\mathbb{R}\times V$, and $V^* \ni \phi \mapsto \phi_1+\phi_2$, the pull back of the dual addition map $V\times V \xrightarrow{+} V$, $V^*\times V^* \xleftarrow{+} V^*$. Now I've told you where the linear maps go, and this determines the whole algebra map.

[[break]]

Now let's compute $h(V,b) := Lie(H(V,b))$. As a super vector space, it is $\mathbb{R} \oplus V \in \mathsf{SVect}$. The Lie bracket is given by

$$\frac{1}{2}[v,w] = b(v,w) \cdot c_{\mathbb{R}}$$

where $c = 1 \in \mathbb{R}$ is the central element.

How do we prove this? If $v \in V$, we want a derivation $\psi_v \in h(V,b)$ by the procedure in the proof of Theorem 8.10. Note that $v \in V$ extends to a global vector field $\partial_v \in Vect(V)$. $\xi_v \colon (t,\phi) \xmapsto{\mu^*} (t_1 + t_2 + b\ [[b(v_1,v_2)]], \phi_1 + \phi_2\ [[\phi_1(v_1) + \phi_2(v_2)]]) \xmapsto{\mathrm{id} \otimes v\ \ [[1 \otimes \partial_{v_2}]]} \big((-1)^{|v_1|\cdot|v_2|} b(v_1,-)\ [[= v_1^*]], \partial_{v_2}\phi_2\big)$. The conclusion is that $\xi_v = \partial_v \pm v^*\partial_t$. I should have done the right invariant case so that I don't pick up the sign. So $\xi_v \colon (t,\phi) \xmapsto{\mu^*} (t_1 + t_2 + b, \phi_1 + \phi_2) \xmapsto{v_1 \otimes \mathrm{id}} \big(v_2^*, \partial_{v_1}\phi_1\big)$, so $\xi_v = \partial_v + v^*\partial_t \in Vect(\mathbb{R} \times V)$.

Now the claim is that $[\xi_v, \xi_w]$, the Lie bracket of derivations, is $2b(v,w)\partial_t$.

$$\begin{aligned}
[\xi_v, \xi_w] &= \cdots \\
&= \underbrace{[\partial_v, \partial_w]}_{0} + \underbrace{[v^*\partial_t, w^*\partial_t]}_{0} + \partial_v(w^*\partial_t) \pm \partial_w(v^*\partial_t) + v^*\partial t\partial_w \pm w^*\partial_t\partial_v \\
&= \partial_v(w^*)\partial_t \pm \underbrace{\partial_w(v^*)}_{b(w,v)=v^*(w)} \partial_t \\
&= 2b(v,w)\partial_t
\end{aligned}$$

The ∂_v terms commute so they fall out and the ∂_t stuff commutes. If you believe the signs are good, then some more stuff cancels out. If you think about this some more, you get the right result.

Now you can take the universal enveloping algebra $U(h(V,b))$ defined as usual $(T^*(h(V,b))/(\alpha\beta \pm \beta\alpha - [\alpha,\beta]))$. Inside of this tensor algebra, you have this central part $\mathbb{R}c_{\mathbb{R}}$ (which is different from the unit of the tensor algebra). You can quotient further by this central piece. If V is completely odd, then the quotient $U(h(V,b))/(c_{\mathbb{R}} - 1)$ is the Clifford algebra $Cl(V,b)$. All of this stuff comes from the super Lie group. When you quantize, you use representations of these Clifford algebra, which really come from representations of the super Lie groups. ◇

5 RB 09-25

Propagators

If you look in a physics book, a propagator is supposed to be an "amplitude for a particle to go from one point to another". I don't know that this means, so I'll ignore it. Six types of propagators (for the hermitian scalar field theory)

1. 2 Feynman propagators
2. 2 Cut propagators
3. 2 advanced/retarded propagators

If you think that's a lot, each propagator can be viewed in position space, but it also has a Fourier transform which lives in energy/momentum space, so all together, there are twelve things. Moreover, propagators can be either *massless* or *massive*, so there are 24 propagators. You also have to worry about spin 0, $\frac{1}{2}$, or 1 propagators, so there are 72 of them. These are propagators in Lorentzian space. We could also look a propagators in Euclidean space (his doesn't quite double it). We can also look at propagators in other dimensions, so you should multiply this 72 by ∞. In dimension ≥ 2, massless propagators have special properties. They also behave differently in odd or even dimensions.

That gives you a bird's-eye view of propagators. What is a propagator Δ? Let's consider the case of a Hermitian scalar field in 4-dimensional Lorentzian space.

(1) Δ is a distribution on $M \times M$, where M is spacetime. If you think of Δ as a function (which it isn't), $\Delta(x,y)$ is the "amplitude of propagation from x to y".

(2) It is translation invariant: $\Delta(x+z, y+z) = \Delta(x,y)$, so it is much easier to think of it as a distribution on one variable: $\Delta(x,y) = \Delta(x-y)$ is now a distribution on M.

(3) Δ is a solution to $(\partial_i^2 + m^2)\Delta(x) = c\delta(x)$ (this first operator is the Klein-Gordan operator), where $\delta(x)$ is just a Dirac delta function at 0. If $c = 1$ (Feynman, advanced/retarded), Δ is a Greens function for the K-G equation. If $c = 0$ (Cut propagators), Δ is a solution of the K-G equation.

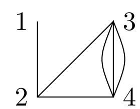
(4) Δ is invariant under rotations (preserving time).

(5) Wave front set of Δ should be as small as possible (see later).

These are the properties that characterize the propagators we'll be interested in. Before we find some propagators, let's say what they are used for.

1. Cut propagators appear as the two point functions of a free field theory. More generally, all the n-point functions can be written in terms of the cut propagator $\Delta(x, y)$.

2. They are needed to define Feynman diagrams. A Feynman diagram is going to appear as a piece of an asymptotic expansion.



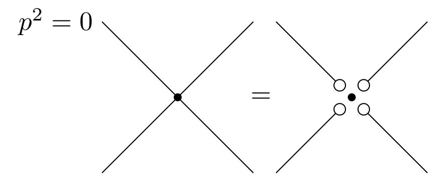
Each edge in the diagram represents a Feynman propagator, and the diagram represents their product, so the picture above represents the term $\Delta(x_1, x_2)\Delta(x_2, x_3)\Delta(x_2, x_4)\Delta(x_3, x_4)^3$. This is NOT DEFINED because of ultraviolet singularities.

Let's try to solve $(\partial_i^2 + m^2)\Delta = \delta(x)$ (i.e. Δ is a Greens function for the K-G operator). Why are we interested in the K-G operator, by the way? Because it appears as the E-L equation for the "free" part of the Lagrangian. For this lecture, assume somebody has given you the K-G operator.

First let's solve it in Euclidean space where it is rather easier. Let's try to solve $(\partial_i^2 - m^2)\Delta = \delta(x)$ with Δ tempered (i.e. don't behave too badly, so closed under Fourier transforms). Taking the Fourier transform, we get $(p^2 + m^2)\tilde{\Delta} = 1$ (where $p^2 = p_1^2 + p_2^2 + \cdots$). The solution is that $\tilde{\Delta} = 1/(p^2 + m^2)$. Note that $p^2 + m^2 > 0$, so this is well-defined. So Δ is the Fourier transform of this $\tilde{\Delta}$. You can write this as a Bessel function if you want, but this doesn't completely specify it as a distribution because of something with singularities. So this is the only reasonable thing to use as a propagator in Euclidean space, so we don't get 6 different things. Note that if $m = 0$ you have some problems, but let's not worry about it.

Now let's try it in Lorentz space. As before, we get $(p^2 + m^2)\tilde{\Delta} = 1$. Now we have a problem because $p^2 + m^2 = 0$ on a two sheet hyperboloid (or a cone if $m = 0$). Anywhere off these surfaces, you can invert $p^2 + m^2$. How do we define $1/(p^2 + m^2)$ (as a distribution) when $p^2 + m^2 = 0$? First of all, the solution of $(p^2 + m^2)\tilde{\Delta} = 1$ is not unique. This equation has solutions given by any function (times measure) on the hypersurface $p^2 = m^2$ (more precisely, any distribution on this hypersurface gives you a solution). So there is an infinite-dimensional space of solutions. We can cut down the dimension by adding the condition that Δ is invariant under time-preserving rotations. In the Euclidean case, it was automatically invariant under rotations, but in the Lorentzian case, this really is a new condition. The rotation invariant solutions of $(p^2 + m^2)\tilde{\Delta} = 0$ form a 2-dimensional space (one for each component of $p^2 + m^2 = 0$). No, that's a bit misleading. This 2 is the number of orbits of the group of time-preserving rotations on the space $p^2 = m^2$; this is not the number of topological components.

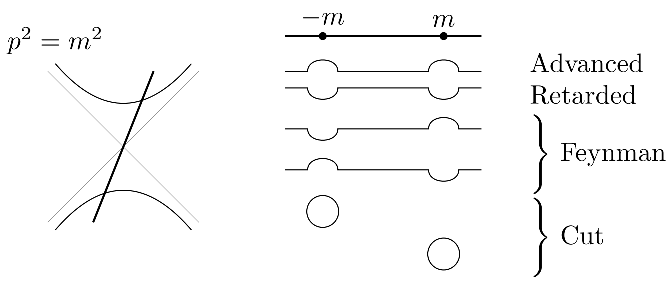
Remark 5.1. If the dimension is 2 and $m = 0$, then $p^2 = 0$ has FIVE components under the connected component of the rotation group.



This corresponds to left moving and right moving things on a string, so this shows up in string theory. $\diamond$

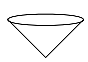
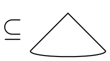
The cut propagators are the two solutions to $(p^2 + m^2)\tilde{\Delta} = 0$ corresponding to these two components. So a cut propagator is the Fourier transform of the invariant measure on a component of $p^2 = m^2$.

Now we want some "good" solutions to $(p^2 + m^2)\tilde{\Delta} = 1$. The solution is essentially $1/(p^2 + m^2)$, but we have to explain what this means when $p^2 = -m^2$ [[There is a sign convension which I can't ever rememeber]]. There are four reasonable things it could mean. The Fourier transform of $1/(p^2 + m^2)$ is $\int_{\mathbb{R}^4} \frac{e^{ipx}}{p^2+m^2} d^4p$. when $p^2 = m^2$, this blows up, but if you think of p as complex, then there is a pole at $p^2 + m^2 = 0$, and you can go around the pole.



There are four reasonable ways to go around the poles. These four choices give the remaining propagators. To get the cut propagators, you're really taking the residue at the pole (i.e. integrating around the little circles). Notice that you can read off lots of linear relations from linear relations between these 1-cycles. For example, if you take the difference between the advanced propagator and a Feynman propagator, you get a cut propagator. The space of solutions is only three dimensional, so if you take any four of these, there will be a linear relation among them.

We're integrating e^{ipx}. This integral goes to zero if the imaginary part of p goes to infinity. So if you have an advanced propagator, you can move your line of integration up to get that it vanishes. The conclusion: The advanced and retarded propagators vanish if x is not in some closed cone.

Supp(Adv) $\subseteq$  Supp(Ret) $\subseteq$ 

The advanced and retarded propagators aren't used very often, except to show that some Feynman propagator is equal to some cut propagator except inside a certain cone.

What is special about these six propagators? They have very small WAVE FRONT sets. I said that there is a 3-dimensional space of Greens functions. Why not integrate along some path that loops around the poles a few times? That gives you a perfectly good propagator, but they have large wave front sets.

Motivation: we want to multiply propagators together (because we want to make sense of Feynman diagrams). The problem is that we cannot multiply distributions. For example, there is no reasonable way to make sense of $\delta(x)\cdot\delta(x)$. We can multiply distributions with DISJOINT SINGULAR SUPPORT because you can always multiply a distribution by a smooth function. The problem is that for Feynman diagrams we want to multiply propagators without disjoint singular support. Wave front sets give a more refined obstruction to multiplying distributions.

Example 5.2. $\delta(x)\cdot\delta(x) = ?$. However, (think of δ as eating functions and giving values) $\delta(f) = f(0)$ which is the residue of $f(z)/z$ at zero, which is given by the integral $\int_C \frac{f(z)}{z}dz$, where C goes around 0 counterclockwise. Alternatively, we could integrate $\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f(z)}{z}dz$ in two different ways (going a little above and a little below the origin), and the difference of these two give you $\delta(f)$. So $\delta(x) = \frac{1}{2\pi i}\frac{1}{x^+} - \frac{1}{2\pi i}\frac{1}{x^-}$. These distributions $\frac{1}{x^\pm}$ are equal to $\frac{1}{x} = \frac{d}{dx}\log(x)$ for $x \neq 0$. $\frac{1}{x^+}$ is $\frac{d}{dx}\log(x)$ for the imaginary part of x greater than zero.

Now note that $\left(\frac{1}{x^+}\right)^2$ is perfectly well defined because it is the boundary value of $\frac{1}{x}$ on the upper half plane. However, $\frac{1}{x^+}\times\frac{1}{x^-}$ is not well-defined because you cannot multiply functions on the upper and lower half planes. The point is that the wave fronts of these two are small enough. $\diamond$

Next time we'll say what wave fronts are and why you can use them to tell if you can multiply propagators.

13 NR 09-26

Unfortunately, this class is going slower than I expected. Something should be done about it or we won't get to the point of the class (Chern-Simons theory). I was going to spend two more classes on connections, but I'll just make a handout which will be part of the notes. So we'll skip the stuff about connections for the time being. I'll return to this part when I'll be discussing guage theories. So connections will be on paper, and later we'll spend a bit more time on connections when we talk about Yang-Mills theory and Chern-Simons theory.

If M is a space of fields with an action of a Lie group G, you get guage theory. Another important part of this is classical field theory with degenerate Lagrangians. Recall that when we went from Lagrangian mechanics to Hamiltonian dynamics, we assumed the Lagrangian is non-degenerate. In general, you can expect that the Hessian will be of constant rank less than the dimension of the manifold. The setup we care about is infinite-dimensional. The nature of the problem can be understood in finite dimensions.

First, let's ignore the most important examples and focus again on the very basic structures.

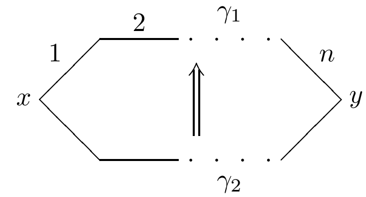
First let's say something about discrete versions of connections. Let M_T be a cell decomposition of M and let $\Gamma = (M_T)^1$ be the 1-skeleton (a graph). A principal G-bundle $P \to M_T$ on M_T is $P \to V(M_T) \cong G \times V(M_T)$ (non-canonically). Two different trivializations are related by

$$\begin{tikzcd} P \arrow[r, "\sim"] \arrow[dr, "\sim"'] & G \times V(M_T) \arrow[d, "\wr"] & (h,v) \arrow[d, mapsto] \\ & G \times V(M_T) & (g(v)h, v) \end{tikzcd}$$

$G_{M_T} = \{g\colon V(M_T) \to G\}$ is the discrete version of the guage group.

$\mathcal{A}^G_{M_T} = \{\text{connections on } P\}/G_{M_T}$. A connection on P, $\alpha = \{\alpha(e)\}_{e\in E(\Gamma)}$, with $\alpha(e)\colon P_{e_+} \xrightarrow{\sim} P_{e_-}$ a G-equivariant (with respect to the right G action) isomorhism given by $p \mapsto \alpha(e)p$. Then we see that $\mathcal{A}^G_{M_T} \cong G^{E(\Gamma)}/G^{V(\Gamma)}$; this isomorphism is the trivialization. Herer $G_{M_T} = \{\beta(v)\colon P_v \xrightarrow{\sim} P_v \ G\text{-equivariant}\}$.

<u>Flatness</u>. Suppose we have two graph paths which are related by a homotopy in the cell complex.



Given α, we can define parallel transport along γ as $h_\gamma(\alpha) = \alpha(e_n)\cdots\alpha(e_2)\alpha(e_1)$. We say that α is *flat* if $h_{\gamma_1}(\alpha) = h_{\gamma_2}(\alpha)$. In particular, if you had a connection on M which was flat in the sense of differential geometry, the induced connection on the cell deomposition is flat. This gives a subspace $\mathcal{M}^G_{M_T}$, the moduli space of flat connections, inside of $\mathcal{A}^G_{M_T}$, the moduli space of graph connections.

[[★★★ HW: $\mathcal{M}^G_{M_T} \cong (\pi_1(M) \to G)/G$ (the representation variety of the manifold) when G is a finite group.]]

These moduli spaces will appear again and again. Moduli spaces that will be common will be [[★★★ something]] of 2-dimensional manifolds M.

back to classical field theory

Let's look at the examples of classical field theories which are finite-dimensional but still have all the important features.

<u>Classical Bose field on surface graphs</u>. A surface graph $\Gamma = (M_T)^1$ is the 1-skeleton of a cell decomposition of a compact oriented manifold, possibly with boundary. When there is a boundary, we will assume that the edges of Γ do not include the edges of $(M_T)^1$ which are on the boundary. The vertices of Γ will be all vertices of $(M_T)^1$.

Recall Hamiltonian field theory. We assign to ∂M (possibly with some special structure, such as marked boundary points) a symplectic manifold $S(\partial M)$ and to M we assign a Lagrangian subspace of $S(\partial M)$ so that some axioms are satisfied. We want to construct this from a Lagrangian field theory. That is, we fix fields on M and a Lagrangian and the variational problem will suggest this structure and the Lagrangian subspace will be the space of solutions to the Euler-Lagrange equations.

In our case, the special structure will be the choice of cell decomposition M_T and the special structure on the boundary is the induced cell decomposition of ∂M.

The space of fields F will be maps from $V(\Gamma)$ to $\mathbb{R}$, $v \mapsto \phi(v)$. The Lagrangian will be

$$\mathcal{A} = \sum_{e \in E(\Gamma)} \frac{\big(\phi(e_+) - \phi(e_-)\big)^2}{\ell(e)^2} v(e)$$

where $\ell, v\colon E(\Gamma) \to \mathbb{R}_{>)}$ are a part of the special structure on M ("part" of the Riemannian metric on M; the length of the edges. $v(e)$ is the volume of the dual to e). For now these are just some functions.

[[★★★ picture]]

If ϕ is a smooth function on M, then we have that $\phi(e_+) - \phi(e_-) \approx \ell(e) \cdot \partial_e \phi$. So the action is (in the limit where the graph fills up the surface)

$$\sum_{e \in E(\Gamma)} (\partial_e \phi(x))^2 \longrightarrow \int_M (d\phi, d\phi)\, d^2 x$$

E-L equations of this form are solutions to the equation $\Delta\phi = 0$. It is easy to see that

$$\begin{aligned} \delta \int_M (d\phi, d\phi) &= 2 \int_M (d\delta\phi, d\phi) \\ &= - \int_M (\Delta\phi)\, \delta\phi \end{aligned}$$

This action is invariant with respect to transformations $x \mapsto \lambda x$ (and in fact invariant with respect to all conformal (angle-preserving) transformations).

What are the Euler-Lagrange equations for this Lagrangian? It should be some different version of the Laplacian.

$$\begin{aligned} (d\mathcal{A}[\phi], \delta\phi) &= \sum_{e \in E(\Gamma)} \frac{\phi(e_+) - \phi(e_-)}{\ell(e)^2} (\delta\phi(e_+) - \delta\phi(e_-)) v(e) \\ &= \sum_{v \in V(\Gamma)} \delta\phi(v) \sum_{e \in S_v} v(e) \frac{(-1)^{(e,v)}}{\ell(e)^2} \big(\phi(e_+) - \phi(e_-)\big) \\ &= \sum_{v \in V^{int}} \delta\phi(v)[\cdots] + \sum_{v \in V^{bdry}} \underbrace{\delta\phi(v) \frac{\phi(e_+) - \phi(e_-)}{\ell(e)^2}}_{*} v(e)(-1)^{(e,v)}. \end{aligned}$$

Where S_v is the collection of edges adjacent to v. $\delta\phi(v)$ is a "vector field", an element of $\bigoplus_{s \in E(\Gamma)} \mathbb{R} = \mathbb{R}^{V(\Gamma)}$, with $\phi(v) \in \mathbb{R}$. $(-1)^{(e,v)} = 1$ if e starts at v and -1 if e ends at v.

So the Euler-Lagrange equations are

$$\begin{aligned} 0 &= \sum_{e \in S_u} v(e) \frac{(-1)^{(e,v)}}{\ell(e)^2} (\phi(e_+) - \phi(e_-)) \\ &= \sum_{w - u} v(w,u) \frac{(-1)^{(w,u)}}{\ell(w,u)^2} (\phi(u) - \phi(w)) \end{aligned}$$

If the adjacency matrix is $a_{u,w} = 0$ if disconnected, -1 if connected (or some other weight), and p if $u = w$. This is the determinant of some weighted adjacency matrix. Under some assumptions, this discrete Laplace-Beltrami operator converges to the smooth Laplace-Beltrami operator.

On solutions to the Euler-Lagrange equations,

$$(d\mathcal{A}, \delta\phi) = \text{boundary terms}$$

$$= (\alpha, \delta\phi)$$

$\partial M \mapsto S(\partial M) = (\bigoplus_{v \in \partial\Gamma} \mathbb{R}) \oplus (\bigoplus_{bdryedges \Pi(e)} \mathbb{R})$, with $\omega = \sum_{v \in \partial\Gamma} d\phi(v) \wedge d\pi(e)$. The Lagrangian L_M is $\{(\phi(v), \pi(v)) | \phi(v)$ is the boundary of a discrete harmonic function such that $\pi(e_v) = *\}$.

This is a finite-dimensional approximation to a very important example in conformal field theory, the Bose field something.

(1) $\omega = d\tilde\alpha$ $\tilde\alpha = \sum_v \pi(e_v) d\phi(v)$, and (2) $\tilde\alpha|_L = d\mathcal{A}$.

10 PT 09-27

More examples of super Lie groups

Problem: If $G \not\cong \mathbb{R}^{p|q}$ (as a super manifold), how does one *write down* $\mu\colon G \times G \to G$? Use the following trick.

Trick: For any super manifold $S \in \mathsf{SMan}$, you can study the *S-points of* G, $G(S) = \mathsf{SMan}(S, G)$. For any S, $G(S)$ is an ordinary group. I hope the group structure is clear:

$$G(S) \times G(S) = \mathsf{SMan}(S, G) \times \mathsf{SMan}(S, G) = \mathsf{SMan}(S, G \times G) \xrightarrow{\mu_*} G(S).$$

Similarly, you get the inverse and identity.

Yoneda's lemma tells us that this determines G. If you want to describe G, it is enough to give a bunch of ordinary groups (with some naturality). To determine $GL_{p|q}$, we just need to write down the (ordinary) groups $GL_{p|q}(S)$.

Definition 10.1. $GL_{p|q}(S) := GL_{p|q}(C^\infty S)$. That is, thinking of $C^\infty S$ as a commutative super algebra, this is the set of invertible even endomorphisms of $(C^\infty S)^{p|q}$. ⋄

Now I claim that this defines the super Lie group $GL_{p|q}$. We have to check naturality.

$$\begin{array}{cccc} S & G(S) & = & \mathsf{SMan}(S, G) \\ \downarrow f & \uparrow f^* = G(f) & & \\ S' & G(S') & = & \mathsf{SMan}(S', G) \end{array}$$

This is clear because we get $C^\infty S' \xrightarrow{f^*} C^\infty S$.

Given such a collection $\{G(S), G(f)\}$, you get a functor $G\colon \mathsf{SMan}^\circ \to \mathsf{Gp}$, but we don't know that the induced $G\colon \mathsf{SMan}^\circ \to \mathsf{Set}$ is representable.

Example 10.2. We have the functor $GL_{p|q}\colon \mathsf{SMan}^\circ \to \mathsf{Gp}$. Is $GL_{p|q}(S) \cong \mathsf{SMan}(S, GL_{p|q})$ for some super manifold $GL_{p|q}$? This looks like we're back to where we started, but the point is that now we're just looking for a super manifold (because the group structure is determined). ◇

General setting: Yoneda embedding. If we have a category $\mathcal{C}$ (SMan in our case), then we get a functor $Y\colon \mathcal{C} \to \mathsf{Fun}(\mathcal{C}^\circ, \mathsf{Set}) =: \hat{\mathcal{C}}$ given by $M \mapsto (S \mapsto \mathcal{C}(S,M))$ and $(g\colon M \to M') \mapsto \big(\mathcal{C}(S,M) \xrightarrow{g_*} \mathcal{C}(S,M')\big)$. $\hat{\mathcal{C}}$ is a kind of completion of $\mathcal{C}$; it has all limits.

Lemma 10.3. $\mathcal{C}(M,M') \underset{\sim}{\xrightarrow{Y}} \hat{\mathcal{C}}(Y(M),Y(M'))$ *(i.e. Y is fully faithful).*

But it is not true that every object in $\hat{\mathcal{C}}$ is in the image of Y.

Definition 10.4. An object in $\hat{\mathcal{C}}$ is *representable* if it is isomorphic (in $\hat{\mathcal{C}}$) to something in the image of Y. ◇

If $\mathcal{C}$ is the category of finite-dimensional vector spaces, then an infinite-dimensional vector space will give you a functor which will not be representable.

Claim. *$S \mapsto GL_{p|q}(S)$ is representable as a domain in $\mathbb{R}^{p^2+q^2|2pq}$ (the space of all even endomorphisms of $(C^\infty S)^{p|q}$).*

We get a functor $S_{red} \mapsto GL_{p|q}(S_{red}) \cong \mathsf{Man}(S_{red}, GL_p\mathbb{R} \times GL_q\mathbb{R})$ induced by $GL_{p|q}$, and $GL_p\mathbb{R} \times GL_q\mathbb{R} \subseteq \mathbb{R}^{p^2+q^2}$ is an open subset.

We define the domain by $U_{red} := \big\{(a,d) |\, \det(a_{red}), \det(d_{red}) \neq 0\big\}$. This gives us a manifold representing our functor [[★★★ hmmm …flesh out the hidden effective descent statement and topology]].

Remark 10.5. Given a super manifold M, as soon as $Y(M)$ factors through Gp, you have the group structure.

$$\begin{tikzcd} \mathsf{SMan} \arrow[r, "Y(M)"] \arrow[dr, dashed, "\widetilde{Y(M)}"'] & \mathsf{Set} \\ & \mathsf{Gp} \arrow[u, "\text{forget}"'] \end{tikzcd}$$

Need $\mu\colon M \times M \to M$, and you have $M(S) \times M(S) \to M(S)$ for each S. Taking $S = M \times M$ and looking at the image of the identity, you get μ. ◇

Definition 10.6. $SL_{p|q}(S) = \{\alpha \in GL_{p|q} |\, \mathrm{Ber}(\alpha) = 1\}$. ◇

You can then check that $SL_{p|q}$ is representable.

[[★★★ There is Project 3 on the website: The K-theory of a super manifold is the same as the K-theory of the underlying manifold]]

[[★★★ Project 4: describe super Lie groups corresponding to Kac's list of simple super Lie algebras over $\mathbb{C}$. After the break, we'll see that there always is some super Lie group giving you the super Lie algebra.]]

[[break]]

There is a theorem that any super Lie group embeds into $GL_{p|q}$, so why do need all this machinery if we're just multiplying matrices? Well, why do you need conceptual mathematics? If you really want to define something precisely (physicists don't define what a field theory is, they just know what it is), you need some conceptual stuff. There is no natural embedding into $GL_{p|q}$.

Theorem 10.7. *There is an equivalence of categories $\mathsf{SMan} \ni G \mapsto (G_{red}, \mathfrak{g}, \mathfrak{g}_{red} \cong Lie(G_{red}), G_{red} \times \mathfrak{g} \xrightarrow{a} \mathfrak{g}$ extending the adjoint action on $\mathfrak{g}^e)$.*

Theorem 10.8 (Stated in [DEF+99, Deligne-Morgan])**.** *Fix a super Lie group G and a super manifold M. Then there is a natural bijection* $\{\textit{actions } M \times G \xrightarrow{r} M\} \leftrightarrow \Big\{\textit{actions } M_{red} \times G_{red} \xrightarrow{\rho} M_{red},\ \textit{Lie homo } \mathfrak{g} \xrightarrow{\phi} Vect(M) \textit{ such that } \mathfrak{g}^e \xrightarrow{\phi|_{\mathfrak{g}^e}} Vect(M)^e \to Vect(M_{red}) \textit{ is } d\rho\Big\}$

$dr\colon \mathfrak{g} \to Vect(M)$.

The idea of the proof: Frobenius theorem [[★★★ Project 5: prove the two theorems using the Frobenius theorem. You have to read between the lines in Deligne and Morgan]]

If X is an ordinary manifold and $\xi \in Vect(X)$, then you get a local $\mathbb{R}$ action (the flow of ξ). This action is global if X is compact. So if $G = \mathbb{R}$, then the reduced action is actually redundant.

Corollary 10.9. *If M is a compact supermanifold, then odd vector fields are in bijective correspondence with actions $(\mathbb{R}^{1|1}) \times M \to M$. (The $\mathbb{R}^{1|1}$ has the Heisenberg group $H(\mathbb{R}^{0|1}, b)$ where b is a non-zero bilinear form. $(t_1,\theta_1)\cdot(t_2,\theta_2) = (t_1 + t_2 + \theta_1\theta_2, \theta_1 + \theta_2)$).*

Proof. An odd vector field $\xi \in Vect(M)$ is a Lie homomorphism from $B(1)$ (free Lie algebra on one odd generator, which is $(1|1)$-dimensional) to $Vect(M)$. $[\xi, \xi]$ is an even vector field on M, so it induces a vector field on M_{red}. Integrating this in the usual way, we get an action of $\mathbb{R}$. □

This is a funny way to do it. You should really solve some ODE with even and odd elements. For project 5, I think you'll have to prove this corollary directly.

Reimundo: how do you know $[\xi, \xi] \neq 0$? PT: it could be zero. This is a good case. Given $\xi \in Vect(M)^o$ such that $[\xi, \xi] = 0$ leads to an action $\mathbb{R}^{0|1} \times M \to M$ and vice versa.

So odd vector fields correspond to $\mathbb{R}^{1|1}$ actions and odd vector fields whose bracket with themselves is zero correspond to $\mathbb{R}^{0|1}$ actions. Later, we'll get $d \in Vect(\pi TX)$ (the de Rham d) from the obvious $\mathbb{R}^{0|1}$ action on $\pi TX = \mathsf{SMan}(\mathbb{R}^{0|1}, X)$ induced by the action of $\mathbb{R}^{0|1}$ on itself.

14 NR 09-28

Last time I gave an example of the classical Bose field on a surface graph. Today we'll generalize a bit and slightly modify. Last time I arranged the theory in such a way that the fields are defined on vertices and the edges are not on the boundary. Another version of the same theory is where $\Gamma = (M_T)^1$ is the 1-skeleton of a cell decomposition on a 2-dimensional compact oriented manifold. Last time I made some mistakes in the signs. What I described last time was Gaussian field theory (or Linear field theory, because the Euler-Lagrange equations were linear; or Free bose field on Γ, because free propagation is described by linear equations).

Now let's consider a more general theory. The fields are the same (maps $V(\Gamma) \to \mathbb{R}$), and the Lagrangian is a collection of smooth functions $\mathcal{L}_{v,w} \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ ((v, w) is an edge in Γ), which is assumed symmetric. The action is given by

$$\mathcal{A}[\phi] = \sum_{neighboring\ vertices\ v,w} \mathcal{L}(\phi(v), \phi(w))$$

Last time, $\mathcal{L}_{v,w}(x, y) = (x - y)^2 \frac{a(v,w)}{\ell(v,w)^2}$. It could be $\mathcal{L}_{v,w}(x, y) = (x - y)^2 \frac{a(v,w)}{\ell(v,w)^2} + V(x) + V(y)$, in which case $V(x)$ is called the self-interacting potential, and this describes a wave interacting with itself.

Last time I talked about the continuum limit. In this case, $\ell(e) \to 0$ and $a(e) \to 0$. We assume $\phi(v)$ is the restriction to v of a smooth function ϕ on $\Sigma = M$. Then $\sum_{(v,w)} \mathcal{L}(\phi(v), \phi(w)) \to \int_\Sigma \big(\frac{1}{2}(d\phi(x))^2 + V(\phi(x))\big)$. In this case, we can compute the E-L equations.

$$\delta\mathcal{A}[\phi] = \sum_v \Big(\sum_{(v,w)} \frac{\partial \mathcal{L}}{\partial x}(\phi(v), \phi(w)) \Big) \delta\phi(v)$$

The E-L equations are then

$$\sum_{w : w - v} \frac{\partial \mathcal{L}}{\partial x}(\phi(v), \phi(w)) = 0$$

for $v \in V(\Gamma^{int})$. A solution to the E-L equations satisfies

$$\delta\mathcal{A}[\phi] = \sum_{v \in V(\partial\Gamma)} \Big(\sum_{(v,w)} \frac{\partial \mathcal{L}}{\partial x}(\phi(v), \phi(w)) \Big) \delta\phi(v)$$

The Hamiltonian interpretation. I'm changing things so that this will work for all dimensions. The symplectic manifold is

$$S(\partial M) = \bigoplus_{v\in V(\partial M)} (\mathbb{R}[[\pi(v)]] \oplus \mathbb{R}[[\phi(v)]])$$

with $\omega = \sum_{v\in V(\partial\Gamma)} d\pi(v) \wedge d\phi(v)$ (this notation is a little different from last time). It is clear that $\omega = d\alpha$ with $\alpha = \sum_v \pi(v) d\phi(v)$. The Lagrangian submanifold is

$$L_M = \{(\pi(v), \phi(v)) \in S(\partial M) | \phi(v) \text{ is the boundary value of a soln to E-L eqns, } (*)\}$$

assuming the E-L equation has a unique solution φ for give boundary values of ϕ

$$\pi(v) = \sum_{(v,w)} \frac{\partial\mathcal{L}}{\partial x}(\varphi(v), \varphi(w)) \qquad (*)$$

The 1-form α and $\mathcal{A}$:

$$\alpha|_{L_M} = d\mathcal{A}[\varphi].$$

Remark 14.1. M_T can be of any dimension. ⋄

I wanted to do discrete Yang-Mills theory, but this would just be another example of a classical field theory demonstrating that the Lagrangian can be invariant with respect to the action of a big group. So if you want the lagrangian submanifold L_M to make sense, you have to reduce the symplectic manifold. If it reduces to a point, you have a topological field theory, which is what happens in discrete Yang-Mills.

Quantization

I forget who said this, but some great person said, "you cannot really understand quantum mechanics; you can only get used to it". Let's accept this point of view. It's an experimental fact. What is the main concept? You have classical observables (smooth functions) on the phase space M (a symplectic manifold). Somehow, this $C^\infty M$ should be replaced by some non-commutative associative algebra. Why it should be this way is a long story, and there are still people who disagree with it. It was formulated by Dirac that there should be a correspondence. Given a classical observable f, there should be an operator $\hat{f}$ which would represent f. This is the quantum-classical correspondence. Dirac (I think) said that these operators should satisfy

$$[\hat{f}, \hat{g}] = i\hbar\widehat{\{f,g\}}$$

This is a version of the quantum-classical correspondence, but it is too naïve to be right. Mathematically, quantization is a deformation of $C^\infty M$ (commutative algebra with $\{\cdot,\cdot\}$) to a family of associative algebras.

Let me make a detour into some obvious facts about (formal) deformations of algebras. Suppose A is a commutative algebra over $\mathbb{C}$. B over $\mathbb{C}[[h]]$. A_h for $h \in [-\varepsilon, \varepsilon]$ is *a family of deformations* of A if

- A_h is an associative algebra for each h,
- $A_0 = A$, and
- there is an isomorphism of vector spaces $A_h \simeq A$.

We fix such an isomorphism $A_h \simeq A$ as part of the data. Then we have

- A a vector space over $\mathbb{C}$, and
- $a *_h b$ a family of associative multiplications such that $a *_0 b = ab$ (multiplication in A).

Q: is there some naturality condition? NR: no, right now it is very generic. Which deformations should we consider equivalent? We say $*_h$ and $\tilde{*}_h$ are equivalent if there exists $\phi_h \colon A \to A$ a linear isomorphism such that $a *_h b = \phi_h^{-1}(\phi_h(a) \tilde{*}_h \phi_h(b))$.

Natural question: given a commutative associative algebra A, describe equivalence classes of deformations.

All of these space are infinite-dimensional, and the question is too general. There should be the condition that the multiplication is continuous and something else is continuous and smooth. One case where it can be answered is if A is finitely generated and has some nice properties. The other thing we can do is do formal deformations (i.e. use formal power series). $*_h$ as function of h is too difficult, so replace it by a formal power series and study the resulting moduli space. Formal deformations are not studied because they are interesting (they are quite boring), but because you can say quite a lot about them. The problem of formal deformations was resolved in the last 15 years (first by Kontsevich, then several people filled in the picture).

Formal deformations of commutative algebras. $B = A[[h]]$ over $\mathbb{C}[[h]]$ is a *formal deformation* of a commutative algebra A if $(B, *)$ has an associative multiplication (called a $*$-*product*)

$$a * b = ab + \sum_{n=1}^{\infty} m_n(a,b)h^n.$$

where $m_n\colon A \otimes_{\mathbb{C}} A \to A$ extended h-linearly to $B \otimes_{\mathbb{C}[[h]]} B \to B$. $*$ is equivalent to $\tilde{*}$ if there exists $\phi\colon B \to B$ such that $\phi(a) = a + \sum_{n=1}^{\infty} h^n \phi_n(a)$ such that

$$a * b = \phi^{-1}(\phi(a) \tilde{*} \phi(b)).$$

Claim. *If $*$ is as above, then $\{a,b\} := \frac{1}{2}m_1(a,b) - \frac{1}{2}m_1(b,a)$ is a Poisson structure on A.*

We can say that $(A, \{,\})$ is classical mechanics (or at least one of the ingredients). The formal quantization deformation problem is: given $(A, \{,\})$, classify equivalence classes of $*$-products such that this Poisson bracket is induced by the $*$-product.

Let's check that an equivalence doesn't change the induced Poisson bracket. Say $a * b = ab + hm_1(a,b) + O(h^2)$, then

$$\begin{aligned} a \tilde{*} b &= \phi^{-1}(\phi(a) * \phi(b)) \\ &= ab + hm_1(a,b) + \underbrace{\big(-\phi_1(ab) + \phi_1(a)b + a\phi_1(b)\big)h}_{\text{symmetric}} + O(h^2) \end{aligned}$$

since the extra linear stuff is symmetric, it doesn't affect the Poisson bracket.

Say (M,p) is a Poisson manifold. Say $A = C^\infty M$, with $\{f,g\} = (p, df \wedge dg)$. Let's assume we want to study *symmetric* $*$-products, meaning

$$m_n(f,g) = (-1)^n m_n(g,f)$$

Theorem 14.2 (Kontseveich, $\mathbb{R}^d$)**.** *The space of such $*$-products modulo equivalence is in bijection with formal deformations of p modulo formal diffeomorphisms.*

A formal deformation of p is where you try to construct

$$\{f,g\}_h = \{f,g\} + \sum_{n=1}^{\infty} h^n p_n(f,g)$$

such that $\{f,g\}_h$ is still a Poisson bracket on $C^\infty M$. A formal diffeomorphism is: given $\alpha\colon M \to M$, you get $\alpha^*(f)(x) = f(\alpha(x))$, and we forget that this comes from a map. A formal diffeomorphism is $\alpha\colon C^\infty(M)[[h]] \to C^\infty(M)[[h]]$ so that $\alpha(f) = f + \sum_{n=1}^{\infty} h^n \alpha_n(f)$.

Next time I'll continue a bit about formal deformation quantization. Then we'll see that there are actual examples of family deformation quantization. Then we'll return to the quantization procedure and construct quantum obervables.

15 NR 10-01

I'll continue with deformation quantization today. Recall that if you have a Poisson algebra $(A, \{,\})$, then a deformation quantization of A is a family of associative algebras A_h so that

- $A_h \cong A$ (as a vector space)[1]
- Assuming the identification $A_h \cong A$, $\{a, b\} = \lim_{h\to 0} \frac{a*b-b*a}{h}$.

This is very hard because these spaces are typically infinite-dimensional and it is hard to construct a family, so there is an easier version, called *formal deformation quantization.*

Let M be a Poisson manifold (assume $M = \mathbb{R}^d$ with some poisson vector field $p \in \bigwedge^2 TM$), with $\{f, g\} = \langle p, df \wedge dg\rangle$. Consider $A = C^\infty\mathbb{R}^d$. Then a *bidifferential $*$-product* on A is a collection $\{m_n : A \otimes_{\mathbb{R}} A \to A\}$ where the m_n are bidifferential operators ($m_n(f, g) = \sum m_n^{\alpha\beta}\partial^\alpha f(x)\partial^\beta g(x)$ for multi-indices α and β of degree $\le n$), such that after extending m_n to $A[[h]] \otimes_{\mathbb{C}[[h]]} A[[h]] \to A[[h]]$ by linearity,

$$f * g = fg + \sum_{n=1}^{\infty} h^n m_n(f, g)$$

is associative. We also require that the $*$-product is *symmetric*, meaning $m_n(f, g) = (-1)^n m_n(g, f)$. Finally, we require $m_1(f, g) = \frac{1}{2}\{f, g\}$.

Let $\phi: A[[h]] \to A[[h]]$, with $\phi(f) = f + \sum_{n=1}^{\infty} h^n\phi_n(f)$, where ϕ_n is a differential operator of degree at most n. We say $* \simeq \tilde{*}$ if $f * g = \phi^{-1}\big(\phi(f)\tilde{*}\phi(g)\big)$ for some such ϕ.

Theorem 15.1 (Kontsevich)**.** *Bidifferential $*$-products up to equivalence are in bijection with formal deformations of the Poisson bracket up to equivalence.*

A deformation of the Poisson bracket is a Poisson bracket of the form $\{f, g\}^\sim = \{f, g\}+\sum_{n=1}^{\infty} h^{2n}p_n(f, g)$, where the p_n are bidifferential operators of order $(1, 1)$. We say that $\{,\}^\sim \simeq \{,\}^\approx$ if $\{\phi f, \phi g\}^\sim = \phi(\{f, g\}^\approx)$ for some ϕ.

[1] This is sometimes called a *torsion free* deformation quantization.

What was surprising about this theorem is that the bijection was constructed completely explicitly with things that look like Feynman diagrams. It turns out that there is a topological quantum field theory interpretation of this result.

The value $\hbar$ is supposed to be an actual number, not a formal parameter ($\hbar$ is Planck's constant after all, it can't "go to zero"). Can something measured in meters, kilograms, or whatever go to zero? No, what we mean when we say that is that the value goes to zero relative to some unit measure. Poincaré in some sense worked out special relativity before Einstein, but his units were $1 = c = 2\pi$ and he didn't have the physical interpretation. Anyway, the point is that there are relative scales, and $\hbar$ changes value based on which scale you're using.

Family deformations.

Example 15.2. Take $M = \mathbb{R}^2$, $A = Pol_{\mathbb{C}}(\mathbb{R}^2) = \mathbb{C}[p, q]$ with the standard symplectic form $dp\wedge dq$ giving the bracket $\{p, q\} = 1$ (this determines the bracket). We have a natural monomial basis p^nq^m on A. Define

$$A_h = \langle p, q | pq - qp = h\rangle$$

It is clear that this is a family of algebras. To say that this is a deformation of A, observe that p^nq^m is a basis A_h, and identifying the bases gives an isomorphism $\theta: A_h \cong A$. Note that we could have chosen a different basis and we would get a different $*$-product, but it would be equivalent. This is why it doesn't make sense to talk about individual $*$-products (rather than equivalence classes). We also have to check that this multiplication is compatible with the bracket. Let's verify that $\{a, b\} = \lim_{h\to 0} \frac{a*b-b*a}{h}$ where $a * b = \theta(\theta^{-1}(a) \cdot_{A_h} \theta^{-1}(b))$. It is enough to check it on generators.

$$\lim_{h\to 0} \frac{p * q - q * p}{h} = \lim_{h\to 0} \theta\Big(\frac{pq - qp}{h}\Big) = 1$$

$\diamond$

Example 15.3. Take $M = T^*\mathbb{R}^d = \mathbb{R}^d \times \mathbb{R}^d$ (with coordinates p_i and q^i, respectively), $\omega = \sum_i dp_i \wedge dq^i = d\alpha$, and $A = Pol_{\mathbb{C}}(\mathbb{R}^d_p) \otimes_{\mathbb{C}} C^\infty\mathbb{R}^d_q$. Then $\{p_i, p_j\} = 0 = \{f, g\}$ (for functions f, g) and $\{p_i, f\} = \frac{\partial f}{\partial q^i}$.

We get the deformation quantization

$$A_h = \langle p_1, \dots, p_n, f \in C^\infty\mathbb{R}^d | [p_i, p_j] = 0, p_if - fp_i = h\frac{\partial f}{\partial q^i}, fg - gf = 0\rangle$$

Take $\theta\colon A_h \xrightarrow{\sim} A$ given by the common basis $p_1^{a_1}\cdots p_n^{a_n} f(q)$.

[[★★★ HW: check that this is a deformation quantization. That is, check that $\{a,b\}$ is the usual limit]]

It is easy to see that A_h can be identified with the algebra of differential operators $\mathit{Diff}_h(\mathbb{R}^d)$ (you have to scale derivatives by h, which is a non-canonical operation).

[[★★★ HW (which could become a projec): if M is d-dimensional and smooth, then $C^\infty_{pol}(T^*M)$ (polynomial in the cotangent direction) has a natural deformation quantization which is the sheaf of differential operators on M]] $\diamond$

Q: what is the multiplication on A_h? NR: I'm defining A_h as a quotient of the free algebra. I can consider the free associative algebra $T(x_1,\dots,x_n)$ and quotient it by some ideal, giving me an associative algebra. We can deform the ideal and get a family $T(x_1,\dots,x_n)/I_h$. Then you have to show that the different algebras you get are isomorphic; this is why we were chosing bases. Q: by $C^\infty\mathbb{R}^d$ is infinite-dimensional. NR: you define it as an algebra over $C^\infty\mathbb{R}^d$. I actually assumed A_h as a space is $Pol(p_1,\dots,p_d)\otimes_{\mathbb{C}} C^\infty(\mathbb{R}^d)$. [[★★★ what is the problem? why can't we just say there are an infinite number of generators?]]

Example 15.4. Let $\mathfrak{g}$ be a Lie algebra, and consider $Pol(\mathfrak{g}^*)=\mathbb{C}[\mathfrak{g}]$. If $\{e_i\}$ is a basis for $\mathfrak{g}$, then we can think of the e_i as coordinate functions x_i on $\mathfrak{g}^*$. A theorem of Kostant, $\{f,g\}(x)=\langle x,[df(x),dg(x)]\rangle$, $\{x_i,x_j\}=\sum_k c_{ij}^k x_k$. $Pol(\mathfrak{g}^*)$ then gets a Poisson bracket.

We can get a deformation quantization

$$A_h=\langle x_1,\dots,x_n|x_ix_j-x_jx_i=h\sum_k c_{ij}^k x_k\rangle$$

Note that $A_h\cong U\mathfrak{g}$ for any $h\neq 0$ (you just have to rescale the x's by h). On the other hand, Chosing the monomial basis $x_1^{a_1}\cdots x_n^{a_n}$ in $\mathbb{C}[x_1,\dots,x_n]$ and the PBW basis in A_h. Identifying them, we get $A_h\cong\mathbb{C}[x_1,\dots,x_n]$, which is how the PBW theorem is usually formulated.

$$U\mathfrak{g}\cong Pol(\mathfrak{g}^*)\cong \mathrm{Sym}(\mathfrak{g})$$

We get a linear isomorphism $\theta\colon A_h\cong A$. It is easy to check that this is a deformation quantization. $\diamond$

There are plenty of examples related to the universal enveloping algebras. [[★★★ Project: report about various aspects of deformation quantization.]]

Quantization of classical mechanics

I didn't talk about all aspects of quantization, just about deformation quantization. You may have heard about geometric quantization and other things. I'll return to them. I want to indoctrinate you that deformation quantization is some how primal (well, not really, there is a way to go back and forth). Geometric quantization gives you representations of deformation quantization. There is a wonderful theorem (the GNS construction) which does something.

Remember that in classical mechanics, we have a symplectic manifold (M,ω) and observables $C^\infty(M)$ (or some algebraic analogue). The deformations we were doing were over $\mathbb{C}$. So if we want to deform, we should complexify: $C^\infty(M)_{\mathbb{C}}=C^\infty M\otimes_{\mathbb{R}}\mathbb{C}$. Then we can recover the classical obervables as fixed points of complex conjugation σ. If you open a textbook, it will say that obervables are hermitian operators. In this case, they are elements of this algebra. You run into the problem that the product of two hermitian operators is not hermitian.

11 PT 10-02

Project 5 (on super group actions) use the following Fröbenius theorem: Every involutive distribution (on a super manifold) is integrable (to a foliation). A *distribution* is a sub-bundle of the tangent bundle. On a super manifold, we have $Der(\mathcal{O}_M)$, which is a locally free $\mathcal{O}_M$-module, and a distribution $\mathcal{D}$ is just a locally free submodule. If you have such a thing, there is the Fröbenius map $\mathcal{D}\otimes_{\mathcal{O}_M}\mathcal{D}\xrightarrow{[,]} Der(\mathcal{O}_M)/\mathcal{D}$, $[X,fY]=X(f)Y+(-1)^{|X||f|}f\cdot[X,Y]$. The first term is an "error term" which is in $\mathcal{D}$, so modulo $\mathcal{D}$, this map is well defined. A distribution is *involutive* if the Fröbenius map is zero. A distribution $\mathcal{D}$ on a manifold M is *integrable* if locally $M\cong M_1\times M_2$ where $\mathcal{D}\cong Der(\mathcal{O}_{M_1})$. This is exactly analogous to the classical version. You can find the proof in [DEF+99, Deligne-Morgan].

Project 6: Formulate a theory of G-principal bundles (where G is a super Lie group) and their connections. This is really a joint project with Kolya's class. A connection picks out a horizontal distribution in $P\to M$. A connection is *flat* exactly when the corresponding horizontal distribution is involutive. The curvature is exactly the Fröbenius map. This should be done in such a way that a representation $G\to GL_{p|q}$ takes a G-principal bundle P to a vector bundle $\mathcal{E}$ and a connection on P to a connection on $\mathcal{E}$ in the following sense. A vector bundle $\mathcal{E}$ is a locally free sheaf of $\mathcal{O}_M$-modules (you should think of this as sections of a total space … making this precise will be your homework). A connection on $\mathcal{E}$ is an $\mathbb{R}$-linear map $\nabla\colon\mathcal{E}\to\Omega^1M\otimes\mathcal{E}$ (remember that $\Omega^1M=\mathrm{Hom}_{\mathcal{O}_M}(Der(\mathcal{O}_M),\mathcal{O}_M)$) such that for a section $s\in\mathcal{E}$ and $f\in\mathcal{O}_M$,

$$\nabla(f\cdot s)=df\otimes s+f\cdot\nabla(s).$$

Usually there would be a sign, but we require ∇ to be even. In particular, if you have a vector field, you can use ∇ to differentiate a section of $\mathcal{E}$ along the vector field:

$$\nabla_X(s)=\langle X,\nabla(s)\rangle\in\mathcal{E}.$$

In my first class, I used the word "Quillen connection". Let me explain that. Given a connection ∇, you can extend it uniquely to a derivation $\widetilde{\nabla}\colon\Omega^*M\otimes_{\mathbb{R}}\mathcal{E}\to\Omega^*M\otimes_{\mathbb{R}}\mathcal{E}$ such that

$$\widetilde{\nabla}(\alpha\otimes s)=d\alpha\otimes s+(-1)^{\text{c-deg}\,\alpha}\alpha\cdot\nabla(s).$$

The sign comes from cohomological degree (which is different from the sign coming from parity). In the ordinary setting, you need the sign so that the curvature $\widetilde{\nabla}^2$ is $\mathcal{O}_M$-linear.

Warning 11.1. Ω^*M has two gradings: (1) cohomological degree "c-deg"$\in\mathbb{N}_0$, and (2) parity $|\alpha|\in\mathbb{Z}/2$. The de Rham d has odd cohomological degree, but even parity. Deligne and Morgan have a good sign convention which doesn't mix the two gradings: Ω^*M is a $\mathbb{Z}$-graded object in the category of super algebras. ⌟

Definition 11.2. A *Quillen connection* on a super vector bundle $\mathcal{E}$ is a $\widetilde{\nabla}$ satisfying $\widetilde{\nabla}(\alpha\otimes s)=d\alpha\otimes s+(-1)^{\text{c-deg}\,\alpha}\alpha\cdot\widetilde{\nabla}(s)$. ◇

Today I want to explain zero dimensional quantum field theories, but you won't know it yet because we haven't said what a quantum field theory is. Recall that if you have a category $\mathcal{C}$ (which will be SMan soon) and if you have $Y,Z\in\mathcal{C}$, then you'd sometimes like to have an inner home $\underline{\mathcal{C}}(Y,Z)\in\mathcal{C}$ so that you get natural isomorphisms

$$\mathcal{C}(X\times Y,Z)\cong\mathcal{C}(X,\underline{\mathcal{C}}(Y,Z)).$$

(here we assume $\mathcal{C}$ has products; in general you can use some other monoidal structure).

Remark 11.3. The adjunction defines a functor $\mathcal{C}^\circ\to\mathsf{Set}$ for a given Y and Z. Then you can ask if the functor is representable. ◇

Take $\mathcal{C}=\mathsf{Man}$. Is there an inner hom? Only when one of the manifolds has dimension 0. If you extend your category to include infinite-dimensional manifolds, then you get an inner hom.

Theorem 11.4. *If $M\in\mathsf{SMan}$, then $\underline{\mathsf{SMan}}(\mathbb{R}^{0|1},M)$ exists and is isomorphic to πTM (by the way, it exists for $\mathbb{R}^{0|q}$ for any q). That is, there are natural isomorphisms*

$$\mathsf{SMan}(S\times\mathbb{R}^{0|1},M)\cong\mathsf{SMan}(S,\pi TM).$$

We'll call $\mathbb{R}^{0|1}$ a "super point", and we'll call πTM the "odd tangent bundle". The caveat is that you need to know what πTM is as a super manifold. [[★★★ current HW2 covers this. If $\mathcal{E}$ is an $\mathcal{O}_M$-module, you'll figure out how this leads to a super manifold E with a morphism $E \to M$ so that E is locally isomorphic to $p_1 : U \times \mathbb{R}^{p|q} \to U$.]]

[[break]]

Statement of HW2: $\mathcal{E}$ a locally free sheaf of dimension $(p|q)$ over $\mathcal{O}_M$ (think $\mathcal{E} = \Gamma(\pi TM)$). Define a morphism $E \xrightarrow{p} M$ which locally looks like $p_1 : U \times \mathbb{R}^{p|q} \to U$. There are two approaches:

(Approach a) $\mathcal{O}_E$ is a completion of $\mathrm{Sym}^*_{\mathcal{O}_M} \mathcal{E}$. If M is an ordinary manifold and $\mathcal{E}$ is an ordinary vector bundle, then we did something which produced a super manifold whose sections were $\mathrm{Sym}^* \mathcal{E}$.

Definition 11.5. If $\mathcal{E}$ is an A-module, then you can form $\mathrm{Sym}_A \mathcal{E}$, the free (commutative) A-algebra on $\mathcal{E}$. That is,

$$\mathsf{SAlg}(\mathrm{Sym}_A \mathcal{E}, B) \cong A\text{-}\mathsf{mod}(\mathcal{E}, B_{\text{forget}}). \qquad \diamond$$

This gives me things which are polynomial on the fibers. $\mathrm{Sym}_{\mathbb{R}}(V^*)$ is polynomials on V. This is why you need to complete: the smooth functions are the completion of polynomial functions (in the Frechét topology).

If F and M are ordinary things, then we defined a super manifold πF. We said that $C^\infty(\pi F) := \bigwedge^*_{\mathcal{O}_M}(F)$.

Definition 11.6. $\bigwedge^*_A \mathcal{E} := \mathrm{Sym}_A(\pi \mathcal{E})$. $\diamond$

Q: Something is funny .... PT: I'm using the sign convention where there is only one grading, which mixes the cohomological degree and the parity.

Sign convention for today: add the parity and cohomological degree to get a single sign. For example, $\bigwedge^*_A \mathcal{E}$ is just a commutative super algebra.

Now the two π's are consistent. What we did a month ago is the special case where M and F are ordinary things, and $F := \pi E$. For the projection, I just have to remember the $\mathcal{O}_M$-module structure.

(Approach b) Functor of points approach. Define

$$\mathsf{SMan}(S, E) := \{(f, s) | f \in \mathsf{SMan}(S, M), s \in f^*(\mathcal{E})^e\}.$$

$$\begin{tikzcd} f^*E \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & E \arrow[d, "p"] \\ S \arrow[r, "f"'] \arrow[ur, dashed] \arrow[u, bend left, "s"] & M \end{tikzcd}$$

Proof of Theorem 11.4. Pick $S \in \mathsf{SMan}$.

$$\begin{aligned} \mathsf{SMan}(S \times \mathbb{R}^{0|1}, M) &\cong \mathsf{SAlg}(C^\infty M, C^\infty S \otimes C^\infty \mathbb{R}^{0|1}) \qquad \text{(for convenience)} \\ &\cong \{(f, s) | f \in \mathsf{SAlg}(C^\infty M, C^\infty S), \\ &\qquad s : C^\infty M \to C^\infty S \text{ odd derivation w.r.t } f\} \end{aligned}$$

Since $C^\infty \mathbb{R}^{0|1} \cong \mathbb{R}[\theta]/\theta^2$, the tensor in the first line is an algebraic tensor product. Given $\phi \in \mathsf{SMan}(S \times \mathbb{R}^{0|1}, M)$, I can write $\phi = f + \theta s$. The condition that ϕ is an algebra map: $\phi(ab) = f(ab) + \theta s(ab)$ for functions a, b, so I have $\phi(a)\phi(b) = (f(a) + \theta s(a))(f(b) + \theta s(b)) = f(a)f(b) + \theta(s(a)f(b) + (-1)^{|a|} f(a)s(b)$. So f must be an algebra map and s is a derivation as desired.

We'll have to finish the last step on Thursday. $\square$

6 RB 10-02

Last lecture we looked at the following problem: (1) a Feynman diagram represents a product of propagators (one for each line). The problem is that propagators have singularities, which make it difficult ot multiply them. There is one easy case when you can multiply them (when they have disjoint singular supports), but this is not refined enough for us. We saw that you can sometimes multiply two distributions even when they are singular in the same place.

Question: when can you multiply two distributions f and g at a point x? Answer: whenever the wave front sets at x travel in roughly the same directions.

What is a waved front set of a distribution f? A crude measure of singularities: look at singular points of f. The wave front set is a subset of the cotangent space at each singular point. PT: how are you thinking of these distributions? As functions which are allowed to blow up? RB: yes, that is a good approximation, and the singular set is the set of points where f cannot be written as a smooth function. A compactly supported distribution f on $\mathbb{R}^n$ is smooth (at all points) is equivalent to saying that the Fourier transform $\hat{f}$ is rapidly decreasing. Now we can ask, "in which directions is $\hat{f}$ NOT rapidly decreasing?" These directions somehow tell you the "directions in which the singularities are going". Such a direction is an element of the dual of $\mathbb{R}^n$.

Definition 6.1. The *wave front* set of f at $x \in \mathbb{R}^n$:

1. localize f at x: multiply f by a smooth compactly supported bump function u with $u(y) = 1$ for y near x.

2. look at the Fourier transform $\widehat{fu}$ of fu. This is a function on the cotangent space of x

The wave front set of f at x is given by intersection over all u of the directions in T_x^* near which $\widehat{fu}$ is NOT rapidly decreasing. ◇

This makes the wave front set of f a conical subset of the cotangent space of $\mathbb{R}^n$

The singular points are just those such that there is a non-zero element of the wave front set in the cotangent space at that point. Some authors say you shouldn't count the zero covector as being in the wave front set, but sometimes it is handy to include the zero covectors.

We can multiply f and g provided there is no point x and vectors $v \in WF(f)_x$ and $w \in WF(g)_x$ with $v + w = 0$ with $v, w \neq 0$. $WF(f)_x$ is a cone and $WF(g)_x$ is a cone. If the cone generated by $WF(f)_x$ and $WF(g)_x$ is a proper cone, then we can multiply f and g. This also works for collections of distributions [[★★★ because the wave front set of the product is contained in the cone generated by the wave front sets of the factors?]].

What are the wave front sets of our propagators? There are six different propagators because there are six interesting choices of wave front sets. The singular points all lie on norm zero vectors (points in the usual cone). We can think of the cotangent space as the tangent space, which inherits the metric. It turns out that the vectors of the wave front sets are cotangent vectors of norm zero (they lie on the cone *in the cotangent space*).

Possible wave front vectors p: $(x > 0$ or $x < 0) \times (p > 0$ or $p < 0)$ (there is also $x = 0$, but that is a really bad guy). We can tweak our propagators so that that the wave front set vanishes in two of these four regions. This gives us $\binom{4}{2} = 6$ possibilities.

(1) Advanced, Retarded propagators. These had the property that the support is contained in a closed cone, so all the singularities are in that cone, so these are the cases where the wave front sets lie in $(x < 0)$ or $(x > 0)$. These don't occur in quantum mechanics because the wave front sets where these are singular are going in all directions, so you can't multiply them together on the light cone. So advanced and retarded propagators are good for classical mechanics, but not for quantum.

(2) Cut propagators. These have the property that the Fourier transforms $\widetilde{\Delta}$ have support on one of the two hyperboloids of revolution of $p^2 = -m^2$ (i.e. one of the sheets of the usual hyperboloid). This is rapidly decreasing except in directions $p^2 = 0$ with $p > 0$ (or $p < 0$) (this is because we're supported on the sheet, so in any other direction, you're eventually zero). So the wave front sets have $p > 0$ (or $p < 0$). Note that you can multiply a cut propagator by itself as much as you want because the wave front sets always go in the same direction.

(3) Feynman propagators Δ. In this case, Δ is equal to a cut propagator except in the negative cone (since Δ is a cut propagator plus the retraded

propagator), so Δ has the same singularities as the cut propagator except in the negative cone. However, Δ is equal to the other cut propagator except in the positive cone (using another relation), so in the positive cone it has the same singularities as the other cut propagator.

Possible wave front sets of propagators:

[[★★★ picture]]

We can multiply Feynman propagators by themselves everywhere except at zero, which is why we get ultraviolet divergences.

Now let's try to evaluate some Feynman diagrams: put a Feynman propagator at each edge and try to multiply them together. By the way, we haven't specified the directions of the edges, but since the Feynman propagators are invariant under multiplication by -1, we're fine.

[[★★★ x y connected by two edges]]

This is $\Delta(x-y)^2$, which is defined except at $x = y$. If we require translation invariance, this implies that the ambiguity is a distribution on $\mathbb{R}^n \times \mathbb{R}^n/\mathbb{R}^n$ with support at a point, given by the diagonal mod $\mathbb{R}^n$. Distributions supported at a point are really easy to deal with.

[[★★★ complicated picture]]

Suppose not all points are the same, and suppose it is connected. We can choose x and y which are joined by a line L. Let F be the diagram minus this line, so the diagram is $F \cup L$. A distribution of $F \cup L$ should be the distribution of F times $\Delta(x-y)$. When is this well defined? Check it is well defined by looking at the wave front sets. For this we need to know about the wave front set of a diagram, which we should be able to do inductively.

Suppose $(p_1, \ldots, p_k)$ is in the wave front set of a Feynman diagram at the point $(x_1, \ldots, x_k) \in (\mathbb{R}^n)^k$.

Theorem 6.2. *If x_i is a minimal*[1] *point with $\sum_{x_j=x_i} p_j \neq 0$, then $\sum_{x_j=x_i} p_j < 0$. If x_i is a maximal point with $\sum_{x_j=x_i} p_j \neq 0$, then $\sum_{x_j=x_i} p_j > 0$*

Wave front set of Δ over a typical point $(x_1, \ldots, x_n)$ (with $x_1 > x_2$) looks like $(p, -p, 0, 0, \ldots, 0)$, $p > 0$. Wave front set of a Feynman diagram F must contain $(-p, p, 0, \ldots, 0)$ for the product not to be defined, but this contradicts the condition satisfied by the wave front set of a Feynman diagram (the theorem).

We also need to check that the wave front set of $F \cdot \Delta$ also satisfies this condition. For this you need to know the wave front set of a product. Use the fact that the wave front set of a product of distributions at a point is contained in the sum of their wave front sets. It is then a fairly easy exercise to check the condition [[★★★ HW: do this exercise, proving the Theorem]].

The result is that we can define each Feynman diagram *up to* addition of a distribution that is (1) supported on the diagonal and (2) translation invariant. This is effectively a distribution supported at a point (the point diagonal/translations in $(\mathbb{R}^n)^k$/translations). PT: so there are only problems when all the points of the Feynman diagram are the same? RB: That's right.

Distributions with support at $0 \in \mathbb{R}^n$ are just given by polynomials in $\frac{\partial}{\partial x_i}$ applied to the Dirac delta function. The Fourier transform is therefore a polynomial in $p_1, \ldots, p_n$. So the ambiguity in the result is a polynomial in momentum. It will turn out that the amibuity will be closely related to what physicists called "counterterms".

[1] Space time is partially ordered: $x < y$ means we can send a signal from x to y.

16 NR 10-03

Last time I gave examples of deformation quantization. Now I want to discuss the states in quantum mechanics. Before that let's discuss one more example.

By $C(M)$ we mean the algebra of obervables (if M is T^*N, this will be polynomial in the cotangent direction and smooth on N; if M is algebraic, this will be algebraic functions; of it might be $C^\infty M$). This is an algebra over $\mathbb{R}$, but all of our deformations were over $\mathbb{C}$. The first step in quantization is to complexify the algebra of obervables, then $C(M)_\mathbb{R} \subseteq C(M)_\mathbb{C}$ is the fixed point set of complex conjugation σ. Then we form the deformation A_h (with the first jet given by the Poisson bracket). We need one more ingredient in the quantum case, which is the $*$-involution. There is a bit of confusion here; this is different from the $*$-product, so I'll use σ when there could be confusion. $\sigma\colon A_h \to A_h$ is an anti-$\mathbb{C}$-linear anti-involution, so $\sigma(fg) = \sigma(g)\sigma(f)$, $\sigma(\lambda f) = \bar\lambda\sigma(f)$, and $\sigma^2 = \mathrm{id}$. Recall that we imposed the assumtion that $A_h \cong C(M)_\mathbb{C}$. This is actually a very strong assumption. In general you get a sequence of matrix algebras and the best you can hope for is that as h goes to zero, you get some kind of isomorphism. Let's ignore this for the moment and assume this torsion free hypothesis. Finally, we need to require that $A_h^\sigma \cong C(M)$ as a real vector space (this should be the restriction of the isomorphism $A_h \cong C(M)_\mathbb{C}$), so we get a deformation of the whole structure. In this case, A_h^σ is called the *quantum space of obervables*. PT: it's not an algebra any more. NR: that's right. If A, B are hermitian operators on a Hilbert space H, then AB is not hermitian, but $AB + BA$ is and $i(AB - BA)$ is. These are the two structures on A. This is the structure of a *Lie-Jordan algebra* on A_h^σ (I think just the first one gives a Jordan algebra). The traditional abuse of language is to say that A_h is the quantum algebra of observables, but not all of it's elements are observables. PT: why do we classically want obervables to be an algebra? Bruce/NR: for example, energey $E = \frac{p^2}{2m}$.

Now we have a family of associative algebras A_h. The first question you should ask is, "what are isomorphism classes of irreducible representations, and what is the structure of its representations?" We have more than an algebra structure, we also have σ. In this setting, there is a natural notion of Hermitian (or unitary, or $*$-) representation. A representation is a homomorphism $\pi_h\colon A_h \to End(V)$. To have a notion of a hermitian conjugate in V, we have to choose a hermitian bilinear form (a bilinear form on V so that $\langle x, y\rangle = \overline{\langle y, x\rangle}$ and $\langle x, x\rangle > 0$ for $x \neq 0$). This gives us a norm $\|x\| = \langle x, x\rangle$. We can complete V with respect to this norm to get a Hilbert space $H = \overline{V}$.

Definition 16.1. A σ-representation of A_h in $(H, \langle,\rangle)$ is a representation $\pi_h\colon A_h \to End(H)$ such that $\pi_h(\sigma(a)) = \pi_h(a)^*$, where A^* of A is defined by $\langle A^*x, y\rangle = \langle x, Ay\rangle$. ⋄

If f is a classical observable, we represent it as $\pi_h(\hat f)$, a hermitian operator on H, where $\hat f$ is the image of f under the isomorphism $C(M) \cong A_h$.

Assume that $\pi_h(A_h) \subseteq B(H)$ (bounded operators on H). This is a common assumption (but rather brave, and not usually true). $B(H)$ is an algebra with $*$-involution (hermitian conjugation). This is the motivating example for the notion of a C^*-algebra. Andy: are you assuming you have a π_h for each h? NR: yes, and we're assuming $h \in \mathbb{R}$. In geometric quantization, $h = 1/m$ for $m \in \mathbb{N}$.

States

Now let's move on to states. I want to deliver the intuitive notion of a state in quantum mechanics. This can be extended to the notion of a state on a C^*-algebra.

Definition 16.2. A is a *trace-class operator* in H, if $\sum_{n=1}^\infty |(Ae_n, e_n)| < \infty$ for any orthogonal basis $\{e_n\}$ of H. In this case, define $\operatorname{tr} A := \sum_{n=1}^\infty (Ae_n, e_n)$. ⋄

Theorem 16.3. —

1. *$B_1(H)$, the space of trace-class operators on H, is a Banach space with $\|A\| = \operatorname{tr}(\sqrt{A^*A}$.*
2. *$B_1(H) \subseteq B(H)$ is a two sided ideal.*
3. *$\operatorname{tr}(AB) = \operatorname{tr}(BA)$ for $A \in B_1(H)$ and $B \in B(H)$.*

Any $A \in B_1(H)$ defines a linear functional on $B(H)$, given by $\ell_A(B) = \mathrm{tr}(AB)$. [[★★★ check that AB is a trace-class operator]]

Definition 16.4. For $A \in B(H)$, the *spectrum* $\sigma(A) = \{z \in \mathbb{C} | A - zI$ is not invertible$\}$. ◇

If $A = A^*$ (i.e. if A is *hermitian*), then $\sigma(A) \subseteq \mathbb{R}$.

Definition 16.5. A hermitian operator A is *positive* if $\sigma(A) \subseteq \mathbb{R}_{\geq 0}$. ◇

Let $\rho \in B_1(H)$ be a positive trace-class operator which is *normalized* (i.e. $\mathrm{tr}\,\rho = 1$).

Definition 16.6. The linear functional $\ell_\rho(A) = \mathrm{tr}(\rho A)$ is a *state* on $B(H)$ and ρ is called the *density matrix* of this state. ◇

You can extend this definition to any operator for which this linear functional is finite.

Definition 16.7. ℓ_ρ is a *pure state* if $\rho = P_\psi$ is the orthoginal projection to $\psi \in H$ (i.e. $P_\psi(\phi) = \frac{\langle\phi,\psi\rangle}{\|\psi\|}\psi$). ◇

So every vector $\psi \in H$ defines a pure state. There is a natural action of S^1 on H, given by $x \mapsto zx$ where $|z| = 1$. The orthogonal projection is invariant with respect to this action, so when you pass to pure states, you get this extra structure.

Let $S_{un}(H)$ be the space of states (i.e. the subspace of $B(H)$ of positive trace class operators).

Proposition 16.8. *$S_{un}(H) \subseteq B(H)$ is a positive cone. That is, (1) if $\rho_1, \rho_2 \in S_{un}(H)$ (the ρs are the density matrices), then $\rho_1 + \rho{-}2 \in S_{un}(H)$ (for $0 < \alpha < 1$, (2) if $\rho \in S_{un}(H)$, then $t\rho \in S_{un}(H)$ for $t \in \mathbb{R}_{>0}$, and (3) $S_{un}(H) \cap (-S_{un}(H)) = \{0\}$*

$S(H) \subseteq S_{un}(H)$, the space of normalized positive trace-class operators (states) is a convex subset in $S_{un}(H)$. That is, $\rho_1, \rho_2 \in S(H)$ implies $\alpha\rho_1 + (1-\alpha)\rho_2 \in S(H)$ for $0 \leq \alpha \leq 1$. Pure states are the extremal points of $S(H)$.

Example 16.9. Assume $\psi_1, \ldots, \psi_N$ is an orthonormal system. Assume that $\rho = \sum_{n=1}^N \rho_n P_{\psi_n}$. ρ is a state if and only if $\rho_n > 0$ and $\sum \rho_n = 1$. In this case, we can interpret ρ_n as the probability that the mixed state ρ is in the pure state P_{ψ_n} during the obervation.

Definition: the *expectation value* of an observable A in a state with density ρ is $\langle A\rangle_\rho = \mathrm{tr}(\rho A)$.

$A = A^*$ is an observable. Say that $A = \sum_{n=1}^M a_n P_{\phi_n}$ (A has finitely many eigenvalues for simplicity). Then $\langle A\rangle_\rho = \sum_{n=1}^N \rho_n (A\psi_n, \psi_n) = \sum_{n=1}^N \sum_{k=1}^M \rho_n a_k |(\phi_k, \psi_n)|^2$. This says that ρ_n is the probability that the system will be found in the pure state P_{ψ_n}. $|(\phi,\psi)|^2$ (≤ 1 since ϕ and ψ are normalized) is the probability that a system in the pure state ψ can be found in the pure state ϕ.

Then $\sum_{n=1}^N \rho_n |(\phi, \psi_n)|^2$ is the probability in a mixed state ρ can be found in the pure state ϕ. ◇

Under certain reasonable assumptions, we will have that if $\rho = e^{-\beta H_h}$ (H_h hermitian converging to something), then $\mathrm{tr}(\rho \cdot \pi_h(\hat{f}))$ converges as h goes to zero to $\frac{1}{(2\pi h)^n} \int_M e^{-\beta H(x)} f(x)\omega^n$.

12 PT 10-05

Theorem 12.1. *If M is a super manifold, then $\underline{\mathsf{SMan}}(\mathbb{R}^{0|1}, M) \cong \pi TM$. That is, there is a natural isomorphism $\mathsf{SMan}(S, \pi TM) \cong \mathsf{SMan}(S \times \mathbb{R}^{0|1}, M)$.*

This theorem will explain zero dimensional quantum field theory. I haven't quite explained what πTM is, but the homework shows that for any locally free sheaf of modules $\mathcal{E}$, there is a total space $E \xrightarrow{p} M$ such that $\mathsf{SMan}(S, E) = \{(f, g) | f \in \mathsf{SMan}(S, M), g \in (f^*\mathcal{E})^e\}$ for any super manifold S. In particular, taking $\mathcal{E} = \pi Der(C^\infty M)$, we get $E = \pi TM$. In particular, the homework shows that these functor are actually representable by a super manifold. We did part of the proof last time, but let's start over.

Proof. Pick S.

$$\begin{aligned}
\mathsf{SMan}(S \times \mathbb{R}^{0|1}, M) &\cong \mathsf{SAlg}(C^\infty M, C^\infty S \otimes C^\infty \mathbb{R}^{0|1}) \\
&\cong \{(f, g) | f \in \mathsf{SAlg}(C^\infty M, C^\infty S), \\
&\qquad g \colon C^\infty M \to C^\infty S \text{ odd derivation } \} \\
&\cong \{(f, g) | f \in \mathsf{SMan}(S, M), \\
&\qquad g \in \Gamma(f^* TM)^{odd} = \Gamma(f^* \pi TM)^{ev}\} \\
&\cong \mathsf{SMan}(S, \pi TM)
\end{aligned}$$

Let $\phi \in \mathsf{SAlg}(C^\infty M, C^\infty S \otimes C^\infty \mathbb{R}^{0|1})$, then $\phi = f + \theta g$ for $f, g \colon C^\infty M \to C^\infty S$. Saying that ϕ is a super algebra homomorphism says that

$$\begin{aligned}
f(ab) + \theta g(ab) &= \phi(ab) \\
&= \phi(a)\phi(b) \\
&= \big(f(a) + \theta g(a)\big)\big(f(b) + \theta g(b)\big) \\
&= f(a)f(b) + \theta\big(g(a)f(b) + (-1)^{|a|} f(a)g(b)\big)
\end{aligned}$$

Comparing coefficients, we get that $f \in \mathsf{SAlg}(C^\infty M, C^\infty S)$, and that $g(ab) = g(a)f(b) + (-1)^{|a|} f(a)g(b)$, which is equivalent to saying that $g \colon C^\infty M \to {}_{f^*}C^\infty S_{f^*}$ is an odd derivation (we're thinking of $C^\infty S$ as a $C^\infty M$-bimodule via f).

For the third isomorphism, we're using the fact that

$$\begin{tikzcd}
f^* TM \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner" very near start] & TM \arrow[d] \\
S \arrow[r, "f"] & M
\end{tikzcd}$$

$\Gamma(f^* TM) = \Gamma(TM) \otimes_{(C^\infty M, f^*)} C^\infty S$. This works for any bundle, but there is a lemma that $\Gamma(TM) \otimes_{(C^\infty M, f^*)} C^\infty S \cong Der(C^\infty M, {}_{f^*}C^\infty S_{f^*})$.

The last isomorphism follows from the homework. I'm not going to check the naturality, but everything we did was totally obvious. □

πTM is the odd tangent bundle. Writing it as in the theorem, we get an action of $\mathbb{R}^{0|1}$ on πTM. Today we'll try to understand the action of $\underline{\mathrm{Aut}}(\mathbb{R}^{0|1})$ on πTM. Let's first understand the endomorphisms group.

Note that by the theorem $\underline{\mathsf{SMan}}(\mathbb{R}^{0|1}, \mathbb{R}^{0|1}) \cong \pi T \mathbb{R}^{0|1} = \mathbb{R}^{1|1}$ (over $\mathbb{R}^{0|1}$ there are no twisted bundles; one of the projects is to show that bundles over a super manifold are the same as bundles over the reduced manifold). Now the super Lie group $\underline{\mathrm{Aut}}(\mathbb{R}^{0|1})$ is $\mathbb{R}^{0|1} \rtimes \mathbb{R}^\times$ (odd translations times even nonzero dilations).

Theorem 12.2. *For any super manifold M, there is an embedding $\Omega^* M \hookrightarrow C^\infty(\pi TM)$ with image the fiberwise polynomial functions such that*

(a) the infinitesimal generator of the odd translations extends the de Rham d (using the convention that adds the two gradings). This is the first time that this super stuff has given us a better understanding of something classical.

(b) the (even) $\mathbb{R}^\times$ action restricts to the grading operator on $\Omega^* M$. That is, for $\lambda \in \mathbb{R}^\times$ and $\omega \in \Omega^k M$, we have that $\lambda_*(\omega) = \lambda^k \omega$.

If M is an ordinary manifold, this is the definition [[★★★ I missed the explanation]]. In general, $C^\infty(\pi TM)$ is the completion of $\bigwedge^*_{\mathcal{O}_M}(\pi TM)$ (the theorem is that this is isomorphic to $\Omega^* M$).

If you don't know anything about the de Rham d, this defines it for you. The best property it has is that $d^2 = 0$. This means that $[d, d] = 0$, which just says that translation commutes with translation. From (b), we can see that d has cohomological degree 1 (this is equivalent to saying that dilations and translations commute the way they do).

[[break]]

Now we'll prove the theorem using local coordinates, and leave it as an exercise that the proof works globally.

Proof. Consider the case $M = U \subseteq \mathbb{R}^{p|q}$ a domain, with coordinates x_i and η_j on $\mathbb{R}^{p|q}$. Then [[★★★ PT: this is actually a third way to understand the total space; by gluing]] $\pi TU \cong U \times \mathbb{R}^{q|p}$, with coordinates $\hat{\eta}_j$ and $\hat{x}_i$ on $\mathbb{R}^{q|p}$ (these are just totally different coordinates, not operators or anything like that). Note that x_i and $\hat{\eta}_j$ are even and $\hat{x}_i$ and η_j are odd. If we don't need to separate the xs and ηs, then we'll write y_k to mean x_k or η_{k-p} (if $k > p$). Now I'll write the isomorphism from the first theorem in local coordinates.

$$\begin{aligned}
\mathsf{SMan}(S, \pi TU) &\overset{HW}{\cong} \{(X_i, H_j, \hat{H}_j, \hat{X}_i) | X_i, \hat{H}_j \in (C^\infty S)^e \text{ such that} \\
&\qquad X_i(s) \in |U|, \hat{X}_i, H_j \in (C^\infty S)^o\} \\
\mathsf{SMan}(S \times \mathbb{R}^{0|1}, U) &\overset{HW}{\cong} \{X_1 + \theta \hat{X}_1, \ldots, X_p + \theta \hat{X}_p \in C^\infty(S \times \mathbb{R}^{0|1})^e \\
&\qquad H_1 + \theta \hat{H}_1, \ldots, H_q + \theta \hat{H}_q \in C^\infty(S \times \mathbb{R}^{0|1})^o \text{ s.t. } \ldots\}
\end{aligned}$$

The notation presents the isomorphism of the first theorem for you.

(a) (Right) translation action of $\mathbb{R}^{0|1} \subseteq \underline{\mathrm{Aut}}(\mathbb{R}^{0|1})$ (coord η, the coordinate on the $\mathbb{R}^{0|1}$ in $S \times \mathbb{R}^{0|1}$ is θ). The group law $\mathbb{R}^{0|1} \times \mathbb{R}^{0|1} \to \mathbb{R}^{0|1}$ is given by $(\theta, \eta) \mapsto \theta + \eta$. On the coordinates of πTU, you get $\pi TU \times \mathbb{R}^{0|1} \to \pi TU$, given by $(y_k, \hat{y}_k, \eta) \mapsto (y_k + \eta \hat{y}_k, \hat{y}_k)$. $X_i + \theta \hat{X}_i \mapsto X_i + (\theta + \eta)\hat{X}_i = X_i + \theta X_i + \eta \hat{X}_i$. y_k is just X_i, which goes to $X_i + \theta 0 \mapsto X_i + (\theta + \eta) 0 \mapsto X_i$. if y_k is $\hat{X}_i$, then it goes to $\theta \hat{X}_i \mapsto (\theta + \eta)\hat{X}_i \mapsto$[[★★★]].

We have to differentiate this action with respect to η. This gives a vector field D on πTU, which is given by $D(y_k, \hat{y}_k) = (\hat{y}_k, 0)$. So in local coordinates, we get $D = \sum_k \hat{y}_k \partial_{y_k}$.

Claim. *The embedding $i \colon \Omega^* M \hookrightarrow C^\infty(\pi TM)$ is given by taking $y_k \mapsto y_k$, and $dy_k \mapsto \hat{y}_k$. These two determine the map i.*

From this we can see that D comes from $\sum_{k=1}^{p+q} dy_k \partial_{y_k}$, which is just the de Rham d. □

Maybe I shouldn't start this again on Tuesday. Let's call this done. Part (b) is way easier, so I'll leave it as an exercise (I don't want to make it a homework).

17 NR 10-05

Last time I introduced the notion of a state with density matrix ρ. The definition should be that a state with density ρ is the linear functional $\ell_\rho(A) = \operatorname{tr}(\rho A)$, with ρ positve .... More generally, we have $\mathcal{A}_h^\sigma \subseteq \mathcal{A}_h$. Under extra assumptions, we can define a postive subspace $\mathcal{A}_h^+ \subseteq \mathcal{A}_h^\sigma$ (e.g. in a C^*-algebra, this could be the set of A with $\sigma(A) > 0$). In this case, a state is a positive linear functional on $\mathcal{A}_h^\sigma$, which means that (1) $\ell(A) \geq 0$ for every $A \in \mathcal{A}_h^+$, and (2) $\|\ell\| = 1$, which implies that $\ell(1) = 1$. The truth is that we will never use this more general context. In the context of C^*-algebras, the GNS construction constructs a representation out of linear functionals, so these are more or less the same [[★★★]].

More on the probablistic interpretation

Remark 17.1. Suppose P_ϕ and P_ψ are two pure states (orthogonal projections to $\phi, \psi \in H$, respectively), then we can compute $\langle P_\phi \rangle_\psi = \operatorname{tr}(P_\phi P_\psi) = |(\phi,\psi)|^2 = \langle P_\psi \rangle_\phi$, the probability that a system in pure state P_ϕ is found in the state P_ψ. This is the nature of quantum mechanics, that even when you're in a pure state, there is still a probability that you're in some other pure state. ⋄

If $A = A^*$ on $\mathbb{C}^n$, it has n eigenvalues, which we can order increasingly. Then we have the orthogonal projections P_{ϕ_i} to the corresponding eigenvectors: $P_{\phi_i} P_{\phi_j} = P_{\phi_i} \delta_{ij}$. The theorem says that $A = \sum_{i=1}^n a_i P_{\phi_i}$. How do we generalize this to some Hilbert space instead of $\mathbb{C}^n$.

When the spectrum is discrete, we should replace the finite sum by an infinite sum. When the spectrum is continuous, we should get a direct integral.

(1) Projection valued measures on $\mathbb{R}$ are maps $P\colon$ (Borel subsets)$\to$ $B(H)$ such that

- $P(E)^* = P(E)$, $P(E)^2 = P(E)$ for $E \subseteq \mathbb{R}$,
- $P(\varnothing) = 0$, $P(\mathbb{R}) = I$, and
- if $E = \bigcup_{n=1}^\infty E_n$, then $P(E) = \lim_{n\to\infty} P(E_n)$.

P has the meaning: it is the sum of those projections for which the eigenvalue fall into the set E.

(2) Projection valued distributions. Given such a P, $P((-\infty,\lambda)) = P(\lambda)$ has the properties

- $P(\lambda)P(\mu) = P(\min\{\lambda,\mu\})$,
- $\lim_{\lambda\to-\infty} P(\lambda) = 0$, $\lim_{\lambda\to+\infty} P(\lambda) = I$,
- $\lim_{\mu<\lambda,\mu\to\lambda} P(\mu) = P(\lambda)$.

Theorem 17.2 (von Neumann)**.** *For any self-adjoint $A\colon H \to H$, there exists a unique P such that*

1. $\mathcal{D}(A) = \{\phi \in H | \int_{\mathbb{R}} \lambda^2 d(P_A(\lambda)\phi, \phi) < \infty\}$ *(the domain of A, the subspace of H where A is defined), and*
2. $A\phi = \int_{\mathbb{R}} \lambda dP_A(\lambda)\phi$ *for $\phi \in \mathcal{D}(A)$.*

The moral: if we are careful, we can treat operators on Hilbert space as if they are self adjoint something [[★★★]].

Given a state $\ell_\rho(A) = \operatorname{tr}(\rho A)$, we have the distribution of values of an observable A in the state ℓ_ρ defined as follows:

- $\mu_{A,\rho}$ is a measure on $\mathbb{R}$,
- $\mu_{A,\rho}(E)$ is the probability that "A takes values in E".
- (definition) $\mu_{A,\rho}(E) := \operatorname{tr}(\rho \cdot P_A(E))$.

Classical case: we have M_{2n} with symplectic form ω, so we have a measure ω^n. A state is a probabilistic measure on M_{2n}. A state with density ρ (ρ a distribution on M_{2n}) is $\ell_\rho(f) = \int_{M_{2n}} f(x)\rho(x)\omega^n$. Then I defined $\mu_\rho(E) = \int_{f^{-1}(E)} \rho(x)\omega^n = \int_M \chi_{f^{-1}(E)}(x)\rho(x)\omega^n$.

For a pure state, ρ is just a delta distribution at a point. In statistical mechanics, a typical ρ is $\rho(p,q) = e^{-E(p,q)/T}$.

That's not the end of the probabistic interpretation. There is something called the *Uncertainty principle*. We already defined the expectation value of A in state ρ, $\langle A\rangle_\rho = \operatorname{tr}(\rho\cdot A) = \ell_\rho(A)$. We can define the dispersion $\sigma^2 = \langle (A - \langle A\rangle_\rho)^2\rangle_\rho$.

Theorem 17.3. *If ρ is a pure state, $\rho = P_\psi$ for $\psi \in H$, then* $\sigma^2_\rho(A)\sigma^2_\rho(B) \geq \frac{1}{4}\langle i[A,B]\rangle^2_\rho$.

Proof. [[★★★ very nice HW exercise.]] □

If we have two quantum observables, then you cannot measure them at the same time. Once we measure, then we know the value of the observable. This theorem tells us that we cannot narrow the disperssions. Two noncommuting observables cannot be in the same pure state.

Assume $A\phi = a\phi$ for some $A = A^*$, so $a \in \mathbb{R}$. Then $\langle A\rangle_{P_\phi} = \mathrm{tr}(P_\phi A) = a$, and $\sigma^2_{P_\phi}(A) = \langle (A-a)^2\rangle_{P_\phi} = \mathrm{tr}(P_\phi(A-a)^2) = 0$. This means that if a pure state is an eigenvector of the observable, then A has the precise value a in this state—there is no dispersion. If some other B doesn't commute with A, then since $\sigma^2_{P_\phi}(B)$ is finite, we get that $\langle i[A,B]\rangle_{P_\phi} = 0$.

If you have p_i, q_i coordinates on $T^*\mathbb{R}^n$, we have $[p_k, q_\ell] = -i\hbar\delta_{k\ell}$.

Summary: We have the notion of quantization of classical observables. $C^\infty M$ (or $C(M)$) can be quantized to $\mathcal{A}_h$, a family of associative algebras over $\mathbb{C}$, with a $\mathbb{C}$-anti-linear anti-involution σ. We have the real subspaces $\mathcal{A}^\sigma_h \subseteq \mathcal{A}_h$. We have states, linear functionals on the algebra of observables which are positive on the positive cone. States are defined by density matrices for a given representation of $\mathcal{A}_h$ in H.

Quantization is not a functor; the functor goes the other way. You can take classical limits. The states that survive when you take the classical limit are the classical states.

Qunatization of Hamiltonian dymanics

So far I completely ignored the dymanics. On the classical level, with $C^\infty M$, we are given $H \in C^\infty M$. This gives us $v_H = \omega^{-1}(dH)$, and evolution $\frac{df_t}{dt} = \{H, f_t\}$, with $f_0 = \mathrm{id}$ and $f_t(x) = f(x_t)$. This gives us $\mathcal{A}_h$, with $H_h \in \mathcal{A}^\sigma_h$, and evolution $ih\frac{df_t}{dt} = [H, f_t]$, with $f_0 = f$. This is $R \times \mathcal{A}_h \to \mathcal{A}_h$, $f \mapsto f_t$, is an automorphism of $\mathcal{A}_t$. If $\exp(\frac{it}{h}H_h) = U(t)$ makes sense, then $f(t) = U(t)fU(t)^{-1}$. This is the Heisenberg picture.

If we have a representation $\pi_h\colon \mathcal{A}_h \to End(H)$, then the Hamiltonian evolution induces the Schrödinger picture: evolution of vector in H: $\psi \mapsto \psi(t) = e^{\frac{it}{h}\pi(H)}\psi$. More precisely, it is the solution to the infinite dimensional ODE $ih\frac{d\psi_t}{dt} = \pi_h(H)\psi_t$ with $\psi_0 = \psi$.

What is the moduli space of these deformations? We can answer this for formal deformations.

We can require that as h goes to 0, H_h goes to H. However, therer is still no canonical quantization of a given system. If you have an integral system, then you do get some kind of functoriality in quantization.

This finishes up quantum mechanics in general. Next time we'll talk about quantum mechanics on $\mathbb{R}^{2n}$.

18 NR 10-08

Remember that if you have a pure state ψ, then $\sigma_\psi^2(A)\sigma_\psi^2(B) \geq \frac{1}{4}\langle i[A,B]\rangle_\psi^2$. There was a question last time. Imagine that $M = \mathbb{R}^2$, so the corresponding algebra is $\mathcal{A}_h = \langle p = -ih\frac{\partial}{\partial q}, C^\infty(\mathbb{R})\rangle$, so $\hat{p} = -ih\frac{\partial}{\partial q}$, $\hat{q} = q$, and H is $L^2(\mathbb{R})$. We can choose pure states to be $\psi_\varepsilon(q) = c_\varepsilon \exp\left(\frac{-(q-q_0)^2}{\varepsilon}\right)$. This is a sequence of functions which concentrate to q_0 (converge to the delta distribution at q_0, which is not in the Hilbert space), with $\|\psi_\varepsilon(q)\|^2 = 1$. We have that $\langle q\rangle_{\psi_\varepsilon} = q_0$, and $\langle\sigma^2(q)\rangle_{\psi_\varepsilon} \to 0$ as $\varepsilon \to 0$. So by the uncertainty principle, $\sigma_{\psi_\varepsilon}(p)\sigma_{\psi_\varepsilon}(q) \geq \frac{h}{2}$. As $\varepsilon \to 0$, $\sigma_{\psi_\varepsilon}(p) \to \infty$ and $\sigma_{\psi_\varepsilon}(q) \to 0$.

If you open a physics text book, $e^{ipx} = |p\rangle$ is presented as a state with momentum p, but this doesn't make sense because it is not normalized. What you should really do is make a sequence of L^2 states that have more and more localized momenta. If you have an operator on the real line, say $\frac{d^2}{dx^2}$, then you have eigenfunctions, like e^{ipx}, but [[★★★ something about turning the operator into the operator p^2]]. You can have such states if you are working on $\ell^2(\mathbb{Z})$ instead of $L^2(\mathbb{R})$.

There are two types of evolution: Heisenberg and Schrödinger evolution. The Heisenberg evolution is a dynamics on $\mathcal{A}_h$ given by $ih\frac{\partial f}{\partial t} = [H_h, f_t]$, where H_h is the quantum hamiltonian, and $H_h \to H$ as $h \to 0$ via the identification $\mathcal{A}_h^\sigma \cong C(M)$. Schrödinger evolution is a dynamics on a representation H of $\mathcal{A}_h$, given by $ih\frac{\partial\psi_t}{\partial t} = \pi(H_h)\psi_t$, where $\pi\colon \mathcal{A}_h \to End(H)$ is a $*$-representation on a hilbert space H.

If you take the Heisenberg algebra and impose the natural $*$-involution ($p^* = p$, $q^* = q$), then there is some theorem which gives you an equivalence between these two. In general, Heisenberg evolution induces a Schrödinger evolution, but they are equivalent on $T^*\mathbb{R}^n$.

Standard problems in quantum mechanics. In general, we don't know what Hamiltonian to choose. If H is the classical hamiltonian, we could choose anything like $H + o(h)$. Using some context, there is usually a natural choice.

- Given H_h, find the spectrum of H_h. This is the quantum analogue of describing values of H. This is a stationary problem; there is not time dependence.
- Scattering. This is the quantum analogue of a classical scattering problem. Imagine $H = \frac{\vec{p}_1^2}{2m_1} + \frac{\vec{p}_2^2}{2m_2} + \cdots + V(\vec{q}_1, \vec{q}_2) + \cdots$, so there are some interaction terms. You can imagine some particles (or stars, for example) coming in, doing something, and then some particles (or stars) are trapped in the interaction area, and some particles fly out. This is a non-stationary problem. In the quantum case, we should assume that as $t \to -\infty$, $\psi_t \to \psi_{ih}(t)$ should be some states with non-interacting particles (a typical example of such a state is $\exp(i\vec{p}\vec{x} + iEt)$; since we want to satisfy the Schrödinger equation, $E = \frac{p^2}{2m}$). Now describe the outgoing asymptotics (as $t \to +\infty$).

Quantization of $T^*\mathbb{R}^n$

In this case, $M = R^*\mathbb{R}^n \cong \mathbb{R}^{2n}$, with coordinates p and q as usual, with the standard symplectic form $\omega = \sum dp_i \wedge dq^i$. The algebras of functions are $Pol(\mathbb{R}^{2n}) \subseteq C^\infty_{pol}(\mathbb{R}^{2n}) \subseteq C^\infty(\mathbb{R}^{2n})$ (the middle one is polynomial functions in the cotangent direction). In this situation, we can actually construct families of $*$-products. Remember that a $*$-product is not unique; you can apply any automorphism which becomes the identity at $h = 0$. Let me describe two of these star products.

(1) Weyl quantization (Weyl $*$-product).

Theorem 18.1. *The operation* $(f_1 * f_2)(p,q) = \frac{1}{(\pi h)^2}\int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^{2n}} f_1(p_1,q_1)f_2(p_2,q_2)\exp\left(\frac{4i}{\ell}\int_\Delta \omega\right)dp_1 dq_1$ *where* Δ *is the Euclidean triangle with vertices* (p,q), (p_1,q_1), *and* (p_2,q_2), *is a family of associative products on* $C^\infty(\mathbb{R}^{2n})$.

Proof. [[★★★ HW: prove it (i.e. prove associativity)]] □

[[★★★ HW; prove that $f_1 * f_2 = f_1 f_2 - \frac{ih}{2}\{f_1, f_2\} + o(h^2)$.]]

Similar thing can be used to quantize Kähler manifolds, but for compact Kähler manifolds, you don't get a family. There is something called deformation quantization with torsion. You have a sequence of $*$-products for $h = \frac{1}{n}$. This is Berezin-Toeplitz quantization. There is $C^\infty(M)$, with a map ϕ_n to $End(H_n)$ (family of representations), and maps $End(H_n) \xrightarrow{\psi_n} C^\infty(M)$. The statement is that $\lim_{n\to\infty}(\psi_n \circ \phi_n) = \mathrm{id}$ and $\lim_{m\to\infty} i[\psi_n, \phi_n(f), \psi_n \circ \phi_n(g)]m = \{f, g\}$. So the case of $\mathbb{R}^{2n}$ is very lucky because we have a torision-free deformation.

Theorem 18.2. *There is an isomorphism of algebras* $\left(Pol(\mathbb{R}^{2n}), *\right) \xrightarrow[\sim]{\Phi} Pol(\hat{p}, \hat{q})$*, where* $\hat{p}_j = -ih\frac{\partial}{\partial q_j}$ *and* $\hat{q}_j = q_j$ *(or think of it as generated by the* $\hat{p}$ *and* $\hat{q}$ *with* $[\hat{p}_j, \hat{q}^k] = -ih\delta_j^k$*), where (for multi-indices* α *and* β*)* $\Phi(p^\alpha q^\beta) = \mathrm{Sym}(\hat{p}^\alpha \hat{q}^\beta)$*, where the symmetrization is the sum of all the things you get by letting the symmetric group act on* α *and* β *(or* $(\sum_i u^i \hat{p}_i + \sum_j v_j \hat{q}^j)^k = \sum_{|\alpha|+|\beta|=k} \frac{k!}{\alpha!\beta!} u^\alpha v^\beta$ *non-commutative binomial formula).*

How to construct a deformation of $Pol(\mathbb{R}^{2n}$ with $\{p_i, q^j\} = \delta_i^j$, ps and qs commute? $\langle \hat{p}_i, \hat{q}^j | [\hat{p}_k, \hat{q}^j] = -ih\delta_k^j$, others commute$\rangle$, $\sigma(\hat{p}_i) = \hat{p}_i$ and $\sigma(\hat{q}_i) = \hat{q}_i$. We need to choose an isomorphism. Choose $P(p,q) \xrightarrow{\Phi} P(\hat{p}, \hat{q})$, then $(P * Q)(p,q) = \exp(\frac{\#}{h} \sum_i (\frac{\partial}{\partial p_{(1)}} \wedge \frac{\partial}{\partial q_{(2)}} - \frac{d}{dq_{(1)}} \wedge \frac{\partial}{\partial p_{(2)}}) P(p_{(1)}, q(q)) Q(p_{(2)}, q_{(2)})$.

A remarkable property of this $*$-product: $\mathrm{tr}(f) = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^{2n}} f(p,q)\omega^n$ is cyclic for this $*$-product (i.e. $\mathrm{tr}(f * g) = \mathrm{tr}(g * f)$).

Next two lectures there will be a guest lecturer who will talk about quantum Calabi-Yau manifolds, then he'll continue the talk 5-6 in the RTGC seminar. Friday, either I'll continue or he'll continue if there is something left.

13 PT 10-09

Last week we proved that $\underline{\mathsf{SMan}}(\mathbb{R}^{0|1}, M) \cong \pi TM$. This implies that $\underline{\mathrm{Aut}}(\mathbb{R}^{0|1})$ ats on πTM. This leads to a $\mathbb{Z}$-grading on $C^\infty(\pi TM) \supseteq \Omega^* M$, with $d^2 = 0$ and d of degree 1. So the easiest super point leads to the de Rham differential.

Project 7: $\underline{\mathsf{SMan}}(\mathbb{R}^{0|2}, M)$ leads to the structure of differential gorms on M. There is a paper to read. These gorms have more structure because the automorphism group of $\mathbb{R}^{0|2}$ is bigger. (Research project: keep going ...structure on $\underline{\mathsf{SMan}}(\mathbb{R}^{0|q}, M)$)

On the homework: nobody proved Batchelor's theorem. You have a super manifold $M^{m|n}$, and you functorially construct $Gr(M) := \pi E_M$, where $E_M \to M_{red}$ is an ordinary vector bundle over M_{red}, given by $\Gamma(E_M) = J_M/J_M^2$, where J_M is the ideal of nilpotents (the ideal generated by the odd elements). Notice that $C^\infty(Gr(M)) = \Gamma(\bigwedge^*(J_M/J_M^2)) = \bigoplus_{k=0}^n \bigwedge^k(J_M/J_M^2)$, but $\bigwedge^k(J_M/J_M^2) \cong J_M^k/J_M^{k+1}$ canonically, given by $a_1 \wedge \cdots a_k \mapsto asym(a_1 \cdots a_k)$. A theorem: there is an isomorphism $M \xrightarrow{\phi} Gr(M)$ such that $Gr(M) \xrightarrow{Gr(\phi)} Gr(Gr(M)) = Gr(M)$ is the identity map. These will be much easier to glue (actually, extend).

We were tied up for the last weeks trying to make super manifolds precise. Now let's loosen up and try to describe where we are and how it is related to Kolya's class. Today will be a kind of a survey of the rest of the class.

Classical Field Theory

Data:

(1) a spacetime M (of dimension d, if this is a d-dimensional field theory), which is usually a Lorentz manifold (has a non-degenerate metric with one negative). This is often too hard, so you do a "Wick rotation" to turn it in to a Riemannian metric (this is called "Euclidean field theory").

(2) Fields $\Phi(M)$. One example is scalar fields (spin 0), in which case $\Phi(M) = C^\infty(M; V)$ (linear case: V a vector space), or $C^\infty(M; X)$ (the nonlinear case, X is the "target"). Another example is guage fields (spin 1), where a field is a G-principal bundle with a connection. In the easiest case (where you have a line bundle, like in electrodynamics), you have

$\Omega^1 M/d\Omega^0 M$. There are p-form fields where the fields are p-forms. Another research project could be "noncommutative p-form fields" for which you'd have to go into gerbes and their connections. Another example of fields is gravitational fields (spin 2), where fields are metrics on M. These integral spin fields will correspond to bosons. There are also spinor fields (spin 1/2), which are sections of the spinor bundle. The last thing is spin 3/2.

(3) Classical action $A\colon \Phi(M) \to \mathbb{R}$ (or $\mathbb{C}$), which describes the physics on the fields.

Remark 13.1. The structure on M (and on the bundle over it) is chosen such that A can be defined. $\diamond$

The examples Kolya studied:

- Classical mechanics ($d = 1$), where $M = \mathbb{R}$, and $\Phi(M) = C^\infty(M; N)$ ($N = X$ is configuration space). M is also sometimes called the world-line. AJ: world-line usually refers to the embedding, so a field would be a world line. If you take $N = V$ a vector space with the simplest possible action (the action from classical mechanics), you call this the *linear σ-model*; if N is something else, it is the *non-linear σ-model*. $A = \int_M L(\gamma)$, where $L(\gamma) = |\dot\gamma|_g^2$, so you need a metric g on N; you also need a measure on M so you can integrate. You could add a potential. This is special because the lagrangian only depends on the 1-jet.

- Classical string theory ($d = 2$), where M is a surface (the world-sheet)

- Electromagnetism (Yang-Mills).

Classically, you get a space of classical solutions $\mathcal{M}$, which is a symplectic manifold, defines as solutions to the Euler-Lagrange equations for the given action A (with respect to variations vanishing near boundary).

If you have $M = [a, b]$, and $A_{ab}\colon \Phi([a, b]) \to \mathbb{R}$ is the 1-form $dA_{ab}(\phi, \delta\phi) = \int_a^b \big(\frac{\partial L}{\partial \phi}\delta\phi dt\big) + \alpha(\phi_b, \delta\phi_b) - \alpha(\phi_a, \delta\phi_a)$ (the "bulk term" and the "boundary terms"; here we do not require any vanishing on $\delta\phi$). Classical solutions are the ones for which the bulk term vanishes. If we restrict to classical solutions $\phi \in \mathcal{M}$, we get that $dA_{ab} = \alpha_b - \alpha_a$. This implies that $\omega = d\alpha_b = d\alpha_a \in \Omega^2\mathcal{M}$ is well defined. In good cases, this is a symplectic form.

Now I want to get to what data are associated to the boundary itself. If we choose a boundary point (a or b in this case), it gives in addition this 1-form α_a (or α_b). This is the kind of input you need if you want to do geometric quantization (here you take the trivial line bundle, and $d + \alpha_a$ is the connection; the non-trivial line bundles come from the case where your action is $\mathbb{R}/\mathbb{Z}$-valued).

[[break]]

Example 13.2. Kinetic energy only. $L(\phi)(t) = |\dot\phi(t)|_g^2$ so $A(\phi) = \int_{\mathbb{R}} |\dot\phi(t)|^2 dt$ for $\phi\colon \mathbb{R} \to (N, g)$. Then $\mathcal{M}$ is the space of geodesics in (N, g). You could identify these with TN. You could also use the metric to identify this with T^*N. Then this 1-form α_0 becomes the canonical 1-form α on T^*N (and α_t is given by pushing forward by the flow). $\diamond$

In general, the dimension of $\mathcal{M}$ is $2r \cdot \dim N$ if L depends only on the r-jets of the fields. In general, you won't get the cotangent bundle, but you will get a symplectic manifold under good conditions.

Time translation gives a vector field ξ on $\mathcal{M}$. This leads to the Hamiltonian function $h\colon \mathcal{M} \to \mathbb{R}$ by $h(\phi) := i_\xi\alpha_t(\phi) - L_t(\phi)$. Independence of t: $i_\xi\alpha_a - i_\xi\alpha_b = i_\xi A_{ab} = \xi(A_{ab}) = L_b - L_a$ by the fundamental theorem of calculus. What we're using is the extra structure that once we fix a time t, we get his 1-form, and for different ts, the difference of the two 1-forms is d of a function.

Qunatization

We have $(M, \Phi(M), A)$. There are different opinions about what a quantization should be. In Kolya's class, we quantize the classical observables (the algebra $C^\infty(M)$) to some $\mathcal{A}_h$ (sometimes you can't set h to Planck's constant). Then we study representations $\mathcal{A}_h \to B(H)$.

I want to go a slightly different route, where we get the state space H directly. The states (positive linear operators on H) only see the projective space on H, and physically, you only expect this much (since scaling a state doesn't change it). We get H by choosing a *polarization*. If you make this extra choice, then you actually get a Hilbert space, and you have to discuss how the Hilbert space changes when you change your choice. In the cases we'll do, it will only change by scaling, so in the end of the day, we'll get a projective space. Good choices of polarization usually lead to isomorphic "Hilbert spaces" (they'll actually be Frechét spaces) and the isomorphism will be canonical up to phase. Moreover, we want operators on H (e.g. we want to quantize functions, like the Hamiltonian h). One way to get these is to go through the deformation quantization (where the operators come first, and then you take a representation). The other way is to choose a polarization and use the path integral approach, but I don't want to start this today. We'll do it Thursday.

If you start with a classical field theory $(M, \Phi(M), A)$, then you can take the classical solutions to pick up $(\mathcal{M}, \omega, \alpha_t, A_{ab})$. Kolya explained how to think of these data (as a "Hamiltonian field theory"): if $\dim M = d$, then you associate to ∂M a symplectic manifold $\mathcal{M}$ (with $\omega = d\alpha$) and to M you associate a Lagrangian submanifold $L \subseteq \mathcal{M}$ (so that $\alpha|_L = dA$). Then you can do geometric quantization to get a Hilbert space and a Hamiltonian operator. Or you could go through deformation quantization and pick a representation to get this stuff. It is a little silly to only look at the classical solutions and then try to quantize. It would be better to look at the whole field theory and then try to quantize. This was Feynman's idea: particles don't travel along the classical solution, but there is some probability which peaks at the classical solution (but is non-trivial outside of the classical solutions).

Supersymmetric (Susy) classical field theories

We're trying to change the data to something that makes sense for super manifolds. We need

– super spacetime $M^{p|q}$ (maximal susy in physics literature is $M^4|16$; here we'll only go up to $M^{2|1}$, which is all you need for elliptic cohomology). There will be something like a metric g, which we'll get by trying to define a classical action.

– space of fields $\Phi(M)$, which you can do by adding the word super a few times to everything we did. We'll concentrate on the scalar field case where $\Phi(M) = \mathsf{SMan}(M, X)$ "Susy σ-model with target X".

– classical action $A\colon \Phi(M) \to \mathbb{R}$. Here we run into trouble. The integral is ok (you integrate sections of the Berezinian line bundle).

To get dA_{ab}, we differentiated the classical action. For this, we needed some kind of smooth structure on $\mathsf{Man}([a, b], N)$. What we really want, therefore, is $\Phi(M) = \underline{\mathsf{SMan}}(M, X)$, which is not a finite-dimensional super manifold (unless M or X is zero dimensional). We understand this in terms of the functor of points. For every super manifold S, we have $A_S\colon \underline{\mathsf{SMan}}(M, X)(S) = \mathsf{SMan}(S \times M, X) \to \mathbb{R}(S) = C^\infty(S)^e$, and this should be natural in S.

These are the theories that we'll quantize via path integrals.

7 RB 10-09

Last lecture, we said what a Feynman diagram is. It represents some product of propagators. We saw that the product of propagators is well defined UP TO addition of a translation invariant distribution with support on the diagonal. The Fourier transform of a distribution with support on the diagonal will be a polynomial in momentum. In physics, you get these things called counterterms associated to Feynman diagrams, and the fact that they are polynomial essentially comes from the fact that the product of propagators is defined up to this distribution on the diagonal. One problem is that we have to specify the product precisely (this is the problem of renormalization). We won't do this today.

Why are we interested in defining Feynman diagrams? Answer: they come from expanding Gaussian integrals $\int \text{polynomial}(x)\cdot e^{-\text{quadratic}(x)}dx$. Whenever you have such an integral, you'll probably get Feynman diagrams popping up. Pretend you're a 1A student.

1-dimensional case. (1) what is $\int e^{-x^2}dx$ (we know it is $\sqrt{\pi}$, but let's say we don't; the tricks won't work in infinite dimensions). We're stuck. (2) What about $\int xe^{-x^2}dx$? This is easier because it is $-\int \frac{1}{2}\frac{d}{dx}e^{-x^2}dx = 0$ since e^{-x^2} vanishes at $\pm\infty$. (3) How about $\int x^2e^{-x^2}dx$? Integrating by parts, we can reduce to the case $\int e^{-x^2}dx$.

In general, $\int x^n e^{-x^2}dx = -\frac{1}{2}\int(n-1)x^{n-2}e^{-x^2}dx$. Diagrammatically, we have

[[★★★ picture]]

The whole integra is the sum of $-\frac{1}{2}$ times what you get if you pair off two xs and cross them out (leaving $n-2$ other xs), and sum over all possible pairings. Consider the case of x^6, we get a sum over all Feynman diagrams like

[[★★★ picture]]

multiplied by the integral $\int e^{-x^2}dx$. We put the "propagator" $-\frac{1}{2}$ on each edge (this $-\frac{1}{2}$ is a distribution on 0-dimensional space).

Now let's try to make some Feynman diagrams on more points.

Example 7.1. $\int x^4\times x^4\times x^6\times e^{-x^2}dx$. Pretend we're particularly stupid 1A students and we haven't realized we can multiply these together. Then we sum over the diagrams

[[★★★ picture]] ◇

In higher dimensional cases, you can't always stick all the points together.

Now let's try to work out $\int e^{-m^2\phi^2-\lambda\phi^4/4!}$, which is a kind of 0-dimensional version of $\int e^{\int(m^2\phi^2(x)+(\partial\phi)^2+\lambda\phi^4)dx}$ for $\phi\colon \mathbb{R}^n\to\mathbb{R}$. This is defined for $\lambda>0$, and you can define it for all complex $\lambda\neq 0$ by changing $\phi\mapsto\phi\cdot\lambda^{-1/4}$, but it has a branch point AND an essential singularity at $\lambda=0$. At the point $\lambda=0$, the integral converges, but there is still an essential singularity (if you approach from positive λ, it's ok, but if you approach from negative λ, you run into trouble). Expanding as a power series in λ at $\lambda=0$ is totally stupid because you can't expand an essential singularity like this, but we're going to do it anyway. The integral becomes $\sum\frac{(-\lambda)^n}{n!}\int(\phi^4/4!)^n e^{-m^2\phi^2}d\phi$. Each term is something we can expand using Feynman diagrams. It is

$$\sum_n \frac{(-1)^n}{(4!)^n n!}[[\bigstar\bigstar\bigstar\text{ picture}]]$$

with the propagator $1/2m^2$ for each edge, and a factor of λ for each vertex. There are a lot of such diagrams and we'd like to reduce the number a bit. Instead of summing over diagrams, we can sum over isomorphism classes of diagrams

$$\sum_{isoclasses}(-1)^n[[\bigstar\bigstar\bigstar\text{ squiggle}]]\frac{\#\text{ diagrams in isoclass}}{(4!)^n n!}$$

and this last factor is one over the size of the automorphism group of the diagram. The reason is that the denominator is the number of automorphisms of the vertices, and if you go to a corner and think about it you'll see that it's right. So we get that what we want is

$$\sum_{isoclasses}(-1)^n\times\frac{\text{value of diagram}}{\text{order of automorphism group}}$$

This kind of weighting a diagram by one over the size of its automorphism group occurs all over the place in mathematics.

Back to our example of $\int e^{-m^2\phi^2-\lambda\phi^4}d\phi$" $=$ "$\int e^{-m^2\phi^2}d\phi\times$ [[★★★ picture]]

But the power series can't possibly converge because there is an essential singularity, so what is the meaning of this power series. The power series is an *asymptotic expansion* of the integral valid for $\lambda > 0$. This means that the integral is asymptotic to $a_0 + a_1\lambda + a_2\lambda^2 + \cdots$, so it is equal to $a_0 + \cdots + a_n\lambda^n + O(\lambda^{n+1})$. So the approximation (for a given n) gets better as λ goes to zero [[★★★ or infinity?]], but the approximation gets worse as you increase n.

Finite dimensional case is pretty similar: $\int P(x)e^{-Q(x)}dx$ where $x \in \mathbb{R}^n$, P is polynomial and Q is quadratic. The calculation is similar; you just need to compute the propagator for something like $\int x_i x_j e^{-Q(x)}dx$. It is the bilinear form Q^{-1} applied to x_i, x_j.

The infinite dimensional case is somewhat trickier. What is (say) $\int P(\phi)e^{-Q(\phi)}D\phi$, where ϕ is a field on $\mathbb{R}^3$. For example, you could have $\int \phi(x_1)\phi(x_2)\phi(x_3)^3 e^{-\int \phi(x)^2+(\partial\phi)^2 dx}D\phi$ (ignoring the complication that ϕ is really a distribution). Trying to expand like before, we run into a severe problem right away.

Problem 1: What is $D\phi$? it should be a translation invariant measure on an infinite dimensional space and this is a big problem because translation invariant measures tend to exist only on locally compact spaces and infinite dimensional spaces are not locally compact. Such a measure doesn't exist. It turns out it is possible to make sense of the Gaussian measure $e^{-Q(\phi)}D\phi$. There are two approaches.

(Analytic approach) On any (real) finite dimensional Hilbert space, we have a canonical Gaussian measure $e^{-\pi x^2}d^n x$ of total mass 1. These are all compatible in the following sense. If we have a finite dimensional vector space $H_1 = H_2 \oplus H_3$ and we have projection $H_1 \to H_1/H_2 = H_3$, then the projection of the Gaussian measure of H_1 to H_3 is the Gaussian measure of H_3. This is obvious from the simple example: $\int f(x)e^{-\pi(x^2+y^2)}\,dx\,dy = \int f(x)e^{-\pi x^2}\,dx$.

Now suppose that H is infinite dimensional. Looking at all finite dimensional quotients, we see that all the Gaussian measure on the quotients are compatible, so we can define the measure of any cylindrical set (the inverse image of a measurable set of some finite dimensional quotient). This ought to give us a nice Gaussian measure on all of H. If it did, quantum field theory would be easy. It is not obvious what goes wrong with this construction. (1) $\mu(H)$ should be 1. (2) Suppose B_r is a ball of radius r in H, we get that $\mu(B_r)$ is less than or equal to the measure of a ball of radius r in $\mathbb{R}^n$, which is less than or equal to the measure of a cube of side $2r$ in $\mathbb{R}^n$, which is less than or equal to $\left(\int_{-r}^{r} e^{-\pi x^2}\,dx\right)^n$, which tends to zero. So a ball of radius r has measure zero. Since H is a countable union of such balls, we get that $\mu(H) = 0$.

So what is wrong? The problem is that we have a "measure" defined on all cylindrical sets, which don't form a σ-algebra. It turns out that you cannot extend this to the σ-algebra generated by cylindrical sets in a countably additive way.

You can define a Gaussian measure on larger spaces of distributions. You have to replace your Hilbert space by something called a rigged Hilbert space, and then do some other stuff. It turns out that this bigger space consists of distributions rather than functions and this is bad because ϕ^4 doesn't make sense for a distribution ϕ.

Next week well see an algebraic approach.

19 NR 10-10

Today's speaker is Yan Soibelman.

Today's seminar talk will be related to holomorphic Chern-Simons theory. This talk will be an elementry background talk about this. References: `math/0606241` (A_∞-algebras and categories), draft of book with Kontsevich "Deformation theory" Vol. 1 can be downloaded from `www.math.ksu/~soibel`.

Two main players in today's lecture: A_∞-algebras and L_∞-algebras over a fixed field k of characteristic 0. I will present very similar points of view on these two structures; the point of view of noncommutative geometry.

Definition 19.1 (Preliminary)**.** An L_∞*-algebra* is a formal pointed manifold in the category of $\mathbb{Z}$-graded vector spaces over k together with a vector field Q of degree $+1$ such that $[Q, Q] = 0$ and $Q(pt) = 0$ (vanishes at the marked point).

An A_∞*-algebra* is a noncommutative formal pointed manifold in the category of $\mathbb{Z}$-graded vector spaces over k together with a vector field Q of degree $+1$ such that $[Q, Q] = 0$ and $Q(pt) = 0$ (vanishes at the marked point). ⋄

In this preliminary definition we see that there is some kind of space. The point of view going back to Grothendieck is that a "space" is a functor $F : \mathcal{C} \to \mathsf{Set}$ [[★★★ not $\mathcal{C}^\circ$]] with some properties. Most of what I'll be talking about can be said for any k-linear symmetric monoidal category $\mathcal{C}$ which admits infinite sums and products. If you don't want to think in such abstract terms, the two main examples will be $\mathcal{C} = \mathsf{Vect}_k$ and $\mathcal{C} = \mathsf{Vect}_k^{\mathbb{Z}}$ ($\mathbb{Z}$-graded vector spaces with grade-preserving morphisms, with ordinary tensor product with induced grading, with commuter $V \otimes W \to W \otimes V$ given by $v_n \otimes w_m \mapsto (-1)^{nm} w_m \otimes v_n$). NR: but these are not super vector spaces . . . this is some kind of hybrid. YS: yes.

You can talk about algebras, coalgebras, and all sorts of other things in $\mathcal{C}$ (c.f. Peter Teichner's lectures). An algebra is an object A with morphisms $m : A \otimes A \to A$ and $1 : \mathbb{1} \to A$ (we are assuming $End(\mathbb{1}) = k$) so that the usual diagrams commute. We have $\mathsf{Alg}_{\mathcal{C}f} \subseteq \mathsf{Alg}_{\mathcal{C}}$ (finite-dimensional or finite length algebras) and $\mathsf{Coalg}_{\mathcal{C}f} \subseteq \mathsf{Coalg}_{\mathcal{C}}$.

Theorem 19.2. *Let $F : \mathsf{Alg}_{\mathcal{C}f} \to \mathsf{Set}$ be a functor that commutes with finite projective limits. Then F is represented by a counital coalgebra. that is, there is a $B \in \mathsf{Coalg}_{\mathcal{C}}$ such that $F(R) \cong \mathrm{Hom}_{\mathsf{Coalg}_{\mathcal{C}}}(R^*, B)$.*

Similar to the category of vector spaces, you can talk about cocommutative coalgebras (this is just throwing in another diagram, which says that the opposite coproduct Δ' is equal to the usual coproduct Δ). If instead of $\mathsf{Alg}_{\mathcal{C}f}$, we take $\mathsf{Alg}_{\mathcal{C}f}^{com}$, then the theorem is still true, with B cocommutative.

If we have a coalgebra, we can take its dual to get an algebra (the opposite isn't true). For this reason, I prefer to work with coalgebras. You can dualize, but then you have to speak about topological coalgebras. If we have a commutative algebra, then we have an affine scheme; if it is not commutative, then we can imagine that there is some noncommutative scheme corresponding to our algebra. That is, we have some "generalized space" according to Grothendieck's point of view.

Definition 19.3. The category of *noncommutative thin schemes in* $\mathcal{C}$ is the category equivalent to $\mathsf{Coalg}_{\mathcal{C}}$. The category of *thin schemes in* $\mathcal{C}$ is the category equivalent to $\mathsf{Coalg}_{\mathcal{C}}^{cocom}$ ⋄

For today's lecture, if we have a coalgebra B, we will denote the corresponding "geometric object" by X_B, and given a thin scheme X, we denote the corresponding coalgebra B_X, so $B_X^* = \mathcal{O}(X)$.

Example 19.4. Fix $V \in \mathcal{C}$, and consider $T(V) = \bigoplus_{n \ge 0} V^{\otimes n}$. We can make it into a coalgebra (the *cofree coalgebra*) by $\delta(v_1 \otimes \cdots \otimes v_n) = \sum_{i=1}^{n-1} (v_1 \otimes \cdots \otimes v_i) \bigotimes (v_{i+1} \otimes \cdots \otimes v_n)$. The noncommutative thin scheme corresponding to $T(V)$ is a *noncommutative formal graded manifold*.

Similarly, we have the cocommutative version $C(V) = \bigoplus_{n \ge 0} \mathrm{Sym}^n(V)$. [[★★★ Exercise: figure out the Δ (you'll need characteristic zero). It should be cocommutative.]] The corresponding thin scheme is called a *formal graded manifold*. ⋄

From now on, let's fix $\mathcal{C} = \mathsf{Vect}_k^{\mathbb{Z}}$. We can almost make sense of the preliminary definition. To implement marked point, we should either fix a morphism from k to the coalgebra, or we can take our direct sums starting at $n = 1$ instead of $n = 0$. We will denote these things by $T_+(V)$ and $C_+(V)$, and the geometric objects will be (noncommutative) formal pointed graded manifolds (or NCfpg manifold). So a NCfpg manifold corresponds to $B \cong T_+(V)$ for some V. PT: you aren't going to allow things like this to be glued together? YS: no, these are really formal manifolds, with just one closed point. A formal pointed graded manifold corresponds to $B \cong C_+(V)$.

You can't do too much differential geometry on formal manifolds, but you can do something. For example, you can speak about vector fields, which correspond to derivations of the algebra or coalgebra. If X is a (noncommutative) formal pointed graded manifold, then $Vect(X)$ corresponds to $Der(T_+(V))$ as a coalgebra (or without the + if not pointed). A derivation is an element of $\mathrm{Aut}(M \otimes k[\varepsilon]/\varepsilon^2)$ [[★★★]]. Since we are working with a graded coalgebra $T_+(V)$, we can look for derivations of different degrees, and define $[\mathcal{D}_1, \mathcal{D}_2] = \mathcal{D}_1\mathcal{D}_2 - (-1)^{|\mathcal{D}_1||\mathcal{D}_2|}\mathcal{D}_2\mathcal{D}_1$, making $Vect(X)$ into a graded lie algebra. Note that a vector field need not commute with itself because for an odd vector field Q, $[Q, Q] = 2Q^2$.

Definition 19.5. A *noncommutative formal pointed differential graded manifold* is a pair $((X, pt), Q)$, where (X, pt) is a NCfpg manifold and Q is a vector field on X vanishing at the marked point such that $\deg Q = +1$ and $[Q, Q] = 0$. A vector field Q of degree 1 with $[Q, Q] = 0$ will be called a *homological vector field*. If you drop noncommutativity, then you get the notion of a formal pointed differential graded manifold. ◇

Let $A \in \mathsf{Vect}_k^{\mathbb{Z}}$, then we denote the shifting of the grading by 1 by $A[1]$, so $A[1]^n = A^{n+1}$. Consider $T_+(A[1])$, a noncommutative formal pointed graded manifold. Given a coalgebra B, let's denote the corresponding space $\mathrm{Spc}\, B$.

Definition 19.6. An A_∞ *structure on* A is given by a structure of a noncommutative formal pointed differential graded manifold on $\mathrm{Spc}\, T_+(A[1])$. If you drop noncommutative and change T_+ to C_+, you get the definition of an L_∞-structure on A. ◇

Algebraically, we have $T_+(A[1]) = \bigoplus_{n\geq 1} A[1]^{\otimes n}$ with $Q^2 = 0$, so the derivation Q respects the coproduct. In order to define a derivation on a free algebra, it is enough to define it on generators. So to define such a Q, it it equivalent to have a collection of maps $Q_n \colon A^{\otimes n} \to A[2-n]$ (for $n \geq 1$). Geometrically, the Q_n are Taylor coefficients of the vector field Q. Abusing language a bit, we write $Q = Q_1 + Q_2 + \cdots$ (we start with Q_1, which corresponds to the fact that $Q(pt) = 0$). The condition $Q^2 = 0$ imposes infinitely many quadratic relations on the Q_n.

We'll finish on Friday.

14 PT 10-11

Classical field theory consists of the data

– space time Σ^d,

– Fields $\Phi(\Sigma)$, and

– a classical action $A\colon \Phi(\Sigma) \to \mathbb{R}$.

The easiest case (so far), which we saw in NR's class, is $d = 1$, with $\Sigma = [a,b]$, $\Phi(\Sigma) = \mathcal{C}^\infty(\Sigma, N)$ for some configuration space N, and action $A(\phi) = \int_\Sigma |\dot\phi(t)|^2\, dt$ (using some Riemannin metrics g on N and something on Σ). Take $N = \mathbb{R}$. We want to quantize thisby taking the hilbert space $\mathcal{H} = L^2(N, vol_g)$. The Hamiltonian operator is $H = -\Delta_g$. In the case $N = \mathbb{R}$, $H = -\partial_x^2$. I want to explain how to get this from path integrals instead of deformation quantization. This case is actually very precise; the measure in question actually do exist (it is called the Weiner(?) measure).

Quantum mechanical evolution on $\mathcal{H}$ is given by e^{itH}. In other words, if you know H, you can apply this operator to a state to determine how the state evolves in time. This is one way to solve the Schrödinger equation. That is, solutions of the Schrödinger equation are $e^{itH}(\psi)$.

I want to do the Euclidean version of this, which is the heat equation. I want to study the operator e^{-tH}. We see that this is the heat equation because $\frac{d}{dt}e^{-tH}(\phi) = -H(e^{-tH}\phi) = \partial_x^2(e^{-tH}\phi)$, so $e^{-tH}\phi$ is a solution to the heat equation. If ϕ gives the distribution of heat on $\mathbb{R}$ at time 0, then $e^{-tH}\phi$ tells you the distribution of heat at time t. This is the easiest way to solve the heat equation if you happen to know how to write down the operator e^{-tH}. In particular, if we put a unit of heat at $y \in \mathbb{R}$ (roughly, "$\phi = \delta_y$", in quotes because $\delta_y \notin L^2$), then we get the "integral kernel" of e^{-tH}.

Because H is self-adjoint, you can write an eigenspace decomposition of $\mathcal{H}$. e^{itH} will have the same eigenspaces, but with eigenvalues $e^{it\lambda}$. Even though the operator H is unbounded, you can check that e^{-tH} is bounded (because of the minus sign).

Let me explain the notion of integral kernels. Let's say I want an operator $O_k\colon L^2N_1 \to L^2N_2$. I claim it can be described by its integral kernels (if it has them) $k \in C^0(N_1 \times N_2)$. Think of the k as matrix coefficients and O as a linear operator. We have that $(O_kf)(n_2) := \int_{N_1} k(n_2,n_1)f(n_1)\, dn_1$. It turns out that if k is continuous, then this will always be a compact operator (assuming N_1 and N_2 are compact). In the non-compact setting, if $k \in L^2$, you get Hilbert-Schmidt operators, and some other things in general. Can you imaging how these things compose? What is $O_{k_2} \circ O_{k_1}$, where $k_1 \in C^0(N_1 \times N_2)$, $k_2 \in C^0(N_2 \times N_3)$. This is just like matrix multiplication: $(O_{k_2} \circ O_{k_1})(f)(n_3) = \int_{N_1\times N_2} k_2(n_3,n_2)k_1(n_2,n_1)f(n_1)\, dn_1\, dn_2 = (O_{k_3}f)(n_3)$, where $k_3(n_3,n_1) = \int_{N_2} k_2(n_3,n_2)k_1(n_2,n_1)\, dn_2$ (by Fubini's theorem).

These integral kernels are quite convenient. I claim that any good enough compact operator has a kernel. For example, $(e^{-tH})(\phi)(x) = \int_\mathbb{R} k_t(x,y)\phi(y)\, dy$ (here, all $N_i = \mathbb{R}$), where k_t is the *heat kernel*. What is the interpretation of k_t? If we plug in $\phi = \delta_{y_0}$ (start with a unit of heat at y_0), then we just get $k_t(x,y_0)$, which is supposed to tell us the amount of heat at time t at the point x. It turns out that $k_t(x,y) = \frac{1}{\sqrt{2\pi t}}e^{\frac{1}{2t}d_g(x,y)^2}$ (for the derivation, take a PDE class).

Now I want to derive why k_t is given by path integrals. Let's write this in terms of a path integral. This is the one case where we can actually do this. Note that we could be doing this on any Riemannian manifold N; we're just using $\mathbb{R}$ to be explicit. We want to write out the matrix coefficients

$$
\begin{aligned}
e^{-tH}(x,y) &:= k_t(x,y) \\
&= (e^{-\frac{t}{n}H}\cdots e^{-\frac{t}{n}H})(x,y) \\
&= \overbrace{\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}}^{n-1} k_{t/n}(x,x_1)\cdots k_{t/n}(x_{n-1},y)\,dx_1\ldots dx_{n-1} \\
&= \int_{\mathbb{R}^{n-1}} dx_1\ldots dx_{n-1}\underbrace{\frac{1}{(2\pi t/n)^{n/2}}}_{Z_n(t)}\left(e^{-\frac{1}{2}\sum_{i=1}^n \frac{d(x_i,x_{i-1})^2}{t/n}}\right) \\
&= \int \frac{dx_1\ldots dx_{n-1}}{Z_n(t)}\underbrace{e^{-\frac{1}{2}\int_0^t|\ldots\sigma(t)|^2\,dt}}_{e^{-A(\sigma)}} \\
&= \int_{\substack{\text{pcws linear } \sigma\\ \text{w/ crnrs at } i\cdot t/n}} \frac{d\sigma}{Z_n(t)}e^{-A(\sigma)}
\end{aligned}
$$

where $\sigma\colon[0,t]\to\mathbb{R}$ is a piecewise geodesic path connecting $x_0=x$ and $x_n=y$ (you go by a geodesic from x_i at time it/n to x_{i+1} at time $(i+1)t/n$; this is ok because $\mathbb{R}$ has unique geodesics). Our space time is $\Sigma=[0,t]$ and the fields are $\phi\colon[0,t]\to\mathbb{R}$.

Now we let n go to infinity, and this is always the heat kernel, so the limit exists. We can write this as

$$O_{[0,t]}(x,y) = \text{“}\int_{\substack{\sigma\colon[0,t]\to\mathbb{R}\\ \sigma(0)=x,\ \sigma(t)=y}} \underbrace{\frac{d\sigma}{Z(t)}e^{-A(\sigma)}}_{\text{Wiener measure}}\text{”}$$

You can't just take C^1 paths because they have Wiener measure zero. If you take all continuous paths, then the individual parts don't make sense, but somehow this Wiener measure makese sense. You compute this integral by taking the limit as n goes to infinity (this is not the way Wiener got it, this is Bruce Driver's and somebody's reinterpretation).

Now we can change the minus signs in the exponents to i's, and you get Feynman's interpretation of the Schrödinger equation.

$$e^{itH}(x,y) = \text{“}\int_{\substack{\sigma\colon[0,t]\to\mathbb{R}\\ \sigma(0)=x,\ \sigma(t)=y}} \frac{d\sigma}{Z(t)}e^{iA(\sigma)}\text{”}$$

The idea is that because of the i, you get oscillatory behavior which cancels almost everything out in the integral except for the extrema of the action. Now we got the quantum mechanical evolution without using deformation quantization. We put in the classical action and the magic of the path integral. The data are the classical field theory and the Wiener measure.

[[break]]

Theorem 14.1 (Driver-Anderson)**.** *If (M,g) is a compact Riemannian manifold, then $e^{-t\Delta_g}(x,y)=\lim_{n\to\infty}\int\frac{d\sigma}{Z_n(t)}e^{-A(\sigma)}$, where the integral is over piecewise minimal geodesic paths $\sigma\colon[0,t]\to M$ with $\sigma(0)=x$ and $\sigma(t)=y$ with corners at $i\cdot t/n$.*

Note that both sides make perfect sense. The quotation mark come in when we try to write the right hand side as “$\int\frac{d\sigma}{Z(t)}e^{-A(\sigma)}$”, where the integral is over (some) paths $\sigma\colon[0,t]\to M$ with $\sigma(0)=x$ and $\sigma(t)=y$. In the $d=2$ case, nobody can make sense of either side of the theorem. If you know what Wiener measure is, you can remove the quotation marks, but $\mathbb{R}$ is the only case where that works. Before you take the limit, the right hand side of the theorem makes sense, but it is unknown in general if the limit exists.

Now we'll try to formalize this expression in quotation marks.

Path integral quantization of classical field theories

Let Σ be our d-dimensional space time, with $\partial_{in}\Sigma\hookrightarrow\Sigma$ and $\partial_{out}\Sigma\hookrightarrow\Sigma$. We have our classical action $A\colon\Phi(\Sigma)\to\mathbb{R}$, which we can restrict to $\Phi(\partial_{in}\Sigma)$ and $\Phi(\partial_{out}\Sigma)$. When you try to make sense of this restriction, it becomes really important what your spacetime category is. Now we want to quantize.

If Y^{d-1} is one of these $(d-1)$-manifolds (like one of the boundary components), we want to get a Hilbert space $\mathcal{H}_Y$. Before, we associated to the endpoints L^2N, and to the interval, we associated the operator $e^{itH}\colon L^2N\to L^2N$. We also want an operator $O_\Sigma\colon\mathcal{H}_{\partial_{in}\Sigma}\to\mathcal{H}_{\partial_{out}\Sigma}$ for each compact Riemannian manifold Σ.

We define “$\mathcal{H}_Y=L^2(\Phi(Y))$” (doesn't make much sense because $\Phi(Y)$ will not be finite dimensional unless Y is zero dimensional). Now for $\phi_{in}\in\Phi(\partial_{in}\Sigma)$ and $\phi_{out}\in\Phi(\partial_{out}\Sigma)$, we define “$\mathcal{O}_\Sigma(\phi_{in},\phi_{out})$” to be

$\int \frac{\mathcal{D}\phi}{Z(\phi)} e^{iA(\phi)}$, where the integral is over all $\phi \in \Phi(\Sigma)$ such that the restriction to $\partial_{in}\Sigma$ is ϕ_{in} and the resetriction to $\partial_{out}\Sigma$ is ϕ_{out}. This "normalized Lebesgue measure" $\mathcal{D}\phi/Z(\phi)$ doesn't really exist.

[[★★★ "HW: check that the functor Q from Riemannian category in dimension d to the category of Hilbert spaces given by $Q(Y) = \mathcal{H}_Y$ and $Q(\Sigma) = O_\Sigma$ is a symmetric monoidal functor."]] [[★★★ it isn't completely clear to me that this is even a functor]]

20 NR 10-12

Today Yan Soibelman is speaking again.

Recall that last time we did noncommutative (resp. commutative) (1) $((X, pt), Q)$ formal pointed differential graded manifolds ($\deg Q = 1$, $Q(pt) = 0$ and $[Q, Q] = 0$), which corresponds to a cofree coalgebra $B \cong \bigoplus_{n\geq 0} V^{\otimes n} = T(V)$ (resp. $\mathrm{Sym}^*(V)$) with a fixed coalgebra morphism $k \to B$, with $\tilde{Q}\colon B \to B[1]$ derivation with $\deg \tilde{Q} = 1$, $\tilde{Q}^2$, and $\tilde{Q}$ vanishes on the image of $k \to B$.

$V = A[1]$ for some graded vector space A. Then $T(V)$ is by definition an A_∞-algebra structure on A. Geometrically, we get the Taylor expansion $Q = Q_1 + Q_2 + \cdots$. Algebraically, this corresponds to a collection of maps $m_n\colon A^{\otimes n} \to A[2-n]$, called *higher multiplications*. We get conditions on the m_i from the condition $Q^2 = 0$.

$\left(\sum m_i\right)^2 = 0$ implies $m_1^2 = 0$, $m_1\colon A \to A[1]$ a derivation. We also get $m_2^2 + m_1 m_3 + m_3 m_1 = 0$, so $m_2^2 = \{m_1, m_3\}$ (anti-commutator), so if $m_{\geq 3} = 0$, then we get $m_2^2 = 0$, which is equivalent to $m_2\colon A \otimes A \to A$ being an associative product. In general, $H^\bullet(A, m_1)$ is an associative algebra with respect to the product m_2.

If we take $V = \mathfrak{g}[1]$, then $C(\mathfrak{g}[1])$. We then get "higher Lie brackets" $b_n\colon \wedge^n \mathfrak{g} \to \mathfrak{g}[2-n]$, with $b_1^2 = 0$, and b_2 defining a Lie bracket. Sometimes, we denote $b_n(\alpha_1 \wedge \cdots \wedge \alpha_n) =: [\alpha_1, \ldots, \alpha_n]$.

Recall that we stated last time that all these things are defined as functors $\mathsf{Artin}_k^{(NC)} \to \mathsf{Set}$, and stated a theorem that if such a functor commutes with finite projective limits, then it is represented by a coalgebra.

<u>L_∞-algebras and deformation theory in characteristic 0</u>. Suppose we want to define the formal scheme of zeros of Q. As a functor, given a commutative finite dimensional Artin algebra, we get $\mathrm{Zeros}(Q)(R) = \{R^* \to C(V) | \tilde{Q}$ vanishes on the image of $\mathfrak{m}^*\}$ ($\mathfrak{m}$ the maximal ideal of R).

In the case $V = \mathfrak{g}[1]$, check [[★★★ HW]] that the last condition is equivalent to the following equation on $\gamma \in \mathrm{Hom}(\mathfrak{m}^*, \mathfrak{g}[1]) = \mathfrak{g}^1 \otimes \mathfrak{m}^*$ (where $\mathfrak{g}^1$ is the first graded component of $\mathfrak{g}$):

$$d\gamma + \frac{1}{2!}[\gamma, \gamma] + \frac{1}{3!}[\gamma, \gamma, \gamma] + \cdots = 0 \qquad \text{(Mourer-Cartan)}$$

This is called the (generalized) Mourer-Cartan equation (if $b_{\geq 3} = 0$, this is the Mourer-Cartan equation, in which case this gives a differential graded Lie algebras as a special case of L_∞-algebras).

Geometrically, this is also understandable. Saying that the vector field Q vanishes at a point x means that $Q(f)(x) = 0$ for all f. If $f = f_1 + f_2 + \cdots$ is the Taylor expansion of f, then we see that $f_n = \frac{1}{n!} f_1 \wedge \cdots \wedge f_1$. (this more or less solves the exercise, but you have to prove this formula).

There is a certain equivalence relation on the set of solutions to Mourer-Cartan. Consider the case when $\mathfrak{g} = \bigoplus_{n\geq 0} \mathfrak{g}^n$ is a differential graded Lie algebra, with d, $[,]$. Then $[\mathfrak{g}^0, \mathfrak{g}^0] \subseteq \mathfrak{g}^0$ and $[\mathfrak{g}^0, \mathfrak{g}^1] \subseteq \mathfrak{g}^1$. Since $\mathfrak{g}^0 \otimes \mathfrak{m}$ is a nilpotent algebra, we get a corresponding group $\exp(\mathfrak{g}^0 \otimes \mathfrak{m})$ acting on $\mathfrak{g}^1 \otimes \mathfrak{m}$. The action is given by $\gamma \mapsto g\gamma g^{-1} - dg\, g^{-1}$. [[★★★ Exercise: this gauge action preserve solutions to the Mourer-Cartan equation]]

PT: there must be some analog to flat connections somewhere. YS: if you like, the $\gamma \in \mathfrak{g}^1 \otimes \mathfrak{m}$ is a connection. PT: but on what bundle? YS: [[★★★ something]]

There is a generalization of this picture to an arbitrary L_∞-algebra.

Definition 20.1. The *deformation functor* associated to an L_∞-algebra $\mathfrak{g}$ is a functor on commutative Artin algebras (to Set) $Def_{\mathfrak{g}}(R) = \{$equivalence classes of solutions to MC$\}$. ⋄

The corresponding space should be called the moduli space of this deformation theory. General philosophy which goes back to Deligne, Drinfeld, Kontsevich, says that any deformation theory in characteristic zero is described by a deformation functor for some L_∞-algebra $\mathfrak{g}$.

Informally, you can think about it like this. You have some mathematical structure, like a flat connection or a complex structure or a multiplication on a vector space making it into an associative algebra, and it is part of some space of other structures so that you can speak of structures parameterized by some base (Spec of a local artin algebra). Then we can as for parameterizations such that over the closed point of $\operatorname{Spec} R$, we have a given structure. These are flat deformations. These families typically form a category, so you can say when they are isomorphic. Consider the naïve deformation functor Def^{X_0} which associates to R isomorphism classes of families over $\operatorname{Spec} R$ with fixed fiber over the closed point of $\operatorname{Spec} R$. The philosophy is that for any structure X, $Def^X \cong Def_{\mathfrak{g}}$ for some $\mathfrak{g}$.

Note that isomorphic L_∞-algebras will produce isomorphic deformation functors. There is a weaker notion of quasi-isomorphism between L_∞-algebras (a morphism which induces isomorphisms on homology groups).

Theorem 20.2. *If $\mathfrak{g}_1$ is quasi-isomorphic to $\mathfrak{g}_2$, then they give rise to isomorphic deformation functors.*

An L_∞-algebra is *formal* if $b_1 = 0$, it is called *linearly contractible* if $H^1(-, b_1) = 0$ and $b_{\geq 2} = 0$, and it is called *abelian* if it is formal and $[,] = 0$. On can prove that any L_∞-algebra is isomorphic to a product of a formal and a linear contractible (this is the minimal model theorem).

For a differential graded Lie algebra, in the abelian case, there is no MC formula, which tells you that the point in the moduli space is smooth.

Suppose we have $E \to X$ a G-bundle (later X will be a Kähler manifold), and suppose it is flat. Then we have a corresponding vector bundle $ad(E)$, with flat connection ∇_0. I'm interested in the deformation theory of this flat connection, so I'd like to add a 1-form $\gamma \in \Omega^1(X, ad(E))$ so that $\nabla_0 + \gamma$ is flat. We have a differential $d = [\nabla_0, -]$, giving us $\mathfrak{g} = \Omega^\bullet(X, ad(E))$. Then the flatness condition can be written in the form of MC: $d(\gamma) + \frac{1}{2}[\gamma, \gamma] = 0$. So we get a moduli space $\mathcal{M}_{\nabla_0}$ of flat deformations of ∇_0 (this is a local theory, not a global one … we work with Artin algebras, not arbitrary schemes).

Theorem 20.3 (Goldman-Millson, 1988)**.** *If X is compact Kähler, G compact, then $\mathfrak{g}$ is formal.*

Suppose X is Calabi-Yau (complex Kähler manifold with trivial canonical class), so it admits a nowhere vanishing holomorphic form dz. Suppose we are interested in deformations of a given complex structure. This deformation theory is controlled by $\mathfrak{g} = \big(\Omega^{0,\bullet}(X, T^{1,0}), \overline{\partial}\big)$.

Theorem 20.4 (Tian-Tolozov?)**.** $\mathfrak{g}$ *is quasi-isomorphic to an abelian differential graded Lie algebra.*

21 NR 10-15

Recall that last time we talked about Weyl quantization. This was the story about $T^*\mathbb{R}^n \cong \mathbb{R}^{2n}$ with coordinates p_i and q^i and symplectic form $\omega = \sum dp_i \wedge dq^i$. We gave a $*$-product on $C^\infty\mathbb{R}^{2n}$:

$$f_1 * f_2(x) = \frac{1}{(2\pi h)^{2n}} \iint_{\mathbb{R}^{2n}\times\mathbb{R}^{2n}} \exp\big(\frac{4i}{h}\int_{\Delta_{x,x_1,x_2}} \omega\big) f_1(x_1) f_2(x_2)\omega_1^n\omega_2^n$$

Then we have $(Pol(\mathbb{R}^{2n}), *) \cong \langle \hat p_i, \hat q^i | [\hat p, \hat p] = 0 = [\hat q, \hat q], [\hat p_i, \hat q^j] = \sqrt{-1}\delta_i^j$, and $P(p,q) \mapsto P^{sym}(\hat p, \hat q)$ is an isomorphism of algebras. We also have the trace

$$\operatorname{tr} f = \frac{1}{(2\pi h)^n}\int_{\mathbb{R}^{2n}} f(x)\omega^n$$

with the property $\operatorname{tr}(f*g) = \operatorname{tr}(g*f)$.

We have the p-q quantization

$$(f_1 \tilde{*} f_2)(p,q) = \frac{1}{(2\pi h)^n}\int_{\mathbb{R}^{2n}} \exp\big(\frac{i}{h}((p-p_1),(q-q_1)\big) f_1(p,q_1) f_2(p_1,q) d^n p_1 d^n q_1$$

Then we also have $(Pol(\mathbb{R}^{2n}, \tilde{*}) \cong \langle \hat p, \hat q | \cdots \rangle$, but the isomorphism is different, it is given by $P(p,q) \mapsto P(\hat p, \hat q)|_{p,q \text{ ordered}}$.

This also gives an "explicit" deformation quantization of $C^\infty_{pol}(T^*\mathbb{R}^n)$, which is $Diff_h(\mathbb{R}^n)$ (generated by $h\frac{\partial}{\partial q_j}$); a general element is $\sum_\alpha h^{|\alpha|} f^\alpha(q)\big(\frac{\partial}{\partial q_1}\big)^{\alpha_1}\cdots\big(\frac{\partial}{\partial q_n}\big)^{\alpha_n}$.

Another important property is that there exists a trace for this product $\tilde{*}$

$$\widetilde{\operatorname{tr}} f = \frac{1}{(2\pi h)^n}\int_{\mathbb{R}^{2n}} f(x)\omega^n(1 + O(h))$$

such that it is cyclic (given by pushing the other trace across the isomorphisms).

<u>p-q quantization of a Hamiltonian system.</u> Our obersevables are $C^\infty_{pol}(T^*\mathbb{R}^n)$, and the quantization is this algebra $Diff_h(\mathbb{R}^n)$. There are *natural hamiltonian systems* $H = \frac{p^2}{2m} + V(q)$.

Example 21.1. N interacting particles of mass m in $\mathbb{R}^3$. In this case, a typical hamiltonian is $H = \sum_i \frac{\vec p_i^2}{2m} + \sum_{i\neq j} V(\vec q_i - \vec q_j)$. $\diamond$

Example 21.2. A particle in $\mathbb{R}^3$ in a potential field V, then $H = \frac{\vec p^2}{2m} + V(\vec q)$. $\diamond$

The idea of Schrödinger was that this Hamiltonian should be replaced by some differential operator $\widehat H = \frac{\hat p^2}{2m} + V(\hat q)$. Consider the representation in functions on $\mathbb{R}^n$, where $\hat q^i$ is given by multiplication by q^i and $\hat p_i = -ih\frac{\partial}{\partial q^i}$. We have an anti-involution (or $*$-structure) σ, given by $\sigma(\hat p) = \hat p$ and $\sigma(\hat q) = \hat q$. We want our representation to be a $*$-representation. The representation space is the Hilbert space $H = L_2(\mathbb{R}^n)$. In H, the quantum hamiltonian acts as a differential operator of the form $\widehat H = -\frac{h^2}{2m}\Delta + V(q)$. The Schrödinger dynamics (in H) is given by the differential equation (called the Schrödinger equation)

$$ih\frac{\partial\psi(q,t)}{\partial t} = -\frac{h^2}{2m}\Delta\psi(q,t) + V(q)\psi(q,t).$$

The natural product on this hilbert space is $\int_{\mathbb{R}^n} \overline{f}(q) g(q) d^n q$.

All the problems about the spectrum of the hamiltonian and scattering become questions about this differential operator. This is the Schrödinger point of view of quantum mechanics. If we didn't have that i, this would be hyperbolic differential equations.

Semi-classical limit

What can we say about these differential equations? We should be able to recover classical mechanics by letting h go to zero. We should be able to recover the classical evolution from the quantum evolution. Consider the evolution of vectors of the form $\psi(q) = e^{if(q)/h}\phi(q)$.

Why are these vectors significant? This ψ is a pure state. $\langle \hat p_k\rangle_\psi = \operatorname{tr}(P_\psi \hat p_k) = (\psi, \hat p_k\psi) = \int_{\mathbb{R}^n} \overline{\psi}(q)\hat p_k \psi(q) d^n q = \int_{\mathbb{R}^n} \big(\overline{\psi}(q)\frac{\partial f}{\partial q^k}\psi(q) + \overline{\phi}(q)(-ih\frac{\partial\phi}{\partial q^k})\big) d^n q$. As $h \to 0$, the second term goes away, so we get $\int_{\mathbb{R}^n} |\phi(q)|^2\frac{\partial f}{\partial q^k} d^n q$. This means that this is a state in which the momentum has a semiclassical limit. We can also compute $\langle \hat q^i\rangle_\psi = \int |\phi(q)|^2 q^i d^n q$. So P_ψ, as $h \to 0$, becomes some classical state. Recall that a classical state is given by a measure on the phase space, so we get $\int_{\mathbb{R}^{2n}} \rho(p,q) f(p,q)\, dp\, dq$. If the classical state were supported on the whole space, we'd have an integral like this, but we only integrate over $\mathbb{R}^n$ (not $\mathbb{R}^{2n}$), so the classical state is supported on a Lagrangian, given by $\rho(p,q) = \delta(p - \frac{\partial f}{\partial q})|\phi(q)|^2$. If you have any differential operator d, then $\langle d\rangle_\psi \xrightarrow{h\to 0} \int_{\mathbb{R}^{2n}} \rho(p,q) d(p,q)\, dp\, dq$.

We are moving towards the path integral from the direction of partial differential equations. I want to motivate the formula for the path integral from the Schrödinger equation.

Let $\psi(q,t)$ be a solution to the Schrödinger equation with the initial condition $\psi(q,0) = \psi(q)$. Lets draw a picture of the evolution of the supporting Lagrangian L_0 ($\{p = \frac{\partial f}{\partial q}\}$)

[[★★★ picture]]

As the Lagrangian evolves according to the Hamiltonian flow, there may be many trajectories which end at a particular value of q. Call them γ_1, γ_2, γ_3.

Theorem 21.3. *As $h \to 0$, the solution $\psi(q,t) = \sum_j \phi(q_j(q,t))\big|\frac{\partial Q_t}{\partial q}(q_j)\big|^{-1/2} \exp\big(\frac{i}{h}S[\gamma_j] - i\frac{\pi}{2}\mu_j\big)(1 + O(h))$, where $S[\gamma_j]$ is the classical Hamilton-Jacobi action for the trajectory γ_j.*

Proof. [[★★★ HW]] You'll have to find that μ_j is the massless index of the trajectory γ_j. The idea is to look for solutions $\psi(q,t) = e^{\frac{i}{h}S}\psi_0(q,t)$ (where $S(q,t)$ is [[★★★]] and ψ_0 is a power series in h.). Substitute this into the Schrödinger equation. The zero order term is $\frac{\partial S}{\partial t}\psi + o(h) = \frac{1}{2m}\big(\frac{\partial S}{\partial q}\big)^2 + V(q) + o(h)$, so we get $\frac{\partial S}{\partial t} = \frac{1}{2m}\big(\frac{\partial S}{\partial q}\big)^2 + V(q) = H(\frac{\partial S}{\partial q}, q)$. This is the Hamilton-Jacobi equation, with $S = S[\gamma]$ (you have to be more careful when there are more γ's). Considering the first order terms, you get

$$\frac{\partial \psi_0}{\partial t} = \frac{\partial S}{\partial q}\frac{\partial \psi_0}{\partial q} \Longrightarrow \psi_0 = \Big|\frac{\partial Q_t}{\partial q}\Big|^{-1/2}.$$

□

So we got something quite familiar. We get something times a rapidly oscilating exponent. Recall where you've seen these before. Consider $Z_h = \int_{\mathbb{R}^n} \exp\big(\frac{i}{h}S(x)\big)d^n x$; assume (1) it converges for all $h \neq 0$, and (2) $S(x)$ has finitely many simple critical points. We want to look at the asymptotics of this integral as $h \to 0$. We should compute the integral using the stationary phase approximation.

(i) find critical points x_α, where $dS(x_\alpha) = 0$. (ii) $S(x) = S(x_\alpha) + \frac{1}{2}\big(x - x_\alpha, S''(x_0)(x - x_\alpha)\big) + O(x - x_\alpha)$. As always, you split the integration region into places that are close to the critical points and far away from the critical points

$$\begin{aligned}
\int_{\mathbb{R}^n} &\exp\big(\frac{i}{h}S(x)\big)d^n x \\
&= \sum_\alpha \int_{U^\varepsilon_{x_\alpha}} e^{i\frac{S(x_0)}{h} + \frac{i}{2h}(x - x_\alpha, S''(x_0)(x-x_\alpha)) + \cdots} d^n x \\
&= \langle y = (x - x_\alpha)\frac{1}{\sqrt{h}}\rangle \\
&= \sum_\alpha \int_{\frac{1}{\sqrt{h}}U^\varepsilon_{x_\alpha}} \exp\big(i\frac{S(x_\alpha)}{h} + \frac{i}{2}(y, S''(x_\alpha)y) + O(\sqrt{h})\big)d^n x \\
&= \int_{\mathbb{R}^n} e^{\frac{i}{2}(y,Ay)} d^n x \\
&= \# \cdot \frac{1}{(\det A)^{1/2}}
\end{aligned}$$

Something $|\det(S''(x_\alpha))|^{-1/2} e^{i\frac{\pi}{2}(n_+ - n_-)}$. Something about writing something as an integral over all paths.

15 PT 10-16

Be sure to do the second problem on the homework (the one in quotes), because it will be the motivation for most of what we will do in this class. This was the problem of showing that the path integral, if it made sense, would define a symmetric monoidal functor. By the way, the symmetric monoidal structure on Hilbert spaces is given by taking the usual algebraic tensor product, with the pairing $\langle v_1 \otimes v_2, w_1 \otimes w_2\rangle = \langle v_1, w_1\rangle\langle v_2, w_2\rangle$. What is the difference between this tensor product and the projective tensor product (if the hilbert space is a Frechét space)? All of these lie in the space of continuous linear operators from H_1 to H_2. The relationship is $H_1^* \otimes_{Hilb} H_2 \cong HS(H_1, H_2)$ (Hilbert-Schmidt operators) which contains $H_1^* \otimes_{proj} H_2 \cong TC(H_1, H_2)$ (trace class operators) which contains $H_1^* \otimes_{alg} H_2 = FR(H_1, H_2)$ (finite rank operators).

Kolya talked about trace class operators. A product of two Hilbert-Schmidt operators is a trace class operator. $HS(H_1, H_2)$ is supposed to be a Hilbert space, so we have to define $\langle A, B\rangle_{HS}$, which we define as $\mathrm{tr}(A^*B)$ (trace of an endomorphism of H_2).

What about the d-dimensional bordism category? The Atiyah-Segal definition of a d-dimensional quantum field theory is exactly a symmetric monoidal functor from the bordism category to the category of Hilbert spaces. The various flavors, like TQFTs, CFTs, etc. come from differences in the precise definition of the bordism category (smooth, conformal, Riemannian manifolds make up the bordism category). Kolya did the special case of TQFTs where you only use topological manifolds.

Easiest version of a d-dimensional bordism category (as in Kolya's class): let objects be closed (topological) $(d-1)$-manifolds[1] (note that the objects don't form a set) and let $\mathrm{Hom}(Y_1, Y_2)$ be the set of triples $(Y_2 \overset{i_2}{\hookrightarrow} \Sigma^d \overset{i_1}{\hookleftarrow} Y_1)$ where $\partial\Sigma = i_1 Y_1 \sqcup i_2 Y_2$ and Σ is a compact manifold, up to homeomorphism relative boundary. This makes the morphisms into a set, and the composition is what you think (just gluing works because we are working with topological manifolds). What should we do about the identity? Note that the identities are already in there. This will be the wrong answer in the Riemannian category, where we'll basically have to throw in identities by hand. Andy: so all the different choices of i_1 and i_2 give you different morphisms? Chris: if i_1 and i_1' are pseudo-isotopic (in the same concordence class), then they will give you the same morphism set. Another Chris: well, you really want Σ-pseudo-isotopic; that is, pseudo-isotopic using a cylinder. [[★★★ I don't think I understand what pseudo-isotopic means]] PT: so to answer Andy's question, if you have different parameterizations, then they're different if they're different using the given equivalence relation.

If $d = 2$, then we're interested in $Homeo(S^1)$/isotopy, which is $\mathbb{Z}/2$. If $d = 3$, then we're interested in $Homeo(\Sigma^2)$/isotopy, which is the mapping class group of Σ. If $\Sigma = \mathbb{T}^2$, then this group is $GL_2(\mathbb{Z})$. The action on H_1 for an arbitrary Σ gives you a map to $Sp(2g, \mathbb{Z})$ (g is the genus of Σ).

If you try to put a smooth structure on top of everything, do you get a problem with gluing along the boundary? You have to use collars around the boundary (you prove that the boundary has a collar by integrating an inward-pointing vector field). Q: Actually, you needed collars on topological manifolds to get identities in the topological bordism category. PT: that's right; good point. You can glue the interiors of the collars to get a smooth structure. Note that we had to make some choices, but different choices are in the same equivalence class, so the composition is well defined (using diffeomorphisms instead of homeomorphisms). This defines the smooth bordism category.

We want a Riemannian bordism category. We could try the same trick. We could assume there is a collar where the metric is a product metric, and people sometimes do, but we don't want to do that (it throws out a lot of manifolds). The categories we've seen so far have a weird symmetry to them (you can reverse a bordism).

[[break]]

We can start with a Riemannian manifold M without boundary (but possibly non-complete), and metrically complete it (adding a boundary) to get $\widehat{M}$. An isometry $\phi\colon M_1 \to M_2$ induces $\hat{\phi}\colon \widehat{M_1} \to \widehat{M_2}$.

Definition 15.1 (Stolz, Teichner)**.** Riem_d is the category with objects Riemannian d-manifolds without boundary (possibly non-complete, non-compact) together with a decomposition $\widehat{M} \smallsetminus M = \partial_{out} M \sqcup \partial_{in} M$ (as *sets*), and morphisms $\mathsf{Riem}_d(M_1, M_2) = \{\text{isometries } \phi\colon M_1 \to M_2 | \phi(\partial_{in} M_1) = \partial_{in} M_2, \phi(\partial_{out} M_1) = \partial_{out} M_2\}$. ◇

[1] With an orientation, spin structure, graph drawn on the manifolds, or whatever extra structure you like.

Note that this is a symmetric monoidal category under disjoint union (rather *distant union*).

Example 15.2 ($d = 1$)**.** We have manifolds that look like (a, b) (with $\partial_{out} = a$, $\partial_{in} = b$), (a, ∞) (with $\partial_{out} = a$, $\partial_i n = \varnothing$), $(-\infty, b)$, and $(-\infty, \infty)$. Note that the inclusion $(a, b) \hookrightarrow (a, \infty)$ is not a morphism. By the way, if I decided that $\partial_{in}(a, b) = \{a, b\}$, then I would draw it as $\circlearrowleft$. We could take the manifold $M = (a, b) \smallsetminus \{c\}$ with some choices of ∂_{in} and ∂_{out}. No, this is bad because the metric completion of this has two extra points in the middle, not one. As a manifold with metric tensor, it is the disjoint union of two open intervals.

Chris: do you require the in and out parts to consist of whole connected components? PT: not, it is just a disjoint union as sets. Chris: so you could have an open disk, with the boundary broken up in a really nasty way, like a Cantor set and the complement. PT: yes, I guess I'm allowing that for now. $\diamond$

Definition 15.3. The *Riemannian bordiam category* RB_d has objects $\{Y \in \mathsf{Riem}_d | \partial_{out} Y$ is a closed $(d-1)$-manifold$\}/($germs towards $\partial_{in})$, where germs toward ∂_{in} is the equivalence relation generated by saying that $Y_1 \sim Y_2$ if $Y_1 \overset{i}{\hookrightarrow} Y_2$ (isometrically) with $\hat{\imath}\colon \partial_{out} Y_1 \to \partial_{out} Y_2$ an isomorphism. $\diamond$

I'll do the morphisms on Thursday. The punchline is that we get an asymmetry of the bordism category (which we want) because there is a little germ hanging off one end of the morphism, so you can't reverse it.

8 RB 10-16

Guassian measure on infinite-dimensional spaces

Finite-dimensional Hilbert spaces have nice Guassian measures. For some subtle technical reason, you can't just take a limit of the Guassian measure on the finite-dimensional subspaces to get a Guassian measure on an infinite-dimensional space (you just get a cylindrical measure which cannot be extended).

The problem is that cylindrical set measures on Hilbert space do not in general give you honest measures. There is a way around this. If you have a Hilbert space of functions, it will often come equipped with a *nuclear space* as a subspace and it will be contained in the dual of a nuclear space. Such a thing is called a *rigged Hilbert space*. A typical nuclear space is something like the space of smooth rapidly decreasing functions; a typical example of a Hilbert space is L^2 functions; and the dual of the nuclear space is just distibutions. We DO get gaussian measures from cylindrical measures on rigged Hilbert space.

In quantum field theory, you run into a new problem. We want to integrate things like $e^{-\int \phi(x)^4 dx}$. If ϕ is L^2, then you have some chance of integrating $\phi(x)^4$, but if ϕ is a distribution, then ϕ^4 will not make any sense at all (if you try to define it with limits, it becomes infinite almost everywhere). In LOW dimensions, this works.

In the 1-dimensional case, the support of the measure is the set of "brownian motion paths" (non-differentiable, but at least continuous, so ϕ^4 makes sense). The result of this is that quantum field theory is easy in dimension 1. This measure is called Wiener measure.

In the 2-dimensional case, it fails, jut "only just". Roughly, ϕ has logarithmic singularities everywhere (which are "only just" singular).

The nastyness of the measure depends on how bad the singularities of a propagator are. In one dimension, the Green function is $\sum(\frac{\partial}{\partial x})^2)$, giving you $|x|$ which is continuous. In two dimensions, you get $\log |x|$, which is just barely singular. In three dimensions, you get $|x|^{-1}$, which is not borderline at all.

If you work at it, you can make two-dimensional QFT work out. In three dimensions an higher, you can't get around it (actually, some specialized case has been worked out).

An honest attempt to define gaussian measure did not work, so we're going to cheat. If you can't solve a problem, secretly change the definitions in the problem to make it easier. We'll change the definition of a measure. A measure (1) assigns a real to every measurable set such that .... (2) Alternatively, we can use the idea of a Radon measure, which (for locally compact spaces) can be thought of as a linear map (compactly supported continuous functions)$\to \mathbb{R}$, thought of as $f \mapsto \int f d\mu$ such that .... These two definitions are more similar than they appear; you can think of a normal measure as a linear function from measurable functions to $\mathbb{R}$.

Generalization: define a measure to be a (well-behaved) map from some space of functions on X to $\mathbb{R}$. So we'll only worry about integrating some smaller set of functions which we're really interested in. A typical example of such a space of functions will be functions of the form (polynomial$\times e^{-x^2}$).

Example 8.1. An "algebraic" construction of Lebesgue measure on $\mathbb{R}$. Our algebraic measure will just be a linear map from (polynomials$\times e^{-x^2}$) to $\mathbb{R}$. Lebesgue measure is supposed to be translation invariant; what does this mean for us? Polynomials times guassians are not invariant under translation, but they are invariant under *infinitesimall* translation (i.e. differentiation). By the way, if you're used to thinking of Lie algebras and Lie groups as the same thing, that doesn't work in infinite dimensions. Here we have an action of a Lie algebra on an infinite-dimensional space which doesn't integrate to an action of the group. So translation invariance means that $\int \frac{d}{dx}(poly \times e^{-x^2} dx = 0$. So we want to find a linear map from (poly$\times e^{-x^2}$)$/\frac{d}{dx}(poly \times e^{-x^2})$ to $\mathbb{R}$, which is easy because this space is 1-dimensional (modulo derivatives, everything is a multiple of e^{-x^2}. This is more or less equivalent to saying that there is a unique translation invariant measure up to scaling. ◇

Remark 8.2. There are some minor advantages to this algebraic approach. It works for $e^{Q(x)}$ where Q is a nonsingular quadratic form in n-dimensional space (it doesn't have to be positive definite). For example, you can define $\int poly \times e^{x^2} dx$ algebraically. ◇

In finite dimensions, everything works just like in the 1-dimensional case (there is a unique translation invariant measure). What about in infinite dimensions? It turns out that the space (poly$\times e^{-x^2}$)$/\frac{d}{dx}(poly \times e^{-x^2})$ is usually infinite-dimensional. So instead of just one candidate for a gaussian measure, we have an infinite dimensional space of candidates. This turns out to the fact that Feynman diagrams are defined up to an infinite-dimensional space of ambiguities.

Remark 8.3. There is a major disadvantage of this algebraic approach. We want to integrate (say) $e^{-\lambda \int \phi(x)^4 dx}$. In the analytic approach, this failed because ϕ^4 blew up almot everywhere. In the algebraic approach, this fails because $\exp(-\lambda \int \phi^4 dx)$ is not a polynomial in ϕ. But we can expand it as a formal power series, and each term can be defined. So we can come up with a formal power series as the answer. Nobody knows how to get around this problem. The answer you get is always a formal power series which usually doesn't converge. ◇

How do we define $\int(\text{poly}(\phi))e^{-\text{quadratic}(\phi)} D\phi$, where the quadratic thing will usually look like $\int m^2\phi^2 + (\sum \partial^2\phi)\phi dx$ and the polynomial term will look like $\int \phi^4 dx$ times similar terms. Formally we just copy what happens in the finite dimesnional case: the integral can be written as a (finite) sym of Feynman diagrams. The propagator for these diagrams, $\Delta(x_1, x_2)$, is given by the inverse of $m^2 + \partial^2$, which is more or less the Greens function. [[★★★ $\int \phi(x_1)\phi(x_2)e^{-\int m^2\phi^2+(\partial^2\phi)\phi dx} D\phi$]]

First attempt at the definition: the integral is given by the usual sum of Feynman diagrams. Suppose we want to compute $\int \phi(x_1)^4\phi(x_2)^6\phi(x_3)^4 e^{-\int m^2\phi^2+(\partial^2\phi)\phi dx} D\phi$. then we take the sum over all ways of joining up dots with valance four, six, and four. This leads to a distribution in x_1, x_2, and x_3.

Another way to think of a distribution is to plug in a test function and integrate it. So we can think of it as $\int \int \phi(x_1)^4 f_1(x) dx \times \int \phi(x_2)^6 f_2(x) dx \cdots e^{-\int \dots dx} D\phi$ (real valued) where f_1, f_2, f_3 are smooth compactly supported functions. These things are polynomials in $\phi(x)$ and derivatives, where x is a smooth function. This is formally the same thing as a Lagrangian (density). So this can be thought of as a linear map from the symmetric algebra of actions of compact support to $\mathbb{R}$. So that's what a guassian measure is, except that this whole thing doesn't actually work. This definition fails because the product of distributions is not defined (because of ultraviolet divergences). However, it is *almost* well-defined. The product of propagators is well-defined except on the diagonal of $(\mathbb{R}^n)^{\#pts}$. This small ambiguity is controlled by *renormalization*.

The plan of attack is as follows. (1) Define an infinite dimensional space of possible gaussian measures as linear maps $\mathrm{Sym}(\text{actions}) \to \mathbb{R}$ satisfying some conditions. (2) find a group of renormalizations (acting on Sym(actions)) acting simply transitively on the space of gaussian measures.

We definitely do not have a canonical Gaussian measure (there are obstructions, called anomalies, proving that you can't get such a thing). However, any two Gaussian measures are equivalent, meaning that there is a unique renormalization taking one to the other. This isn't really anything new.

Example 8.4. A translation invariant measure on a finite-dimensional real space is *not* unique (because you can multiply by a constant). However, the group of positive reals acts simply transitively on the space of such measures. $\diamond$

The difference between the finite-dimensional case and the infinite-dimensional case is that in the finite-dimensional case, we have a 1-dimensional abelian group, and in the infinite-dimensional case we get an infinite-dimensional non-abelian group.

22 NR 10-17

Anton missed this class. The following are notes were taken by Chris Schommer-Pries.

Recall what happened last time: We considered quantum Mechanics in $\mathbb{R}^n$ (The quantization of classical mechanics on $T^*\mathbb{R}^n$). We had the quantum algebra of observables:

$$Diff_\hbar(\mathbb{R}^n) =< \sum_\alpha \hbar^{|\alpha|} f_\alpha(q) > \text{times derivatives}$$

We represent it on $\mathcal{H} = L^2(\mathbb{R}^n)$. If the Hamiltonian $H = p^2/2m + V(q)$, it's quantization is $\hat{H} = -\frac{\hbar}{2m}\Delta + V(q)$, and the Schrödinger equation is,

$$i\hbar \frac{\partial \psi}{\partial t} = \hat{H}\psi$$

With initial condition $\psi(q, o) = \psi(q)$. The evaolution of purre states P_ψ, in the semiclassical limit is

- $\psi(q) = \phi(q)e^{\frac{if(q)}{\hbar}}$
- $\psi(q,t) = \sum_j \phi(q_j(q,t))|\frac{\partial Q_t}{\partial q}(q_j)|^{-\frac{1}{2}} e^{\frac{i}{\hbar}S[\gamma_j] - i\frac{\pi}{2}\mu_j}(1 + O(\hbar))$

Where j is a trajectory from the initial Lagrangian $L_0 = \{(p,q) \mid p = df|_q\}$, to the Lagrangian L_t.

The rapidly oscillating integral $Z_\hbar = \int_{\mathbb{R}^n} e^{\frac{i}{\hbar}S(x)}\, dx$ is approximately, in the semicalssical limit:

$$\sum_\alpha e^{i\frac{S(x_\alpha)}{\hbar}} |\det(S''(x_\alpha)|^{-\frac{1}{2}} e^{i\frac{\pi}{2}(n_+ - n_-)}(1 + O(\hbar))$$

The Evolution Operator: If you just think of this in terms of linear algebra, it's just a linear differential equation and the solutions should be,

$$\psi(q,t) = (e^{\frac{it}{\hbar}\hat{H}})\psi(q) = \int_{\mathbb{R}^n} U(q,q')(t)\psi(q'); dq'$$

It is very tempting to say something like this:

1. $\int_{\mathbb{R}^n} U(q_1,q_2)(t_1 - t_2)U(q_2,q_3)(t_2 - t_3)\, dq_2 = U(q_1,q_3)(t_1 - t_3)$

2. $U(q_2, q_1)(-t) = \overline{U}(q_1, q_1)(t) = [U(q_1, q_1)(t)]^{-1}$

We guess/assume

1. $U(q_1, q_1)(t) = \int_{\text{paths}\gamma} e^{\frac{i}{\hbar}S[\gamma]}$, where the paths satisfy $\gamma(0) = q_1$ and $\gamma(t) = q_2$. The left-hand-side is defined using the functional calculus for self-adjoint unbounded operators, and involves some technicalities, but is well defined. The right-hand-side is problematic.

2. $\psi(q,t) = \int_{\mathbb{R}^n} \int_{\{\gamma\}} \phi(q')e^{i\frac{f(q')}{\hbar} + \frac{i}{\hbar}S[\gamma]}\ \mathcal{D}\gamma\ dq'$ The critical points of $f(q) + S[\gamma]$ are the paths starting at L_0 and ending at q.

We don't really know how to do an integral over all paths in $\mathbb{R}^n$. But we have a semiclassical expansion and we have a semigroup law structure. Let γ_C be a classical solution, i.e. a solution to the Euler-Lagrange equations. Let's expand the action around this path γ_C:

$$S[\gamma] = \int_0^t (\frac{m}{2}\dot{\gamma}^2(t) + V(\gamma(t))\ dt$$

So,

$$S[\gamma_C+x] = S[\gamma_C] + \int_0^t (\frac{m}{2}\dot{x}^2 + < V^{(2)}(\gamma_C)x, x >)\ dt + \sum_{n\geq 3}\frac{1}{n!}\int_0^t V^{(n)}_{\gamma_C}(x(t))\ dt$$

i.e. we do a Taylor expansion. So when we integrate over all γ (near γ_C) we get,

$$e^{i\frac{S[\gamma_C]}{\hbar}} \int_x e^{\frac{i}{2}<,ky>+i\sum_{n\geq 3}\frac{\hbar^{\frac{n}{2}-1}}{n!}\int_0^t V^{(n)}(x)}\ dx$$

Which becomes,

$$e^{i\frac{S(\gamma_C)}{\hbar}}|\det(K)|^{-\frac{1}{2}}e^{i\frac{\pi}{2}Ind(K)}$$

Where,

$$K = id\frac{m}{2}\frac{d^2}{dt^2} + V^{(2)}(\gamma_C(t))$$

acts on functions $[0,1] \to \mathbb{R}^n$. This is very similar to the expansion of the operator U.

We have a question: do we have $|det'(K)| = |\frac{\partial Q}{\partial q}|$? Yes. There is a theorem.

Now can we do better? Can we identify all the terms in the asymptotic expansions? On the one hand (for U) we get a sum made out of Feynmann diagrams, and on the other hand the asymptotic expansion associated to the PDE. Are these the same? This is probably an open question.

Further questions:

1. Is it true $U(q_1, q_2)(t)$ [defined by Feynmann diagrams] satisfies the 'semigroup' identities?

If so, then we can assign a vector space to the endpoints on an interval, and to the interval itself we can assign the power series $U(q_1, q_2)(t)$. We don't know if it converges or anything like that.

16 PT 10-18

Homework question: what is the "Hilbert space"? You have a classical field theory $\Phi(\Sigma^d) \xrightarrow{A} \mathbb{R}$, with restriction maps $\Phi(\Sigma) \to \Phi(\partial_{in/out}\Sigma)$. A quantum field theory would be a functor Q from the bordism category to hilbert spaces. You take $Q(Y^{d-1}) =$"$L^2(\Phi(Y))$".

I'm going to start over again with the Riemannian category Riem_d of Riemannian d-manifolds.

Definition 16.1. The objects are Riemannian d-manifolds (M^d, g) without boundary with finitely many connected components M_i and finitely many ends[1] such that $\widehat{M} := \sqcup \widehat{M_i}$, the metric completion of M (completing the connected components), is compact. And together with the decomposition $\delta M := \widehat{M} \smallsetminus M = \delta_{perm} M \sqcup \delta_{germ} M$ (perm for permanent) so that $\delta_p M$ is closed. Note that the boundary of $\widehat{M}$ will be contained in δM. Note also that $\widehat{M}$ need not be a manifold (for example, take M to be the open cone on a torus; then $\widehat{M}$ adds in the point of the cone, which doesn't have a neighborhood that looks like $\mathbb{R}^n$). The morphisms are $\mathsf{Riem}_d(M_1, M_2) = Isom(M_1, M_2)$, isometric embeddings (these don't have to preserve the decomposition of δM_1). ⋄

Remark 16.2. Such an embedding induces a map $\widehat{M_1} \to \widehat{M_2}$, but this map doesn't have to have any nice properties (like injectivity). ⋄

Definition 16.3. The objects of RB_d are objects Y of Riem_d such that $Y \cup \partial_p Y$ is a (topological) manifold with boundary $\delta_p Y$ (note that $\delta_p Y$ is a closed $(d-1)$-manifold, topologically), modulo a germ equivalence relation generated by $\sim$ saying $Y_1 \sim Y_2$ if $Y_1 \subseteq Y_2$ (open isometric inclusion) such that $\delta_p Y_1 = \delta_p Y_2$. We denote the equaivalence class by $[Y]$.. $\mathsf{RB}_d([Y_1], [Y_2]) = \{(\Sigma, [i_1], [i_2]) | \Sigma \in \mathsf{Riem}_d, i_k \in \mathsf{Riem}_d(Y_k, \Sigma)^2$ such that i_2 induces homeomorphism $\delta_p Y_2 \cong \delta_p \Sigma$ and i_1 induces a homeomorphism $\hat{i}_1 \colon \delta_g Y_1 \to \delta_g \Sigma$ for some representative $Y_1\}$/germ equivalence of $(Y_1 \overset{i_1}{\hookrightarrow} \Sigma)$. You also mod out by isometries rel boundary to get a category (actually, I want to make a bicategory where the 2-morphisms are isometries). ⋄

Chris will lecture next Wednesday at 2 on bicategories.

Maybe next time I'll give you an equivalent definition without germs.

[1] The *ends* of a space is $Ends(X) = \varprojlim_{K \subseteq X} \pi_0(X \smallsetminus K)$ where the limit is over compact subsets of X. For example, if you have $\mathbb{R}^2$ minus some points, each deleted point is an end (and one end "at infinity").

[2] For some choice of representative Y_k in $[Y_k]$.

23 NR 10-19

$U(q_1,q_2|t) = e^{itH/h}(q_1,q_2) = \left(\frac{1}{2\pi i h}\right)^{n/2}\left|\frac{\partial^2 S}{\partial q_1 \partial q_2}\right|^{1/2}\exp(\frac{i}{h}S(q_1,q_2,t))(1+O(h))$

Assuming a single trajectory connecting two points.

$$\psi(q,t) = \int U(q,q'|t)\phi(q')e^{i\frac{f(q')}{h}}dq'$$
$$= \phi(q_0)\left|\frac{\partial\gamma(t)}{\partial q_0}\right|^{-1/2}\exp(iS(q,q_0,t)/h)(1+O(h))$$

Where γ is a path with $\gamma(0)=q_0$ and $\gamma(t)=q$. The Legendre transform of $\dot\gamma(t)$ is $df(q)$.

Proof of formula. Assume $T^*\mathbb{R}$. We have

$$ih\frac{\partial\psi}{\partial t} = -\frac{h}{2m}\frac{\partial^2\psi}{\partial q^2} + V(q)\psi$$

Try solutions of the form $\psi(q,t) = e^{iS(q,t)/h}\psi_0(q,t)(1+O(h))$ where ψ_0 is a power series in h. Then the equation becomes

$$-S_t\psi_0 + ih\frac{\partial\psi_0}{\partial t} = -\frac{h^2}{2m}\frac{\partial^2\psi_0}{\partial q^2} - i\frac{h}{m}\frac{\partial\psi_0}{\partial q}\frac{\partial S}{\partial q} + \frac{1}{2m}(\frac{\partial S}{\partial q})^2\psi_0 - i\frac{h}{2m}\frac{\partial^2 S}{\partial q^2}\psi_0 + V(q)\psi_0.$$

Looking at the terms

h^0: $-\frac{\partial S}{\partial t} = \frac{1}{2m}(\frac{\partial S}{\partial q})^2 + V(q)$. This is the Hamilton-Jacobi equation $S = \mathcal{A}_f[\gamma_{cl}]$, where $\mathcal{A}_f[\gamma_{cl}] = \int_0^t(\frac{1}{2m}\dot\gamma_{cl}^2 + V(\gamma))d\tau + f(q_0)$.

h^1: $i\frac{\partial\psi_0}{\partial t} = -\frac{i}{m}\frac{\partial\psi_0}{\partial q}\frac{\partial S}{\partial q} - \frac{i}{2m}\frac{\partial^2 S}{\partial q^2}\psi_0$, whic we can write as

$$(\frac{\partial}{\partial t} + \frac{1}{m}\frac{\partial S}{\partial q}\frac{\partial}{\partial q})\log\psi_0 = -\frac{1}{2m}\frac{\partial^2 S}{\partial q^2}$$

From Hamilton-Jacobi, $\frac{\partial\mathcal{A}[\gamma_{cl}]}{\partial q} = p = m\frac{d\gamma(t)}{dt}$ and $\frac{\partial\mathcal{A}[\gamma_{cl}]}{\partial q_0} = p_0$. This is just because $\mathcal{L}(\xi,q) = \frac{\xi^2}{2m} - V(q)$ and $p = m\xi$. This means that this formula is

$$(\frac{\partial}{\partial t} + \frac{1}{m}\frac{\partial\gamma(t)}{\partial t}\frac{\partial}{\partial q})\log\gamma_0(q,t) = -\frac{1}{2}\frac{d}{dt}\log\frac{\partial q}{\partial q_0}$$

because the right hand side is

$$\begin{aligned}\frac{\partial^2 S}{\partial q^2} &= \frac{\partial}{\partial q}\frac{\partial\mathcal{A}[\gamma_{cl}]}{\partial q} = m\frac{\partial}{\partial q}\frac{d\gamma(t)}{dt}\\ &= m\frac{\partial q_0}{\partial q}\frac{\partial}{\partial q_0}(\frac{d\gamma(t)}{dt})\\ &= m\frac{\partial q_0}{\partial q}\frac{d}{dt}(\frac{\partial\gamma(t,q_0)}{\partial q_0})\end{aligned}$$
$$= m(\frac{\partial q}{\partial q_0})^{-1}\frac{d}{dt}(\frac{\partial q}{\partial q_0}) = \frac{d}{dt}\log\frac{\partial q}{\partial q_0}$$

$$(\frac{\partial}{\partial t} + \frac{1}{m}\frac{\partial\gamma(t)}{\partial t}\frac{\partial}{\partial q})\log\gamma_0(q,t)\Big|_{q=\gamma(t)} = \frac{d}{dt}(\log\psi_0(\gamma(t),t))\underbrace{\frac{\partial q}{\partial q_0}\Big|_{t=0}}_{=1}$$

so

$$\frac{d}{dt}\log\psi_0(\gamma(t),t) = -\frac{1}{2}\log\frac{\partial q}{\partial q_0}$$
$$\psi_0(\gamma(t),t) = \phi(q_0)\Big(\frac{\partial q}{\partial q_0}\Big)^{-1/2}$$

□

From the Schrödinger picture, we have $U(q_1,q_2|t) = C\cdot e^{i\frac{S(q_1,q_2,t)}{h}}\left|\frac{\partial^2 S}{\partial q^2}\right|^{1/2}(1+O(h))$. From the path integral, we have

$$U(q_1,q_2|t) = C\cdot e^{i\frac{S(q_1,q_2,t)}{h}}|\det K|^{-1/2}(1+O(h))$$

Where $K = -\frac{1}{m}(\frac{d}{ds})^2 + V^{(2)}(\gamma(s))$ for $0\le s\le t$ on $L^2[0,t]$ with $x(0) = x(t) = 0$.

Computing these coefficients is painful (and grows more painful with higher order terms). Using the path integral, we have

$$\begin{aligned}
U(q_1,q_2|t) &= \int_{\gamma(0)=q_1,\gamma(t)=q_2} \exp(\frac{i}{h}\mathcal{A}[\gamma])\,\mathcal{D}\gamma \\
&= \int_{x(0)=x(t)=0} \exp\Big(\frac{i}{h}\mathcal{A}[\gamma_c+x]\Big)\,\mathcal{D}x \\
&= e^{i\mathcal{A}[\gamma_c]/h}\int_{y(0)=y(t)=0} \exp(i(y,Ky)+\sum_{n\geq 0}\frac{h^{n/2-1}}{n!}(V^{(n)},y^n))\,\mathcal{D}y \\
&= Ce^{i\mathcal{A}[\gamma_c]/h}|\det K|^{-1/2}e^{i\pi\nu(K)/2}\cdot\sum_{n\geq 0}h^n c_n[\gamma_x]
\end{aligned}$$

Say $\gamma = \gamma_c + x$, then use $\mathcal{A}[\gamma_c+x] = \mathcal{A}[\gamma_c] + \int_0^t(\frac{1}{2m}(\dot{x}(t))^2 + V''(\gamma_c(\tau))x^2(\tau))\,d\tau + \sum_{n\geq 3}\frac{1}{n!}\underbrace{\int_0^t V^{(n)}(\gamma_c(\tau))x^n(\tau)\,d\tau}_{=:(V^{(n)},y^n)}$, and $x = \sqrt{h}y$.

$\nu(K)$ is the index of the operator (difference of positive and negative eigenvalues)

Look at

$$\int e^{i(x,Bx)+\sum_{n\geq 3}\frac{1}{n!}(V^{(n)},x)h^{n/2-1}}\,dx = \sum_{n_3\geq 0,n_4\geq 0\cdots}\frac{h^{(3n_3+4n_4+\cdots)/2-n_3-n_4-\cdots}}{n_3!(3!)^{n_3}n_4!(4!)^{n_4}\cdots}\int_{\mathbb{R}^n} e^{i(x,Bx)}(V^{(3)},x^3)^{n_3}(V^{(4)},x^4)^{n_4}\cdots$$

Assume B has some positive imaginary part.

$$\begin{aligned}
\int e^{i(x,Bx)}x_{i_1}\cdots x_{i_n}\,d^nx &= \frac{1}{n!}\frac{\partial}{\partial y_{i_1}}\cdots\frac{\partial}{\partial y_{i_n}}\int e^{i(x,Bx)+(y,x)}d^nx\Big|_{y=0} \\
&= C\cdot\frac{1}{n!}\frac{\partial}{\partial y_{i_1}}\cdots\frac{\partial}{\partial y_{i_n}}(e^{-i/4\cdot(y,B^{-1}y)})\Big|_{y=0}
\end{aligned}$$

You can write this as the sum over all perfect matchings on $(i_1,\dots,i_n)$ of the product of the $(-\frac{\sqrt{-1}}{4}B^{-1})_{ij} =: G_{ij}$ (where i and j are paired).

Example 23.1. $n = 4$, then this integral will be equal to $G_{12}G_{34} + G_{13}G_{24} + G_{14}G_{23}$. ◇

The power series can be written as

$$= \sum_{n_i\geq 0,i\geq 3}\frac{h^{\#}}{\cdots}\sum_{\text{perfect matchings}}\text{perfect matchings with } V^{(3)},\ V^{(4)}, \text{etc attached to vertices}$$

This is the sum over all graphs Γ with valence at least 3 at each vertex of $\frac{1}{|\operatorname{Aut}(\Gamma)|}F(\Gamma)$, where $F(\Gamma)$ assigns a $V_{\cdots}^{(k)}$ to each k-valent vertex and G_{ij} to each edge between i and j.

As far as I know, nobody has bothered to prove that $\int U(q_1,q_2|t)U(q_2,q_3|s)\,dq_2 = U(q_1,q_2|s+t)$.

24 NR 10-22

Last time: I explained that the amplitude in quantum mechanics can be considered as a sum of Feynman diagrams. As far as I know, it is open to verify that the composition law is satisfied. Recall that

$$\int_{\mathbb{R}^n} \exp\big(-\frac{1}{2}(x,Kx)+V(x)\big)d^n x = \sum_\Gamma \frac{F(\Gamma)}{|\operatorname{Aut}\Gamma|}$$

Where $F(\Gamma)$ is computed by assigning elements of your potential, $V_{i_1,i_2,\dots}$, and to each vertex and to each edge assigning $(K^{-1})^{i_1 j_1}$, and then multiplying everything together. Then

$$U(q_!,q_2|t) = \int_{\gamma(0)=q_1,\gamma(t)=q_2} \exp\big(\frac{i}{h}S[\gamma]\big)$$

which we expand near a classical path γ_c, with $\gamma=\gamma_c+x$. We get $C\exp\big(i\frac{\mathcal{A}[\gamma_c]}{h}\big)|\det'(K_{\gamma_c})|^{-1/2}\sum_\Gamma \frac{F(\Gamma)}{|\operatorname{Aut}(\Gamma)|}$ with

$$\mathcal{A}[\gamma_c+x] = \int_0^t \big(\frac{1}{2m}\dot\gamma(\tau)^2+V(\gamma(\tau))\big)d\tau = \mathcal{A}[\gamma_c]+\underbrace{\int_0^t \big(\frac{1}{2m}\dots x^2+V''(\gamma_c)x^2\big)}_{-\frac{1}{2}(x,Kx)}$$

From now on I'll assume $n=1$. We have that the regularized determinant is $\det'(K)=\prod_{n=1}^\infty \frac{\lambda_n}{\lambda_n^0}$.

What is $F(\Gamma)$? If we have n vertices, and $0\le s\le t$, then K^{-1} will have kernel $K^{-1}(s,t)$, so it will satisfy the differential equation

$$\big(-\frac{1}{m}\frac{d^2}{ds^2}+V''(\gamma_c(s))\big)K^{-1}(s,u)=\delta(s-u)$$

This $K^{-1}(s_i,s_j)$ is the weight we assign to an edge between vertices i and j in the diagram, and we assign $V'''(\gamma(s_i))$ to the vertices. As far as I know, nobody has bothered to prove that $\int_{\mathbb{R}^n} U(q_1,q_2|t)U(q_2,q_3|s)dq_2 = U(q_1,q_3|s+t)$. Phisicits didn't do this because it is obvious. Mathematicians didn't bother to do this because it doesn't solve any fundamental problem. So this problem, which is probably not very hard, is open.

This is more or less the end of the path integral in classical mechanics. Now we ask the following questions.

1. How is this related to the deformation quantization?
2. What to do if $\frac{\partial^2\mathcal{L}}{\partial\xi_i\partial\xi_j}$ is degenerate? If you have a classical system with a Lie group action so that the Lagrangian is invariant, then you'll run into this problem. In this case, you should reduce the dynamics to the orbits. This is known as Hamiltonian reduction. This is where all these gadgets like Fadeev-Popov ghosts, BRST quantization, and BV quantization appear.

Since I already gave an example of a classical field theory, let me give and example of a quantum field theory which quantizes this scalare Bose field.

Quantum field theory of a scalar Bose field (perturbative)

Classically

- M is Riemannian
- Fields are $\mathbb{R}$-valued functions on M, so the space of fields is $C^\infty(M)$.
- $\mathcal{A}[\phi]=\int_M\Big(\frac{1}{2}\big(d\phi(x),d\phi(x)\big)+V(\phi(x))\Big)d^n x$
- Critical points of $\mathcal{A}[\phi]$ with fixed $\phi|_{\partial M}=\varphi$ have $\delta\mathcal{A}[\phi_c]=0$. $\delta\phi$ is a vector field on the space of fields $C^\infty(M)$. Given a functional F on fields, $\delta F[\phi]:=\frac{d}{ds}F[\phi+s\delta\phi]|_{s=0}$. So

  $$\delta\mathcal{A}[\phi]=\int_M\big(-\Delta\phi(x)+V'(\phi(x))\big)\delta\phi(x)\,d^n x+\int_{\partial M}\delta\phi(x)\big(d\phi(x),d^n x\big)$$

  This second term will be zero anyway because we assume the fields are φ at the boundary. This is where you run into the problem of renormalization. So the Euler-Lagrange equations are

  $$-\Delta\phi(x)+V'(\phi(x))=0$$

  with the boundary condition $\phi|_{\partial M}=\varphi$. For good potentials V, this problem has unique solutions. You can kind of see this from $\frac{1}{2}(d\phi,d\phi)+V(\phi)$; if V is good, there is a unique minimum.

We can try to define the amplitude U_M, a functional on the space of possible values of boundary values φ. In a quantum field theory, this would be exactly $\mathcal{H}(\partial M)$. Since this space is very bad, I'll try to define what I can. So we try

$$U_M(\varphi) = \int_{\phi|_{\partial M}=\varphi} \exp\big(\frac{i}{h}\mathcal{A}[\phi]\big)\mathcal{D}\phi$$

How can we define this in a meaningful way? I know how to make sense of formal oscillating integrals like I did before: $\int e^{\frac{i}{h}S(x_c+y)}dy$ using Feynman diagrams. To deal with that, we wrote $S(x_c + y) = S_(x_c) + \frac{1}{2}(y, Ky)) + \sum_n V_n(y)$ ... then everything turned out to $e^{i\frac{S[x_c]}{h}}\det(d^2S(x_c)^{-1/2}\sum_\Gamma \frac{F(\Gamma)}{|\operatorname{Aut}\Gamma|}$. In this case, we will just define this U_M as a sum of diagrams. In the previous case, I could actually "do the diagram integrals", but now we're over an infinite-dimensional space.

In quantum mechanics, we can define $U_M(\varphi)$ as

1. the kernel $(e^{i\frac{t}{h}H})(x, y)$ (this is the honest definition). We can derive this as the $h \to 0$ limit of the Schrödinger equation.

2. A wild project:

$$\begin{aligned} U(q_1, q_2|t) &= \lim_{h\to 0}\int_{x(0)=x(t)=0} \exp\big(i\frac{S[x_c+x]}{h}\big)\mathcal{D}x \\ &\overset{\text{def}}{=} \exp(i\frac{S[x_c]}{h}\,{\det}'(K_{x_c})^{-1/2}\sum_\Gamma \frac{F(\Gamma)}{|\operatorname{Aut}\Gamma|} \end{aligned}$$

Problem 1: Prove that 2 gives the same notion as 1 (as far as I know, it's open).
Problem 2: $U_s^{p,int} * U_t^{p,int} = U_{s+t}^{p,int}$ [[★★★ I don't know what those superscripts are]].

The idea is to approximate the infinite-dimensional integral as a finite-dimensional integral and hope that there is a limit of these expressions. But this is very hard. I want to go through known facts as much as possible, but this area is like a mine field; you often step on something unproven. We'll use the perturbative approach as much as we can. We'll see how it works in this very simple example. Then we'll do this in Chern-Simons theory. The perturbative approach will give you knot invariants and other good things, but the limitations of this perturbative approach will become clear, so we'll defined things as these power series.

In quantum mechanics we're in very good shape; we have an honest definition. Anywhere else, we only have these guesses. The goal is to organize these guesses as much as we can.

So I define

$$U_M(\varphi) \overset{\text{def}}{=} \exp\Big(i\frac{\mathcal{A}[\phi]}{h}\Big)\det(K_{\phi_c})^{-1/2}\sum_\Gamma \frac{F(\Gamma)}{|\operatorname{Aut}(\Gamma)|}$$

$$\int \exp\Big(i\frac{\mathcal{A}[\phi_c+x]}{h}\Big) = \mathcal{A}[\gamma_c] + \frac{1}{2}(x, K_{\phi_c}x) + \sum_n V^{(n)}(\phi_c)x^n.$$

You can already see a problem with this definition. This $K(x, y)$ is singular at $x = y$ (ultraviolet divergences). This is what Richards course is about. I will make a few comments about this next time. Then we will completely ignore all divergence problems in this course.

17 PT 10-23

What Kolya called spacetime categories, I'll call bordism categories. Let B_d be the category whose objects are closed $(d-1)$-manifolds, with $\mathsf{B}_d(Y_{in}, Y_{out}) = \{Y_{out} \xhookrightarrow{j_{out}} \Sigma \xhookleftarrow{j_{in}} Y_{in} | \Sigma \text{ compact}, \partial\Sigma = \operatorname{im} j_{in} \sqcup \operatorname{im} j_{out}\}$/equivalence. Really, you shouldn't mod out by equivalence. $\mathsf{B}_d(Y_{in}, Y_{out})$ is really a category with objects $(Y_{out} \xhookrightarrow{j_{out}} \Sigma \xhookleftarrow{j_{in}} Y_{in})$ and morphisms are isomorphisms $\Sigma \to \Sigma'$ respecting the embeddings $j_{in/out}$. Thus, B_d should really be thought of as a bicategory. A bicategory $\mathcal{C}$ has a class of objects, hom *categories* $\mathcal{C}(x,y)$ for each pair of objects $x, y \in \mathcal{C}$, and composition functors $\mathcal{C}(y,z) \times \mathcal{C}(x,y) \to \mathcal{C}(x,z)$ (called *horizontal composition*). Furthermore, we reqire an associator natural transformation satisfying the pentagon identity. Note that the composition in $\mathcal{C}(x,y)$ is associative (because it is a category); this composition is called the *vertical composition*. Furthermore, there is some $\mathrm{id}_x \colon pt \to \mathcal{C}(x,x)$, which are *weak identities*.

Definition 17.1. A *2-category* is a category enriched over Cat. That is, a 2-category is where the composition functors are associative on the nose. ◇

Mac Lane proved that any bicategory is equivalent (as a bicategory) to a 2-category, but this rigidification is unnatural (there are more functors between bicategories than between 2-categories).

Let me tell you the horizontal composition $\mathsf{B}(Y_2, Y_3) \times \mathsf{B}(Y_1, Y_2) \to \mathsf{B}(Y_1, Y_3)$. If $Y_3 \xleftarrow{\Sigma'} Y_2 \xleftarrow{\Sigma} Y_1$, then the horizontal composition is the pushout $\Sigma \cup_{Y_2} \Sigma' \colon Y_1 \to Y_3$. Since union of sets is not associative, this composition is not associative. Note that this gluing doesn't work in the Riemannian category. Even in the smooth category, you need to use collars. This is ok in the topological category.

Claim. *This pushout defines a composition in* RB_d.

Let me first give an equivalent version of RB_d, with no germs. Recall that Riem_d has objects Riemannian d-manifolds without boundary Y^d with three tameness conditions:

- finitely many ends,
- finitely many components, [[★★★ follows from next condition]] and
- $\widehat{Y}$ is compact,

together with the decomposition $\widehat{Y} \smallsetminus Y = \delta_p Y \sqcup \delta_g Y$. Then $\mathsf{Riem}(Y_1, Y_2)$ consists of isometric embeddings $Y_1 \hookrightarrow Y_2$.

RB_d has objects $Y \in \mathsf{Riem}_d$ such that $Y \cup \delta_p Y$ is a topological d-manifold with boundary $\delta_p Y$. Last time we used a germy equivalence relation, but we aren't doing that this time. $\mathsf{RB}_d(Y_{in}, Y_{out}) = \{Y^1_{out} \xhookrightarrow{j_{out}} \Sigma \xhookleftarrow{j_{in}} Y^1_{in} | \Sigma \in \mathsf{Riem}_d, Y^1_{in/out}$ an open subset of $Y_{in/out}$ such that

- $\delta_p Y^1_{out} \cong \delta_p Y_{out}$,
- $\hat{j}_{out} \colon \delta_p Y^1_{out} \xrightarrow{\sim} \delta_p \Sigma$, $\hat{j}_{in} \colon \delta_g Y^1_{in} \xrightarrow{\sim} \delta_g \Sigma\}$.

Note that the germ boundary of Y^1_{in} doesn't have to match the germ boundary of Y_{in}. Given $Y \in \mathsf{RB}_d$, $\mathrm{id}_Y \in \mathsf{RB}_d(Y,Y)$ is $Y \hookrightarrow Y \hookleftarrow Y$. Given another morphism $Y^2_{out} \hookrightarrow \Sigma' \hookleftarrow Y^2_{in}$, a 2-morphism $\Sigma \to \Sigma'$ is an isometry $\Sigma \to \Sigma'$ such that *there exist* $Y^3_{in/out} \subseteq Y^1_{in/out}, Y^2_{in/out}$ so that $\Sigma \to \Sigma'$ respects the inclusions of $Y^3_{in/out}$. Composition is just the usual composition.

I still have to define the (horizontal) composition. Say we have $Y_3 \hookrightarrow \Sigma \hookleftarrow Y_2 \hookrightarrow \Sigma' \hookleftarrow Y_1$, then we can glue Σ and Σ' along Y^3_2 (intersecting the representatives Y^1_2 and Y^2_2).

Variations on the theme: You can't just choose Y^3_2, you have to take it ot be the intersection $Y^1_2 \cap Y^2_2$. Another point: for 2-morphisms, you have to use $\Sigma' \smallsetminus (Y^2_{in} \smallsetminus Y^3_{in})$ instead of Σ'.

There is a bit of a subtlety with the gluing. Consider two copies of $\mathbb{R}^1$ glued together along $(0,1)$. This is still locally $\mathbb{R}^1$, but the result is non-Hausdorff (the two 0's and the two 1's are indistinguishable). We can remove $(-\infty, 0)$ from the first $\mathbb{R}$ and $[1, \infty)$ from the other $\mathbb{R}$, and then the problem is gone.

More generally, if you have embeddings $U \xhookrightarrow{i_1} M_1$ and $U \xhookrightarrow{i_2} M_2$ such that $i_1 \times i_2 \colon U \to M_1 \times M_2$ is proper, then $M_1 \cup_U M_2$ is Hausdorff. [[★★★ HW. You probably only need that U, M_1, and M_2 are Hausdorff.]]

Now we have to check properness in the horizontal composition in RB_d, which I claim follows from our conditions (this is where you use that $\widehat{Y}$ is compact). [[★★★ HW: show that if $Y^1 \subseteq Y$ open, then $Y_1 \cong Y$ in RB_d]]

9 RB 10-23

We're going to continue trying to figure out what a gaussian Feynman measure. Goals:

1. *Define* what we mean by a Gaussina Feynman measure.

2. Define a group of renormalizations that acts simply transitively on Feynman measures.

The setup: 1 hermitian scalar field ϕ on (say) Minkowski space $M = \mathbb{R}^{1,d-1}$. We assume we're given a well-behaved propagator Δ that is a distribution on $M \times M$. Most of the constructions of Feynman measure really only depend on the choice of this Δ.

Recall that a Feynman measure is a map from Sym*(compactly supported actions) to $\mathbb{R}$. A compactly supported action is something that looks roughly like $\int f(x)\phi(x)^4\,dx$ where f is a smooth compactly supported function. The reason for making it compactly supported is to eliminate infra red divergences. The result is not translation invariant, but we'll see how to get that back. This $\phi(x)^4$ can be any polynomial in derivatives of ϕ. PT: you don't have a preferred action corresponding to the propagator? RB: no; it's a little misleading to call them actions, actually. It looks like you're looking at the space $C_0^\infty M \otimes$ (polys in ϕ and derivatives), but you have to mod out by the images of derivatives ∂_i to get actions. You can do

$$\int \underbrace{\int f\phi^4 dx \int f\phi^6 dx}_{\in \mathrm{Sym}^*(\text{actions})} e^{i\int m^2\phi^2+\phi^2\partial^2\phi dx}\mathcal{D}\phi \to \mathbb{R}$$

We require that a Feynman measure have some property. Formally, this integral can be written as a sum of Feynman diagrams (products of propagators, well-defined up to distribution on the diagonal).

Definition 9.1 (First)**.** A *Feynman measure* is a linear function Sym*(actions) $\to \mathbb{R}$ which can be obtained by summing over Feynman diagrams satisfying the conditions mentioned earlier (that for any edge between x and y, you can remove the edge at the cost of adding a factor of $\Delta(x,y)$). ◇

It would be nice to have a definition which doesn't explicitly talk about summing over Feynman diagrams.

Definition 9.2 (Second)**.** A *Feynman measure* is a linear function $\int\colon$ Sym*(actions) $\to \mathbb{R}$ with the following property. If $a = \int\phi^3\int\phi^4\int\phi^2$ and $b = \int\phi^4\int\phi^6$ are in Sym*(actions) *have disjoint supports* (there are hidden functions with compact support, which I'm too lazy to write), then $\int ab$ is

$$\sum \int a' \times \prod(\Delta) \times \int b'$$

summing over all ways to joing a ϕ in a to a ϕ in b, where b' is b without the ϕ's joined to a (and a' similar), and where $\prod(\Delta)$ is a product over propagators where something in a is joined to something in b. ◇

Whichever definition you like, the result is an infinite-dimensional space of Gaussian Feynman measures. Instead of trying to find a canonical element of this space (which is not possible in general), we try to find a group acting transitively on it.

Construction of group of ("finite") renormalizations

What should this group look like?

(1) First of all, it is a subgroup of GL(Sym*(actions)), invertible linear maps Sym*(actions) $\to$ Sym*(actions) (which don't have to preserve the grading). This acts (by definition) on Sym*(actions) and on its dual (which contains Feynman measures). By the way, the reason for using these huge spaces is that the formulas become more transparent; we'll see an example of this in a moment.

(2) We need renormalizations to action on Lagrangians. The reason is that we want a map from Lagrangians×measures to quantum field theories. In physics books, they pretend like there is only one measure, but this is false. Both Lagrangians and measures are acted on by renormalizations, and we want the map to QFTs to be invariant under the renormalization action. Lagrangians are more or less the same as actions, and we've already got an action on actions, but there is something very tricky. The renormalization action on Lagrangians is NONLINEAR. This is one of the reasons renormalization is so hard to understand. The reason

it's nonlinear: if you look inside a Feynman integral, it contains a factor of $e^{i\mathcal{L}}$, and it turns out that the action of the renormalization group on these things is linear. $e^{i\mathcal{L}}$ is more or less in $\mathrm{Sym}^*(\text{actions})$ (ignoring convergence problems). We have that the Lagrangians are mapped into $\mathrm{Sym}^*(\text{actions})$ by the exponential map, making Lagrangians a subspace of actions. This induces the action of the renormalization group on Lagrangians.

There is a problem, because there is no reason the action on $\mathrm{Sym}^*(\text{actions})$ should preserve things of the form $e^{i\mathcal{L}}$. Problem: suppose V (which will be the space of actions) is a module over a $\mathbb{Q}$-algebra, and suppose a group G acts on $\mathrm{Sym}^* V$. When does G preserve the subset of elements of the form $e^{\lambda v} \in \mathrm{Sym}^*(V)$ (where λ is infinitesimal, meaning nilpotent). Solution: pretend V is an abelian Lie algebra (you can ignore this if you like), so $\mathrm{Sym}^* V$ is the universal enveloping algebra of V (again you can ignore this if you like), which is, in particular, a Hopf algebra with *coproduct* given by $\Delta(v) = v\otimes 1 + 1\otimes v$ and extended to make it an algebra homomorphism. A coproduct is a map $\Delta\colon \mathrm{Sym}^* V \to \mathrm{Sym}^* V \otimes \mathrm{Sym}^* V$. The coproduct roughly tell you how something in $\mathrm{Sym}^* V$ acts on the tensor product of two things. Now we can define two special sorts of elements.

Definition 9.3. $g \in \mathrm{Sym}^* V$ is *primative* (Lie-algebra-like) if $\delta(g) = 1 \otimes g + g \otimes 1$. $g \in \mathrm{Sym}^* V$ is *group-like* if $\delta(g) = g \otimes g$. $\diamond$

For example, the primitive elements of $\mathrm{Sym}^* V$ are exactly the elements of V. If G is a group, then the group-ring $\mathbb{C}[G]$ is a Hopf algebra with $\delta(g) = g\otimes g$ for $g \in G$ and the group-like elements of $\mathbb{C}[G]$ can be identified with G.

If g is primative and nilpotent, then $\exp(g)$ is group-like. Conversely, if g is group-like and unipotent (1 plus something nilpotent), then $\log(g)$ is primative.

So elements of the form e^L (if we ignore convergence) are exactly the group-like elements of $\mathrm{Sym}^*(V)$. If we go back to thinking of V as the space of actions, then we want a renormalization group action which preserves the group-like elements of $\mathrm{Sym}^* V$. How can we ensure that an endomorphism preserves the set of group-like elements? An obvious way to do it is to require it to preserve the coproduct $\Delta\colon \mathrm{Sym}^* V \to \mathrm{Sym}^* V \otimes \mathrm{Sym}^* V$.

What on earth are the coproduct preserving maps? Well, a good way to think about this is to dualize everything. What are maps $\mathrm{Sym}^* V \to \mathrm{Sym}^* V$ preserving the product $\mathrm{Sym}^* V \otimes \mathrm{Sym}^* V \to \mathrm{Sym}^* V$. These are easy to identify; they are the same as linear maps $V \to \mathrm{Sym}^* V$, because any such map can be uniquely extended to an algebra homomorphism. Dualizing, you find that maps preserving coproduct can be identified with linear maps $\mathrm{Sym}^* V \to V$.

We've cut down our space a bit, but this space of linear maps is still too big, so we need some more conditions. You may be thinking, "why not require that the product is preserved as well?" Well, then it would be too small to act transitively on all measures.

(3) Renormalizations ALMOST preserve products: $g(ab) = g(a)g(b)$ if a and b have disjoint support.

(4) Renormalizations fix $1 \in$actions (not the same thing as $1 \in \mathrm{Sym}^*(\text{actions})$).

(5) Renormalizations commute with the group of sections of the vector bundle which I'll explain next week.

This more or less defines what a renormalization is. It is clear that renormalizations form a group. This group acts transitively on measures (I'll explain this next week).

25 NR 10-24

Today I wanted to continue talking about the Scalar Bose field. I want to focus on a problem which doesn't exist in any TQFT, but does exist in any realistic QFT: divergences due to ultraviolet divergences and renormalization. RB should be doing this in his class. In this case, fields ϕ are elements of $C^\infty(M)$. The action functional is $\mathcal{A}[\phi] = \int \big(\frac{1}{2}(d\phi, d\phi) + V(\phi)\big)\, dx$, where $V(\phi)$ is the self-interaction term. We assume M is Riemannian with boundary ∂M. Let ϕ_{cl} be the solution to the Euler-Lagrange equations, assuming we fix $\phi|_{\partial M} = \varphi$, so $\delta\phi|_{\partial M} = 0$. Then we have

$$\delta\mathcal{A}[\phi] = \int_M \big(-\Delta\phi(x) + V'(\phi(x))\big)\delta\phi(x)\, dx$$

For this to vanish, we must have $-\Delta\phi_c + V'(\phi_c) = 0$. We have $U_M(\varphi)$, the analogue of $U(q_1, q_2|t)$ (where $q_1 = \gamma(0)$, $q_2 = \gamma(t)$). We define

$$U_M(\varphi) = \int_{\phi|_{\partial M} = \varphi} \exp\big(\frac{i}{h}\mathcal{A}[\phi]\big)\mathcal{D}\phi$$

We want to have some analogue of the composition law $\int_M U(q_1, q_2|t)U(q_2, q_3|s)dq_2 = U(q_1, q_3|s+t)$, so we require U_M to satisfy the following axiom. If M_1 and M_2 are manifolds, with part of their boundaries identified, say $\partial_1 M_1 \xrightarrow[\sim]{f} \partial_1 M_2$, then

$$\int_{\varphi|_{\partial_1 M_1 = \partial_1 M_2}} U_{M_1}(\varphi)U_{M_2}(\varphi)\mathcal{D}\varphi = U_{M_1 \#_f M_2}(\varphi) \qquad (*)$$

We can only make sense of $U_M(\varphi)$ perturbatively as

$$c\sum_{\phi_c} \exp\big(\frac{i}{h}\mathcal{A}[\phi_c]\big)(\det{}' K_{\phi_c})^{-1/2} \sum_\Gamma \frac{F(\Gamma)}{|\operatorname{Aut}(\Gamma)|}$$

The problem: Does $U_M(\varphi)$ satisfy $(*)$?

Remark 25.1. Though the integral itself doesn't make sense, we can try to make sense of it perturbatively, so $(*)$ should be regarded as an identity involving series of Feynman diagrams. ⋄

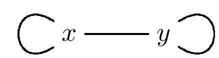
Let me say what are the rules of these Feynman diagrams. The intuitive idea is that we should consider $\mathcal{A}$ near ϕ_c, so

$$\mathcal{A}[\phi_c+\psi] = \mathcal{A}[\phi_c] + \frac{1}{2}\int_M \psi\big(-\Delta + V''(\phi_c(x))\big)\psi\, dx + \sum_{n\geq 2}\frac{1}{n!}V^{(n)}(\phi_c(x))\psi(x)^n\, dx$$

When you have a vertex with valence n, you assign $x \in M$ and assign $V^{(n)}(\phi_c(x))$ to the vertex. To an edge between x and y, you assign $(K_{\phi_c})^{-1}(x, y) =: G(x, y)$.

Let's look at all diagrams of order 1.

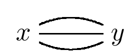
Example 25.2. To

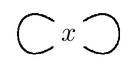


we assign

$$\iint_{M\times M} G(x,x)V^{(3)}(\phi_c(x))G(x,y)V^{(3)}(\phi_c(y))G(y,y)\, dxdy$$

Also, we have





[[★★★ add formulas]] ⋄

$K_{\phi_c} = -\Delta + \overbrace{V''(\phi_c(x))}^{U}$. The problem with K is that eigenvalues λ_i blow up as $n \to \infty$, so $\det K_\phi$ has no chance to exist. But we also have the operator $-\Delta$, whose eigenvalues λ_i^0 also diverge if you order them like $\lambda_1^0 \leq \cdots$. But λ_n/λ_n^0 converges to 1, so we have some hope of making sense of $\det(\frac{-\Delta+U}{-\Delta}) = \det(I - U\Delta^{-1})$. It turns out that this determinant exists.

Let's see how these expressions behave. For this, we have to understand how $K_{\phi_c}^{-1}(x,y) = G(x,y)$ behave as x goes to y. "x goes to y" means that in the Fourier transform, we should take p to ∞. If x and y are close, then the distance to the boundary is much larger than the distance between them. Let's say M is m-dimensional. I claim that as x goes to y, the asymptotics of this Greens function is $G(x,y) \to G_{\mathbb{R}^m}(x-y) = \int_{\mathbb{R}^m}\frac{\exp(ip(x,y))}{p^2}d^m p = |x-y|^{-m+2}\int_{\mathbb{R}^m}\frac{\exp(iq(\frac{x-y}{|x-y|}))}{q^2}$. In the short distance asymptotics, we're picking up large eigenvalues of the Laplacian Δ, so we can pretend like $U = 0$. We can ignore the integral $\int_{\mathbb{R}^m} \frac{\exp(iq(\frac{x-y}{|x-y|})}{q^2}$ [[★★★ for some reason]], so $G_M^U(x,y) \to \frac{c}{|x-y|^{m-2}} + \cdots$.

If $m = 1$ (quantum mechanics), $G(x,y) \sim |x-y|$, so no divergences.

If $m = 2$, it turns out that $G(x,y) \sim \log|x-y| + \cdots$, so there divergences, but they aren't that bad.

If $m \geq 3$, then $G(x,y) \sim \frac{1}{|x-y|^{m-2}}$, with $m - 2 > 0$, so you have to do something to make the formulas corresponding to Feynman diagrams make sense; all the integrals have divergences at short distances. These are ultraviolet divergences.

So we've failed to define the integral $U_M(\varphi)$ using Feynman diagrams. In any QFT in dimension at least 2, there will be singularities. If we believe that something like this should exist, we can try to regularize.

How should we fix the problem? You could integrate over some neighborhood outside the diagonal in $M \times M$. Another idea is as follows. Let's do something with the action to make these Greens functions non-singular at the diagonal. In $\mathbb{R}^m$, instead of considering $-\Delta$, take $-\Delta + \sum_{i\geq 4} \varepsilon_i \Delta^i$, where we've added some higher order differential operators. If you do the Fourier transform, this becomes $p^2 + \sum_{i=1}^k \varepsilon_i(p^2)$, so as $p \to \infty$, this behaves like $\varepsilon_k p^{2k}$. Then $G_M^{U,\varepsilon}(x,y) \sim \frac{c(\varepsilon)}{|x-y|^{m-2k}}$. So we can kill off the singularities, but at a very heavy price; we'll have to let ε_i go to zero eventually or something. The idea of changing the Laplacian like this is known as Pauli-Williams regularization. Physicists used this for more than 50 years. The idea is: regularize $\mathcal{A}[\phi]$, and then $U_{M,\varepsilon}^{pert}(\varphi)$ is defined. The big question is then, "what happens when $\varepsilon \to 0$?" We know the answer: each of the Feynman diagrams (which now depend on ε) diverges as $\varepsilon \to 0$. The hope is that as $\varepsilon \to 0$, we can choose a modification $V_\varepsilon(\phi)$ of our potential $V(\phi)$ so that the coefficients in $F(\Gamma)$ will be finite. Richard is doing this more carefully.

Q: how does this address the problem of $G(x,x)$? NR: just as in the case of mechanics, we had a power in the denominator of $m - 2k$, so if k is large, this will be negative, so you get a positive power of $|x-y|$. Q: if I regularize the action, I'm really changing the entire problem; the determinant changes as well. Will we have to reregularize that part too? NR: with the determinant we should be more careful. $\det' K_\phi^\varepsilon = \det(K_\phi^\varepsilon / K_0^\varepsilon)$.

So now we have to adjust V_ε to compensate the divergences. It is not at all clear that this is ever possible, but it turns out it is.

Theorem 25.3 (BZZ)**.** *If $m = 2$, then $V(\phi)$ can be any polynomial and the renormalization procedure exists. This is a rather involved statement already. If $m = 3$, then $V(\phi)$ should be a polynomial of degree 6, or it won't be renormalizable. If $m = 4$, then V must be a polynomial of degree 4. A renormalizable theory doesn't for $m > 4$.*

There is another complicated question: there are many many renormalization procedures; Theo proposed the momentum cutoff regularization. And we can produce many more. Will the answer depend on the regularization procedure or not? The statement at the moment is that it depends up to a certain finite renormalization. There is the (infinite-dimensional) group of renormalization schemes. There is a series of papers by Connes and Krimer. They invented some kind of cocommutative Hopf algebra of diagrams, a candidate for the universal enveloping algebra of some Lie algebra. It simplified lots of computations in the proof of this theorem. Observables should be invariant with respect to the action of this group. I don't understand if this has really been resolved, and I haven't gotten a straight answer to it. It is probably not an issue for phisicits.

These lectures will sometimes be quite pessimistic: I'm telling you that people know very little about quantum field theory. Though if you look in textbooks, you'll see more optimistic statements.

Today we did an example of a QFT with non-degenerate Lagrangian (that is, $(d\phi, d\phi)$ was non-degenerate). Next time we'll start talking about systems with degenerate Lagrangian. One question is to make the Hamiltonian formulation, and the other question is how to quantize using the path integral.

18 PT 10-25

Definition 18.1 (Reminder Chris' talk)**.**

(a) A *(strict) 2-category* is a category enriched over $(\mathsf{Cat}, \times)$.

(b) A *bicategory* is a "weak" version of a 2-category where one only has canonical "assiciators" and "identitators". $\diamond$

This means that in a 2-category $\mathcal{C}$, we have

- objects $x, y \in obj(\mathcal{C})$,
- morphism categories $\mathcal{C}(x, y)$,
- composition functors $\mathcal{C}(y, z) \times \mathcal{C}(x, y) \to \mathcal{C}(x, z)$, and
- identity functors $pt \to \mathcal{C}(x, x)$ for each $x \in \mathcal{C}$

such that the identity is an identity and the composition is associative. In a bicategory, instead of imposing these conditions on the identity and the composition, we add the extra data of the associators and identitators. The associator must satisfy the pentagon identity, which (by Mac Lane's theorem) tells you that any two ways you associate are canonically isomorphic. There is a similar thing for the identitators; they satisfy the triangle identity (which also tells you that anything you want to do is canonical).

Example 18.2. Alg is a bicategory, where the objects, 1-morphisms, and 2-morphisms are algebras, bimodules and intertwiners, respectively. Tensor product gives you the horizontal composition, and the universal property of tensor product gives you the associators and identitators.

There are also more subtle versions of this. For example, you could take Frechét algebras and Frechét bimodules. $\diamond$

A QFT will be a bifunctor from a geometric bicategory to this algebraic bicategory.

Example 18.3. B_d, whose objects, 1-morphisms, and 2-morphisms are closed (topological) $(d-1)$-manifolds, compact d-manifolds (with a decomposition of the boundary into inclusions of incoming and outgoing $(d-1)$-manifolds), and homeomorphisms of compact d-manifolds respecting the inclusions.

RB_d has a forgetful bifunctor F to B_d. The objects, 1-morphisms, and 2-morphisms of RB_d are germs of Riemannian structures around $(d-1)$-manifolds Y, agreeable bordisms Σ between them (with some germiness at the in-boundary), and isometries. $F(Y) = \delta_{perm} Y$, $F(\Sigma) = \Sigma_{core}$ (where you've removed the germiness) and F of an isometry is the isometry restricted to the cores.

Is this F a 2-functor? Well, Chris hasn't discussed 2-functors yet, so we won't get into it too much. It is weaker than you might think it is. A 2-functor doesn't have to respect (horizontal) composition on the nose, but only up to natural isomorphism.

Theo: This functor doesn't send identities anywhere. PT: If you add homeomorphisms to the bordisms in B_d, then it becomes ok. You have to make sure that you still have a bicategory; you have to know how to compose these homeomorphisms with bordisms. You do this by composing the embedding with the homeomorphism (or its inverse, depending on which side you're composing). It turns out that this bigger B_d is equivalent. Chris S-P: no, now you have some more automorphisms ... [[★★★ some stuff about the cylinder not playing like it should]]. $\diamond$

I want to give you some examples of 1-morphisms in RB_d.

Assume we have a geometric bi-collar Σ of $Y = \delta_p \Sigma$ [[★★★ Σ is a cylinder, with the whole boundary declared to be germ boundary]]. Let's cut it in half (along Y) and call the (open) two sides Y_R and Y_L. We have that $F(Y_R) = F(Y_L) = Y$. I claim that we can make a nice 1-morphism out of this. You can think of it as living in $\mathsf{RB}_d(Y_L \sqcup Y_R, \varnothing)$ by letting $j_{in} \colon Y_L \sqcup Y_R \hookrightarrow \Sigma$ be the inclusion.

By the way, I'm going to give up the right-to-left notation because it gets us confused; you just have to label the boundary components as in or out. In RB_d, we'll draw collars on the outside for the in boundary and collars on the inside for the out boundary.

Can you think of this as an element of $\mathsf{RB}_d(Y_L, Y_R)$? Can you think of this as an element of $\mathsf{RB}_d(\varnothing Y_L \sqcup Y_R)$? No. $\delta_p \Sigma$ is empty, so the target must be the empty set.

Analogy: in Frechét spaces, it is much easier to get bilinear maps $V \otimes W \to \mathbb{C}$ than to get vectors in $V \otimes W$. For example, take $W = V'$; you get a canonical pairing, but you only get a canonical vector if V happens to be finite dimensional. This kind of corresponds to the fact that in B_d, you can flip bordisms around to get a “reversed morphism”.

Given a monoidal bifunctor $Q\colon \mathsf{RB}_d \to \mathsf{Frechét}$, we get two Frechét spaces $Q(Y_L) = V_L$ and $Q(Y_R) = V_R$, and a pairing $Q(\Sigma)\colon V_L \otimes V_R \to \mathbb{C}$. In TQFTs, this morphism doesn't exist, so it is sometimes added as an axiom that the two guys are dual.

[[break]]

A couple of remarks. If you apply F to this Σ, you're going to get nonsense: $F(\Sigma) = Y$, which is supposed to have an inclusion from $Y \sqcup Y$. This doesn't make sense, but I invitet you to make a bigger category where this works. Chris convinced me that the enlarged B_d is not equivalent to the smaller B_d because the cylinder is not the identity on the homeomorphisms. Another thing you can do is use the axiom of choice to choose slightly larger cores for all bordisms (so in our case, the core of Σ would be a cylinder on Y, not Y). Then you don't have to throw in homeomorphisms either.

The other thing I should announce is that Bruce was volunteered to talk about how path integrals connect the three different classes next week.

I think of a quantum field theory as a representation of the category RB_d. A representation of a group is exactly the same thing as a functor from G (thought of as a 1-object category) to the category of vector spaces.

About HW2: You can enrich B_d by adding a space X (which you think of as the target of some classical field theory) to get $\mathsf{B}_d(X)$, where the objects are continuous maps $f\colon Y^{d-1} \to X$, bordisms have maps to X, and homeomorphisms must respect the maps to X. If you do this for RB_d (you probably want to take smooth maps to X). Another thing you could do is equip Y with a bundle and a section (a crazy way to think of a map $Y \to X$ is to think of it as a section of the trivial bundle $X \times Y \to Y$, but there is no reason to take the trivial bundle). When I wrote $\Phi(\Sigma^d)$ in the homework, this is what I had in mind; Σ was equipped with a bundle $P \to \Sigma$, and the fields $\Phi(\Sigma)$ is the space of sections of this bundle. This part was precise, the imprecise part was that the Hilbert space associated to this Σ was supposed to be $L^2(\Phi(\Sigma))$.

Consider $d = 0$ (this is below mechanics, where $d = 1$. This is called instanton theory because there is no time). What are symmetric monoidal functors $\mathsf{B}_0(X) \to \mathsf{Vect}$. The objects are (-1)-manifolds mapping to X. There is only one (-1)-manifold, which is $\varnothing$.

If you have a functor between two monoidal categories $F\colon (\mathcal{C}, \otimes) \to (\mathcal{D}, \otimes)$, you can require a map $F(x \otimes_{\mathcal{C}} y) \to Fx \otimes_{\mathcal{D}} Fy$, and you usually require it to be an isomorphism (or quasi-isomorphism), but I'll require it to be an equality for this class.

So there is one object of $\mathsf{B}_0(X)$, which is $Q(\varnothing) = Q(\mathbb{1}) = \mathbb{C}$. $\mathsf{B}_0(\varnothing, \varnothing)$ is sets of circles, so $x \in \mathsf{B}_0(X)(\varnothing, \varnothing)$ is a map from a bunch of circles to X, and x get's mapped to some $Q(x) \in \mathbb{C}$.

We need to add requirements on the Q's such that

- Q is smooth. Recall that $H^n_{dR}(X) \overset{HW1}{=} \Omega^n_{cl}(X)/\text{concordence}$. $\Omega^n_{cl}(X) = $ 0-dimensional susy QFTs. $\pi TX = $ super points in X. $C^\infty(X)$, $C^\infty(\pi TX) = \Omega^*(X)$.
- susy Q's leading to functions on super points, which is $\Omega^*(X)$.
- understand closedness and degree.

26 NR 10-26

Last time, we took M Riemannian and $\mathcal{L} = \int_M \big(\frac{1}{2}(d\phi)^2 + V(\phi)\big)dx$. I talked about a regularization scheme where you first regularize the propagators $G(x,y) = (-\Delta + V''(\phi_c))^{-1}(x,y)$ which you assign to edges in the Feynman diagrams, where ϕ_c is a solution to the Euler-Lagrange equation with given boundary conditions. For $m \geq 2$, you get singularities, so the integrals which we assign to Feynman diagrams don't make sense. To take care of the singularities at $x = y$, you replace $-\Delta$ by $-\Delta + \sum_{i=2}^n \varepsilon_i \Delta^i$ and eventually send ε_i to 0. The first problem is that $F(\Gamma)$ are singular as $\varepsilon_i \to 0$.

The second part of the procedure is renormalization. $V_\varepsilon(\phi) = \sum_{i\geq 3} g_i(\varepsilon)\phi^i$, and adjust $g_i(\varepsilon)$ such that $F_\varepsilon(\Gamma) \to F_0(\Gamma)$. There are many questions, like "what if we regularize the propagators some other way? Will you get different results?" This is what Richard is doing and will be doing. The answer is that there is a group of renormalizations (not "the renormalization group" you find in physics literature). Different regularization schemes are related by the transitive action of the group of renormalizations.

You will always have problems with ultraviolet divergences when you do perturbation theory. In this case, fortunately, they arise in a controllable way.

The work by Kevin Costello; BV quantization, as I understand it (or don't understand it, as the case may be), the goal is to have "$d^2 = 0$" description of working with ultraviolet divergences. The other name for this is BRST, and secretly, it is the same as the Fadeev-Popov trick. All of these involve super analysis, so I'll do a complementary introduction to super geometry.

Grassman algebra

Recall some facts about $\mathbb{R}^{n|k}$ and $\mathbb{C}^{n|k}$. The Grassman algebra is the algebra $\langle c_i, \dots, c_n | c_i c_j + c_j c_i = 0\rangle$. We can consider odd derivations $\frac{\partial}{\partial c_i} c_{i_1}\cdots c_{i_n} = \begin{cases} 0 & i \notin \{i_1,\dots,i_n\} \\ (-1)^k c_{i_1}\cdots \hat{c}_{i_k}\cdots c_{i_n} & i = i_k \end{cases}$. This is the "left derivative" and you get the right derivative using the sign $(-1)^{n-k}$ instead.

Ingegral over $G_n = \bigwedge^\bullet \mathbb{C}^n$: Choose an orientation of $\mathbb{C}^n$, a basis in $\bigwedge^n \mathbb{C}^n$. Choose $c_1 \wedge \cdots \wedge c_n$ in $\bigwedge^n \mathbb{C}^n$. If you have $P \in G_n$, you can write it as $p^{top} c_1 \wedge \cdots \wedge c_n +$ lower terms. Then we define

$$\int_{C^{0|n}} P\, dc := p^{top}.$$

Now let's see if this is a useful definition. So far, we've only been using integration very crudely, we only care about integrating gaussians.

Example 26.1. $P = \exp\big(\frac{1}{2}(c,Bc)\big)$, where $(c,Bc) = \sum_{ij} c_i B_{ij} c_j$. Since the c_i anti-commute, we need $B_{ij} = -B_{ji}$, and we should assume n is even (otherwise, we'll never get something in the top degree by exponentiating an even degree function). If n is even, then

$$\int \exp\Big(\frac{1}{2}(c,Bc)\Big)\, dc = \frac{(1/2)^{n/2}}{(n/2)!}.$$

To see this, not that

$$(c,Bc)^{n/2} = \sum_{i_k, j_k, 1\leq k\leq n/2} B_{i_1 j_1}\cdots B_{i_{n/2} j_{n/2}} c_{i_1}\cdots c_{i_{n/2}} c_{j_1}\cdots c_{j_{n/2}}$$

and we get that

$$(c_{i_1}\cdots c_{i_{n/2}} c_{j_1}\cdots c_{j_{n/2}})^{top} = (-1)^{\sigma(i|j)} c_1 \wedge \cdots \wedge c_n$$

(incidently, $\sigma(i|j)$ is the number of perfect matchings on n elements)

$$= \frac{(1/2)^{n/2}}{(n/2)!} \sum_{\sigma(i|j)} B_{i_1 j_1}\cdots B_{i_{n/2} j_{n/2}} (-1)^{\sigma(i|j)}$$

Note that the sign doesn't change when you switch i_a with j_a because the signs come in pairs. Also, the sign doesn't change when you apply a given permutation to both $\{i\}$ and $\{j\}$, so

$$= \sum_{\substack{\sigma(i|j) \\ i_a < j_a \\ i_{a_1} < \cdots < i_{a_n}}} (-1)^{\sigma(i|j)} B_{i_1 j_1}\cdots B_{i_{n/2} j_{n/2}} = Pf(B)$$

which is the Pfaffian of B. This is the formula that every physicists know for $\int_{\mathbb{C}^{0|n}} \exp((c,Bc)/2)\,dc$. This is the combinatorial definition of the Pfaffian. More conceptually, you can define the Pfaffian as

$$\Big(\sum_{i<j} x_i \wedge x_j B_{ij}\Big)^{\wedge \frac{n}{2}} = Pf(B) x_1 \wedge \cdots \wedge x_n$$

over $\bigwedge^\bullet \mathbb{C}^n$. It depends on the basis, but only on the orientation of the basis (i.e. it depends on a choice of orientation of $\bigwedge^n V$). [[★★★ HW: Prove that $Pf(B)^2 = \det B$. All you have to do is take $\bigwedge^{2n}(\mathbb{C}^n \oplus \mathbb{C}^n)$, and compute something like this in two way, one of which is the determinant and one of which is the Pfaffian.]] Given a basis $c_1 \wedge \cdots \wedge c_n \in \bigwedge^n \mathbb{C}^N$, $x_i = \sum_{j=1}^n A_{ij} c_j$, $x_1 \wedge \cdots \wedge x_n = \det A c_1 \wedge \cdots c_n$. ◇

What if we only know determinants, but not Pfaffians. Let $c_1, \dots, c_n, \bar c_1, \dots, \bar c_n$ be a basis for $\mathbb{C}^{0|2n}$ (don't think complex conjugation, these are independent), then

$$\begin{aligned}\int_{\mathbb{C}^{0|2n}} \exp((\bar c, Ac))\,d\bar c\,dc &= \frac{1}{n!}\int (\bar c, Ac)^n\,d\bar c\,dc\\ &= \frac{1}{n!}\int \sum_{\{i,\},\{j\}} A_{i_1 j_1}\cdots A_{i_n j_n} \bar c_{i_1} c_{j_1} \cdots \bar c_{i_n} c_{j_n}\,d\bar c\,dc\\ &= \cdots\\ &= (-1)^{n(n-1)/2+n^2} \frac{1}{n!} \sum_{\sigma,\tau\in S_n} (-1)^{\sigma+\tau} \prod_i A_{\sigma(i)\tau(i)}\\ &= \cdots = \pm \det A\end{aligned}$$

On the other hand, we can write this as

$$\int_{\mathbb{C}^{0|2n}} \exp\Big(\frac{1}{2}(x, \left(\begin{smallmatrix} 0 & A \\ -A^t & 0\end{smallmatrix}\right) x)\Big) dx + \pm Pf\left(\begin{smallmatrix} 0 & A \\ -A^t & 0\end{smallmatrix}\right)$$

So we get $\det B = Pf(B)^2$ and $Pf\left(\begin{smallmatrix} 0 & A \\ -A^t & 0\end{smallmatrix}\right) = \det A$. This is what you need to know.

So let's derive the formula for the Berezinian using this voodoo. So far, I was integrating over $\mathbb{C}^{0|n}$, but what if I want to integrate over $\mathbb{C}^{n|m}$? Choose a hermitian bilinear form $(,)$ with even coordinates x (with complex conjugates $\bar x$) and odd coordinates c and $\bar c$ (not related by complex conjugation). Let $(\bar x, Bc) = \sum_{i,a} \bar x_i B_{ia} c^a$. Consider the following real integral over $\mathbb{R}^{2n}$:

$$\int_{\mathbb{C}^{n|2m}} \exp\big(-(\bar x, Ax) + (\bar x, Bc) + (B^* \bar c, x) + (\bar c, Dc)\big)\,d\bar c\,dc\,d\bar x\,dx$$

$$\begin{aligned}&+(\overbrace{\bar x - A^{-1}B^*\bar c}^{\bar y}, A(\overbrace{x - A^{-1}Bc}^{x})) + (B^*\bar c, A^{-1}Bc) + (\bar c, Dc)\\ &= \int \exp\big(-(\bar y, Ay) + (\bar c, (D + B^* A^{-1} B)c)\big)\\ &= \det A^{-1} \det(D + B^* A^{-1} B)\end{aligned}$$

$$\int \exp\Big((\bar x, \bar c)\left(\begin{smallmatrix} A & B \\ B^* & D\end{smallmatrix}\right)\left(\begin{smallmatrix} x \\ c\end{smallmatrix}\right)\Big)\,d\bar x\,dx\,d\bar c\,dc = Ber\left(\begin{smallmatrix} A & B \\ B^* & D\end{smallmatrix}\right)^{-1}$$

on $GL(n|m)$.

27 NR 10-29

Last time: $\int_{\mathbb{R}^{0|n}} P(c)\,dc := (P(c))^{top}$ is the definition of the integral over the Grassman algebra $G_n = \bigwedge^\bullet \mathbb{R}^n$. Note that we need to choose a basis for the top degree part (we choose basis $c_1 \wedge c_n$ in G_n, where c_i are the usual basis for $\mathbb{R}^n$). We used this to compute that

$$\int_{\mathbb{R}^{0|n}} \exp\Big(-\frac{1}{2}(c, Bc)\Big)\,dc = Pf(B).$$

I want to complexify, so I look at $G_{2n} = \langle a_s, b_s | 1 \le s \le n, \cdots\rangle = \bigwedge^\bullet(\mathbb{R}^n \oplus \mathbb{R}^n)$. Then we consider $\mathcal{G}_{2n,c} = G_{2n} \otimes_{\mathbb{R}} \mathbb{C} = \langle c_s, \bar{c}_s | c_s = a_s + ib_s, \bar{c}_s = a_s - ib_s\rangle = \bigwedge^\bullet \mathbb{C}^{2n}$. In this algebra, we have the following identity for a complex $n \times n$ matrix:

$$\int \exp((\bar{c}, Ac))\,d\bar{c}\,dc = \pm \det A$$

Last time I showed how the Berezinian comes up, but I started in the complexified case. Last time I showed that if $A^* = A > 0$, and if B and C are odd elements,

$$\begin{aligned}\int_{\mathbb{C}^{n|2m}} \exp\big(-(\bar{x}, Ax) - (\bar{x}, Bc) - (\bar{c}, Cx) - (\bar{c}, Dc)\big)\,d\bar{x}\,dx\,d\bar{c}\,dc \\ = \det(A)^{-1}\det(D - CA^{-1}B)\end{aligned}$$

This is an identity in $\bigwedge^\bullet\big(M_{n\times m}(\mathbb{C}) \oplus M_{m\times n}(\mathbb{C})\big)$. In general, this is an identity in the corresponding exterior algebra. If $P(c)$ is a polynomial in c with coefficients in $\bigwedge^\bullet V$, then this identity is in $\bigwedge^\bullet V$. In this case, this is the algebra $\mathcal{A}$ generated by odd elements C_{ia} and B_{dj}. The integrand is in $\mathcal{A} \otimes \langle c_s, \bar{c}_s | \cdots\rangle$.

So we get $\mathrm{Ber}\left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right) = \det A \cdot \det(D - CA^{-1}B)^{-1}$. Notice that the $\exp(\cdots)$ in the integrand can be written as $\exp\Big((\bar{x}, \bar{c})\left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right)\left(\begin{smallmatrix} x \\ c\end{smallmatrix}\right)\Big)$. It is clear that for unitary super matrix U, $U\left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right)U^{-1}$ is change of basis by $\left(\begin{smallmatrix} x \\ c\end{smallmatrix}\right) \mapsto U\left(\begin{smallmatrix} x \\ c\end{smallmatrix}\right)$. The "measure" is invariant under such transformations. This explains why the Berezinian is invariant under such transformations. PT: what do you mean by a unitary super matrix? Is it some super Lie group? NR: $U(n|m) \subseteq GL(n|m)$ is the compact real form. It should be the super matrices that preserve the hermitian inner product. It acts on $\mathbb{C}^{n|m}$, with hermitian product something like $\langle x, y\rangle = \sum_i \bar{x}_i y_i + \sum_a \bar{c}_a c_a$. Consider polynomial functions $Pol(\mathbb{C}^{n|m}) = \mathrm{Sym}(\mathbb{C}^n) \otimes \bigwedge(\mathbb{C}^m)$. The scalar product $\langle\cdot,\cdot\rangle$ is an element in $Pol(\mathbb{C}^{n|m} \oplus \mathbb{C}^{n|m}) = Pol(\mathbb{C}^{n|m})^{\otimes 2}$. The usual $\mathbb{C}$-bilinear scalar product is $(\cdot,\cdot) = \sum_i x_i \otimes x_i + \sum_a c_a \otimes c_a$. The hermitian product is $\langle\cdot,\cdot\rangle = \sum_i \bar{x}_i \otimes x_i + \sum_a \bar{c}_a \otimes c_a$. $a \mapsto \bar{a}$ is an anti-linear anti-involution of the algebra $Pol(\mathbb{C}^{n|m})$. PT: this is different from the bar you used before …I take it back, maybe it's the same bar as before. I think we're doing a functor of points description of $U(n|m)$. NR: you're right. There are no super groups, there are Hopf algebras which are the functions on super groups. This thing that looks like an action is a coaction of the Hopf algebra.

You have the braided monoidal category SVect. In this category, you have an algebra object $Pol(\mathbb{C}^{n|m})$. In the same category, you have $Pol(GL_{n|m})$, a Hopf algebra object. There is a coaction $Pol(\mathbb{C}^{n|m}) \to Pol(\mathbb{C}^{n|m}) \otimes Pol(GL_{n|m})$, making $Pol(\mathbb{C}^{n|m})$ into a comodule. To make this clearer, we'll talk about this last time. PT: in some sense, we talked about this in my class. If you have a super group acting on a super manifold, then this is really a coaction on the level of algebras. You're doing the universal case, where you're taking your base algebra to be generated by all the things that appear in the formula. NR: yes.

You should all know that SVect_k is an abelian monoidal category with braiding given by $v \otimes w \mapsto (-1)^{|v|\cdot|w|} w \otimes v$. You should also know from Peter's class what an algebra object in such a category is. You should also know what a Hopf algebra object is. Recall what is the meaning of a Hopf algebra. We want a Hopf algebra H to coact on an algebra A.

Example 27.1. If Γ is a finite group, then $H = Maps(\Gamma, k)$ is a Hopf algebra object in Vect_k, with $(f \cdot g)(x) = f(x)g(x)$, $(\Delta(f))(x,y) = f(xy)$, $S(f)(x) = f(x^{-1})$, and $\varepsilon(f) = f(e)$. ◇

From a completely algebraic point of view, (functions on) a *super group* is a commutative Hopf algebra object in SVect_k.

Example 27.2. $H = C(M_{n|m}) = \langle a_{ij}, b_{i\beta}, c_{\alpha j}, d_{\alpha\beta} | 1 \le i,j \le n, 1 \le \alpha, \beta \le m$, supercommutative with b, c odd$\rangle$. This has a bialgebra structure. The co-algebra structure is given by $\Delta a_{ij} = \sum_k a_{ik} \otimes a_{kj} + \sum_\alpha b_{i\alpha} \otimes$

$c_{\alpha j}$, and the counit is $\varepsilon(a_{ij} = \delta_{ij}$, $\varepsilon(b) = \varepsilon(c) = 0$, $\varepsilon(d_{\alpha\beta} = \delta_{\alpha\beta}$. You can think of this as

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \boxtimes \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} \Delta a & \Delta b \\ \Delta c & \Delta d \end{smallmatrix}\right)$$

Then you get $\tilde{H}$, the Hopf algebra of $GL_{n|m}$, as $H \otimes \langle A, D\rangle/(A \det a - 1, D \det d - 1)$. In this algebra, there exists an antipode, which you can think of as

$$S\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^{-1}. \qquad \diamond$$

Next time I'll say a few more words about actions and coactions. The eventual goal for this week is to get the Feynman diagrams for fermions. The goal for next week is to get

$$\int_{\mathbb{R}^n} \exp(\frac{i}{h} S) dx$$

where S is invariant under some group action. One way to deal with this is with the Fadeev-Popov trick. The mathematical meaning of this trick is revealed by the BRST quantization or BV quantization.

19 PT 10-30

Today's lecture was given by Bruce Driver.

Today is a bosonic day. I'll start with some finite-dimensional calculation and hopefully get to the point where you'll see the connection with Borcherds' class.

Let $A > 0$ be an $N \times N$ real matrix. The invariant way to do this is not to introduce a matrix at all; just use an inner product, but we've always been using this A. Consider the partition function $Z_A = \int_{\mathbb{R}^N} \exp\left(-\frac{1}{2} Ax \cdot x\right) dx = \sqrt{\det(2\pi A^{-1})}$. Let $d\mu_A(x) = \frac{1}{Z_A} \exp\left(-\frac{1}{2} Ax \cdot x\right) dx$ be the associated Guassian probability measure. The measure you've been seeing in Borcherds' class is

$$d\mu(\phi) = \frac{1}{Z_A} \exp\left(-\frac{1}{2} \int \left((\nabla\phi(x))^2 + m^2 \phi(x)^2\right) dx\right. \qquad (*)$$

which we've been writing as $\mathcal{D}\phi$. To get the operator A, you do integration by parts on this integral. You can rewrite the thing in the exp as $\left((-\Delta + m^2)\phi, \phi\right)_{L^2(\mathbb{R}^d, dx)}$, so our A is $-\Delta + m^2$.

<u>Integration formulas</u>. Now I'll give you some ways to integrate against this measure. The first one is

$$\int_{\mathbb{R}^N} e^{\lambda \cdot x}\, d\mu_A(x) = e^{\frac{1}{2} A^{-1} \lambda \cdot \lambda} \qquad (1)$$

You do this by completing the square. Define the operator $L = L^A := \sum_{i,j=1}^N A^{-1}_{ij} \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j}$. If A is the identity matrix, this is the Laplacian. Next, we have, for a positive number t,

$$\int f(x - \sqrt{t} y)\, d\mu_A(y) = (e^{tL/2} f)(x) = u(t, x). \qquad (2)$$

By definition, u solves the heat equation: $\frac{\partial u}{\partial t} = Lu/2$, $u(0, , x) = f(x)$. This measure is completely determined by this, by the way. Again, I won't prove this; you could do it with Fourier transforms. Let me at least do an example.

Example 19.1. Take $f(x) = e^{\lambda \cdot x}$. Compute $Lf = \left(\sum_{i,j=1}^N A^{-1}_{ij} \lambda_i \lambda_j\right) f = (A^{-1}\lambda \cdot \lambda) f$. Since f is an eigenfunction for L, it is easy to write down $e^{tL/2} f = e^{t(A^{-1}\lambda \cdot \lambda)/2} f$.

On the other side, we have (recall that $d\mu_A(y)$ is invariant under $y \mapsto -y$)

$$\begin{aligned}\int f(x-\sqrt{t}y)d\mu_A(y) &= \int e^{\lambda(x-\sqrt{t}y)}d\mu_A(y)\\ &= \overbrace{e^{\lambda\cdot x}}^{=f} \overbrace{\int e^{\sqrt{t}\lambda)\cdot y}d\mu_A(y)}^{e^{\frac{1}{2}A^{-1}(\sqrt{t}\lambda)\cdot\sqrt{t}\lambda}}\\ &= e^{\frac{t}{2}A^{-1}\lambda\cdot\lambda}f\end{aligned}$$

◇

Next, by setting $x = 0$ and $t = 1$ or $\log(2)$, we get

$$\int_{\mathbb{R}^N} f(y)d\mu_A(y) = (e^{L/2}f)(0)$$

This is a good way to compute these integrals on polynomials.

Example 19.2. $\int(\lambda\cdot x)^2 d\mu_A(x) = (e^{L/2}(\lambda\cdot x)^2|_{x=0}$. Since L is nilpotent here, you can just use the power series to get $(I + L/2 + (L/2)^2/2! + \cdots)(\lambda\cdot x)^2$, which at $x = 0$, you get $\frac{L}{2}(\lambda\cdot x)^2|_{x=0} = (A^{-1}\lambda\cdot\lambda)$. ◇

You can do this for any polynomial. The other way to do this is to use formula (1) and differentiating with respect to λ to get new formulas.

If you want to see Feynman diagrams coming out, you can do integration by parts. Suppose we have $\int\partial_v f(x)d\mu_A(x)$, where $v \in \mathbb{R}^N$ and ∂_v is the directional derivative. You can compute this as

$$\begin{aligned}\int \partial_v f(x)\frac{1}{Z_A}e^{-\frac{1}{2}Ax\cdot x}dx &= \int f(x)(Av\cdot x)\frac{1}{Z_A}e^{-\frac{1}{2}Ax\cdot x}dx\\ &= \int (Av\cdot x)f(x)d\mu_A(x)\end{aligned}$$

You're using that some things go to zero fast enough. If you replace f by fg, then we have that the adjoint ∂_v^* with respect to the inner product given by the integral is $-\partial_v + M_{(Av\cdot x)}$. You can write this as

$$M_{(v,x)} = \partial^*_{A^{-1}v} + \partial_{A^{-1}v}.$$

It is this sum of creation and annihilation operators which is interesting.

Example 19.3. Apply the formula to the function 1, giving $\int(v_1\cdot x)\cdots(v_4\cdot x)d\mu_A(x)$, where $v_1\cdot x = \partial^*_{A^{-1}v_1}1$, which is

$$\begin{aligned}&\int \partial_{A^{-1}v_1}[(v_2\cdot x)(v_3\cdot x)(v_4\cdot x)]d\mu_A(x)\\ &\qquad = \int(v_2\cdot A^{-1}v_1)(v_3\cdot x)(v_4\cdot x)d\mu_A(x) + \cdots (2\text{ more terms})\end{aligned}$$

The way to compute this is by drawing a dot for each pairing of four dots, and for each such pairing you have to assiciate the weight $(A^{-1}v_i, v_j)$ when i and j are paired, then multiply the weights together. This simply comes from integration by parts. ◇

Suppose $c\mu(\phi)$ as before, and let $f, g \in L^2(\mathbb{R}^d)$ or $C_c^\infty(\mathbb{R}^d)$. Then, by analogy with the two-vertex formula, we get

$$\int(\phi, f)_{L^2}(\phi, g)_{L^2}d\mu(\phi) = (A^{-1}f, g)_{L^2}.$$

The point of Gaussian measure is the once you know it for two, you can get everything else as products. I really want to stick in delta functions for f and g to compute $\int\phi(x)\phi(y)d\mu(\phi)$, but there are problems.

$(A^{-1}f)(x) = \int\Delta_m(x-y)f(y)dy$. Here, Δ_m is the propagator, which you can write as a function of one variable (usually you need a Greens function). Here are some of the properties of the function Δ_m:

1. $\Delta_m(x) \sim \begin{cases}|x| & d = 1\\ -\log|x| & d = 2\\ |x|^{2-d} & d > 2\end{cases}$ for $x \sim 0$.
2. $\Delta_m(x) \sim e^{-m|x|}$ for $x \gg 1$ independent of dimension.

If we take $f = \delta_x$ and $g = \delta_y$, then we get $\int\phi(x)\phi(y)d\mu(\phi) = \Delta_m(x-y)$.

The only reasonable interpretation of $\int\phi(x)^2 d\mu(\phi)$ is $+\infty$ for $d \geq 2$. If you thought the measure lived on functions, you'd expect this integral to be finite. Fact: there exists a measure μ on some space of distributions (depending on dimension) that does deserve to be thought of as $(*)$.

Let me go over one more theorem in preparation for Richard's class today.

Theorem 19.4. *Let f and g be polynomials on $\mathbb{R}^N$. $\mu_A(f \cdot g) := \int_{\mathbb{R}^N} f \cdot g \, d\mu_A = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\substack{i_1,\dots,i_n=1 \\ j_1,\dots,j_n=1}}^{N} A^{-1}_{i_1 j_1} \cdots A^{-1}_{i_n j_n} \mu_A(\partial_{i_1} \cdots \partial_{i_n} f) \cdot \mu_A(\partial_{j_1} \cdots \partial_{j_n} g)$*

This is the formula that Richard said a Lagrangian has to satisfy, which he said was hard to describe. think of the i's as x's and the j's as being y's. If you assume the i's and j's have disjoint support, you can make sense of all the $A^{-1}_{i_k j_k}$.

[[break, and back to PT talking]]

We don't have that much time. It would be good to somehow see how the different classes are related. Path integrals are at the heart of all three classes, which is why I asked Bruce to give the analytic side of the story. Are there any questions about how the classes are related? Let me try to say what happens in all three classes.

We're trying to "quantize" a classical field theory. Remember that a classical field theory consists of the data

- Space-time M. In RB's class, he works in flat Minkowski space $\mathbb{R}^{d-1,1}$. In this class, we've been trying to keep this generic; the space-time is a bordism (which we usually draw as surfaces). One of the differences between the classes is that we think of the boundary as very important; that's where you get your Hilbert space. RB keeps looking at compact support things, and you have a distinguished time direction, so you can kind of think of it as a bordism from $\mathbb{R}^{d-1}$ to $\mathbb{R}^{d-1}$. NR talked about mechanics, where the space-time is $[0,t]$, and quantum mechanical evolution is e^{itH}.
- Fields $\Phi(M)$. In RB's class, we think of it as $C_c^\infty(\mathbb{R}^d, \mathbb{R})$. These are the things we're going to integrate over. We just learned from Bruce that the Gaussian measure makes these things measure zero, so you have to go into distributions ... you have to enlarge your space of fields. In this class, we haven't specified this yet, but we've been thinking of fields as sections of some bundle over space-time. The easiest case is the σ-model, where the bundle is trivial, so $\Phi(M) = C^\infty(M, X)$ for some target space X. In mechanics (NR's class), the fiber in this bundle is a specification of a configuration space N.
- Action $A\colon \Phi(M) \to \mathbb{R}$. In RB, $\Phi(M) = C_c^\infty(\mathbb{R}^d, \mathbb{R})$, and the action is given by $\int_{\mathbb{R}^d} (-\Delta + m^2)\phi(x)\, dx$. This $-\Delta + m^2$, which Bruce called A, is a positive operator on $C_c^\infty(\mathbb{R}^d, \mathbb{R})$. In this class, we haven't specified any action. We'll do the supersymmetric version of the classical action later. To do that, we'll have to explain the integral over super manifolds using the Berezinian. In NR's class and in mechanics, a field is a path in configuration space, and $A[\gamma] = \int_0^t \big(\dot\gamma(\tau)^2 + V(\gamma(\tau))\big)\, d\tau$.

To get quantum mechanics, you want $Q\colon \mathsf{RB}_1 \to \mathsf{Vect}$. It is enough to evaluate Q on a point to get it on objects; we say $Q(pt)$ is the Hilbert space of states. An interval $[0,t]$ goes to the operator e^{itH}, where H is the Hamiltonian operator. This is the quantum mechanical evolution (the fact that it is an evolution is built into the assumption that Q is a functor).

If $V = 0$, you can quantize by taking $Q(pt) = L^2(N)$, and $H = -\Delta$. Remember that N must have a Riemannian metric, so we have a Laplacian Δ. In this class, we'll study space-times of dimensions $d|1$, with $d = 0, 1, 2$.

If N is a spin manifold, we'll see that if we take space-time to be $M = [0,t] \times \mathbb{R}^{0|1}$, then geometric quantization will give you $Q(pt \times \mathbb{R}^{0|1})$ will be the L^2-spinors on M, and H will be the square of the Dirac operator $\not\nabla_N$.

Let's say that you believe that the Hilbert space is right (there is the subtle issue of polarization, telling you why you can just consider functions of position and not momentum). You'd like to know how the system evolves, so you want to understand the operator $e^{it\Delta}$. This operator has a kernel $e^{it\Delta}(x,y)$, which you think of as matrix coefficients. We get that

$$e^{it\Delta}(x,y) = \int_{\substack{\gamma\colon [0,t] \to N \\ \gamma(0)=x, \gamma(t)=y}} \frac{e^{iA(\gamma)}}{Z_\gamma} \mathcal{D}\gamma$$

This is the Feynman-Kac formula. In all the classes, we're trying to compute this integral. The i in the exponent makes this highly oscillatory. If we get rid of it and assume A is positive, then you get rapid decay, and these integrals are quite computable, as Bruce showed us today.

In RB's class, we're struggling with integrating $e^{\lambda\phi(x)^4}$ because it is not Gaussian, which doesn't make sense, so we expand as a power series in λ so that we get terms which are polynomials multiplying the Guassian part, which we know how to do. So when the thing in the exponent is not purely quadratic, you have to do perturbative stuff. If you have such a term in the classical action, you'll run into this problem.

10 RB 10-30

Last time I was explaining Gaussian Feynman measure, which is formally something like $e^{i\text{quadratic Lagrangian}}\mathcal{D}\phi$. This is more or less a linear map from the symmetric algebra of compactly supported actions to $\mathbb{R}$. Remember that a compactly supported action is something which looks like $\int f(x)\phi(x)^4\,dx$ where f is compactly supported.

A *renormalization* is a linear map $\mathrm{Sym}^*(\text{actions}) \to \mathrm{Sym}^*(\text{actions})$ preserving certain structures:

1. Renormalizations preserve the coproducts Δ of $\mathrm{Sym}^*(\text{actions})$. Recall that $\mathrm{Sym}^* V$ always has a coproduct $\Delta(v) = v\otimes 1 + 1\otimes v$ extended as an algebra homomorphism. The reason we want renormalizations to preserve Δ is that we want renormalizations to act on the set of things of the form $e^{(\text{action})}$. Remember that this is how we got a (nonlinear) action of the group of renormalizations on the space of actions. PT: do you have the renormalization act on the quadratic Lagrangian or do you fix it once and for all? RB: normally you split the Lagrangian into a quadratic bit and the self-interaction part, and this splitting is non-canonical. Depending on how you split it, the answer might be yes or no.

2. Renormalizations almost preserve the product $\mathrm{Sym}^*(\text{actions}) \times \mathrm{Sym}^*(\text{actions}) \to \mathrm{Sym}^*(\text{actions})$. More precisely, $g(ab) = g(a)g(b)$ whenever a and b have disjoint supports. The reason we want this is to get an action on Feynman measures.

3. Renormalizations commute with the action of sections of the vector bundle ϕ (whose sections are fields) on actions. This is a boring condition. The reason for putting it in is that it cuts down the size of the group of renormalizations so that it acts *simply* transitively on Feynman measures. If you don't put it in, everything works, but your group is too big. BD: "Feynman measure" is just associated to the quadratic part of the Lagrangian? RB: yes. For the non-quadratic part, you have to expand as a power series.

4. Renormalizations preserve 1 in the space of actions. Again this is a boring condition.

The definition of the group of renormalizations looks really technical and hairy, but all you care about is that it acts transitively on measures.

There is a problem here: writing down explicit renormalizations is rather hard. There is an easy construction of explicit renormalizations: take exp(infinitesimal renormalization). To write down a renormalization, you have to know what it does on all Feynman diagrams, which is a pain, but an infinitesimal renormalization only has to be specified on Feynman diagrams of a given order (which you can make zero on all but one of them). An infinitesimal renormalization is something which satisfies infinitesimal versions of all the conditions above.

Now I want to give a vague sketch of why renormalizations act transitively on Gaussian Feynman measures. A proper proof of this would require many technical details which are boring. Suppose M_1 and M_2 are two Feynman measures, and suppose they differ on some Feynman diagram F, but are the same on all smaller Feynman diagrams (and all others of the same size). The idea is to find a renormalization g which fixes all smaller Feynman diagrams such that $g_1(M_1)(F) = M_2(F)$. If we can do this, then we can repeat this an infinite number of times to get the measures to agree on all Feynman diagrams (you can check that the renormalizations converge because there are only finitely many diagrams of a given size).

What is the difference between $M_1(F)$ and $M_2(F)$. Since M_1 and M_2 are the same on all smaller diagrams, the difference $M_1(F) - M_2(F)$ is a distribution with support on some diagonal (remember that the value of a measure on a Feynman diagram is determined by the value on smaller diagrams up to some distribution with very limited support). PT: How are you thinking of these Feynman diagrams as elements of Sym^*(actions). RB: you can think of measures as functions $\mathrm{Sym}^*(\text{actions}) \to \mathbb{R}$, but you can also think of it as a map $\mathrm{Sym}^*(\text{polys in } \phi \text{ and derivatives}) \to$ (distributions), which you can expand in terms of Feynman diagrams, though I really didn't mean to do that. Since I'm only giving a vague sketch, I'll pretend like this problem doesn't exist.

We're going to construct this renormalization as something of the form $e^{\text{infinitesimal renormalization}}$. The exponential roughly makes sure that the result is well behaved on larger diagrams. So we need to find an infinitesimal renormalization g with $g(M_1)(F) = M_2(F)$. Notice the following things:

1. g is a map $\mathrm{Sym}^*(\text{actions}) \to \mathrm{Sym}^*(\text{actions})$.
2. g is determined by $\mathrm{Sym}^*(\text{actions}) \to (\text{actions})$ because it respects the coproduct and actions generate Sym^*(actions).
3. Using that g commutes with sections of bundles, we can reduce to a map $\mathrm{Sym}^*(\text{actions}) \to \mathbb{R}$. These are related to distributions on products of space-time.
4. g almost preserves products, which implies that these distributions have support on the diagonal.

So unravelling the definitions, we see that a choice of an infinitesimal renormalization at each step more or less corresponds to the ambiguity, a distribution with support on the Feynman diagram corresponding to the difference between two Gaussian Feynman measures. So we have just enough freedom in choice of g to make $g(M_1)(F) = M_2(F)$. As I said earlier, this is really all you care about when you think about the group of renomalizations.

Actually, there is another problem: I haven't actually constructed a single Gaussian Feynman measure. I've vaguely sketched that given any two measures, there is a renormalization taking one to the other. How can we show that there is at least one Gaussian Feynman measure? There are two proofs, one of which is an existence proof, which I'll vaguely sketch, and the other honestly constructs the measure.

For each Feynman diagram, we need to extend a distribution on $M^n \smallsetminus$diagonal to M^n. If we use translation invariance, this reduces to the following basic problem in distribution theory. If we have a distribution on $\mathbb{R}^k \smallsetminus \{0\}$, can we extend it to all of $\mathbb{R}^k$? The answer in general is no. A typical example is e^{1/x^2}. This is so big near the origin that there is no way to extend it to a distribution, though it can be extended as a hyperfunction. Hyperfunctions are extensions of distribution with the property that funtions can always be extended to hyperfunctions. We don't want to use hyperfunctions because they give me a headache. A better answer is that you usually can extend distributions unless you've built it so that you can't extend it. Any distribution with "mild growth" near 0 can be extended. In practice, any naturally occuring distribution will be good enough. In particular, if the original propagator is "reasonable", then all distributions can be extended.

I won't go into this too much because I want to construct an explicit example of a Feynman measure. In order to do this, I want to discuss extension of distributions in detail.

Example 10.1. Take $f(x) = \frac{1}{|x|}$ on $\mathbb{R} \setminus \{0\}$. This can be extended to $\mathbb{R}$, but there is no canonical way of doing it. For example, $\frac{1}{|x|} = \frac{d}{dx}(\log x \cdot \sin x)$ for $x \neq 0$. This is a locally integrable function, so it is a distribution. If you have any distribution, its derivative is another distribution. This looks perfectly canonical, but there is a catch. $f(x)$ is homogenous of degree -1 (that is, it is invariant under some sort of rescalings of the reals), and we would like the extension to also be of degree -1. If there were a unique way to extend, it would automatically be of degree -1. There is no way to do this. The problem is that $\log|x|$ is "not quite of degree 0". Rescaling adds a constant to $\log|x|$, which messes up rescaling of the extension:

$$\frac{d}{dx}\big((\log x + c)\sin x\big) = \frac{d}{dx}\log(x)\sin(x) + c\frac{d}{dx}\sin(x)$$

So when we rescale we pick up delta functions at the origin. You can see that we're not doing something stupid as follows. All possible extensions of $\frac{1}{|x|}$ differ by a distribution with support at the origin. Such distributions are just spanned by deriviatives of $\delta(x)$. Since the higher derivatives $\big(\frac{d}{dx}\big)^n\delta(x)$ of $\delta(x)$ are homogeneous of degree $-1-n$, we should only be thinking about $\delta(x)$. Degree n means $x\frac{d}{dx}f = nf$, or $\big(x\frac{d}{dx} - n\big)f = 0$. It turns out that for $f(x) = \frac{d}{dx}\log(x)\sin(x)$, $\big(x\frac{d}{dx}+1\big)^2 f = 0$, so it is some kind of generalized eigenvector. All extensions that are generalized of degree -1 are given by $\frac{d}{dx}\log(x)\sin(x) + c\delta(x)$. Rescaling acts *transitively* on these things by $c \mapsto c + \log(\lambda)$. So there are no homogenous extensions of degree -1. ◇

Now let's consider $\frac{1}{|x|^s}$ as a meromorphic-distribution-valued function of s. The idea is going to be that if you send s to 1, you find that there is a pole, and this pole measures the problem. We'll do that next week.

28 NR 10-31

Let me start by reminding you where we got stuck last time. I want to make clear what $GL_{n|m}$ is and what is the action on $\mathbb{C}^{n|m}$. We have the category SVect_k, in which we have the Hopf algebra $H =$"$Pol(GL_{n|m})$"$= \big(\langle 1, a_{ij}, b_{i\alpha}, c_{\alpha j}, d_{\alpha\beta}\rangle \otimes \mathbb{C}[\Delta_a^{\pm 1}, \Delta_d^{\pm 1}]\big)/\langle \Delta_a = \det a, \Delta_d = \det d\rangle$, where a, d are even and b, c are odd. The coalgebra structure is given by

$$\Delta\begin{pmatrix} a & b \\ c & d\end{pmatrix} = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \dot\otimes \begin{pmatrix} a & b \\ c & d\end{pmatrix} \qquad \varepsilon\begin{pmatrix} a & b \\ c & d\end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I\end{pmatrix}$$

which should be read as saying, for example, that $\Delta a_{ij} = \sum_k a_{ik}\otimes a_{kj} + \sum_\alpha b_{i\alpha}\otimes c_{\alpha j}$.

In a usual group $S(f)(x) = f(x^{-1})$. The axiom for S is [[★★★]]. For $GL_{n|m}$, this axiom says that

$$S\begin{pmatrix} a & b \\ c & d\end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d\end{pmatrix} = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \cdot S\begin{pmatrix} a & b \\ c & d\end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I\end{pmatrix}$$

which you should read as saying that $\sum_k S(a_{ij})a_{kj} + \sum_\alpha S(b_{i\alpha})c_{\alpha j} = \delta_{ij}$. I claim that this determines S uniquely. Define $(a^{-1})_{ij} := \frac{M_{ij}^{n-1}(a)}{\Delta_a}$. If b and c are even, then we have

$$S\begin{pmatrix} a & b \\ c & d\end{pmatrix} = \begin{pmatrix} a & b \\ c & d\end{pmatrix}^{-1} = \frac{M(a)}{\Delta a}.$$

In general, we will have that

$$S\begin{pmatrix} a & b \\ c & d\end{pmatrix} = \frac{(SM)_{ij}\begin{pmatrix} a & b \\ c & d\end{pmatrix}}{\mathrm{Ber}\begin{pmatrix} a & b \\ c & d\end{pmatrix}}.$$

We also have $Pol(\mathbb{C}^{n|m}) \in \mathsf{SVect}_{\mathbb{C}}$. As an algebra, this is isomorphic to $Pol(\mathbb{C}^n)\otimes \bigwedge^\bullet \mathbb{C}^m$. $\mathbb{C}^n$ is a GL_n-module, so $Pol(\mathbb{C}^n)$ is a commutative algebra and a $Pol(GL_n)$-comodule. Similarly, $Pol(\mathbb{C}^{n|m})$ is an algebra and an H-comodule. A comodule structure is an even map $Pol(\mathbb{C}^{n|m}) =$

$A \xrightarrow{\delta} H \otimes A$ satisfying the commutative diagram

$$\begin{tikzcd} A \arrow[r, "\delta"] \arrow[d, "\delta"'] & H \otimes A \arrow[d, "\mathrm{id}\otimes\delta"] \\ H \otimes A \arrow[r, "\Delta\otimes\delta"] & H \otimes H \otimes A \end{tikzcd}$$

We have $Pol(\mathbb{C}^{n|m}) = \langle x_i, s_\alpha\rangle$, and the coaction is given by $\delta(x_i) = \sum_j a_{ij} \otimes x_j + \sum_\alpha b_{i\alpha} \otimes s_\alpha$ and $\delta(s_\alpha) = \sum_j c_{\alpha j} \otimes x_j + \sum_\beta d_{\alpha\beta} \otimes s_\beta$.

If we have $H = Pol(G)$, the Hopf algebra of an affine algebraic group G, since it is an infinite-dimensional vector space, we can choose a dual in a couple of different ways. You can choose $H^\vee$ and a pairing $\langle\cdot,\cdot\rangle\colon H^\vee \otimes H \to \mathbb{C}$. Such a triple $(H, H^\vee, \langle,\rangle)$ is called a dual pairing. One of the important dual pairings for $Pol(G)$ is given by taking $H^\vee$ to be distributions supported at the identity. [[★★★ HW: open a textbook on Lie groups and Lie algebras or go to Anton's Lie theory notes and look at the discussion about how this space of distributions can be identified with $U\mathfrak{g}$.]] You can think of $Dist_1(G)$ as left or right invariant differential operators on G.

When $\mathfrak{g}$ is a Lie super algebra, we still have the notion of the universal enveloping algebra and we still have the notion of left and right invariant differential operators. There is *a* dual Hopf algebra to $Pol(GL_{n|m})$ which is $U\mathfrak{gl}_{n|m}$. If you are not familiar with universal enveloping algebras, I strongly encourage you to learn about them. When you have a comodule over one of the guys in the dual, it is always a module over the other one.

PT: I'm a little nervous about evaluating at 1 in the super case. I agree that the universal enveloping algebra still acts on these comodules, but is the pairing still valid? NR: let's do an example. We have $U\mathfrak{gl}_{n|m}$ on one side and $Pol(GL_{n|m})$. I should define $U\mathfrak{gl}_{n|m}$. The standard way to do this is to define $\mathfrak{gl}_{n|m}$. In the usual setting, the notion of the Lie algebra is very natural, it is the subalgebra of left invariant vector fields. In the category of super manifolds, we can still do this and we'll end up with the notion of a Lie super algebra. You can think of $\mathfrak{gl}_{n|m}$ as having a linear basis $e_{ij}, e_{i\alpha}, e_{\alpha j}$, and $e_{\alpha\beta}$, with $[e_{ij}, e_{k\ell}] = \delta_{jk}e_{i\ell} - \delta_{i\ell}e_{kj}$, $[e_{i\alpha}, e_{\beta k}] = \delta_{\alpha\beta}e_{ik} + \delta_{ik}e_{\beta\alpha}$, and so on. You sometimes get a sign. This defines $\mathfrak{gl}_{n|m}$. Now $U\mathfrak{gl}_{n|m}$ is a unital associative algebra generated by the same elements, with the relations $[a,b] = ab - (-1)^{|a|\cdot|b|}ba$. A representation of a Lie super algebra is the same thing as a representation of its universal enveloping algebra. Note, by the way, that $U\mathfrak{gl}_{n|m}$ is just an associative algebra with a $\mathbb{Z}/2$-grading; it doesn't know anything about its super origins.

What is the pairing between $U\mathfrak{gl}_{n|m}$ and $Pol(GL_{n|m})$. First, let's describe the $(n|m)$-dimensional representation of $\mathfrak{gl}_{n|m}$. We have to construct a homomorphism of $\mathbb{Z}/2$-graded algebras $\pi\colon U\mathfrak{gl}_{n|m} \to End(\mathbb{C}^{n|m})$. It is given by taking e_{ij} to $\left(\begin{smallmatrix} E_{ij} & 0 \\ 0 & 0\end{smallmatrix}\right)$, $e_{i\alpha}$ to $\left(\begin{smallmatrix} 0 & E_{i\alpha} \\ 0 & 0\end{smallmatrix}\right)$. and so on. It is easy to check that this π extends to a homomorphism of algebras. Now for an element a, I can associate functions $\pi_{i/\alpha, j/\beta}$.

PT: why do you need the representation? NR: the coordinates on $Pol(GL_{n|m})$ were a_{ij}. I want to make the pairing $\langle x, a_{i_1j_1}\cdots a_{i_nj_n}\rangle = \langle \Delta^{(n)}x, a_{i_1j_1}\otimes\cdots\otimes a_{i_nj_n}\rangle$, where $\langle x, a_{ij}\rangle = \pi_{ij}(x)$.

PT: maybe I should not have been nervous about evaluating at the identity. Part of the homework from my class was that $\mathfrak{gl}_{n|m}$, left invariant vector fields, really is isomorphic to the tangent space at the identity, so you can let a left-invariant vector field act on a function and then evaluate at the identity.

NR: Recall from PT's class that you came across a Lie super algebra $\langle X \text{ odd}, H \text{ even} | [X,H] = 0, [X,X] = H\rangle$, which is somehow related to the de Rham differential. Let me tell you about something related to the Lie algebra $\mathfrak{gl}_{1|1}$. Consider $[X,Y] = H$ central, $[G,X] = X, [G,Y] = -Y$, and $X^2 = Y^2 = 0$, where $X = e_{12}$, $Y = e_{21}$, $H = e_{11} - e_{22}$, and $G = e_{11} + e_{22}$.

V is a representation of $\mathfrak{gl}_{1|1}$, given by $V = \bigoplus_{n\in\mathbb{Z}} V[n]$, $X\colon V[n] \to V[n+1]$, $Y\colon V[n] \to V[n-1]$, $H\colon V[n] \to V[n]$. $XY + YX = H$.

If you have a Riemannian manifold M , with de Rham operator d, and conjugated by the Hodge star, $*d*$, and you have $\Delta = H = d*d* + *d*d$. PT: you could make it a project to explain this $\mathfrak{gl}(1|1)$ action on any Riemannian manifold just like we explained the de Rham differential on a Manifold.

Next time, I will not continue with the representation of the Lie super algebras.

20 PT 11-01

Bruce talks again.
Notation:

- $d\mu_A(x) = \frac{1}{Z_A} e^{-\frac{1}{2}Ax\cdot x}\, dx$, $Z_A = \int_{\mathbb{R}^N} \exp\big(-\frac{1}{2}Ax\cdot x\big)\, dx$.
- $\int e^{\lambda\cdot x} d\mu_Z(x)$
- $\int f(x) d\mu_A(x) = (e^{L/2} f)(0)$

For a while, we'll take $A = \mathrm{id}$. In general, you define a new inner product $(v, w)_A = (v, Aw)$. The thing that tells you independence of A is that $\sum_{i,j=1}^N A_{ij}^{-1} e_i \otimes e_j = \sum_{k=1}^N u_k \otimes u_k$, where $\{u_k\}$ is an orthonormal basis with respect to $(,)_A$. In general, you interpret $Ax \cdot x$ by $(x, x)_A$. We'll set $\mu = \mu_I$.

Before I go to the infinite-dimensional case, let me go back to the theorem from last time.

Theorem 20.1. *If p and q are polynomials, then $\mu(p\cdot q) := \int_{\mathbb{R}^N} pq\, d\mu = \sum_{n=0}^\infty \frac{1}{n!} \sum_{i_1,\dots,i_n=1}^N \mu(\partial_{i_1}\cdots\partial_{i_n} p)\cdot \mu(\partial_{i_1}\cdots\partial_{i_n} q)$, where $\partial_i = \frac{\partial}{\partial x_i}$.*

There are no j_k's because we took A to be the identity.

This is more or less what RB is using as his definition of a Feynman measure. Actually, RB was a little misleading by saying that you can't make sense out of these measures. The thing to have in mind that at the end of the day you want to apply this stuff to $d\mu(\phi) = \frac{1}{Z}\exp\big(-\frac{1}{2}\int(|\nabla\phi|^2 + m\phi^2)dx\big)\mathcal{D}\phi$, where $\phi\colon \mathbb{R}^d \to \mathbb{R}$. Polynomials in $\tilde\phi(f) = \int \phi(x) f(x)\, dx$ where $f \in C_c^\infty(\mathbb{R}^d)$. PT: are there two different ϕ's? BD: we're identifying the function ϕ with the distribution it gives you. The f's are like the i's. We've basically already worked out that $\int \phi(f)\phi(g)\, d\mu(\phi) = \big((-\Delta+m^2)^{-1} f, g\big)$. What RB wants to do is integrate things like "$\phi \mapsto \int f(x)\phi^4(x)\, dx$". If ϕ were defined, then this would be a polynomial and everything would be fine. But this would amount to setting f and g to be delta functions. We want to find an extension of this μ, but in a nice way. PT: how does the notion of polynomials translate? BD: the analogue of $p(x)$ would be $p(\phi) =$polynomial$(\int f(x)\phi^4(x)\, dx, \cdots)$. A polynomial before was something of the form $p(x) = \sum c_\alpha x^\alpha$. Now we think of c_α as $f(\alpha)$.

Proof of Theorem. Let $P(t,x) = (e^{-tL/2}p)(x)$. Expand this as a Taylor series in t, then we get $\sum_{n=0}^\infty \frac{1}{n!}\big((tL/2)^n p\big)(x)$. This is a finite sum because p is of finite degree, so it will eventually be killed by the derivatives. In the litirature, you see the notation $: p := e^{-1/2}p$, the Wick ordering of p. I'm going to drop the x from the notation, then we want to compute

$$\begin{aligned}\frac{d}{dt}e^{tL/2}[P(t)\cdot Q(t)] &= \text{chain rule computation}\\ &= \sum_i e^{tL/2}\partial_i P\cdot \partial_i Q\end{aligned}$$

This essentially completes the proof. Just repeat the process. You see that $\frac{d^n}{dt^n}e^{tL/2}[P\cdot Q] = \sum_{i_1,\dots,i_n} e^{tL/2}\partial_{i_1}\cdots\partial_{i_n}P\cdot \partial_{i_1}\cdots\partial_{i_n}Q$. Since these series always truncate, we get polynomials in t and x, so there is no problem in applying Taylor's theorem. We get

$$e^{L/2}[e^{-L/2}p\cdot e^{-L/2}q] = \sum_{n=0}^\infty \frac{1}{n!}\underbrace{t^n\Big(\frac{\partial}{\partial t}\Big)^n e^{tL/2}[P\cdot Q]\Big|_{t=1}}_{=\sum_{i_1,\dots,i_n} e^{L/2}(\partial_{i_1}\cdots\partial_{i_n}e^{-L/2}p)\cdot(\partial_{i_1}\cdots\partial_{i_n}e^{-L/2}q)}$$

Now let $P\to e^{L/2}p$, $Q\to e^{L/2}q$ then evaluate at $x=0$. Something got screwed up. If Peter gives me five minutes, I'll fix it next time. □

One of the corollaries of this computation is this.

Corollary 20.2. *The mapping $L^2(\mu)\ni p\mapsto (e^{L/2}p)\in \mathrm{Sym}^*(\mathbb{R}^N)$ extends to a unitary map after completion. The right-hand side is usually called the Fock space. The norm on $\mathrm{Sym}^*\mathbb{R}^N$ (with respect to which you complete) is $\|q\|^2 = \sum_{n=0}^\infty \frac{1}{n!}\sum|\partial_{i_1}\cdots\partial_{i_n}q|^2(0)$.*

The way you're supposed to think about these infinite-dimensional wave equations is $\ddot\phi(t) = -\nabla(\phi(t))$, where $\phi(t,\cdot)\in L^2(\mathbb{R}^{\text{space}})$, $\partial_t^2\phi - \Delta_{\text{space}}\phi + \cdots = 0$. In the description of taking L^2 of L^2, you run into trouble… the Fock space description is better.

Question: suppose we have a Hilbert space H. Can you make sense out of $d\mu(x) = \frac{1}{Z}e^{-\frac12(x,x)}\mathcal{D}x$? That is, can you find a measure μ on H such that for each $\lambda\in H$, $\int_H e^{(\lambda,x)}\,d\mu(x) = e^{\frac12(\lambda,\lambda)}$? We know this is supposed to be the answer in finite dimensions. You should be thinking of H as $\{f\in L^2 \mid \int(|\nabla f|^2 + m^2f^2)dx<\infty\}$. The thing that causes all the problems (and all the good things) is that the answer is NO.

The problem is that H is too small. A good analogue is this. If you just have $\mathbb{Q}$ and you want a Lebesgue measure on it. Since a measure must be countably additive, you run into trouble because $\mathbb{Q}$ is countable. This is similar to what is happenning here. Let me state the theorem.

Theorem 20.3. *Suppose H and K are real separable Hilbert spaces such that $H\overset{i}{\hookrightarrow}K$ is a continuous embedding (i.e. $\|i(h)\|_K\le C\|h\|_H$ for all $h\in H$) and so that $\overline{i(H)}^K = K$. If (and only if) i is Hilbert-Schmidt, then there exists a unique (Gaussian) measure μ on K such that for $\lambda\in K^*$, $\int_K e^{\lambda(x)}\,d\mu(x) = e^{\frac12(\lambda|_H,\lambda|_H)_{H^*}}$.*

This tells you what you need to do to enlarge your space. $A\colon H\to K$ is *Hilbert-Schmidt* if $\sum_{j=1}^\infty \|Ae)i\|_K^2 = \|A\|_{HS}^2<\infty$ for any orthonormal basis $\{e_i\}$ of H. This is also $\|A\|_{HS}^2 = \mathrm{tr}(A^*A)$.

Note that $\mu(H)=0$.

[[break]]

For lack of time, let me stick to the following example.

Example 20.4 (Wiener measure)**.** $d\mu(\omega) = \frac1Z\exp\big(-\frac12\overbrace{\int_0^T\dot\omega(t)^2\,dt}^{(\omega,\omega)_H}\big)\mathcal{D}\omega$, where $\omega(0)=0$, $\omega(t)\in\mathbb{R}^d$. When you see a $\mathcal{D}$, it means that this is an informal expression which doesn't make sense. Where can you construct this measure? $H = \{\omega\colon[0,T]\to\mathbb{R}^n|(\omega,\omega)_H<\infty\}$. Note that Z is usually 0 or ∞ … no one term in this expression makes sense.

Claim. *$H\overset{i}{\hookrightarrow}L^2([0,T]^1,\mathbb{R}^n)$ is Hilbert-Schmidt.*

Where is the Hamiltonian operator coming from? In other setups, $\mathbb{R}^n$ is replaced by an infinite-dimensional thing. In our language, $\mathbb{R}^N$ is $L^2([0,T]^1,\mathbb{R}^n)$. It turns out that $\|i\|_{HS}^2 = T^2/2$. You should compute $i^*\colon L^2\to H$.

Claim. *$(i^*f)(s) = \int_0^T s\wedge t\,f(t)\,dt$, where $s\wedge t = \min(s,t)$.*

This is not hard to check. The Hilbert-Schmidt norm $\|i\|_{HS}^2$ is $\operatorname{tr}(ii^*)$, and ii^* is basically i^* since the inclusion doesn't do anything. So we should evaluate $\int_0^T s \wedge s\, ds = T^2/2$.

Now you do another little exercise. You can ask what is $\int_K (x,k_1)_K (x,k_2)_K \, d\mu(x) = \sum_{n=1}^\infty (e_n,k_1)_K (e_n,k_2)_K$, where $\{e_n\}$ is an orthonormal basis for H. This is really $\big((k_1,\cdot)_K,(k_2,\cdot)_K\big)_{H^*}$. Now you may as well replace e_n by $i(e_n)$, and then $(i(e_n),k_j)_K = (e_n, i^*k_j)_H$, so we see that the result is $(i^*k_1, i^*k_2)_H = (k_1, ii^*k_2)_K$.

Now we compute $\int_K (f,\omega)_K^2 \, d\mu(\omega) = (f, ii^* f) = \iint_{[0,T]} s \wedge t\, f(s) f(t)\, ds\, dt$. Formally, you can take f to be a delta function at t_0. The left hand side is $\int \omega(t_0)^2 d\mu(\omega)$ and the right hand side is $t_0 \wedge t_0 = t_0$. More generally, you get $\int \omega(s)\omega(t) d\mu(\omega) = s\wedge t$. We haven't shown that this makes sense. Weiner showed that something something making the total measure of continuous functions 1, so you can just work with continuous functions. $\diamond$

Where is the operator theory? This is supposed to have to do with Quantum mechanics, which has to do with operators. In particular, where is the Hamiltonian? Define

$$(T_t^\vee f)(x) = \text{“}\frac{1}{Z_t} \int_{\omega\colon [0,1]\to\mathbb{R}^n, \omega(0)=x} \exp\Big(-\int_0^t \big(\tfrac{1}{2}\dot\omega(\tau)^2 + V(\omega,\tau)\big)\, d\tau\Big) f(\omega(t)) \mathcal{D}\omega\text{”}$$

If you didn't have the V, this would be the Weiner measure. The rigorous definition of T_t would be

$$T_t^\vee f(x) = \int_{C([0,t],\mathbb{R}^n),\omega(0)=0} f\big(x + t\omega(t)\big) \exp\Big(-\int_0^t V(\omega(\tau + x) d\tau\Big) d\mu(\omega)$$

On of the homework problems is the gluing axiom. From the informal formula, you should check that $T_s^\vee T_t^\vee = T_{s+t}^\vee$ and $T_0^\vee = \mathrm{id}$.

$$\frac{d}{dt}\Big|_{t=0} T_t f = -\hat H f \qquad \hat H = -\frac{1}{2}\Delta + M_V$$

When you do this calculation, you get an $\dot\omega$, which you know doesn't make sense (because the paths aren't smooth), but you pretend that it does make sense. If the paths were smooth, you wouldn't get the $-\Delta$. You have to be a little more careful, you can't just pull the derivative under the integral. The key is

$$\int \omega(t)^2 d\mu(\omega) = t \qquad \int \omega(t) d\mu(\omega) = 0$$

(these paths are rough). When Ito was developing stochastic calucus, this had to do with the fact that

$$df(\omega(t)) = f'(\omega(t)) d\omega(t) + \frac{1}{2} f''(\omega(t) d\omega(t)^2 dt$$

29 NR 11-02

$U\mathfrak{gl}_{n|m}$ is generated as a unital algebra by e_{ij}, $e_{i\alpha}$, $e_{\beta j}$, and $e_{\alpha\beta}$, with the relations you expect (e.g. $e_{\alpha i}e_{j\beta} + e_{j\beta}e_{\alpha i} = \delta_{ij}e_{\alpha\beta} + \delta_{\alpha\beta}e_{ji}$). $\mathbb{C}^{n|m}$ has basis e_i, s_α. If we don't care about the tensor product, this is just a vector space. Once we start dealing with tensor product, we have to decide whether to treat it as a $\mathbb{Z}/2$-graded space or a super vector space. The vector representation on $U\mathfrak{gl}_{n|m}$ in $\mathbb{C}^{n|m}$ is an even linear map $\pi\colon U\mathfrak{gl}_{n|m} \to End(\mathbb{C}^{n|m})$, with $\pi(e_{ij}) = \left(\begin{smallmatrix} E_{ij} & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $\pi(e_{i\alpha}) = \left(\begin{smallmatrix} 0 & E_{i\alpha} \\ 0 & 0 \end{smallmatrix}\right)$, and so on. It is easy to check that the appropriate relations hold. So for every $x \in U\mathfrak{gl}_{n|m}$, we have linear functions $\pi_{ij}(x)$, $\pi_{i\alpha}(x)$, $\pi_{\beta j}(x)$, and $\pi_{\alpha\beta}(x)$ on $U\mathfrak{gl}_{n|m}$. These are "coordinate functions" on $GL_{n|m}$ in the sense that "$Pol(GL_{n|m})$" is the Hopf (super) algebra forming a dual pair with $U\mathfrak{gl}_{n|m}$. This $Pol(GL_{n|m})$ is generated by $\pi_{i/\alpha,j/\beta}$, with the condition that the matrix is invertible. PT: I don't understand how this works for any other group, because you need a fundamental representation. NR: it doesn't. If you wanted to construct some other group, you'd need to pick a representation. PT: so how would you put the unitary or symplectic conditions into the algebra $Pol(Sp)$ or $Pol(U)$. NR: For $SL_{n|m}$, you would add the relation that the Berezinian is 1. For $U_{n|m}$, you have to say a little more. On $\mathfrak{gl}_n(\mathbb{C})$, we have the involution $\sigma(e_{ij}) = e_{ij}$ and $\sigma(\lambda x) = \overline{\lambda}x$. The fixed points of σ is $\mathfrak{gl}_n(\mathbb{R})$. The general construction is that you pick an involution of the Lie algebra (these are all classified), and then the fixed points give you a real form of the Lie algebra. For example, if you take $\sigma(e_{ij}) = -e_{ji}$, then $\mathfrak{gl}_n(\mathbb{C})^\sigma = \mathfrak{u}_n = \{a^* = -a\}$. The claim is that these involutions carry through the whole story. You can get involutions on the dual Hopf algebra and take the fixed points. In the non-super case, you have that $Pol(SU(n)) = \langle u_{ij} | \overline{u}_{ji} = M_{ij}(u)\rangle$. An algebraic version of Peter-Weyl tells us that this is $\bigoplus_{\lambda \text{ irrep}} \bigoplus_{i,j=1}^{\dim V_\lambda} \mathbb{C}\pi^\lambda_{ij}$. I'll leave it as an exercise to work out how it works in the super case. It is trickier because Lie superalgebras are not simple in general (only $\mathfrak{osp}(n|1)$ is simple).

The main reason we've made this detour about super groups is because I'll want to do some manipulations with them later. Let's return to where we started.

$$\begin{aligned}\int \exp\big((\overline{x}, AX) + (\overline{c}, BX) + (C\overline{c}, x) + (\overline{c}, Dc)\big)\, d\overline{x}\, dx\, d\overline{c}\, dc &= \operatorname{Ber}\begin{pmatrix} A & B \\ C & D\end{pmatrix}^{-1} \\ &= \det(A)\cdot\det(D - CA^{-1}B).\end{aligned}$$

Suppose you have

$$\int_{x\in\mathbb{R}^n, \overline{c}, c\in\mathbb{C}^n} \exp\Big(\frac{1}{2}(x, Ax) + P(x) + (\overline{c}, Bc) + Q^{even}(x, \overline{c}, c)\Big)\, d\overline{c}\, dc\, dx = I$$

Where c and $\overline{c}$ are independent. In general, $\int P(\overline{c}, c)\, d\overline{c}\, dc = P^{\overline{top}, top}(\overline{c}, c)$, where we've picked an orientation. I don't know how to compute this I for general $P(x)$, but I'll take $P(x) = \sum_{n\geq 3} V^{(n)}(x)$, where $V^{(n)}$ is a homogeneous polynomial of degree n, and I assume $Q(x, \overline{c}, c) = \sum_{n\geq 1, m\geq k\geq 1} \underbrace{\frac{1}{n!k!} Q^{(n,k)}(x)_{i_1,\dots,i_n}\overline{c}_{j_1}\cdots\overline{c}_{j_k}c^{i_1}\cdots c^{i_k}}_{Q^{(n,k)}}$.

We want to write I as a formal power series in V and Q. Say we live 100 years ago and we don't have any computers, but we want to compute these numbers. We know that we get an asymptotic expansion, so we know that the first coefficients give a good approximation in some areas.

$$I = \sum \int \exp\Big(\frac{1}{2}(x, Ax) + (\overline{c}, Bc)\Big) V^{(n_1)}(x)\cdots V^{(n_k)}(x) Q^{(n_1,k_1)}\cdots Q^{(n_\ell,k_\ell)}\, d\overline{c}\, dc\, dx$$

Each term is well defined, though we know that the series diverges. We can use the Feynman diagram technique. We can say that $V^{(n)}(x)_{i_1,\dots,i_n}$ is a vertex with valence n, with labels $i_\bullet$ on the edges. $Q^{(n,k)}$ is associated to some edges labelled $i_1, \dots i_n$ and some directed edges $a_1, \dots a_\ell$, $b_1, \dots b_\ell$ (a's go in, b's go out). Then we compute (using Wick's theorem)

$$\begin{aligned}&\int \exp((\overline{c}, Bc))\overline{c}_{b_1}\cdots c_{b_\ell}c_{a_1}\cdots c_{a_\ell}\, d\overline{c}\, dc \\ &\qquad = \sum \det(B) \sum_{\substack{\text{bipartite} \\ \text{perfect} \\ \text{matchings } \sigma}} (-1)^\sigma (B^{-1})_{a_1\sigma(a_1)}\cdots(B^{-1})_{a_\ell\sigma(a_\ell)}.\end{aligned}$$

You usually write this with pictures. Let's do an example to see how this shows up.

Example 29.1.

$$\int \exp((\bar{c}, Bc))\bar{c}_b c_a \, d\bar{c} \, dc = \left(\frac{1}{(m-1)!}(\bar{c}, Bc)^{m-1}\bar{c}_b c_a\right)^{top}$$
$$= [[\bigstar\bigstar\bigstar \text{ HW: is a minor}]] \qquad \diamond$$

Example 29.2.

$$\int \exp((\bar{c}, Bc))\bar{c}_{b_1}\bar{c}_{b_2}c_{a_1}c_{a_2} \, d\bar{c} \, dc = \frac{1}{(m-2)!}\big((\bar{c}, Bc)\bar{c}_{b_1}\bar{c}_{b_2}c_{b_1}c_{b_2}\big)^{top}$$

We can at least check that the degrees are both $m-2$. In general, the total degree is $m-\ell$. $\diamond$

$$\int \exp\Big(\frac{1}{2}(x, Ax)\Big)x^{i_1}\cdots x^{i_n} \, dx = Pf(A)^{-1} \sum_{\text{perf match}} \cdots$$

After this long exercise, we have

$$I = \sum_{\Gamma_{b,f}} \frac{(-1)^F F(\Gamma_{b,f})}{|\operatorname{Aut}\Gamma_{b,f}|}$$

where the weight is computed by [[★★★ picture]] and F is computed by the number of loops formed by the fermionic variables.

Next time we'll see how this formal power series can be used to approximate the asymptotics of oscillating integrals. This is the Fedeev-Popov trick.

30 NR 11-05

Last time I wrote the formula for the Feynman diagram expansion of an oscillatory integral which includes some Grassman variables. Let me repeat part of it.

$$\int_{\mathbb{R}^{0|m}} \exp((c, Bc)/2) \, dc = Pf(B).$$

Now I want to do

$$\int_{\mathbb{R}^{0|m}} \exp((c, Bc)/2 + V(c)) \, dc$$

where $V = \sum_{k\geq 4}\sum_{\{a\}} V^{\{a\}}c_{a_1}\cdots c_{a_k}$. If you do the expansion in the integral, you get

$$= \sum_{\ell\geq 0} \frac{1}{\ell!}\exp((c, Bc)/2)\cdot V(c)^\ell \, dc.$$

You can write $V(c)^\ell = \sum V^{\{a\}}V^{\{b\}}c_{a_1}\cdots c_{a_k}c_{b_1}\cdots c_{b_k}$. So the question is, what is

$$\int \exp((c, Bc)/2)c_{a_1}\cdots c_{a_k} \, dc \overset{\text{def}}{=} I_{a_1\cdots a_k} \tag{30.1}$$

This is something like the square root of what we computed last time, so we expect some kind of sum over perfect matchings. As Bruce suggested, we could integrate this by parts. Another thing we could notice is that we could compute the generating function

$$I(\eta) = \int_{\mathbb{R}^{0|m}} \exp\big((c, Bc)/2 + \sum_a \eta^a c_a\big) \, dc \in {\textstyle\bigwedge}^\bullet \mathbb{R}^m = \langle \eta_1, \ldots, \eta_m\rangle \tag{30.2}$$

We can do a change of variables $c = c' - B^{-1}\eta$, completing the square $(\frac{1}{2}(c, Bc) + (\eta, c) = \frac{1}{2}(c + B^{-1}\eta, B(c + B^{-1}\eta)) - \frac{1}{2}(\eta, B^{-1}\eta))$ to get

$$\int \exp\big((c', Bc')/2 - (\eta, B^{-1}\eta)/2\big) = Pf(B)\exp\Big(-\frac{1}{2}(\eta, B^{-1}\eta)\Big) = I(\eta)$$

And we have the relationship

$$I(\eta) = \sum_{k\geq 0}\frac{1}{k!}I_{a_1\cdots a_k}\eta^{a_1}\cdots\eta^{a_k}(-1)^{k(k-1)/2} \tag{30.3}$$

So all we have to do is expand the power series $\exp(-(\eta, B^{-1}\eta)/2)$.

$$\frac{I(\eta)}{Pf(B)} = \sum_{k\geq 0} \frac{(-1/2)^k}{k!} \underbrace{(\eta, B^{-1}\eta)\cdots(\eta, B^{-1}\eta)}_{k}$$
$$= \sum_{k\geq 0} \frac{(-1/2)^k}{k!} \sum_{a_1,\dots,a_k} \eta_{a_1}\cdots\eta_{a_k} \sum_{\text{peft match on } \{a\}} (-1)^\mu (B^{-1})_{i_1 j_1}\cdots(B^{-1})_{i_{k/2} j_{k/2}}$$

where $m\colon (a_1,\dots,a_k) \to (i_1,\dots,i_{k/2}, j_1,\dots,j_{k/2})$, and μ is the sign of the perfect matchings

$$(\eta, B^{-1}\eta)\cdots(\eta, B^{-1}\eta) = \sum_{i.j.} \eta_{i_1}(B^{-1})_{i_1 j_1}\eta_{j_1}\cdots\eta_{i_{k/2}}(B^{-1})_{i_{k/2} j_{k/2}}\eta_{j_{k/2}}$$
$$= \sum_{i.,j.} \overbrace{\eta_{i_1}\cdots\eta_{i_{k/2}}\eta_{j_1}\cdots\eta_{j_{k/2}}}^{\text{odd}} \overbrace{(B^{-1})_{i_1 j_1}\cdots}^{\text{even}}$$

So

$$\frac{I_{a_1\cdots a_k}}{Pf(B)} = (-1)^{k(k+1)/2} \underbrace{(1/2)^k \sum_{\mu \text{ perf match}} (-1)^\mu \prod_\alpha (B^{-1})_{i_\alpha j_\alpha}}_{Pf_{k\times k}((B^{-1})_{a_\alpha a_\beta})}$$

Maybe I should take a break from these computations and explain why I need them. It will be a homework to derive the formula

$$\int_{\mathbb{R}^{0|m}} \exp\bigl((c, Bc)/2 + V(c)\bigr)\, dc = Pf(B)\cdot \sum_\Gamma \frac{(-1)^\ell F(\Gamma)}{|\operatorname{Aut}\Gamma|}$$

where the Γ have only even-valent vertices. The weight of the graph Γ =[[★★★ two vertices connected by 4 edges]] by asigning weights $i_1,\dots,i_4$ and $j_1,\dots,J_4$ to the halves of edges, then the weight is $\sum V^{i_1\cdots i_4}V^{j_1\cdots j_4}\prod_{e=1}^4 (B^{-1})_{i_\ell j_\ell}(-1)^{3+2+1}$ (the $3+2+1$ is the number of loops). Let's leave it as part of the homework to derive this formula for the Grassman integral with the prescription for computing the weights for Feynman graphs.

What is the real reason we want to have all these strange integrals? The physical reason is that there are elementary particles which have fermionic statistics, where each state has at most one particle in it. We model this mathematically by taking generating functions in a Grassman algebra.

Consider the following problem which seems completely irrelevant. Let's try to compute

$$I = \int_{\mathbb{R}^m} \exp\bigl(iS(x)/h\bigr)\, d^m x$$

but let's assume there is a Lie group G acting on $\mathbb{R}^m$ so that $S(gx) = S(x)$. The measure $d^m x$ is also G-invariant. If G is compact, we replace this integral by

$$|G| \int_{\mathbb{R}^m/G} \exp\bigl(iS([x])/h\bigr)\, d[x]$$

where $d[x] = dx/dg$, where dg is the Haar measure on G. Assume the action has trivial stabilizers. The only way we've been able to study these so far is with specific asymptotic expansions.

If G is not compact, then we don't have this, but we know that the original integral is meaningless, so we should still try to study the second integrals.

How can we make sense of the perturbative expansion of such integrals (we want to get some Feynman diagrams and so on). Say a cross section [[★★★ level hypersurface?]] through the space of orbits is given by $f_a(x) = 0$, $a = 1,\dots d = \dim G$. We have that $\dim(\mathbb{R}^m/G) = m - d$ (since we assume the action is free). I claim that

$$J_f(x) \int \delta\bigl(f(gx)\bigr) dg = 1$$

where $\delta(f(x))$ is the distribution on $\mathbb{R}^m$ which is supported at the cross section given by the equations $f_a(x) = 0$, and $J_f(x)$ is the Jacobian $\det\bigl(\frac{\partial f^a}{\partial \xi^b}\bigr)$, where the ξ are the vector fields of the G-action. I'll return to this next time but let me just give the answer.

Assume G is compact, then

$$\begin{aligned}\int_{\mathbb{R}^m} \exp(iS/h)\,dx &= \int_{\mathbb{R}^m} \exp(iS/h)\cdot J_f(x)\int_G \delta(f(gx))\,dg\,dx\\ &= \int_G\int_{\mathbb{R}^m} \exp(iS(x)/h)\delta(f(gx))J_f(x)\,dx\,dg\\ &= \int_G dg\int_{\mathbb{R}^m} \exp(iS/h)J_f(x)\delta(f(x))\,dx \qquad (x\mapsto gx)\end{aligned}$$

But we know that the result of the integral doesn't depend on the choice of the cross section. We can say that $\delta(f(x)) = \int_{\mathbb{R}^d}\exp(i\lambda f(x)/h)\,d\lambda$, and we can write $J_f(x)$ as the Grassman integral $\int \exp\left(\bar{c}^a \frac{\partial f_a}{\partial \xi^b} c^a\right) d\bar{c}\,dc$, so the result is

$$\int_{\mathbb{R}^m/G} \exp(iS)\,dx = \int_{\mathbb{R}^m\times\mathbb{R}^d\times\mathbb{C}^{0|d}} \exp\Big(iS/h + \sum_a \bar{c}^a\frac{\partial f^a}{\partial \xi^b}c_b + \sum_a \lambda_a f^a(x)/h\Big)\,dx\,d\bar{c}\,dc\,d\lambda$$

Theorem 1: this is what we should understand as the integration over the quotient space

Theorem 2: it doesn't depend on the choice of cross section. This uses BRST.

This is the only reason we went through the trouble of Grassman variables.

21 PT 11-06

Homework 5 is due Nov. 20 (before Thanksgiving).Projects:

1. Yoneda for $\mathcal{A}$-enriched categories. [Theo, Dan]
2. Super manifolds via algebra. [Matthias, Dan]
3. K-theory for super manifolds. [Manuel]
4. Simple super Lie algebras. [Andre, Jonah]
5. G-actions on super manifolds.
6. Super principal bundles and connections. [Alan, Dan B.]
7. Differential gorms. [Kevin]

I'll give you at least three more projects in the next weeks.

We're way behind in this class because we spent so much time on super manifolds. We'll go very quickly through geometric quantization. I have to do this very quickly so that we can actually get to cohomology theories.

Start with a symplectic manaifold (M,ω). I explained at some point (as did Kolya) how to go from a classical field theory to such a setup (look at classical solutions). For today, I'll assume M is finite-dimensional, though it could be infinite-dimensional in general. We want to somehow quantize this classical system. We want a vector space V (I won't discuss the inner product for now), which is like the state space of a quantum system, and Lie homomorphism $C^\infty(M,\omega)\to End(V)$. You should think of $C^\infty(M,\omega)$ as classical observables and $End(V)$ as quantum observables. You want to quantize the observables in such a way that Poisson bracket goes to Lie bracket. Furthermore, you want a map that takes the constant function 1 to the identity id_V (this doesn't follow from Lie homomorphism).

Warning 21.1. I will ignore factors of $\hbar$, π, and i (I won't "set π or i to 1"). How do you make sure an operator has real eigenvalues? You require that it is self-adjoint. However, unitary operators are skew-adjoint. This i converts between skew-adjoint and self-adjoint operators.

So far this is *pre-quantization*. We require some more properties to call it *geometric quantization*. These other properties come from physics motivation. To do the geometric quantization, you have to give up part of $C^\infty M$ (the map will not be a Lie homomorphism).

This pre-quantization always exists if ω is *integral*, i.e. $\int_{\Sigma^2} f^*(\omega) \in \mathbb{Z}$ for all $f\colon \Sigma^2 \to M$, where Σ^2 is a closed oriented surface. Using some algebraic topology, this is the same as saying that $[\omega] \in H^2_{dR}(M)$ is actually lies in the image of $H^2(M;\mathbb{Z})$.

Construction of pre-quantization

Step 1: If ω is integral, there exists a hermitian line bundle $L \to M$ with connection such that the curvature of the connection is ω. The reason you need integrality for this is that if $[\omega] = [\text{curvature}(\nabla)]$, then it come from the first Chern class $c_1(L) \in H^2(M;\mathbb{Z})$.

You can discuss how many such line bundles there are up to isomorphism. If you have two line bundles, you can complex conjugate and tensor, so there is a notion of a "difference of line bundles". Any two such bundles will differ by a flat bundle. Thus, isomorphism classes of such (L,∇) forms a torsor under the group of isomorphism classes of flat line bundles. Flat line bundles, up to isomorphism, are classified by their holonomy, and because we assume unitary and assume M connected, the holonomy is just $\mathrm{Hom}(\pi_1 M, S^1) = H^1(M;S^1)$ (by the universal coefficient theorem, thinking of the circle S^1 as a discrete abelian group). So there is this torus which acts simply transitively on the choice of line bundle.

I will skip the construction of the bundle. The name of this line bundle with connection is called the *pre-quantum line bundle*. Choose one such (L,∇).

Remark 21.2. $H^2(M;\mathbb{Z}) \cong H^1(M,\underline{S}^1)$ ($\underline{S}^1$ is sheaf of S^1-valued functions). ⋄

The main diagram of this story is the following. There is an exact sequence of Lie algebras (we get this for any symplectic manifold)

$$\begin{tikzcd}
0 \arrow[r] & H^0(M;\mathbb{R}) \arrow[r] \arrow[d, "\exp"'] & C^\infty(M,\omega) \arrow[r, "d"] \arrow[d, "E"'] & sp(M,\omega) \arrow[r, "\text{cohom class}", "x\mapsto[i_x\omega]"'] \arrow[d, "\exp"'] & H^1(M,\mathbb{R}) \arrow[r] \arrow[d, "\exp"'] & 0 \\
1 \arrow[r] & H^0(M;S^1) \arrow[r, "\text{rot.}", "\text{fibers}"'] & \mathrm{Aut}(L,\nabla) \arrow[r, "\text{diffeo}", "\text{on base}"'] & Sp(M,\omega) \arrow[r, "\text{diff}", "\text{holonomy}"'] & H^n(M;S^1) \arrow[r] & 1
\end{tikzcd}$$

where $sp(M,\omega)$ are *symplectic vector fields*, vector fields X so that $\mathcal{L}_X(\omega) = 0$. If you use the Cartan formula, you see that $\mathcal{L}_X(\omega) = d(i_x\omega)$. Each of these are Lie algebras, all the maps are Lie homomorphisms, and the sequence is exact

I want to exponentiate to an exact sequence of Lie Groups, which is the bottom row (using our choice of (L,∇)). If $\tilde{f}\colon (L,\nabla) \to (L,\nabla)$ (respecting connection), then $f\colon M \to M$ must be a symplectomorphism. The map $E\colon C^\infty(M,\omega) \to \mathrm{Aut}(L,\nabla)$ is given as follows, you can flow along the function in M and simultaneously rotate the fibers.

Step 2: Define $V := \Gamma(L)$, the sections of L. This is a complex vector space. Let $C^\infty(M)$ act on V via the map E (remember we wanted a Lie homomorphism $p\colon C^\infty M \to End(V)$). The diagram is

$$\begin{tikzcd}
C^\infty(M) \arrow[r, "p"] \arrow[d, "E"'] & End(V) \arrow[d, "\exp"] \\
\mathrm{Aut}(L,\nabla) \arrow[r] & GL(V)
\end{tikzcd}$$

More concretely, for $f \in C^\infty M$ and $s \in \Gamma(L)$, $(p(f))(s) = \nabla_{X_f}(s) + f\cdot s$. Remember that we wanted $p(1) = \mathrm{id}_V$. The reason this is true is because of the big diagram above commuting (you get a constant rotation of the fibers, where $1 \in \mathbb{R}$ exponentiates to $1 \in S^1$). You can also see this from the formula.

The other formula I wanted to give you is this. If we have $\omega = d\alpha$ (i.e. if we have a *symplectic potential* $\alpha \in \Omega^1 M$), then we can take $L = \mathbb{C}\times M$ and $\nabla = d + m_\alpha$ (m_α is multiplication by α). We add the 1-form α to get curvature $d\alpha$, which we want. Now we get $p(f)(s) = X_f(s) + (\alpha(X_f) + f)\cdot s$. In the cases that lead to quantum mechanics, $M = T^*N$ (N is the configuration space), so we have the symplectic potential α.

After the break, I'll do the even more special case where M is a vector space.

[[break]]

Linear case (i.e. M is a finite dimensional vector space and $\tilde{\omega}\colon M\times M \to \mathbb{R}$ is a non-degenerate skew pairing). We have $TM \cong M \times M$, and $\omega_m(v_1, v_2) = \tilde{\omega}(v_1, v_2)$ defines ω. Then $\omega = d\alpha$, where $\alpha_m(v) = \tilde{\omega}(m, w)$ is in $\Omega^1(M)$. Now we can evaluate these formulas. I want to evaluate them on linear functions $M \overset{\phi}{\cong} M^* \subseteq C^\infty(M)$. We have the pre-quantization $p(v) = \partial_v + m_{\phi(v)}$. Because I defined the symplectic form to be constant, this symplectic manifold has translational symmetry. That is, M acts on itself by translations, which are symplectomorphisms (since the form is constant).

Remark 21.3. Translations do *not* preserve the 1-form α. $\diamond$

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & \mathbb{R} \arrow[r] \arrow[d, equal] & heis(M,\omega) \arrow[r] \arrow[d, hook] & (M,[,]=0) \arrow[rr] \arrow[d, hook] & & 0 \arrow[d, equal] & \\
0 \arrow[r] & H^0(M;\mathbb{R}) \arrow[r] \arrow[d, "\exp"'] & C^\infty(M,\omega) \arrow[r, "d"] \arrow[d, "E"'] & sp(M,\omega) \arrow[rr, "\text{cohom class}", "x\mapsto[i_x\omega]"'] \arrow[d, "\exp"'] & & H^1(M,\mathbb{R}) \arrow[r] \arrow[d, "\exp"'] & 0 \\
1 \arrow[r] \arrow[d, equal] & H^0(M;S^1) \arrow[r, "\text{rot.}", "\text{fibers}"'] \arrow[d, equal] & \mathrm{Aut}(L,\nabla) \arrow[r, "\text{diffeo}", "\text{on base}"'] & Sp(M,\omega) \arrow[rr, "\text{diff}", "\text{holonomy}"'] & & H^n(M;S^1) \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & S^1 \arrow[r] & Heis(M,\omega) \arrow[r] \arrow[u, hook] & (M,+) \arrow[rr] \arrow[u, hook] & & 0 &
\end{tikzcd}$$

Since translations preserve the de Rham d, but don't preserve α, so they don't preserve the connection, but when we do the extension to get the Heisenberg group, which does respect the connection. This formula leads to a representation of $heis(M,\omega)$ on $V = C^\infty(M;\mathbb{C})$.

We're sort of going back between symmetries an observalbes, so we're really hiding Noether's theorem in this diagram.

You don't expect symmetries to quantize well, you have to get a phase. This corresponds to the fact that you had to take the extension to the $Heis(M,\omega)$.

Fact: This representation is highly reducible! This is bad for quantization from some point of view. [[PT: I'm using some notes I wrote a couple of years ago, which are probably on my web site.]] To get an irreducible representation, we choose a *polarization*.

In the linear case, this is a decomposition $M \cong N \oplus N^*$, where N is a Lagrangian subspace, with $\tilde{\omega}$ corresponding to $\left(\begin{smallmatrix} 0 & -ev \\ ev & 0 \end{smallmatrix}\right)$. In the linear case, this decomposition always exists.

Geometric quantization in this case takes $V := C^\infty(N)$. Then I'd still like to write down an action of all functions, but as I said, you can't always do that. I certainly want to quantize the Heisenberg, so I'll just do that. $heis(M,\omega)$ acts on V via $N \oplus N^* \to End(V)$ given by $(n,\phi) \mapsto \partial_n + m_\phi$. The ∂_n are *annihilators* and the m_ϕ are *creators*. You have to check it, but the commutation relations are exactly the commutation relation in the Heisenberg algebra, so this is really a Lie homomorphism $M \to End(V)$, with the property that the central element (corresponding to the constant function) goes to id_V. It turns out that this V is *the* irreducible representation of (M,ω) for which the central element acts as the identity (Stone-von Neumann theorem tells us that the center either acts trivially, or the irrep is characterized by the action of the center).

11 RB 11-06

Last week we were looking at the following example. $\frac{1}{|x|}$ is a distribution for $x \neq 0$. It can be extended to a distribution for $x = 0$, but not in a canonical way. This is supposed to be a toy example of what is going to happen for Feynman diagrams. First of all, let's look at it in more detail.

Here is an idea for finding a canonical extension: look at $|x|^s$, where $s \in \mathbb{C}$. If $Re(s) > 0$, this is actually continuous, so it gives a distribution. The idea is to analytically continue it to all values of s. What do we mean by that? We think of $|x|^s$ as a function of s taking values in the space of distributions in x. This always gives people a bit of a headache when they first come across the concept. The idea is that for each test function $f(x)$, we look at $\int f(x)|x|^s\,dx$, and think of this as an analytic function of s. Analytically continuing is the same thing as saying you can extend this integral for other values of s.

How do we continue $|x|^s$ to $Re(s) \leq 0$. Here's an idea for doing it: differentiate to get $\frac{d}{ds}|x|^s = s|x|^{s-1}\mathrm{sign}(x)$ for $x \neq 0$, $s > 1$. Thus, we get $|x|^{s-1} = \frac{1}{s}\frac{d}{ds}|x|^s\mathrm{sign}(x)$. This $\mathrm{sign}(x)$ is a headache because it is discontinuous. To get rid of it, we differentiate again to get $\frac{d^2}{dx^2}|x|^s = s(s-1)|x|^{s-2}$ for $Re(s) \gg 0$. So we get $|x|^{s-2} = \frac{1}{s(s-1)}\frac{d^2}{dx^2}|x|^s$, giving an analytic extension of $|x|^s$ to a meromorphic function of s (meromorphic because we may get poles at $s = 0, 1$).

Where are the poles and what are the residues (the residue at a pole is a distribution, not a number). For example, we have $s|x|^{s-1} = \frac{d}{dx}|x|^s\mathrm{sign}(x)$. If we try putting $s = 0$, we see that this right hand side is the residue of $|x|^{s-1}$ at $s = 0$. We can figure out the value by just setting $s = 0$, getting $\frac{d}{dx}\mathrm{sign}(x) = 2\delta(x)$. So the residue is a constant times the Dirac delta.

Now let's go back and try to define $\frac{1}{|x|}$ as a distribution. First attempt: take the value of $|x|^s$ at $s = -1$, which fails because there is a pole at $s = -1$ (we've just worked out the residue at this pole). Second attempt: take the constant term of $|x|^s = a_{-1}(x)s^{-1} + a_0(x) + a_1(x)s + \cdots$ (where the a_i are distributions). This doesn't work because of the following subtle problem. The constant term of a meromorphic function at a singular point is not canonical (for example, it changes with a change of variables). Suppose I take $f(s) = s^{-1} + b + \cdots$, and take $s = t + ct^2 + \cdots$, with $s^{-1} = t^{-1} - c$. Then $f(t) = at^{-1} + b+$constant depending on a plus more stuff. So the constant term is only canonical up to addition of a multiple of the residue. If we were working with $\mathbb{C}$-valued functions, this would be useless, but since we're working with distribution-valued functions, this gives is some information.

Similarly, if $f(s) = a_{-n}s^{-n} + \cdots + a_0$, then the constant term changes by multiples of $a_{-n}, \ldots, a_{-1}$. So we can't pick out the constant term, but we can pick out a subspace of distributions which could be constant terms. So we have a canonical *family* of extensions of $\frac{1}{|x|}$ to $x = 0$ by taking $\frac{1}{|x|}$ =constant term of $|x|^s$ at $s = -1$ plus a complex nmber times the residue at $s = -1$. The residue tells us the ambiguity in extending $|x|^{-1}$. If we have a pole of order k, then we find that there is a k-dimensional space of extensions, controlled by the singularity. $|x|^s$ has poles at $s = -1, -3, -5, \ldots$.

Example 11.1. Consider $\int_{-\infty}^{\infty}|x|^{s-1}e^{-x^2}\,dx$ (think of e^{-x^2} as a test function). If we change x to x^2, this is $2\int_0^\infty e^{-x}|x|^{s/2-1}\,dx = 2\Gamma(s/2)$. We know that Γ has an analytic continuation, and lots of poles all over the place. Since $|x|^{s-1}$ is meromorphic at $s = 0, -2, -4, \ldots$, we see that $\Gamma(s)$ is meromorphic at $s = 0, -1, -2, \ldots$. So the meromorphic continuation of $|x|^s$ is more or less the meromorphic continuation of Γ. If you unwind this, you find that the proof of meromorphic continuation of $|x|^s$ is more or less the same as the proof of meromorphic continuation of Γ. ◇

Summary: to define $|x|^{-1}$, we first look at $|x|^s$ for s large, analytically continue it to $s = -1$, take the constant term plus multiples of the singular parts.

Now let's consider the more complicated case of Feynman diagrams.

Problem: Feynman diagram is a product of propagators $\Delta(x_i - x_j)$, and the product is sometimes not defined if $x_i = x_j$. So we have the same sort of problem as for $|x|^s$.

Strategy:

1. Replace the propagator by some holomorphic family of propagators $\Delta_s(x_i - x_j)$ for $s \in \mathbb{C}$ so that if $Re(s) \gg 0$, then $\Delta_s(x_i - x_j)$ is continuous (say) with $\Delta_0 = \Delta$.

2. For large s, the product of propagators is defined, so we get a function from s to distributions

3. Try to take some analytic continuation in s to $s = 0$. We usually find that it is meromorphic at $s = 0$ with a pole of finite order. This gives us a reasonable space of extensions whose dimension is the order of the pole. The Feynman diagram is given by the constant term plus multiples of the singular parts at $s = 0$.

(4. Try to eliminate poles by applying a renormalization.)

How do we do step (1)? What should we use as $\Delta_s(x)$? The propagator is a bit of a pain in x coordinates. Notice that $\Delta(x)$ is the Fourier transform of $\frac{1}{p^2+m^2}$ (suppose we're doing scalar field theory in Euclidean space). Then $\Delta(x)$ has singularities at $x = 0$. Singularities are caused by the fact that $\frac{1}{p^2+m^2}$ does not decrease fast enough at $p = \infty$ (in reasonably high dimensions). So we should make $(p^2 + m^2)^{-1}$ decrease faster. You can immediately see lots of ways of doing this. Here are a few of the more popular ways.

1. $(p^2 + m^2)^{-1-s}$. This is similar to dimensional regularization. Physicists say that this is "changing the dimension of spacetime to $4 + s$". This has the additional advantage that it actually makes some sense. This still doesn't decrease all that fast as $p \to \infty$.

2. $(p^2 + m^2)^{-1}e^{-(p^2+m^2)}$ (there should be an s in there somewhere), which has several advantages over the previous one.

3. $\int_0^\infty \theta(t)e^{-(p^2+m^2)t}\, dt$.

Let's just use the first one. You might think that this gives you a canonical extension. The trouble is that $(p^2+m^2)^{-1-s}$ is not scale invariant (since it doesn't have the same degree as $(p^2 + m^2)^{-1}$). You can add an extra mass μ, and take $(p^2+m^2)^{-1}\left(\frac{\mu^2}{p^2+m^2}\right)^s$. This extra mass really does have to show up . . . it's not just that you've done something stupid.

Bruce: what exactly does scale invariance mean if $m \neq 0$? RB: it doesn't make a whole lot of sense; I guess you rescale the mass an your spacetime.

Ok, so now we've chosen a family of propagators and we find that (2) works.

Now what about step (3)? This involves a certain amount of work. For simplicity, we take $m = 0$. Then the propagator is $(x^2)^{1-d/2}$ times some constant. Adding in a factor of s means we have a family of propagators $(x^2)^s$ (let's ignore the $d/2$ part for now). So a Feynman diagram like [[★★★ picture]] gives us something of the form (polynomial)^s. This is defined for $Re(s) > 0$ and we want to analytically continue it.

General problem: given a polynomial p (say positive), can we analytically continue $p(x)^s$ to all complex s? The answer is YES! This is a theorem of Bernstein and Sato (at about the same time).

To see how to do this, look at the case of $p(x)^s = x^s$. Consider $\frac{d}{dx}x^s = sx^{s-1}$. Then we have $sp(x)^{s-1} = \frac{d}{dx}p(x)^s$. We can write this as $b(s)p(x)^{s-1} =$ (diff operator with polynomial coeffs)$p(x)^s$. This $b(s)$ is called the *Bernstein polynomial of* $p(x)$. If we can find such a relation, then we can continue $p(s)$ to all complex s.

Next week will be all about Bernstein polynomials and how you can prove they exist.

31 NR 11-07

Last time we started to discuss oscillating integrals $\int_{X=\mathbb{R}^m} \exp(iS(x)/h)\,dx$ as $h \to 0$. On X we have a Lie group G acting so that $S(gx) = S(x)$ and dx is G-invariant. If we do the standard variational analysis, we run into trouble because the Hessian is zero. Fadeev and Popov suggested the following trick.

Suppose G is compact of dimension k, and suppose $f_a(x)$ for $a = 1, \ldots, k$ such that $\{f_a(x) = 0\}$ is a cross section of the G action on X (i.e. $X/G \simeq f_a^{-1}(0)$). Then we have vector fields $\frac{\partial}{\partial \xi^a}$ on X representing the action of the basis elements e_a in $\mathfrak{g} = Lie(G)$. Then we have

$$\det\Big(\frac{\partial f_a}{\partial \xi^b}\Big) \int_G \delta(f(gx)) = 1 \qquad (*)$$

where $\delta(f(x))$ is a δ distribution on X supported at $f^{-1}(0)$.

Consider $\mathbb{R}$, and $f\colon \mathbb{R} \to \mathbb{R}$. Then $\int_{\mathbb{R}} g(x)\delta(x)\,dx = g(0)$ by definition. Changing the variables to $t = f(x)$, we get $\int g(x)\delta(f(x))\,dx = \int g(f^{-1}(t))\delta(t)\frac{dx}{dt}\,dt = g(f^{-1}(0))/f'(0)$.

Similarly, we get a δ distribution on $\mathbb{R}^n$, so $\int_{\mathbb{R}^n} g(x)\delta(x)\,dx = g(0)$. Again changing variables (in such a way that the Jacobian is non-zero), we get $\int_{\mathbb{R}^n} g(x)\delta(f(x))\,dx = g(x_0)\det(df(x_0))^{-1}$, where $f(x_0) = 0$.

Now we have to generalize this to the situation where the distribution is supported on a submanifold. Assume $X = \mathbb{R}^n \supseteq S$ a submanifold with some chosen measure μ on S. We say that $\delta_S(x)$ is *the δ distribution supported on S with measure μ* if $\int_X g(x)\delta_S(x)\,dx = \int_S g(x)\,d\mu$. Ok, sorry; this is not relavent.

$$\begin{aligned}
\int_G \delta(f(gx))\,dg &= \int_{U_e^\varepsilon} \delta\Big(f(x) + \sum_a t^a \frac{\partial f}{\partial \xi^a}(x)\Big)\,dt \\
&= \int \delta\Big(\sum_a t^a \frac{\partial f}{\partial \xi^a}(x)\Big)\,dt \\
&= \det\Big(\frac{\partial f_a}{\partial \xi^b}\Big)^{-1}
\end{aligned}$$

on $f(x) = 0$ where t^a are local coordinates near $e \in G$.

Now

$$\begin{aligned}
\int_X \exp(iS/h)\,dx &= \int_X \exp(iS/h)\det\Big(\frac{\partial f_a}{\partial \xi^b}\Big) \\
&= \int_G \delta\big(f(gx)\big)\,dg\,dx \\
&= \int_G \bigg(\int_X \exp(iS(x)/h)\det\Big(\frac{\partial f_a}{\partial \xi^b}(x)\Big)\delta\big(f(gx)\big)\,dx\bigg)\,dg \\
&= \int_G \bigg(\int_X \exp(iS(x)/h)\det\Big(\frac{\partial f_a}{\partial \xi^b}(g^{-1}x)\Big)\delta\big(f(x)\big)\,dx\bigg)\,dg \\
&= |G|\int_X \exp(iS(x)/h)\det\Big(\frac{\partial f_a}{\partial \xi^b}\Big)\delta(f(x))\,dx
\end{aligned}$$

The Jacobian is G-invariant if the identity $(*)$ is true for all x, not just x for which $f(x) = 0$. X is covered by the set of all $g \cdot f^{-1}(0)$ (because $f^{-1}(0)$ is a cross section of the action). Let's assume the Jacobian is G-invariant so that we don't get stuck here. We'll clear it up next time. PT: it depends on the choice of f. For a general f, it won't be invariant. You're saying that there exists an f so that it is invariant. NR: ok. Let's assume it's true for now. Theo/Bruce: you should should takde $\tilde{x}$ to be the point on $f^{-1}(0)$ which is in the same orbit as x, then $J_f(\tilde{x})\int_G \delta(f(gx))\,dx = 1$ is automatically gauge-invariant.

We can write

$$\det\Big(\frac{\partial f_a}{\partial \xi^b}\Big) = \int \exp\Big(\sum_{a,b} \bar{c}^a \frac{\partial f_a}{\partial \xi^b} c^b\Big)\,d\bar{c}\,dc$$

$$\delta(f(x)) = h^k \int_{\mathbb{R}^k} \exp\Big(i\frac{1}{h}\sum_a \lambda^a f_a(x)\Big)\,d\lambda$$

$$\int_{\mathbb{R}} g(x)\delta(x)\,dx = g(0) = \int_{\mathbb{R}} \hat{g}(\lambda)\,d\lambda$$

$$\delta(x) = \int_{\mathbb{R}} e^{ix\lambda}\,d\lambda$$

$$\hat{g}(\lambda) = \int_{\mathbb{R}} e^{ix\lambda} g(x)\,dx$$

$$\int_X \exp(iS/h)\,dx$$
$$= \int_{X\times\mathbb{R}^k\times\mathbb{R}^{0|k}\times\mathbb{R}^{0|k}} \exp\Big(\underbrace{i\frac{S(x)}{h} + i\sum_a \lambda^a f_a(x)/h + \sum_{a,b}\tau^a \frac{\partial f_a}{\partial \xi^b} c^b}_{S_{FP}}\Big)\,dx\,d\lambda\,d\bar{c}\,dc$$

Now what to do with this integral? If we consider $\widetilde{S}(x,\lambda) = S(x) + \sum_a \lambda^a f_a(x)$, what are the critical points? $\frac{\partial\widetilde{S}}{\partial\lambda^a} = 0$ implies $f_a(x) = 0$, and $\frac{\partial\widetilde{S}}{\partial\lambda^a} = \frac{\partial S}{\partial x^i} + \sum_a \lambda^a \frac{\partial f_a}{\partial x^i} = 0$, so critical points are critical points of S which are on $f^{-1}(0)$. $d^2\widetilde{S}$ is non-degenerate. You can expand near the critical points and evaluate the Gaussian integral as an asymptotic series, so we get

$$S_{FP}(x_0+X;\lambda_0+\lambda,\bar{c},c) = S_{FP}(x_0,\lambda_0) + (d^2S(x_0,\lambda_0)z,z) + \bar{c}K(x_0)c + \sum_k z^k, \bar{c}z^n c$$

where $z = (x,\lambda)$.

$$\int_X e^{iS/h}\,dx = |G|\sum_\Gamma \frac{(-1)^{F(\Gamma)}}{|\operatorname{Aut}\Gamma|}F(\Gamma)$$

Where solid (bosonic) edges get weight $(d^2S_{FP}(x_0))^{-1}$, dashed (fermionic) edges get weight $K(x_0)^{-1}$. There will be vertices coming from the expansion of the action (giving the n-th derivative $\widetilde{S}^{(n)}$) and vertices with dashed edges $K^{(n)}(x_0)$. This $(-1)^F$ will cancel the most severe divergences.

Next time I will probably have to return to some of these questions, but then I want to explain that there is another way to think about all this. This Fadeev-Popov action can we written as $S_{FP} = S(x) + Q\psi$, where $Q^2 = 0$ and ψ is odd. This is the BRST approach to gauge theory. A more sophisticated version is known as BV quantization. You can see that this is true if $Q\bar{c}^a = \lambda^a$, $Q\lambda^a = 0$, $Qc^a = \frac{1}{2}\sum_{a,b,c} c^a_b d^b c^c$, $Qf(x) = \frac{\partial f(x)}{\partial \xi^a}c^a$. You can try to interpret this Q as the derivation in some cohomology theory.

22 PT 11-08

Referecnes for geometric quantization:

- Bates-Weinstein (Berkeley MLN)
- Kirillov (Springer)
- Guilleman-Sternberg "Geometric Asymptotics"

Last time we had (M,ω) (integral) symplectic manifold. Our pre-quantization is $p\colon C^\infty M \to End(V)$ a Lie algebra homomorphism with $p(f) = \nabla_{X_f} + m_f$, where $V := \Gamma_{C^\infty}(L)$ for a particular line bundle with connection (L,∇). The easiest case is the example where M is a vector space and ω is constant $\tilde{\omega}\colon M\times M \to \mathbb{R}$. In that case, we have $M \cong_{\tilde{\omega}} M* \subseteq C^\infty M$, and on that subspace, we have $p(v) = \partial_v + m_v$. Notice that if you write down the commutation relations, then the vectors do not commute. We really want to take the constants as well: $\mathbb{R}\cdot c \oplus M =: heis(M,\omega) \subseteq C^\infty M$, and $p\colon heis(M,\omega) \to End(V)$ is a Lie homomorphism.

I'm purposly allowing ∂_v to act on $C^\infty M$. I'm not completing to a Hilbert space. Remember that $C^\infty M$ has a Frechét topology, and these ∂_v and m_v are continuous, but the spectrum is unbounded. I'm avoiding the problem by taking $C^\infty M$ and now worrying about what happens on the completion. Once you have an inner product, you can ask about adjoints. I'm ignoring some i's; ∂_v is skew-adjoint and m_v is self-adjoint. If you us im_v, then this actually leads to a unitary representation of $Heis(M,\omega)$ on $L^2(M,\omega^n/n!)$. Physicists like self-adjoint rather than skew-adjoint operators, so they would use $i\partial_v + m_v$ instead.

The problem is that this representation is not irreducible, and we expect it to be because it comes from a single particle. Before we cut it down to make it irreducible, let's do the odd version of this. [[★★★ Project 8: geometric quantization for symplectic super manifolds. Everything goes through beautifully. There are some notes for pre-quantization by Kostant]]

Odd (linear) analogue of pre-quantization

Take W a finite dimensional vector space and $b\colon W\times W \to \mathbb{R}$ a non-degenerate symmetric bilinear form. This means that $(\pi W, b)$ is a symplectic super manifold (remember that $(\pi W)_{red} = pt$). What is the analogue of pre-quantization? We have to look at $C^\infty(\pi W) = \bigwedge^*(W^*) \supseteq \mathbb{R}\cdot c \oplus W^* \cong_b \mathbb{R}c \oplus W =: heis(W,b)$ (this is a super Lie algebra). $C^\infty(\pi W)$ is also a super Poisson algebra. The Lie algebra structure and the Poisson structure are compatible, and this compatibility tells us that the Lie algebra structure on $C^\infty(\pi W)$ is completely determined by its behavior on the linear functions because $\{f, gh\} = \{f,g\}h + (-1)^{|f||g|}g\{f,h\}$.

Super pre-quantization is a super Lie homomorphism $C^\infty(\pi W) \to End(C^\infty(\pi W))$. On $heis(W,b) \subseteq C^\infty(\pi W)$, the c goes to $\mathrm{id}_{C^\infty(\pi W)}$ and $w \mapsto \partial_w + m_w$ (again, we're ignoring some i's if you want to work with inner products). So we have a super Lie algebra represented on an associative algebra, and an associative algebra is always a super Lie algebra, so we get a unique extension to a super algebra homomorphism $U(heis(W,b)) \to End(C^\infty(\pi W))$. Our criterion is that c goes to the identity, so we get an algebra homomorphism from $U(heis(W,b))/(c=1) =: \mathrm{Weyl}(W,b)$. All this works in the even case by the way. $p\colon U(heis(W,b))/(c=1) \to End(\bigwedge^*(W^*))$.

Lemma 22.1. *$U(heis(W,b))/(c = 1_U) \cong Cl(W,b)$, with defining relations $w_1w_2 + w_2w_1 = b(w_1,w_2)$.*

Since the w_i are odd, $[w_1,w_2] = w_1w_2 + w_2w_1 = b(w_1,w_2)$. So the way to prove the lemma is to observe that the two algebras have the same defining relations.

So what we've constructed is a representation of the Clifford algebra on the exterior algebra, given by the formula $w \mapsto \partial_w + m_w$. You can get $Cl(W,b) \to \bigwedge^*(W^*)$ by taking $a \mapsto p(a)\cdot 1_{\bigwedge^*}$. This turns out to be an isomorphism of $Cl(W,b)$-modules. This is something you might have seen before.

Corollary 22.2. *$\bigwedge^*(W^*)$ is* not *irreducible as a $Cl(W,b)$-module.*

Example 22.3. If you take $End(V)$, as a representation over itself, it is isomorphic to $V \otimes V^*$ (with the action on the left side, so the V^* is the multiplicity space).

If you take a finite group G, then $k[G] = \bigoplus_\lambda V_\lambda \otimes V_\lambda^*$ (with the action on the left again). ⋄

We see that we kind of have to take some sort of "square root" to extract the irreducible representations. This is what the polarization does

Back to (M,ω)

A polarization of $(M,\tilde\omega)$ is a decomposition $M \cong N \oplus N^*$ such that ω corresponds to $\left(\begin{smallmatrix} 0 & ev \\ -ev & 0\end{smallmatrix}\right)$.

Theorem 22.4. *$heis(M,\tilde\omega) \to End(C^\infty N)$, with $c \mapsto \mathrm{id}$ and $(v,\phi) \mapsto \partial_v + im_\phi$, is an irreducible representation.*

The proof is not hard, but we won't do it. It is proved in all of the references.

Theorem 22.5 (Stone-von Neumann)**.** *This representation exponentiates to an irreducible unitary representation $\rho\colon Hesi(M,\omega) \to U(L^2N)$, with $c \mapsto \mathrm{id}$. Moreover, this is the* unique *irreducible unitary representation of $Heis(M,\omega)$ on a Hilbert space sending c to id.*

This theorem is also not that hard.

I'm very interested in how unique the representation $heis(M,\tilde\omega) \to End(C^\infty N)$ is. Bruce: it isn't always unique. PT: ok.

Corollary 22.6. *$Sp(M,\omega)$ acts projectively on L^2N.*

Note that Sp must fix the origin; earlier we were looking at translations as well.

Proof of Corollary. For $g \in Sp(M,\omega)$, the key formula is $\rho(g(h)) = U_g \circ \rho(h) \circ U_g^*$ for all $h \in Heis(M,\omega)$, where $g(h)$ is the action of $Sp(M,\omega)$ on $Heis(M,\omega)$. That is, we've precomposed the representation with the action of $Sp(M,\omega)$. But by uniqueness of the representation, there must be some unitary operator U_g making the key formula work. The U_g is not quite unique. Two such U_g could differ by a phase (by Schur's lemma). That means that $g \mapsto U_g$ must be multiplicative up to phase, which is what it means to have a projective representation: $Sp(M,\tilde\omega) \to U(L^2N)/S^1$. □

If you differentiate this action, it leads to $\mathfrak{sp}(M,\omega) \to End(C^\infty N)/S^1$. But $\mathfrak{sp}(M,\omega) \subseteq C^\infty(M)$ are exactly the "quadratic" functions (given by $A \mapsto (m \mapsto \tilde\omega(Am,m))$). In fact, quantization gives a Lie homomorphism $\dots heis(M,\omega) \rtimes \mathfrak{sp}(M,\omega) = \mathbb{R}c \oplus M^* \oplus \mathrm{Sym}^2(M^*) \subseteq C^\infty(M)$. If you're careful, you'll see that this representation extends to a Lie homomorphism $heis(M,\omega) \rtimes \mathfrak{sp}(M,\omega) \to End(C^\infty(N))$ (this is quantization of observables). It *does not* extend to $C^\infty(M)$. This is the down side of geometric quantization; you can only quantize some classical observables, not all of them. This Lie homomorphism actually integrates to $Heis(M,\omega) \rtimes Sp(M,\omega) \to U(L^2N)$. This is quantization of symmetries.

[[break]]

Corollary 22.7. *Geometric quantization of $(M,\tilde{\omega})$ does* not *depend on the polarization (up to phase).*

This follows from the uniqueness part of Stone-von Neumann. This "up to phase" is good because physically, changing the phase doesn't change anything.

Example 22.8. $(M,\omega) = (\mathbb{R}^2, dx \wedge dy)$. I choose the polarization $\mathbb{R}^2 \cong \mathbb{R}_x \oplus \mathbb{R}_y = N \oplus N^*$. The two operators are ∂_x and m_x. Let $P = i\partial_x$ and $Q = m_x$ so that everything is self-adjoint. Question: how does $SO(2) \subseteq Sp(2)$ act on $L^2\mathbb{R}$? The answer is that the infinitesimal generator of the circle action is the energy $E = \frac{1}{2}(P^2+Q^2)$ (remember that geometric quantization can only quantize quadratic observables). $L^2\mathbb{R}$ has an orthonormal basis of eigenvectors for this operator and you can get between them with these creation and annihilation operators. The spectrum of E is $\frac{1}{2} + \mathbb{N}_0$. The lowest eigenvector if $\frac{1}{2}$ and the eigenvector is $e^{-x^2/2}$.

When we integrate, we get $e^{2\pi i E} = -1$, so we see that we only get a projectiv action. In fact, you can do this for any symplectic group and you see that it is actually a double cover that acts.

$$\begin{tikzcd}
 & S^1 \arrow[d] \arrow[r, phantom, "="] & S^1 \arrow[d] \\
Mp \arrow[rr, bend left] \arrow[dr, "2"'] & \widetilde{Sp} \arrow[d] & U(L^2N) \\
 & Sp & U(L^2N)/S^1
\end{tikzcd}$$

[[★★★ This is the Weil representation of Mp, yes?]] ◇

The fermionic case is way easier than the bosonic case because the Clifford algebra is finite-dimensional. In the bosonic case, you get this infinite-dimensional universal enveloping algebra of the Heisenberg. The analytic subtleties come from the fact that $Heis(M,\omega)$ is not compact. In the fermionic case, the underlying space is a point which is compact, and the rest is just linear.

Let me tell you the punch line, and I'll do the polarization for fermions on Tuesday.

Let $(\pi W, b)$ be an odd symplectic vector space, as before. We want a polarization $W = N \oplus N^*$, with b corresponding to $\left(\begin{smallmatrix} 0 & ev \\ ev & 0 \end{smallmatrix}\right)$.

Problem: This only exists if the signature of b is zero. If we study Riemannian manifolds, then we want b to be positive definite, so it looks like we're in trouble, but I'll show you some tricks which will allow us to polarize.

If the signature is zero, then $Cl(W,b) \to End(C^\infty(\pi N)) = End(\bigwedge^*(N^*))$. Counting dimensions (say $\dim N = n$), we see that this has dimension $(2^n)^2 = 2^{2n}$ on the right, and $\dim Cl(W,b) = 2^{2n}$. It turns out that this map is an isomorphism in the signature zero case. So in this case, the clifford algebra is a matrix algebra.

Corollary 22.9. *$Cl(W,b)$ has the irreducible representation $\bigwedge^*(N^*)$ and it is unique.*

32 NR 11-09

Last time we had a formula for the asymptotic expansion of $I_h = \int_X \exp(iS(x)/h)\,dx$. As $h \to 0$, this is given by Feynman diagrams. When S is invariant under the action of some Lie group G of dimension k (assume $X = \mathbb{R}^m$ for simplicity). We got that

$$I_h = \int_{X\times\mathbb{R}^k\times\mathbb{R}^{0|2k}} \exp\Big(\frac{iS}{h} + \sum_a \frac{\lambda^a f_a(x)}{h} + \sum_{a,b} \bar{c}^a \frac{\partial f_a(x)}{\partial \xi^b} c^b\Big)\, d\bar{c}\, dc\, d\lambda\, dx$$
$$\overset{h\to 0}{=} \sum_\Gamma \frac{(-1)^F}{|\operatorname{Aut}\Gamma|} F(\Gamma)$$

where $f(x) = 0$ is a cross section of the action of G. In quantum field theory, such expressions cannot be derived, so they are taken as definitions. A main goal (which is still largely open) is to construct a QFT which is not perturbative. The progress has been largely disappointing. You can go beyond perturbation theory in some cases which are not physical but still quite interesting to mathematicians.

PT: where are you hiding the h's in the notation? NR: in the $F(\Gamma)$. Say z is an even coordinate on $X \times \mathbb{R}^k$. Then we expand the action in powers of z. Every vertex with n even (solid) lines will have weight $h^{n/2-1}$. There will also be fermionic edges.

$$S(z)/h = S(z_c)/h + A(z-z_c)^2/h + \frac{1}{h}\sum_n \Big(\frac{z-z_c}{\sqrt{h}}\Big)^n V_n \cdot h^{n/2}.$$

Theorem 32.1. *Let Q be an operation on $\langle \bar{c}^a, c^a, C^\infty(X\times\mathbb{R}^k)\rangle \cong \bigwedge^\bullet(\mathfrak{g}^* + \mathfrak{g}^*) \otimes C^\infty(X\times\mathbb{R}^k)$, given by $Q\lambda^a = 0$, $Qc^a = \frac{1}{2}\sum_{bc} c^a_{bc} c^b c^c$ [[★★★ unfortunate notation, the c^a_{bc} are the structure constants of $\mathfrak{g}$.]], $Q\bar{c}^a = \lambda^a$, and $(Qf)(x) = -ih\sum_a c^a \partial_a f(x)$, where $\partial_a f(x) = \frac{d}{dt} f(e^{te_a}x)|_{t=0}$ for a basis $\{e_a\}$ for $\mathfrak{g}$ (note that $[\partial_a, \partial_b] = \sum_c c^c_{ab}\partial_c$). Then*

$$S_{FP} = S - Q\psi \qquad \psi = \sum_a c^a f_a(x) \qquad Q^2 = 0.$$

You can think of this Q as follows. We have the super manifold $X_{FP} = X \times \mathbb{R}^k \times \mathbb{R}^{0|2k}$, and Q is an odd vector field on X_{FP} such that $Q^2 = 0$. You can write Q as a vector field:

$$Q = \sum_a \lambda_a \bar{c}^a + (-ih)\sum_a c^a \partial_a + \frac{1}{2}\sum_{a,b,c} c^a_{bc} c^b c^c \frac{\partial}{\partial c^a}$$

where $\frac{\partial}{\partial c^a}$ is the "left derivation" in $\langle c^a\rangle$. This is known as the BRST operator.

Let's go over Lie algebra cohomology a bit. Consider the *standard complex $C^\bullet(\mathfrak{g}, M)$ for $\mathfrak{g}$ with coefficients in M* (M a representation of $\mathfrak{g}$, defined as $\sum_{\ell=0}^{k=\dim\mathfrak{g}} \operatorname{Hom}_{\mathfrak{g}}(\bigwedge^\ell \mathfrak{g}, M)$. We define $C^\ell(\mathfrak{g}, M) = \operatorname{Hom}_{\mathfrak{g}}(\bigwedge^\ell \mathfrak{g}, M) = \bigwedge^\ell \mathfrak{g}^* \otimes M$. Then we define $d\colon C^\ell \to C^{\ell+1}$ by

$$dc_\ell(x_1,\dots,x_{\ell+1}) = \sum_{i<j}^{\ell} (-1)^{i+j-1} c_\ell([x_i,x_j], x_1, \dots \hat{x}_i, \dots, \hat{x}_j, \dots, x_{\ell+1})$$
$$+ \sum_{i=1}^{\ell} (-1)^{i-1} x_i \cdot c(x_1, \dots \hat{x}_i \dots x_{\ell+1})$$

[[★★★ HW: check that $d^2 = 0$.]] Now we can define $H^\bullet(\mathfrak{g}, M)$ as the homology of the complex.

Example 32.2. Let $M = \mathbb{C}$ be the trivial representation of $\mathfrak{g}$, so $x \cdot c = 0$. Then let $c_2 \in C^2(\mathfrak{g}, \mathbb{C})$ and require that

$$dc_2(x,y,z) = c_2([x,y],z) - c_2([x,z],y) + c_2([y,z],x) = 0.$$

And we have that for $c_1 \in C^1(\mathfrak{g}, \mathbb{C})$,

$$dc_1(x,y) = c_1([x,y]).$$

So $H^2(\mathfrak{g}, \mathbb{C}) = \{c_2 \in C^2 | dc_2 = 0\}/\{dc_1\}$.

Claim. *Given $c_2 \in H^2(\mathfrak{g}, \mathbb{C})$, we can define a central extension $\hat{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{C}k$ by $[x,y]^\sim = [x,y] + kc_2(x,y)$ and $[k,x] = 0$. The condition $dc_2 = 0$ implies the Jacobi identity.*

We can define a trivial central extension $c_2(x,y) = c_1([x,y])$. [[★★★ HW: find a basis for $\hat{g}$ in this case such that $\hat{g} \cong \mathfrak{g} \oplus \mathbb{C}k$ as a Lie algebra]].

The conclusion is that $H^2(\mathfrak{g}, \mathbb{C})$ classifies central extensions of $\mathfrak{g}$ by $\mathbb{C}$. PT: you can replace $\mathbb{C}$ by any representation M to get extensions of $\mathfrak{g}$ by M (with the abelian Lie algebra structure). NR: yes. ⋄

Example 32.3. Let $\mathfrak{g} = Vect(S^1)$. There is a unique non-trivial central extension of this Lie algebra given by

$$c_2(f(t)\frac{d}{dt}, g(t)\frac{d}{dt}) = \frac{1}{2\pi i}\int_{S^1}(fg''' - f'''g)\,\frac{dt}{t}.$$

If you choose the basis $L_n = t^{-n-1}\frac{d}{dt} = ie^{in\theta}\frac{d}{d\theta}$ (where $t = e^{i\theta}$, then $[L_n, L_m] = (n-m)L_{n+m} + \frac{k}{12}(n^3 - n)\delta_{n,-m}$. $\diamond$

Example 32.4. Let $M = \mathfrak{g}$ with the adjoint action. Then $H^0(\mathfrak{g},\mathfrak{g}) = \{c \in C^0(\mathfrak{g},\mathfrak{g}) \cong \mathfrak{g} | dc = 0\}$. Since $dc(x) = x\cdot c = [x,c]$, $H^0(\mathfrak{g},\mathfrak{g}) = \mathfrak{g}^{inv}$ is the invariant part of the adjoint representation, the center of $\mathfrak{g}$. This is true for any module; $H^0(\mathfrak{g}, M) = M^{\mathfrak{g}}$, the invariant part of M (the part killed by the action of $\mathfrak{g}$).

Now let's consider $H^1(\mathfrak{g}, M) = \{c_1 \in \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g},\mathfrak{g}) | dc_1(x,y) = c_1([x,y]) - x\cdot c_1(y) + y\cdot c_1(x) = 0\}/\{dc_0\}$. This says that $c_1([x,y]) = [c_1(x), y] + [x, c_1(y)]$, so c_1 is a derivation. Elements of the form dc_0 are inner derivations, so $H^1(\mathfrak{g},\mathfrak{g})$ is the space of outer derivations of $\mathfrak{g}$.

One can consider $H^2(\mathfrak{g},\mathfrak{g})$. Remember that I was talking about deformation quantization of Poisson manifolds. I formulated Kontsevich's theorem. There is a commutative product which we don't deform and a Poisson bracket that we do deform. It turns out that $H^2(\mathfrak{g},\mathfrak{g})$ is the space of equivalence classes of infinitesimal deformations of $\mathfrak{g}$.

[[★★★ HW: check that $H^3(\mathfrak{g},\mathfrak{g})$ is the space of possible obstructions to extending an infinitesimal deformation one step further]] PT: what is a deformation? NR: it is a bracket $[x,y]_t = [x,y] + \sum_{n\geq 1} t^n c^{(n)}(x,y)$. We require that the Jacobi identity hold for this bracket and we consider such deformations up to equivalence $\phi\colon \mathfrak{g}[[t]] \to \mathfrak{g}[[t]]$, $\phi = \mathrm{id} + \sum_{n\geq 1} t^n\phi^{(n)}$.

First statement: it turns out that $c^{(2)} \in Z^2(\mathfrak{g},\mathfrak{g})$. Second statement: ϕ equivalences act on the possible c_2 as elements of B_1, $\phi\colon c_2(x,y) \mapsto c_2(x,y) + [\phi_1(x), y]$, so $[c_2] \in H^2(\mathfrak{g},\mathfrak{g})$.

We want that $[[x,y]_t, z_t] + alt = 0$. The coefficient in t^n is $\big\{[c^{(n)}(x,y),z] + alt + c^{(n)}([x,y],z) + alt\big\} + \overbrace{c^{(1)}(c^{(n-1)}(x,y),z) + alt + \cdots}^{\text{other}} = 0$. This is related to the Gerstenhabber algebra and the Schouten bracket, which appear naturally. Only the terms in $\{\}$ have $c^{(n)}$.

This can be written as $dc^{(n)} + \sum_{K+\ell=k}^{n-1}[c^{(k)}, c^{(\ell)}] = 0$ (the bracket is called the Gerstenhabber bracket of Schouten bracket). This "other stuff" (other than $dc^{(n)}$) is a class in H^3, which is an obstruction to the existence of $c^{(n)}$. $\diamond$

23 PT 11-13

Let (M,ω) be a symplectic vector space. Then we have the Lie algebra $h(M,\omega) \subseteq C^\infty(M)$, and a polarization $(M,\omega) \cong (N\oplus N^*, \left(\begin{smallmatrix} 0 & ev \\ -ev & 0\end{smallmatrix}\right))$ and an irreducible representation $h(M,\omega) \to End(C^\infty N)$, where $n \mapsto \partial_n$ and $n^* \mapsto m_{n^*}$. By Stone-von Neumann, we get uniqueness (up to phase) of the unitary representation of $H(M,\omega)$ on L^2N. So we could extend the representation to quadratic functions, but not to all of $C^\infty M$. The outcome is a projective representation of $Sp(M,\omega)$, which is independent of the polarization.

Definition 23.1. This is called the *metaplectic* representation of the *metaplectic group* $Mp(M,\omega) \xrightarrow{2} Sp(M,\omega)$. ◇

By the way, how do you classify covers of $Sp(M,\omega)$? We have to compute $\pi_1(Sp(M^{2n},\omega))$. A maximal compact inside of $Sp(M,\omega)$ is $U(n)$, which must then have the same π_1, which is $\mathbb{Z}$. We all know that $\pi_1(U(1)) \cong \mathbb{Z}$. And $U(n-1) \subseteq U(n)$, and $U(n) \to U(n)/U(n-1) \cong S^{2n-1}$ is a fibration. So there is one non-trivial connected double cover of $Sp(M,\omega)$.

Now for the fermionic side. If (W,b) is symmetric bilinear non-degenerate, then we get a super Lie algebra $h(W,b) \subseteq C^\infty(\pi W) = \bigwedge^*(W^*)$, a polarization $(W,b) \cong (N\oplus N, \left(\begin{smallmatrix} 0 & ev \\ ev & 0\end{smallmatrix}\right))$ (this exists if and only if the signature of b is zero), and an irreducible representation $h(W,b) \to End(C^\infty(\pi N))$. The uniqueness here is obvious because it leads to a representation of the associative algebra $Cl(W,b) \to End(C^\infty N) = End(\bigwedge^* N^*)$. Counting dimensions, we see that this is an isomorphism, so $Cl(W,b)$ is the algebra of endomorphisms of a vector space, which has a unique irreducible representation. The outcome here is a projective representations of the orthogonal group $O(W,b)$ on $\bigwedge^*(N^*)$. I want to get to $SO(W,b)$. Given $g \in Cl(W,b)$, we act on $\bigwedge^*(N^*)$. This action is even if and only if $g \in SO(W,b)$.

Definition 23.2. This is the *spinor representation* of $\mathrm{Spin}(W,b) \xrightarrow{2} SO(W,b)$. ◇

Since we're under the assumption that the signature of b is zero, we should take care of that. It should be clear that we should not have done two different cases. We should have just started with a symplectic super vector space. The claim is that you can do it in this generality [[★★★ Project 8: do this geometric quantization story in this case. There are now 10 projects online.]][[★★★ Another project (10): in the bosonic case, we said that once you have an irrep of the Heisenberg, you can't extend it to all of C^∞. In the odd case, if you have $h(W,b) \rtimes O(b) \cong \bigwedge^{\leq 2} W^*$, which acts on $End(\bigwedge^* N^*)$. But $\bigwedge^{\leq 2} W^* \subseteq \bigwedge^*(W^*) = C^\infty(\pi W^*)$. Can we extend the representation to $C^\infty(\pi W^*) \to End(\bigwedge^* N^*)$. There is a filtration preserving iso $Cl^{\leq 2}(W) \cong \bigwedge^{\leq 2} W^*$. We have a map of associative algebras $Cl(W,b) \to End(\bigwedge^* N^*)$. If we had an iso of super Lie algebras $Cl(W,b) \cong C^\infty(\pi W^*)$, you could extend the representation.]]

What if $sign(b) \neq 0$? There are two answers.

1. (this is the usual one in physics books) Assume $\dim W = 2n$. Then remove the signature by complexifying: $(W\otimes_{\mathbb{R}}\mathbb{C}, b\otimes\mathbb{C})$. Because we assumed even dimension, we can still write this as $(W\otimes_{\mathbb{R}}\mathbb{C}, b\otimes\mathbb{C}) \cong (N_{\mathbb{C}}, N^*_{\mathbb{C}}, \left(\begin{smallmatrix} 0 & ev \\ ev & 0\end{smallmatrix}\right))$, which we call a *complex polarization*. The story continues just as above, but now you have a complex vector space. You're now dealing with CS super manifolds. Then $Cl(W,b)\otimes_{\mathbb{R}}\mathbb{C} \cong Cl(W\otimes\mathbb{C}, b\otimes\mathbb{C}) \to End(C^\infty(\pi N_{\mathbb{C}})) \cong \bigwedge^*_{\mathbb{C}}(N^*_{\mathbb{C}})$. The dimension count argument still works, so this is still an isomorphism. The outcome is that we get a *complex* representation of the real group $SO(W,b) \subseteq SO(W\otimes\mathbb{C}, b\otimes\mathbb{C})$. In particular, we can now talk about $\mathrm{Spin}(2n)$ and $SO(2n)$.

   If you take this representation, you can just pull back

$$\begin{tikzcd} & S^1 \arrow[r, equal] \arrow[d] & S^1 \arrow[d] \\ \mathrm{Spin}(W,b)\times_{\{\pm 1\}} S^1 \arrow[r, phantom, "\cong"] & \mathrm{Spin}^c(W,b) \arrow[r] \arrow[d] & U(\bigwedge^* N^*) \arrow[d] \\ & SO(W,b) \arrow[r] & PU(\bigwedge^* N^*) \end{tikzcd}$$

   It is a little work to show that the maps from Spin^c factor through Spin. But what do we do if W is not even dimensional?

2. If $b \cong \left(\begin{smallmatrix} I_{n_+} & 0 \\ 0 & -I_{n_-}\end{smallmatrix}\right)$, so $sign(b) = n_+ - n_-$ and $rk(b) = n_+ + n_- = n$. Then take the orthogonal direct sum $W \perp \mathbb{R}^n = (W, b) \oplus (\mathbb{R}^n, \left(\begin{smallmatrix} I_{n_-} & 0 \\ 0 & -I_{n_+}\end{smallmatrix}\right)) \cong (N \oplus N^*, \left(\begin{smallmatrix} 0 & ev \\ ev & 0\end{smallmatrix}\right))$, where $\dim N = n = \dim W$. By this trick, we get $Cl(W \perp \mathbb{R}^n) \xrightarrow{\sim} End(\bigwedge^* N^*)$. But $Cl(W \perp \mathbb{R}^n) \cong Cl(W) \otimes Cl_{n_-,n_+}$, where $Cl_{m,n} = Cl(\mathbb{R}^{n+m}, \left(\begin{smallmatrix} I_n & 0 \\ 0 & -I_m\end{smallmatrix}\right))$. Since these b's are diagonalizable, these are all the clifford algebras. Watch out, this is a graded tensor product of *super algebras*. Now you can think about it like this. $\bigwedge^* N^*$ is a graded irreducible $Cl(W)$-Cl_{n_-,n_+}-bimodule (the signs you pick up are exactly cancelled by the signs from the tensor product).

[[break]]

Outcome: $\bigwedge^* N^*$ is graded irreducible $Cl(W, b)$-Cl_{n_-,n_+}-bimodule and $\mathrm{Spin}(W, b) \overset{2}{\to} SO(W, b)$ acts on it, *commuting with the Cl_{n_-,n_+} action.* As a bimodule, this guy is unique, as before (if you preserve gradings at least), then you play the same game to get a projective representation of $SO(W, b)$ which lifts to $\mathrm{Spin}(W, b)$. In this case, the lifting is easy to check because everything is real, so you can only get ± 1 for phases. That's also the way to show it for the metaplectic group.

Now I want to put these things on bundles. There are two generalizations of these linear discussions. (a) Discuss the vector bundle case. This is the "family version" of what we've been doing. (b) Curved case of symplectic super manifolds. If you work purely fermionically, it's all linear. The intersection of these two is the tangent bundle of a symplectic manifold. I want to say a few things about (b), but basically we'll follow (a).

(b) Let's do the bosonic case first. Let (M, ω) be a symplectic manifold [[★★★ Project 9 is to do all this for super symplectic manifolds.]]. Prequantization worked very nicely to give us a Lie algebra homomorphism $C^\infty M \to End(\Gamma_{C^\infty}(L \to M))$ $(f \mapsto \nabla_{X_f} + m_f)$ by writing ω as the curvature of a connection on the line bundle $L \to M$. For geometric quantization, we need a polarization $P \subseteq TM$, a Lagrangian subbundle (which, it will turn out, has to be integrable/involutive). If you have this, then M^{2n} looks locally like a cotangent bundle $\mathbb{R}^n \times \mathbb{R}^n$. In this case, you can half the space $\Gamma_{C^\infty}(L \to M)$ by taking the sections that are covariantly constant with respect to P, i.e. if $X \in \Gamma(P) \subseteq Vect(M)$, then $V_P := \{s \in \Gamma(L \to M) | \nabla_X = 0\ \forall x \in \Gamma(P)\}$.

Now we will get a representation of some subalgebra of $C^\infty(M)$ on V_P. What is this subalgebra? Which functions act? V_P has a caonical action by $C^\infty(M)_P = \{f \in C^\infty M | X_(f) = 0\ \forall X \in \Gamma(P)\}$.

Example 23.3. If $(M, \omega) \cong (T^*N, d\alpha)$, then the typical P is $P_{vert} \subseteq T(T^*N)$. If N had a metric, then we would get a canonical connection, but in general, the vertical subbundle is the only canonical thing around. Now the claim is that $V_P \cong C^\infty N$ (canonically). We are taking sections which are covariantly constant in the vertical direction, i.e. functions which are constant along fibers. If you pick different isomorphisms $(M, \omega) \cong (T^*N, d\alpha)$, then you get different polarizations, which lead to different geometric quantizations. The subalgebra $C^\infty(T^*N)_{P_{vert}} \cong C^\infty N$ acts on $C^\infty N$ by multiplication operators. This is terribly boring.

In our linear example, we took $M \cong N \oplus N^* = T^*N$, which is a special case of this example. If T^*N happens to be linear, then we got the differentiation operators as well. So if you try harder, in some cases, you can extend the action. But in general, the machine doesn't give you much. This is why people say quantization is an art, not a functor.

In fact, on can quantize more functions on $C^\infty(T^*N)$. In particular, those that are linear on the fibers (these are the p's, momenta) and some that are quadratic in fibers (e.g. Riemannian metrics). Remember that $V_{P_{vert}} \cong C^\infty N$. A Riemannian metric g on N acts by the corresponding Laplacian Δ_g. One way to get this is to do the path integral (I'm thinking of only potential energy). ◇

12 RB 11-13 Bernstein polynomials

In the last lecture we ran into the following problem. We wanted to define a Feynman diagram as something like a product of propagators $\prod \Delta_s(x_i - x_j)$, where $\Delta_s(x_i, x_j)$ should be analytic in s and continuous for $s \gg 0$ (so the product is well defined) and we analytically continue the product to the s we're interested in (where there actually turns out to be a pole).

More generally, suppose $p(x)$ is any polynomial in $x = (x_1, \ldots, x_n)$. Can we analytically continue $|p(x)|^s$? The answer is yes; you can use a Bernstein polynomial $b(s)$, which has the property that $b(x)p(x)^{s-1} =$ (some polynomial diff operator)$p(x)^s$. You pick up poles from the zeros of $b(s)$ (the poles of the analytic continuation will be at integer shifts of the zeros of $b(s)$, since you have to repeat).

Example 12.1. $sx^{s-1} = \frac{d}{dx}x^s$. This is essentially how you prove the analytic continuation of the gamma function. ⋄

Example 12.2. $p(x) = x_1^2 + \cdots + x_n^2$. This is what you're interested in if you want to raise Laplacians to various powers. Well, $\frac{d}{dx_1}p(x)^s = 2x_1 p(x)^{s-1}$, so $\frac{d^2}{dx_1^2} = 2sp(x)^{s-1} + (2x_1)^2 s(s-1)p(x)^{s-2}$, so

$$\sum_i \frac{d^2}{dx_i^2} p^s = (2ns + 4s(s-1))p^{s-1}$$

so the Bernstein polynomial is $b(s) = 2ns + 4s(s-1)$. ⋄

Example 12.3. $p(x) = x^2 + y^3$. If you manage to find the Bernstein polynomial of this, I'll be very impressed. It takes a huge amount of calculation. The answer $b(s) = (s+1)(s+5/6)(s+7/6)$. Direct calculation is very hard. ⋄

The point is that finding Bernstein polynomials in general is very difficult. Fortunately, we don't always need to know $b(s)$ explicitly. Bernstein showed how to prove that it exists without actually producing it. The existence result is actually very powerful.

Application to show the strength of the Bernstein polyomial: the Malgrange-Ehrenpreis Theorem. This is the fundamental theorem of differential operators with constant coefficients. People spent about a hundred years trying to prove it. It says that partial differential operators with constant coefficients have a Green function, i.e. $p(b)f = \delta$ has a solution f for any polynomial p in $\frac{\partial}{\partial x_i}$ with constant coefficients. This is actually false if you allow polynomial coefficients, which was surprising.

Proof. Take Fourier transforms. We have to solve $\tilde{P}(p)\tilde{f} = 1$, where $P(p)$ is a polynomial in P. So we have to find a *distributional* inverse of a polynomial $P(p)$. In the case where it is positive, this is trivial because you can invert it. We can assume $P \geq 0$ by replacing it with $P\overline{P}$ (if we can invert this, it will be easy to invert P). The case $P > 0$ is trivial. You might think that you can then take a limit, but there is a problem. The problem is the complexity of a singularity of $P(p)$. Hironaka's resolution of singularities can be used to do this. The nice thing about Bernstein's polynomial is that it is easier to prove and can often be used as a substitute for Hironaka's resolution of singularities.

Look at $P(p)^{s-1} = a_{-n}s^{-n} + \cdots + a_0 + \cdots$ near $s = 0$. We're using Bernstein to analytically continue and look at the Laurent expansion. $P(p)P(p)^{s-1} = P(p)^s = 1 + (\quad)$ near $s = 0$ (if $P \not\equiv 0$). So $P(p)a_{-n} = 0$, $P(p)a_{1-n}, \ldots,$ $P(p)a_0 = 1$, so a_0 is a distributional inverse to $P(p)$ as desired. □

Warning 12.4. The inverse of $P(p)$ is NOT unique. This seems odd because it is easy to prove that it is unique: suppose $aP = 1 = bP$, then $a = aPb = b$. This proof is wrong because multiplication of distributions by polynomials is NOT ASSOCIATIVE! For example, $\left(\frac{1}{x} \cdot x\right) \cdot \delta(x) = \delta(x) \neq 0 = \frac{1}{x} \cdot 0 = \frac{1}{x} \cdot \left(x \cdot \delta(x)\right)$. ⌟

PT: is there a finite-dimensional space of inverses? RB: the space of inverses is usually infinite-dimensional. If you have a function of more than one variable, say $p(x, y)$, then if you have a function on $p(x, y) = 0$, you get something. Even when $p(x, y) = 0$ is a zero-dimensional variety, you can still run into trouble.

Example 12.5. Say you want to solve $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f(x, y) = \delta$, with f rotationally invariant. The solution is $f(x, y) =$const$\cdot \log(x^2+y^2)+$arbitrary constant. This arbitrary constant is exactly arising from the pole. ⋄

The point is that this problem of having unknown functions really does turn up in simple natural examples. This shows that when you try to quantize something something, then dialation invariance is automatically broken.

The proof of Bernstein polynomial has lots of useful ideas which should be in every mathematician's kit. It uses $\mathcal{D}$-modules (in fact, a lot of $\mathcal{D}$-module stuff was invented to prove this). A $\mathcal{D}$-module is a module over the ring $\mathcal{D} = k[x_1, \dots, x_n, \partial_1, \dots, \partial_n]$. This is a non-commuting ring in the sense that $[x_i, x_j] = 0 = [\partial_i, \partial_j]$, and $\partial_i x_j = x_j \partial_i + \delta_{ij}$. This is basically the universal enveloping algebra of a Heisenberg algebra.

Example 12.6. S: all smooth functions on $\mathbb{R}^n$ with x_i acting by multiplication by x_i and ∂_i acting by $\frac{\partial}{\partial x_i}$. ◇

Example 12.7. Suppose you have a system of PDEs D_1, D_2,... with polynomial coefficents. Then we get a $\mathcal{D}$-module $M = k[x_i, \partial_i]$/left ideal generated by D_i. Note that Smooth solutions to the system are exactly the same thing as $\mathcal{D}$-module homomorphisms from M to S. So modules over $\mathcal{D}$ should give you some insight into solutions to systems of PDEs. ◇

Recall some basic commutative algebra. Recall that the *Hilbert polynomial* of a finitely generated graded $k[x_1, \dots, x_n]$-module $M = \bigoplus_{i\geq 0} M_i$ is given by $p(i) = \dim_k \bigoplus_{j=0}^{i} M_j$ for some polynomial p for large i. This polynomial in some sense measures the size of the module. The existence of the Hilbert polynomial is easy using induction on n. For $n = 0$, this is completely trivial because M is a finite dimensional vector space so $M_i = 0$ for $i \gg 0$, so $p(i) = \dim M$ for $i \gg 0$. For $n > 0$, you consider $M_{i-1} \xrightarrow{\cdot x_n} M_i$ and look at the kernel and cokernel:

$$0 \to \ker(x_n)_{i-1} \to M_{i-1} \xrightarrow{\cdot x_n} M_i \to (M/x_n M)_i \to 0$$

Both the kernel and cokernel are modules over $k[x_1, \dots, x_{n-1}]$ so by induction, they both have Hilbert polynomials. So $\dim M_i - \dim M_{i-1}$ is a polynomial for $i \gg 0$ by induction. But then $\dim M_i$ is also a polynomial (of one degree higher) for $i \gg 0$.

If a polynomial $p(n)$ is integral for all large integer n, p need not have integer coefficients. For example, $p(n) = n(n+1)/2$. However, $p(n)$ is an integral linear combination of $\binom{n}{k} x^k$. This is an easy exercise. So if p has degree k, the leading coefficient is $\frac{m}{k!} n^k + \cdots$ for some integer m.

If M is a module with Hilbert polynomial $p(n) = \frac{m}{k!} n^k + \cdots$, then k is called the *dimension* of M and m is called its *multiplicity*. These are the most important measure of size of M. The complete Hilbert polynomial depends on choices, but these values do not.

The ring we want to work with is not commutative so we have the following problem: we want to turn the non-commutative ring $\mathcal{D}$ into a commutative ring so that we can apply the nice theory of Hilbert polynomials to its modules. Well, $\mathcal{D}$ is very close to being commutative. We have the *Bernstein filtration* $B_0 \subseteq B_1 \subseteq \cdots$, where B_i is the set of polynomials in x_i and ∂_i of degree less than or equal to i. You have to be careful talking about degree; there is no such thing as a homogeneous polynomial (e.g. is $x_i \partial_i - \partial_i x_i = 1$ of degree 2 or 1?), but you can talk about thing of degree *at most* i. B_i and B_j commute modulo terms in B_{i+j-1}. So we can form a commutative ring of about the same size: $B_0 \oplus B_1/B_0 \oplus B_2/B_1 \oplus \cdots$. This has the same generators, but is now a commutative polynomial ring.

Now let's take a finitely generated module M over this non-commutative ring $\mathcal{D}$, generated by some finite vector space M_0. Let $M_i = B_i \cdot M_0$, so we have $M_0 \subseteq M_1 \subseteq \cdots$. Notice that $B_i M_j \subseteq M_{i+j}$, so $M_0 \oplus M_1/M_0 \oplus M_2/M_1 \oplus \cdots$ is a graded module over $\bigoplus B_i/B_{i-1}$, which is about the same size as the original module. Now we can start applying Hilbert polynomials and things.

Warning 12.8. This module $\bigoplus M_i/M_{i-1}$ depends on the choice of M_0. The Hilbert polynomial changes if M_0 changes. So we can't talk about the Hilbert polynomial, but it turns out that the multiplicity and dimension of $\bigoplus M_i/M_{i-1}$ do not depend on the choice of M_0, so we can talk about the multiplicity and dimension of M. ⌟

Next week we'll finish off the proof of existence of the Bernstein polynomial.

33 NR 11-14

I updated my website, so now there are references. There is also a link to projects. If you want a project for this class, you should talk to me, preferably this week. Today 2-3, Friday 9-11 and 2-3. You can also find homework problems there.

Last time I was talking about the standard complex $C^\bullet(\mathfrak{g},\mathfrak{g}) = \bigoplus_{n\geq 0}\mathrm{Hom}_\mathfrak{g}(\bigwedge^n \mathfrak{g},\mathfrak{g})$. There is another approach to this complex which is helpful for thinking about BRST quantization. Let X be a finite dimensional vector space and consider $\bigwedge^\bullet X$. Choose a basis $\{a_\alpha\}$ of X. Let $Q\colon \bigwedge^\bullet X \to \bigwedge^\bullet X$ be a derivation of this algebra (i.e. $Q(ab) = Q(a)b + (-1)^{|a|\cdot|Q|}aQ(b)$). Let $Der(X)$ be the linear space of derivations. It has a natural Lie super algebra structure: $[Q_1,Q_2] = Q_1\circ Q_2 + (-1)^{|Q_1|\cdot|Q_2|}Q_2\circ Q_1$.

Proposition 33.1. *$Der(X)\cong \mathrm{Hom}(X,\bigwedge^\bullet X)$ as a vector space.*

The proof is obvious: it is because Q is completely determined by how it acts on X, and any choice of action on X can be extended uniquely to an endomorphism of $\bigwedge^\bullet X$.

If $X=\mathfrak{g}^*$, then $Der(\mathfrak{g}^*)\cong C^\bullet(\mathfrak{g},\mathfrak{g})$ is a vector space isomorphism. Let $[\cdot,\cdot]\colon \mathfrak{g}\wedge\mathfrak{g}\to\mathfrak{g}$. It defines $c\colon \mathfrak{g}^*\to\mathfrak{g}^*\wedge\mathfrak{g}^*$. What does it mean for the bracket to satisfy the Jacobi identity? It defines an odd derivation of $\bigwedge^\bullet \mathfrak{g}^*$.

Proposition 33.2. *The Jacobi identity for $[\,,]$ holds if and only if $[c,c] = 2c^2 = 0$.*

Proof. [[★★★ HW: direct computation]] ☐

Corollary 33.3. *A Lie algebra structure on $\mathfrak{g}$ defines a differential on $Der(\mathfrak{g}^*) = \mathrm{Hom}(\mathfrak{g}^*,\bigwedge^\bullet\mathfrak{g}^*) = C^\bullet(\mathfrak{g},\mathfrak{g}) = \mathrm{Hom}(\bigwedge^\bullet\mathfrak{g},\mathfrak{g})$, given by $d_c(a) = [c,a]$ for $c,a\in C^\bullet(\mathfrak{g},\mathfrak{g})\cong Der(\mathfrak{g}^*)$.*

You can think of $Der(X)$ as $Vect(X[1])$, vector fields on X (thought of as an odd vector space). You can generalize this to the case where X is a super vector space.

Last time I used $C^\bullet(\mathfrak{g},\mathfrak{g})$ to talk about deformation theory, using mysterious words like "Gerstenhabber". The bracket on $Der(\mathfrak{g}^*)$ is actually the Gerstenhabber bracket. The Schouten bracket is another example of such a construction.

So $c\in C^2(\mathfrak{g},\mathfrak{g})$ is a Lie algebra structure on $\mathfrak{g}$. We want to deform it. A formal deformation is $c^h = c+\sum_{n\geq 1}h^n c^{(n)}$ where $c^{(n)}\in C^2(\mathfrak{g},\mathfrak{g}) = \mathrm{Hom}(\bigwedge^2\mathfrak{g},\mathfrak{g})$. We want $[c^h,c^h]=0$ again (i.e. we still want a Lie algebra structure). What does this mean for the coefficients $c^{(n)}$. The coefficient of h^n is

$$\overbrace{[c,c^{(n)}]+[c^{(n)},c]}^{2[c,c^{(n)}]} + \sum_{k=1}^{n-1}[c^{(k)},c^{(n-k)}] = 0 \qquad (*)$$

We write $d_c(c^{(n)}) = [c,c^{(n)}]$.

Lemma 33.4 ([[★★★ HW]])**.** *$d_c\big(\sum_{k=1}^{n-1}[c^{(k)},c^{(n-k)}]\big) = 0$ assuming $c^{(1)},\dots,c^{(n-1)}$ satisfy $(*)$.*

So $(*)$ is $d_c(c^{(n)})+z^{(n)}=0$ for some $z^{(n)}\in Z^3(\mathfrak{g},\mathfrak{g})$. Assume inductively that $[z^{(n)}]=0$ in $H^3(\mathfrak{g},\mathfrak{g})$. The inductive step is to construct a $c^{(n)}$ so that $dc^{(n)} = -z^{(n)}$ such that $[z^{(n+1)}]=0$. If $H^3(\mathfrak{g},\mathfrak{g})=0$, then there is no problem; you can choose $c^{(n)}$ without any trouble. In general it is a very non-trivial problem. The whole deformation theory of associative algebras can be formulated this way.

When Kontsevich classified $*$-products, the result itself wan't the striking thing. He constructed a very explicit $*$-product, which has to do with some field theory.

Now I want to return to quantization of guage systems. Reminder about L_∞-algebras. Say X is a super vector space. Consider $P(X) = X\oplus\mathrm{Sym}^2(X)\oplus\cdots$ (like functions on X^* vanishing at 0). This is a super commutative (non-unital) algebra. Then a super derivation $Q\colon P(X)\to P(X)$ is something satisfying $Q(ab) = Q(a)b+(-1)^{aQ}aQ(b)$.

1. $Der(P(X))\cong\mathrm{Hom}(X,P(X))$ as before (because the action on a monomial is determined by the action on degree 1 elements).
2. $Der(P(X))$ forms a Lie super algebra, with the usual bracket.

Definition 33.5. A *formal pointed differential graded manifold* (or an *L_∞ algebra*) is a pair $\big(P(X),Q\big)$ where Q is an odd differential with $Q^1 = \frac{1}{2}[Q,Q]=0$. ◇

Example 33.6. If $\{c_\alpha\}$ is a basis for X, then if $Q(c_\alpha) = \sum_{\beta,\gamma} c_\alpha^{\beta\gamma} c_\beta c_\gamma$, this is where X^* is a Lie algebra. ⋄

Example 33.7. If $Q(c_\alpha) = \sum_\beta d_\alpha^\beta c_\beta + \sum_{\beta,\gamma} c_\alpha^{\beta\gamma} c_\beta c_\gamma$, where $d\colon X^*[1] \to X^*[1]$, $c\colon X^*[1] \wedge X^*[1] \to X^*[1]$ Lie bracket. Then $Q^2 = 0$ implies $d^2 = 0$, so d is a derivation of c.

This is the notion of X^* being a *differential graded Lie algebra.* $x \in X$, $Q(x) = d(x) + c(x)$. If X is a super vector space then this gives you a super differential graded Lie algebra. ⋄

Suppose $E = E_0 \oplus E_1$ is a super vector space. Then it is natural to assume we have $d\colon E \to E$ with $d^2 = 0$, with $d(E^i) \subseteq E^{1-i}$. Then we can construct the cohomology space $H = \ker d / \operatorname{im} d$. There is a general fact from linear algebra. After some work, it is also known as the Lipschitz theorem.

Assume $A\colon E \to E$ so that $[A, d] = 0$. Then A also defines a map $[A]\colon H \to H$. Fact: $\operatorname{str}_E(A) = \operatorname{str}_H([A])$. We are given the right hand side by some path integral, and we're trying to construct an A with some properties on some larger space with all these fermions. All these things (Fadeev-Popov, BRST, etc.) are different constructions of such an A.

24 PT 11-15

Today we'll put linear quantizations on bundles. Let $\pi\colon E \to X$ be a real vector bundle of fiber dimension k on a topological space X. Let $P(E) \to X$ be the corresponding bundle of frames (this is actually a $GL(k)$-principal bundle).

(1) We want to reduce the structure group from $GL(k,\mathbb{R})$ to a subgroup G. (2) Given some extension $A \hookrightarrow \widetilde{G} \twoheadrightarrow G$, we want to lift G-bundles to $\widetilde{G}$-bundles (this is sometimes called "reducing").

For (1), we can formulate it in terms of (0) putting extra structure on E (a) transition functions, (b) classifying spaces.

Definition 24.1. *Reducing the structure group to* G means finding a G-principal bundle $P \to X$ and an isomorphism of bundles $P \times_G GL(k) \cong P(E)$. ⋄

In terms of transition functions, you take an open covering $\{U_i\}$ of X with trivializations of $P(E)$ (or E if you like). Then the transition functions $\phi_{ij}\colon U_i \cap U_j \to GL(k)$ measure the difference between the two trivializations. An automorphism of the trivial bundle $(U_i \cap U_j) \times GL(k) \to U_i \cap U_j$ is the same as a transition function. Because the bundle is a bundle, the transition functions satisfy the cocycle condition $\phi_{kj} \circ \phi_{ji} = \phi_{ki}$ on $U_i \cap U_j \cap U_k$. If you order the indices and only consider $i < j$, then you don't have to have $\phi_{ii} = \mathrm{id}$ and $\phi_{ij} = \phi_{ji}^{-1}$. On the other hand, these data (covering with transition functions satisfying the cocycle condition) exactly recover the data of the bundle. Restricting is just factoring ϕ_{ij} trough $G \subseteq GL(k)$ (you're allowed to change the ϕ_{ij} by a coboundary, which doesn't change the bundle). This happens when the cohomology class lies in $H^1(X; G)$.

Theorem 24.2. *$H^1(X;G)$ (actually $\check{H}^1(X;G)$) naturally parameterizes isomorphism classes of principal G-bundles over X, which are also parameterized by homotopy classes of maps to BG.*

There is a contractible principal G-bundle $EG \to BG$ so that for a principal G-bundle $P \to X$, there is a unique homotopy class of maps $X \to BG$ so that P is the pullback of EG. Fact: $BGL(k)$ is the Grassmanian of k-planes in $\mathbb{R}^\infty$.

Let $EGL(k)$ be the total space of the universal $GL(k)$-bundle. Given $G \subseteq GL(k)$, I claim that $EGL(k)/G \cong BG$. This is because $EGL(k) \to EGL(k)/G$ is a universal bundle. We can divide out further:

$$\begin{tikzcd}
EGL(k) \arrow[d] & GL(k)/G \arrow[d, "\text{fiber}"] & \\
EGL(k)/G \arrow[d, two heads] \arrow[r, phantom, "\cong"] & BG \arrow[d] & \\
EGL(k)/GL(k) \arrow[r, phantom, "="] & BGL(k) & X \arrow[l] \arrow[ul, dashed]
\end{tikzcd}$$

If $GL(k)/G \cong *$, then $BG \cong BGL(k)$ and therefore $[X, BG] \cong [X, BGL(k)]$. If you wanted to use the transition function point of view, you'd use partitions of unity.

$$\begin{tikzcd}
 & & GL(2n,\mathbb{R}) \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash, "\simeq"] & \\
 & Sp(n) \arrow[dl, dash] \arrow[dr, dash, "\simeq"'] & GL(n,\mathbb{C}) \arrow[d, dash, "\simeq"] & O(2n) \arrow[dl, dash] \\
GL(n,\mathbb{R}) \arrow[dr, dash, "\simeq"'] & & U(n) \arrow[dl, dash] & \\
 & O(n) & &
\end{tikzcd}$$

If $k = 2n$, then $GL(2n,\mathbb{R})$ has three nice subgroups (corresponding to putting symplectic, complex, or inner product structure on a real bundle). Which of these quotients are contractable? The first obstruction to $GL(2n,\mathbb{R})/GL(n,\mathbb{C})$ being contractible is that complex vector bundles have natural orientations; the same problem shows up for Sp. $GL(n,\mathbb{R})$ has two components, but $GL(n,\mathbb{C})$ and $Sp(n)$ are connected. The obstructions to lifting lie in $H^{i+1}\big(X, \pi_i(GL(k)/G)\big)$. So if you have some π_0, you get some obstruction (called the first Steifel-Whitney class) in H^1. In general, the obstructions can be identified with certain characteristic classes of the bundle.

What is the inclusion $GL(n,\mathbb{R}) \to Sp(n)$? It is given by $A \mapsto \begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{pmatrix}$, the symplecomorphisms that preserve the polarization $\mathbb{R}^{2n} \cong N \oplus N^*$, with ω corresponding to $\begin{pmatrix} 0 & ev \\ -ev & 0 \end{pmatrix}$. This is not a homotopy equivalence, so there is some obstruction to finding a polarization on a symplectic vector bundle.

Now let's discuss separately the two cases of $Sp(n)$ and $O(k)$ (k need not be even). I have a bundle $E \to X$. $Sp(n)$ structure would give me a non-degenerate skew form $\omega \in \Gamma(\bigwedge^2 E^*)$ on E. $O(k)$ structure would give me a fiberwise positive definite for $b \in \Gamma(\mathrm{Sym}^2 E^*)$ on E. By the way, I don't assume any integrability condition. Usually a symplectic (or complex) structure has the additional condition that the form be closed, and we're ignoring this for now.

Let's say we got a symplectic structure ω. Next, we look for a polarization (a $GL(n,\mathbb{R})$ structure). Again, we get obstructions (and again we ignore the integrability condition on the polarization).

Claim. *We have the following diagram.*

$$\begin{tikzcd}
\mathbb{R}^\times \arrow[d, two heads, "2", "square"'] & ML(n) \arrow[l] \arrow[d, two heads, "2"] \arrow[r, phantom, "\subseteq"] & Mp(n) \arrow[d, two heads, "2"] \\
\mathbb{R}_{>0} & GL^+(n) \arrow[l, "\det"'] \arrow[r, phantom, "\subseteq"] & Sp(n)
\end{tikzcd}$$

[[★★★ PT: no, this is not right. Neither square is right, but that might cancel out.]] In particular, if $E \to X$ is TX, X is a symplectic manifold, the polarization $P \subseteq TX$ is integrable (i.e. we have a foliation $\mathcal{F}$ of X whose tangent bundle is P), and a metaplectic structure on TX gives a metalinear structure on each leaf of $\mathcal{F}$ (i.e. $\bigwedge^{-top/2} \mathcal{F}$ exists).

The reason I mention this is, as I pointed out on Tuesday, if you have T^*N, you get functions on N ($= X/\mathcal{F}$, the space of leaves), but you're missing an inner product. These things give us "half-forms" and resolve the inner product problem. Instead of looking at functions, we look at these half-forms. Then you can multiply two of them to get a volume form, which you can integrate.

[[break]]

Let's go back to this problem. wWhat does it mean to lift the structure group from G to $\widetilde{G}$. It means finding factoring the ϕ_{ij} through $\widetilde{G} \to G$ in such a way that the lifts still satisfy the cocycle condition. In general, $c_{ijk} = \tilde{\phi}_{kj}\tilde{\phi}_{ji}\tilde{\phi}_{ij}^{-1}$ will be a map $U_i \cap U_j \cap U_k \to \ker(\widetilde{G} \to G) =: A$.

Assume A is abelian (in our case it will be $\mathbb{Z}/2$). This will define a class in $\check{H}^2(X;A)$ (you have to check that c_{ijk} is a cocycle). If this c_{ijk} is a coboundary, then you can change the $\tilde{\phi}_{ij}$ by whatever c_{ijk} is the coboundary of to get it to be zero. The punch line is that

$$\begin{array}{ccccccc} H^1(X;A) & \longrightarrow & H^1(X,\widetilde{G}) & \longrightarrow & H^1(X;G) & \longrightarrow & H^2(X;A) \\ & & [\widetilde{P}] & \longmapsto & [P] & \longmapsto & [c] \end{array}$$

is an exact sequence. You can see this from the fact that

$$BA \to B\widetilde{G} \to BG$$

is a fibration, so

$$H^1(X;A) = [X, BA] \to [X, B\widetilde{G}] \to [X, BG] \to [X, K(A,2)] = H^2(X;A)$$

is exact (this can be extended to the left by taking loop spaces). But it turns out that $BA = \Omega K(A,2)$. $\pi_1(\Omega Y) \cong \pi_2(Y)$, so $BA \cong K(A,1)$.

So if you want to lift a G-bundle to a $\widetilde{G}$-bundle, the obstruction lies in $H^2(X;A)$, and the non-uniqueness is given by $H^1(X,A)$.

Given any H-principal bundle $P_H \to X$ and $\rho\colon G \to H$, we say that P_H has a "reduction" to G (over ρ) if we can find a G-principal bundle and an isomorphism of H-principal bundles $P_G \times_G H \cong P_H$.

Example 24.3. $\mathbb{Z}/2 \to Mp(n) \to Sp(n)$ and $\mathbb{Z}/2 \to \mathrm{Spin}(n) \to SO(n)$. In both cases, it turns out that the obstruction is $w_2(E) \in H^2(X;\mathbb{Z}/2)$, the second Steifle-Whitney class.

The first Steifle-Whitney class $w_1 E \in H^1(X;\mathbb{Z}/2)$ gives you the obstruction to reducing to $O(2n)$.

To get the symplectic structure, you need more things to vanish, but the obstruction to lifting to $Mp(n)$ is the same as the obstruction of lifting from $O(2n)$ to $\mathrm{Spin}(2n)$. ⋄

Definition 24.4. Let $E^k \to X$ be an oriented vector bundle with inner product (i.e. frame bundle has $SO(k)$ structure). Then a *spin structure* on E is a lift of the structure group to $\mathrm{Spin}(k)$. ⋄

Geometric quantization leads to a more concrete picture. We'll rewind the definition on Tuesday and answer the following question: What is a spin structure on a (single) vector space?

34 NR 11-16

Last time I explained a construction from linear algebra. If $[A,d] = 0$, for A and d operators on a super vector space Σ, with d odd and $d^2 = 0$, then A defines an operator $[A]$ on $H := \ker d/\operatorname{im} d$ and $\mathrm{str}_E A = \mathrm{str}_H [A]$.

Remember that the reason I brought this up was to make sense of integrals of the form $\int \exp(iS(x)/h)\,\mathcal{D}x$. For this we need to make a connection between path integrals and traces. But I haven't yet explained what a quantum field theory is.

Classical field theory

We'll talk about Hamiltonian d-dimensional field theory. It is the following assignment

- to a $(d-1)$-dimensional (compact oriented) manifold N, we assign a symplectic manifold $S(N)$,
- to a d-dimensional M, we assign a Lagrangian $L_M \subseteq S(\partial M)$.

The axioms are

- $S(\varnothing) = *$,
- $S(N_1 \sqcup N_2) = S(N_1) \times S(N_2)$,
- $S(\overline{N}) = \overline{S(N)}$ (opposite orientation goes to opposite symplectic structure),
- $L_{M_1 \sqcup M_2} = L_{M_1} \times L_{M_2} \subseteq S(\partial M_1) \times S(\partial M_2)$.
- $L_\varnothing = *$.
- (gluing axiom) $\partial M = \partial_1 M \sqcup \partial_2 M \sqcup \partial_3 M$, with an orientation reversing isomorphism $\partial_1 M \cong_f \partial_2 M$, then let M_f be the gluing, then $S(\partial M) \supseteq L_M \to L_{M_f} \subseteq S(\partial_3 M)$.

This can be formulated as a functor.

Now let's consider Lagrangian classical field theory of first order (i.e. the Lagrangian only depends on the first jet), which is an example of this more general Hamiltonian picture. It consists of

- $F(M)$, the space of fields on M,
- $F(\partial M) = F(M)|_{\partial M}$, and $S(\partial M) = T^*F(\partial M)$,
- $L_M \subseteq S(\partial M)$ is the set of solutions to the Euler-Lagrange equations.

One can then argue that all of the axioms hold assuming the Lagrangian is non-degenerage. This is a very strong assumption; in all guage theories, the Lagrangian will be degenerate.

Ok, this is the classical picture. How do we quantize it? Let's start with classical mechanics. Waht did we discuss? If S is a symplectic manifold, then $C(S)$ is a Poisson algebra, which we can deform (at least formally) to an associative algebra $C_h(S)$, with the first jet of the deformation equal to the Poisson structure. If we have a Lagrangian submanifold L, then it gives us an ideal $I_L \subseteq C(S)$ (it is an ideal in the commutative algebra, and a Lie subalgebra, but not a Lie ideal). This quantizes to I_L^h, a one-sided (say right) ideal in the algebra $C_h(S)$.

So we have the representation (left module) $H_L = C_h(S)/I_L^h$ of $C_h(S)$. Observe also that if the quantization is flat (i.e. $C_h(S) \cong C(S)$ and $I_L^h \cong I_L$ as vector spaces), then $H_L \cong C(L)$ as a vector space (at least in the smooth case). PT: is L a Lagrangian or a polarization? NR: a Lagrangian; this is not geometric quantization. Geometric quantization has the huge advantage that from the geometry and the polarization, you produce a representation. PT: this is quite similar. NR: I think they are related as [[★★★ I didn't catch how]], so they are morally the same thing because in a small neighborhood you can always choose a polarization where these Lagrangians are the fibers.

Classically, we can have a Lagrangian $L \subseteq S$ in a symplectic manifold. Upon quantization, it becomes a left ideal I_L^h in an associative algebra $A(S)$ (which in some sense looks like the algebra of endomorphisms of some vector space, so you expect a trivial center for example, so you don't have families of irreps).

Heisenberg picture of Quantum mechanics

Strictly speaking, in quantization, we have to complexify at some point, but let me not pay attention to this at the moment. We have $C_h(S) = A$ and a family of algebra automorphisms $u_t\colon A \to A$ such that $u_0 = \mathrm{id}_A$, $u_t u_s = u_{s+t}$, and (when it makes sense) $u_t(a) = e^{iHt/h} a e^{-iHt/h}$.

Now let's try to translate this picture as a 1-dimensional quantum field theory. Classical 1-dimensional field theory is an assignment:

- To a point, assign a symplectic manifold S.
- To an interval $[t_1, t_2]$, assign $L_{t_2,t_1} \subseteq S \times \overline{S}$ defined by the Hamilton-Jacobi action (assuming the solutions are unique).

$u_{t_2-t_1}$ is the evolution operator that we assign to the segment $[t_1, t_2]$. It is a mapping from A to A, so we can think of it as an element of $A \otimes A^*$ (where A^* is a dual vector space to A; we don't try to make any completions in any topology). This can be regarded as an assignment:

- To a point (which is a compact orientated 0-manifold), we assign $A = C_h(S)$. To a point with the opposite orientation, we assign A^{op}, the algebra A with the opposite multiplication.
- To an interval $[t_1, t_2]$, assign the ideal $I^h_{L_{t_2,t_1}} \subseteq C_h(S) \otimes C_h(S)^{op}$ (assuming $C(S\times\overline{S}) = C(S) \otimes C(S)^{op}$, which is a very strong assumption; in general, I must take the Frechét completion of the tensor product).

How is this ideal realted to the automorphism u_t? The relation is the following. Given $u\colon A \to A$ an algebra automorphism, I can define $I_u \subseteq A \otimes A^{op}$ by the following formula: I_u is the right ideal generated by elements $u(a) \otimes 1 - 1 \otimes a$. The idea is that if $A = C(X)$, then if $u = \phi^*$ for $\phi\colon X \to X$, then I_u is the ideal of functions generated by $\{(x, y) \mapsto f(\phi(x)) - f(y)\}$ on $X \times X$ which vanish at $(x, \phi(x))$ [[★★★]]. You can write this ideal as $\langle u(f) \otimes 1 - 1 \otimes f\rangle \subseteq A \otimes A$.

Claim. $I_{u_{t_2-t_1}} = I^h_{L_{t_2,t_1}} \subseteq A \otimes A^{op}$.

Where L_{t_2,t_1} is the Lagrangian submanifold generated by the Hamilton-Jacobi function for the classical limit $[H]$ of H (if the deformation is flat, we can consider any element of $C_h(S)$ as an element of $C(S)$, which is what we call the classical limit). PT: we use Kontsevich's result to quantize the associative algebra, does he also show that we can quantize these ideals? Is there some analogous statement? NR: The main strength of his result is that you can quantize any Poisson manifold. For symplectic manifolds, it is easier to quantize, and [[★★★ student of Weinstein]] showed how to quantize a Lagrangian submanifold to a left ideal. PT: what is the statement? NR: for any $L \subseteq S$ Lagrangian, and for a given star product $(C_h(S), *)$ with chosen isomorphism $C_h(S) \cong C(S)$, there exists an ideal $I_L^h \cong I_L[[h]]$ as a vector space. I think there is a stronger statement that by changing the star product [[★★★ to an equivalent one?]], you can leave the ideal constant. PT: do you have something other than formal deformation in mind? NR: you can do it for family deformation in algebraic cases.

Now let's talk about the gluing. It should be the analogue of the property $u_t(a) = e^{iHt/h}ae^{-iHt/h}$. Suppose we have $[t_1, t_2]$ and $[t_2, t_3]$, with ideals $I_{t_2,t_1} \subseteq A \otimes A^*$ and $I_{t_3,t_2} \subseteq A \otimes A^*$. At some point, I did something strange. I said the automorphism u defines an element of $A \otimes A^*$, which is just linear algebra. The property $u_t u_s = u_{s+t}$ is equivalent to the following. In $A \otimes A^* \otimes A \otimes A^*$, we have $u_{t_2-t_1} \otimes u_{t_3-t_2}$; pairing the middle two, this element should map to $u_{t_3-t_1}$. So we should have some composition of ideals $I_{t_2 t_1} * I_{t_3,t_2}$ (which should be $I_{t_3 t_1}$. What is this composition? The only natural thing is trace.

We have a trace in $C^\infty(S)$, given by $\operatorname{tr} f = \int_S f \cdot \omega^n$. Assume it quantizes to a trace on $C_h(S) = A$.

Next time I'll prove that $\mathrm{id} \otimes \mathrm{tr} \otimes \mathrm{id}$ is really the composition map. It will just be the translation of the old gluing property into the new setting. I'm pretty sure this is true, but I haven't seen it written down.

35 NR 11-19

There will be no lecture on Wednesday.

Last time I made an anouncement but I didn't really explain it. Another point of view of Heisenberg evolution is ideals in the algebra of observables. Let's first have an algebraic lemma.

Let A be a unital algebra, and let $I \subseteq A \otimes A^{op}$ be a left ideal. I is also a left-right (LR) ideal in $A \otimes A$ (i.e. a sub-A-bimodule). Let I_1 and I_2 be two such LR ideals in $A \otimes A$. Define $I_1 \circ I_2 = (I_2 \otimes 1)(1 \otimes I_1) \cap A \otimes 1 \otimes A \cong A \otimes A$. [[★★★ This should be the tensor product of bimodules over A followed by intersection , $I_1 \otimes_A I_2 \cap A \otimes 1 \otimes A$]]

Lemma 35.1. *$I_1 \circ I_2$ is a LR ideal in $A \otimes A$.*

Proof. Suppose I_L is a left ideal and I_R is a right ideal in A. Then $I = I_L I_R \subseteq A$ is a two-sided ideal. So $I_{12} = (I_2 \otimes 1)(1 \otimes I_1) = \{\sum x^i \otimes y^i x_i \otimes y_i | x^i \otimes x_i \in I_1, y^i \otimes y_i \in I_2\}$ is a LDR (left, double-sided,right) ideal in $A \otimes A \otimes A$. So $I_{12} \cap A \otimes 1 \otimes A$ is naturally a LR ideal in $A \otimes A$. □

The conclusion is that we have "composition" of LR ideals in $A \otimes A$. [[★★★ nevermind checking associativity. If we define the composition as tensor product over A, it should clear]]

Heisenberg evolution in Quantum mechanics. We have $A = C_h(M)$, $u_t(a) = e^{iHt/h}ae^{-iHt/h}$ gives an algebra automorphism $u_t \colon A \to A$. We define the left ideal $I_t = \langle u_t(a) \otimes 1 - 1 \otimes a\rangle_{\text{left}} \subseteq A \otimes A^{op}$. It quantizes the vanishing ideal I_{L_t}, for $L_t = \{(x, y) \in M \times M | x = \phi_t(y)\}$. L_t is a Lagrangian subspace in $(M, \omega) \times (M, -\omega)$, so this ideal $I_{L_t} \subseteq C(M) \otimes C(M)^{op}$ (opposite Poisson bracket) is an ideal (in the commutative assiciative algebra) and a Lie subalgebra (in the Poisson structure). I_{L_t} quantizes to the left ideal I_t in $C_h(M) = A$. The Heisenberg evolution is a family of LR ideals $I_t \subseteq A \otimes A$ such that $I_t \circ I_s = I_{t+s}$. We can check that this is true:

$$\begin{aligned} I_t \circ I_s &= (I_t \otimes_A I_s) \cap A \otimes 1 \otimes A \\ &= [[\bigstar\bigstar\bigstar \text{ HW}]] \end{aligned}$$

Q: composition of Lagrangians doesn't work unless the intersection is clean; do you have the same sort of issue here? NR: yes. We'll discuss this more later. The intersections here should be clean.

Schrödinger picture. We have (1) our usual algebra of obervables $A = C_h(M)$. We choose a representation $\pi\colon A \to End(\mathcal{H})$ (H should be a Hilbert space if you want the probablistic interpretation). $U_t = \exp(it\pi(H)/h)\colon \mathcal{H} \to \mathcal{H}$, where H is the quantum Hamiltonian. (2) Let's return to the classical picture. Classically, we have the symplectic manifold (M, ω). Let $L_1, L_2 \subseteq M$ be two Lagrangian subspaces. We can choose a modified action functional $\mathcal{A}_{L_1,L_2} = \mathcal{A} + F_{L_1} - F_{L_2}$. It has the extremum on a solution to the Euler-Lagrange equations such that $\gamma(0) \in L_1$ and $\gamma(t) \subseteq L_2$. This is the flow from L_1 to L_2. We also have $L_t \subseteq (M, \omega) \times (M, -\omega)$. If the evolution and boundary conditions are good, $L_t \cap (L_1 \times L_2)$ is a single point. In the case $M = T^*N$, we can choose $L_1 = L_2$ to be the zero section $N \subseteq T^*N$.

Quantization. $L_1 \subseteq (M, \omega)$ gives the left ideal $I^h_{L_1} \subseteq A = C_h(M)$, which gives the representation $\mathcal{H}(L_1) = C_h(M)/I^h_{L_1}$. Similarly, we get $\mathcal{H}(L_2)$, another left A-module. So once we choose boundary conditions (L_1, L_2), there is a unique trajectory connecting them (if things are good). When we quantize, we get two representation of our algebra of observables. $L_t \subseteq (M, \omega) \times (M, -\omega)$ quantizes to $I_t \subseteq A \otimes A$. This produces a vanishing subspace $V_t \subseteq \mathcal{H}(L_1) \otimes \mathcal{H}(L_2)$, the subspace where the ideal I_t acts trivially: $I_t V_t = 0$. The dimension of V_t will be the number of points in $L_t \cap (L_1 \times L_2)$. So in good cases, $V_t = \mathbb{C} v_t$ is one-dimensional, with $v_t \in \mathcal{H}(L_1) \otimes \mathcal{H}(L_2)$. The is related to the Schrödinger picture like this. $\mathcal{H}(L_i)$ are Hilbert spaces, so v_t gives a linear map $\widetilde{U}_t\colon \mathcal{H}(L_1) \to \mathcal{H}(L_2)$. It is an unproven theorem that this is a linear isomorphism coinciding with U_t (in good cases, provided $\mathcal{H}(L_1) = \mathcal{H}(L_2) = \mathcal{H}$).

In the case $L_1 = L_2 = N \subseteq T^*N$, then $\mathcal{H} = C^\infty(M)$ and the evolution operator can be regarded as an integral operator with kernel $U_t(q_1, q_2)$. PT: are you thinking of N as linear or curved? NR: curved, but I'm assuming unique trajectories. PT: I think this may behave poorly in the curved case. NR: when the geodesic is not unique, it should be more complicated. Instead of a function, you get a sheaf, where for generic points you get a function, but I haven't thought about this carefully.

Summary:

1-dimensional QFT (Heisenberg): to a point, we assign A. to the opposite orientation of the point, we assign A^{op}. To the interval $[t_1, t_2]$, we assign an LR ideal $I_{t_2-t_1} \subseteq A \otimes A$. We then get gluing $I_t \circ I_s = I_{s+t}$.

(Schrödinger) to a point we assign $\mathcal{H}(L)$ for $L \subseteq M$ Lagrangian. to the point with opposite orientation, we assign $\mathcal{H}(L)^\vee$. To the interval $[t_1, t_2]$, we get $V_{t_2-t_1} \in \mathcal{H}(L_1) \otimes \mathcal{H}(L_2)$ a 1-dimensional subspace.

25 PT 11-20

Projects 5 and 10 are still open. Please let me know your preferences on

- talk and/or paper version. The papers will be due Dec. 6 so that we can read them.
- date of mini-conference (Fr/Mo afternoons Dec. 7 and 10 or all day Tuesday Dec. 11)

Clean up of the mess. This is about the symplectic group $Sp(n)$ and the metaplectic group $Mp(n)$, the unique connected double cover of $Sp(n)$. Inside $Sp(n)$, there is $GL(n)$, the subgroup of matrices of the form $\begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{pmatrix}$. Pulling back, we get the metalinear group $ML(n)$.

$$\begin{tikzcd}
\mathbb{Z}/2 \arrow[r, equal] \arrow[d] & \mathbb{Z}/2 \arrow[d] \\
ML(n) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & Mp(n) \arrow[d, "2"] \\
GL(n) \arrow[r, hook] & Sp(n)
\end{tikzcd}$$

I thought this was trouble before because $\pi_1(GL_n^+) = \mathbb{Z}/2$, so the map to $\pi_1(Sp(n)) = \mathbb{Z}$ must be the zero map. So the extension $ML(n) \to GL(n)$ is *topologically* trivial, though not group-theoretically trivial. Another example: consider

$$\begin{tikzcd}
\mathbb{Z}/4 \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & S^1 \arrow[d, "2"] \\
\mathbb{Z}/2 \arrow[r, hook] & S^1
\end{tikzcd}$$

As a topological space, $\mathbb{Z}/4 \cong \mathbb{Z}/2 \times \mathbb{Z}/2$, but not group-theoretically.

Conclusion: There is an interesting map $\mathrm{Hom}(\pi_1 G, \mathbb{Z}/2) \to H^2(G_\delta; \mathbb{Z}/2)$ (this H^2 parameterizes group extensions; the δ means discrete topology). If you want to learn more about this, come to 253 next semester. One of the things I'll do is the relation between group cohomology and extensions.

The other thing I did, which was look at $\det\colon GL(n)^+ \to \mathbb{R}_{>0}$, was not right. As a Lie group, $\mathbb{R}_> \times \mathbb{Z}/2 \cong \mathbb{R}^\times$. So we keep all of $GL(n)$:

$$\begin{tikzcd}
\mathbb{Z}/2 \arrow[d] & \mathbb{Z}/2 \arrow[d] & \mathbb{Z}/2 \arrow[r, equal] \arrow[d] & \mathbb{Z}/2 \arrow[d] \\
\mathbb{C}^\times \arrow[d, "2"'] & \{\text{x-axis}\cup\text{y-axis}\setminus 0\} \arrow[l] \arrow[d] & ML(n) \arrow[l] \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & Mp(n) \arrow[d, "2"] \\
\mathbb{C}^\times & \mathbb{R}^\times \arrow[l] & GL(n) \arrow[l] \arrow[r, hook] & Sp(n)
\end{tikzcd}$$

So we get the non-trivial extension $\mathbb{R}_> \times \mathbb{Z}/4$ of $\mathbb{R}^\times \cong \mathbb{R}_> \times \mathbb{Z}/2$.

If $E^n \to X$ is a vector bundle with metalinear structure, then $\bigwedge^n(E^*) \otimes_{\mathbb{R}} \mathbb{C}$ has a canonical square root: half forms (if $E = TX$). $TX \to X$, then $L \to X$ is a line bundle such that $L^{\otimes 2} \cong_\rho \bigwedge^n(T^*X)$. Then we get a pairing $\Gamma(L) \times \Gamma(L) \to \mathbb{C}$, given by $(s_1, s_2) \mapsto \int_X \rho(s_1 \otimes s_2)$ (if X is oriented).

Spin structures on inner product spaces

Let (W, b) be a real *inner product* space (i.e. finite dimensional with positive definite symmetric bilinear form). We know

- $O\big((W_1, b_1), (W_2, b_2)\big) = \{f\colon W_1 \xrightarrow{\sim} W_2 \text{ linear} \mid f^* b_2 = b_1\}$,
- orientations o_1 and o_2 on W_1 and W_2 are equivalence classes of orthonormal bases, so we know what $SO\big((W_1, b_1, o_1), (W_2, b_2, o_2)\big) \subseteq O\big((W_1, b_1), (W_2, b_2)\big)$.

Question: define a notion of "spin structure" σ on (W, b) such that $\mathrm{Spin}\big((\mathbb{R}^n, \mathrm{std}), (\mathbb{R}^n, \mathrm{std})\big) \cong \mathrm{Spin}(n)$ canonically. Bruce: left ideal in $Cl(W, b)$? PT: maybe.

Definition 25.1. A *spin structure* on (W, b) is a graded irreducible $Cl(W, b)$-Cl_n-bimodule (where $Cl_n = Cl(\mathbb{R}^n, \mathrm{std})$, where $e_i^2 = -1$). The $Cl(W, b)$ are $*$-algebras, and modules should come equipped with an inner product such that $Cl(W, b) \to End(M)$ is a morphism of $*$-algebras. ◇

[[★★★ HW 6: there are exactly two isomorphism classses of such bimodules, and they correspond canonically to orientations on W. So an orientation is a choice of isomorphism class of such a bimodule, but a spin structure is an actual choice of bimodule. Since you only care about isoclasses for this homework, it is enough to do it for $\mathbb{R}^n$ with the standard spin structure.]]

Example 25.2. Let $(W, b) = (\mathbb{R}^n, \mathrm{std})$. If you ignore the grading, there is only one irreducible Cl_n-Cl_n-bimodule (if you don't take grading into account). ChrisSP: A is only irreducible as an A-A-bimodule if A is simple. PT: yes, ok. If A if finite dimensional simple, then it is $End(D)$ for some division ring D, and then A is the only irreducible bimodule [[★★★]]. Cl_n is simple as a *super algebra*, so therefore a spin structure on $\mathbb{R}^n$, called the *standard* spin structure on $\mathbb{R}^n$. The isomorphism class is given by πCl_n, the parity reversed bimodule. ⋄

Theo: where is the orientation? Andy: it comes from the right action of Cl_n. The right action by $e_1 \cdots e_n$ (note that we make this one choice once and for all) is equivalent to the left action of some orientation.

Definition 25.3. $\mathrm{Spin}\big((W_1, b_1, \sigma_1), (W_2, b_2, \sigma_2)\big) := \{(f, F) | f \in O(W_1, W_2), F\colon {}_{Cl(W_1,b_1)}\sigma_1{}_{Cl_n} \xrightarrow{\sim} {}_{Cl(W_1,b_1)}Cl(f)_*\sigma_2{}_{Cl_n}$ even bimodule iso$\}$ ⋄

Theorem 25.4. *The map* $\mathrm{Spin}\big((W_1, b_1, \sigma_1), (W_2, b_2, \sigma_2)\big) \to SO\big((W_1, b_1, o_{\sigma_1}), (W_2, b_2, o_{\sigma_2})\big)$*, given by* $(f, F) \mapsto f$ *is a connected double cover.*

[[break]]

Proof. [[★★★ HW 2]] □

The idea is Schur's lemma. In the easiest case, where $W_1 = W_2$, then let's check that over $f = \mathrm{id}$, there are two points. By Schur's lemma, an isomorphism must be a multiple of the identity, and it must be $\pm\,\mathrm{id}$ because it has to respect the inner product on the module.

Classical (Lagrangian) field theory $\mathcal{L}$:

- space-time Σ^d
- classical fields $\Phi(\Sigma)$
- classical action $\mathcal{A}\colon \Phi(\Sigma) \to \mathbb{R}$ (could depend on a finite number of deriviatives at a point)

(Step 1) From this we can get a classical Hamiltonian field theory $H_{\mathcal{L}}$ using the Euler-Lagrange equations. this is a functor from RB_d^Σ (where all the d-manifolds are submanifolds of Σ) to Symp, the category of symplectic manifolds (with a potential) with morphisms Lagrangians (with a function) (not quite a category because to compose Lagrangians, you need them to intersect cleanly). It is give by taking $H_{\mathcal{L}}(Y)$ to be the set of solutions of the Euler-Lagrange equations on Y for variations with compact support. Notice that this is a subset of $\Phi(Y)$ (sections of the pull-back bundle giving $\Phi(\Sigma)$). $H_{\mathcal{L}}(Y)$ comes equipped with a 1-form α, coming from variations on $\delta_{perm}Y$, as NR discussed. In good cases, $d\alpha$ is a symplectic structure. Let's assume that we're in a good case.

Given a morphism M from Y_0 to Y_1, then $H_{\mathcal{L}}(M)$ (solutions to the Euler-Lagrange equations on M) is a Lagrangian in $H_{\mathcal{L}}(\partial M)$. The punch line is that $\alpha|_{H_{\mathcal{L}}(M)} = d\mathcal{A}|_{\Phi(M)}$.

(Step 2) Quantization. Given any functor $H\colon \mathsf{RB}_d \to \mathsf{Symp}$, you want to quantize to a functor $Q_H\colon \mathsf{RB}_d \to \mathsf{ProjHilb}$ (projective Hilbert spaces). Note that we're throwing out the dependence on Σ. On objects, (M, α), you choose a polarization P (this is why it's an art, not a functor). NR picked one or two Lagrangians (L_1 and L_2); I'm picking a whole foliation by Lagrangians. Then you get $V = C^\infty(M/P)$ (where M/P is the space of leaves). If you assume existence of a polarization, then we want any other one will be connected to it (using the Hichen connection, for example).

On morphisms, Q uses the functional integral (path integral, but we're in higher dimensions). It isn't good enough to just compose with $H\colon \mathsf{Symp} \to \mathsf{ProjHilb}$ by taking solutions because you have to subdivide your morphism arbitrarily and then use the functor H. NR: there is functoriality in the other direction; semi-classical limit is functorial.

13 RB 11-20

[[★★★ Using absent-mindedness, I missed this class.]]

36 NR 11-26

[[★★★ Kolya was ill, so there was no class]]

26 PT 11-27

Projects 11, 12: Stuff about spin structures.

Recall that if you have an inner product space W of dimension n, then a *spin structure* is a graded irreducible $Cl(W)$-Cl_n-$*$-bimodule. There are exactly two isomorphism classes of these bimodules. If you pick a spin structure, it makes sense to talk about a two-fold covering. In the case $W=\mathbb{R}^n$, you get the usual $\mathrm{Spin}(n)$. The reason this is good is that you can now define spin strucutures on vector bundles. If $E\to X$ is a vector bundle with fiberwise inner product, then a spin structure on E is an irreducible bimodule bundle $S\to X$ over the algebra bundles $Cl(E)$ and Cl_n (this one is trivial) which gives spin structures S_x on the fibers E_x. The good thing about this is that this bimodule bundle can be equipped with a natural connection as soon as you have a connection on E. Project 11 is to show that these are good definitions, and Project 12 is to show that if ∇ is a metric-preserving connection on E, then there is a unique connections ∇^S on any spin structure S on E (with some compatibility conditions, spelled out in the statement of the project).

The consequence is that it is very easy to write the Dirac operator on a spin manifold. Say X is a Riemannian spin manifold (i.e. we have a spin structure on the tangent bundle TX) and corresponding connection (coming from the Levi-Civita connection) ∇^S. Then the Dirac operator on X $\not{D}_X\colon\Gamma(S)\xrightarrow{\nabla^S}\Omega^1X\otimes\Gamma(S)\cong_g Vect(X)\otimes\Gamma(S)\xrightarrow{\text{left } Cl(TX)\text{ action}}\Gamma(S)$. This is the Cl_n-linear Dirac operator (also called the Atiyah-Singer operator). The nice thing about having a Cl_n-linear operator is that $\ker\not{D}_X$ is a finite dimensional graded Cl_n-module. The Dirac operator is an odd operator, so the kernel is actually graded. Finite dimensional comes from ellipticity. Usually, you define the index to be the dimension of this kernel. Here we take the dimension of the even part of the kernel minus the dimension of the odd part. This is not a good definition, because it totally ignores the Cl_n action. This is the right invariant if you use the usual Dirac operator. The real thing to do is take the kernel itself, and consider it as $\ker\not{D}_X\in\{$isoclasses of finite dimensional graded Cl_n-modules$\}$. This will depend on the metric. What if we change the metric on the same topological manifold? Because you impose this Cl_n action, the isomorphism class of $\ker\not{D}_X$ will only change by restrictions of finite dimensional Cl_{n+1}-modules. Thus, you can think of $[\ker\not{D}_X]\in\{\text{f.d. } Cl_n\text{-mods}\}/\{\text{f.d. } Cl_{n+1}\text{-mods}\}\cong KO_n(pt)\cong KO^{-n}(pt)$, which is $\mathbb{Z}$ if $n|4$, $\mathbb{Z}/2$ if $n\equiv 1,2\text{-}\mathsf{mod}4$, and 0 otherwise.

Theorem 26.1. *If the scalar curvature of g is positive, then* $[\ker\not{D}_X]=0$.

So we get some obstruction to having positive scalar curvature. For example, there are 9-dimensional Riemannian manifolds which are homeomorphic to S^9, but do not have positive scalar curvature.

Now I'd like to go back to the first class, where I told you that this stuff is related to cohomology theories. I'll start in arbitrary dimension (I think we'll eventually figure out how to do everything in arbitrary dimension). Plan for the next lectures.

Explain $(d|1)$-dimensional (if you change the 1 to a 0, you get boring zeros for everything) Euclidean (Euclidean signature) field theories over X of degree n. We'll call this $EFT^n_{d|1}(X)$. Then $EFT^n_{d|1}[X]=EFT^n_{d|1}(X)/\text{concordance}\cong\begin{cases}H^n_{dR}(X) & d=0\\ KO^n(X) & d=1\\ TMF^n(X)? & d=2\end{cases}$. Today, we'll do $n=0$ and $(d|0)$-dimensional stuff (that is, show that it's boring). We'll do $TFT_d(X)$, topological field theory.

Reminder: there is a symmetric monoidal bicategory B_d, whose objects are closed oriented smooth $(n-1)$-manifolds Y with germy collars (but no metric). The 1-morphisms $\mathsf{B}_d(Y_0,Y_1)$ are bordisms with the usual embeddings. The 2-morphisms are diffeomorphisms relative boundary. You can mod out by 2-morphisms to get a category, but I don't want to do that. The symmetric monoidal structure is disjoint union.

There is a very easy variation of the theme. Given a manifold X (which you think of as the target thing that you're trying to compute the cohomology of), we can define $\mathsf{B}_d(X)$ by adding smooth maps to X to everthing in sight (i.e. an object is a manifold with a smooth map to X, a morphism is a bordism with a smooth map to X, and a 2-morphism is a diffeomorphism commuting with the maps to X).

Remark 26.2. A smooth map $\phi\colon X_1\to X_2$ induces a symmetric monoidal bifunctor $\mathsf{B}_d(X_1)\to\mathsf{B}_d(X_2)$ by composing everything with ϕ.

So roughly speaking, a field theory is a representation of this category. Recall that given a group G (or a monoid or something), then a representation is the same thing as a functor from $\mathcal{C}_G \to \mathsf{Vect}$, where $\mathcal{C}_G$ is the category with one object $*$ and $\mathcal{C}(*,*) = G$. Bordism categories are covariant, but we're going to take representations, so we'll get something contravariant. $\diamond$

Instead of Vect, the target bicategory is Fr_2, whose objects are separable Frechét spaces, morphisms are continuous linear maps, and the only 2-morphisms are identities.

Remark 26.3. bifun$(\mathsf{B}_d(X), \mathsf{Fr}_2)$ =fun$(\mathsf{B}_d(X)/$2-morphisms$, \mathsf{Fr})$ (actually an isomorphism of categories), so we're not changing the definition of a QFT, just making things more complicated. But this complication is important. $\diamond$

[[break]]

Over the break, we decided there is an adjunction between taking a category and thinking of it as a bicateogry and collapsing all 2-morphisms. The functors forms a category, and the bifunctors form a bicategory (which turns out just to be a category), and you get an isomorphism of categories.

Now we have to do the major additional step, which is to formulate the smoothness condition on such a bifunctor from $\mathsf{B}_d(X)$ to Fr_2.

Example 26.4. Let's do $d = 0$ right now. $\mathsf{B}_0(X)/$2-mor$\to \mathsf{Fr}$. The unique object in $\mathsf{B}_0(X)$ is $\varnothing$. $\mathsf{B}_0(X)(\varnothing, \varnothing) = \{\Sigma^0 \to X\}/$diffeo$\supseteq X$, where Σ^0 is a compact 0-manifold (a bunch of points). We have that $\varnothing \mapsto \mathbb{C}$ because the functor must be monoidal. If Σ^0 is a single point, then we get a copy of X in the hom set. Since the functor must be monoidal, the whole functor is determined by where the points in X go (they go to some scalar in $\mathbb{C}$). Thus, symmetric monoidal functors from $\mathsf{B}_0(X)/$2-mor to Fr is the same thing as $Map(X, \mathbb{C})$.

We want to do two things. First, we really want to restrict to smooth functors, so we want to restrict to $C^\infty(X)$. Secondly, we really want closed forms on X, and this will come from the supersymmetry. $\Omega^*(X) \cong C^\infty(\pi T X) \cong C^\infty(\underline{\mathsf{SMan}}(\mathbb{R}^{0|1}, X))$ (C^∞ on the *super points* of X). Just the smoothness gives us TFT. $\diamond$

So what does it mean that a functor is smooth? We have to put more structure on these categories. The idea is that a smooth map is something which takes smooth functions to smooth functions. If we enrich the categories a bit to say that some of the morphisms are smooth, then we can use this as a definition. First, we need to make family versions of these (bi)categories $\mathsf{B}_d(X)$ and Fr_2 over Man_2. This is like in algebraic geometry, where you always work over some base scheme.

First we define a Grothendieck site Man_2, and put smooth structures (fiber functors) $\mathsf{B}_d(X)^{fam} \to \mathsf{Man}_2$ and $\mathsf{Fr}_2^{fam} \to \mathsf{Man}_2$. Then a smooth functor is one which respects these extra maps to Man_2. If we look at the fiber over the point in Man_2, we get the old story.

To define Man_2, we have to be a little more clever than the stupid adding of identity 2-morphisms. Man_2 is the bicategory of correspondences between smooth manifolds. The objects are smooth manifolds S (because there will be obvious generalizations to super manifolds). The 1-morphisms from S_0 to S_1 are correspondences between S_0 and S_1. A correspondence is a third manifold S, with maps $f\colon S \to S_1$ and $p\colon S \twoheadrightarrow S_0$ (we require p to be a submersion). The 2-morphisms from $(S_1 \xleftarrow{f} S \xrightarrow{p} S_0)$ to $(S_1 \xleftarrow{f'} S' \xrightarrow{p'} S_0)$ is a morphism $\phi\colon S \to S'$ making the two triangles commute. The vertical composition is easy. The horizontal composition of 1-morphisms is more interesting; you take pull-back:

$$\begin{tikzcd}
 & & S_{20} \arrow[dl] \arrow[dr] & & \\
 & S_{21} \arrow[dl] \arrow[dr] & & S_{10} \arrow[dl] \arrow[dr] & \\
S_2 & & S_1 & & S_0
\end{tikzcd}$$

There is a sub-bicategory where you require $p = \mathrm{id}$. That is the category of smooth manifolds!

14 RB 11-27

Today's lecture will be on the following question: how do you reconstruct a QFT from Feynman diagrams? First I'll explain what the problem is. We've more or less shown how to define Feynman diagrams. For each diagram, we've shown how to construct some kind of distribution (in a way that depends on the choice of Gaussian Feynman measure which is quite complicated to construct). Assume we've solved the problems of constructing the Feynman measure. Then we can construct Green's functions, defined as

$$\int \phi(f_1)\phi(f_2)\cdots e^{iL(\phi)}\mathcal{D}\phi$$

Where L is a Lagrangian of the form (free part) + λ(integral part). This is a formal power series in λ with coefficients which are distributions on M^n.

But a QFT isn't a bunch of distributions. A QFT consists of a space H (which is something like a Hilbert space) which is a module over $\mathbb{C}[[\lambda]]$ with a sesquilinear pairing $(,)$, with operators $\phi(f)$ for every smooth funcion f of compact support. These $\phi(f)$ should satisfy some axioms, which we listed earlier. Once you construct H and $\phi(f)$, it usually isn't too bad to check the axioms. So how do we go from a pile of Green's functions to a Hilbert space with some operators? First of all, we use the GNS construction, or the Wightman reconstruction theorem. Recall that Wightman distributions on M^n are $W(f_1,\dots,f_n) = \langle vac|\phi(f_1)\cdots\phi(f_n)|vac\rangle$. From these distributions, you can reconstruct the QFT. The Wightman distributions give a "state" (morally, but not really) on the free algebra A generated by all operators $\phi(f_i)$ (i.e. a linear map $\omega\colon A\to\mathbb{C}$, given by $\omega(\phi f_1 \dots, \phi f_n)\mapsto W(f_1,\dots,f_n)$. Now we can reconstruct H by the GNS construction. We define the inner product on A by $(a,b)=\omega(ab^*)$. Then $H = A/\ker(,)$. So to reconstruct the field theory, it is enough to reconstruct the Wightman distributions.

Feynman diagrams give you loads of distributions (Green's functions) $G(f_1,\dots,f_n)$. The Wightman reconstruction theorem says that by starting with loads of distributions $W(f_1,\dots,f_n)$, we can get a QFT. This suggests that we construct the QFT by taking $W(f_1,\dots,f_n)$ to be $G(f_1,\dots,f_n)$. This doesn't work for a very simple reason. The Green's functions are obviously symmetric in the f_i. PT: there are two meanings of $\phi(f)$ here. One is an operator on H. What is the other meaning? RB: the integral defining the Green's function is just obtained by integrating the distribution coming from the Feynman diagram against the functions $f_1,\dots,f_n$. On the other hand, the Wightman distributions are not symmetric because the operators $\phi(f_i)$ do not commute (unless the supports are spacelike separated.

However, the idea is not completely stupid and it nearly works, but needs some modification. If f_1 and f_2 have spacelike separated supports, then the operators $\phi(f_1)$ and $\phi(f_2)$ commute. So the Wightman distributions $W(f_1,\dots,f_n)$ are symmetric if the supports of the f_i are spacelike separated. In this case, we can identify Wightman distributions with Green's functions. The key point is that if you know the Wightman distributions on spacelike points, this determines the Wightman distributions at all points as follows.

Wightman distributions $W(x_1,\dots,x_n)$ (pretend they are functions) are in fact boundary values of holomorphic Wightman functions $W(z_1,\dots,z_n)$, where $z_i\in M\otimes\mathbb{C}$ and $Im(z_j-z_i)\in C$ (positive cone) (see some basic book on QFT, like Streater-Wightman). Moreover, these holomorphic functions can be analytically continued to a larger region, including all totally spacelike points, and we know they have to be Green's functions on totally spacelike points.

Schematically, we have (symmetric) Green's functions, which are distributions on M^n. These give you Wightman functions $W(z_1,\dots,z_n)$ by analytic continuation from spacelike points (completely ignore what the Green's functions do on non-spacelike points). These Wightman functions are symmetric (because they are analytic continuations of symmetric functions). Taking certain boundary values, you get Wightman distributions. Finally, applying the GNS construction (Wightman reconstruction theorem), we get the QFT.

The Wightman distributions are not symmetric. Why is the boundary value of a symmetric function not symmetric? Because there is more than one way to take a boundary value. For example, suppose you take $\sqrt{x}$, defined for $x\not\leq 0$ on the complex plane. It has two possible boundary values, $\sqrt{x+0i}$ and $\sqrt{x-0i}$. Similarly, for Wightman functions, you can approach from many different directions (this thing is codimension n, so there are many different ways to approach). PT: which one do you pick? RB: given a point $(x_1, \ldots, x_n)$, approach from $(z_1, \ldots, z_n)$ with $Im(z_i - z_j) \in C$ (positive cone in spacetime tensor $\mathbb{C}$) when $i > j$.

There are several problems with this approach.

1. We need to show that the analytic continuation exists. If you work in Minkowski spacetime, you can go through the mess of writing it out.

2. The analytic continuation doesn't even make sense on curved spacetimes. As far as I know, there is no such thing as the complexification of a Riemannian manifold.

3. Infrared divergences, which I've been brushing under the carpet for most of the semester.

The problem is that Feynman diagrams give distributions, so we can handle things like interactions of the form $\int \lambda(x)\phi(x)^4\,dx$ where λ is smooth of compact support. But we want λ to be 1, which is not of compact support. Interactions usually look like $\lambda \int \phi(x)^4\,dx$. We want it to be 1 so that it is invariant under the action of the Poincaré group. If you made λ of compact support, everything would converge nicely, but your theory wouldn't be Lorentz invariant.

So we want a way of reconstructing the QFT which doesn't involve analytic continuation on the complexification (which may not exist) and also deals with this infrared divergence.

Recall the two cut propagators Δ^+ and Δ^-. The ordinary propagator $\Delta(x)$ is basically the Fourier transform of $((p + i\varepsilon)^2) + m^2)^{-2}$. Δ^+ is the Fourier transform of something with support on $p^2 = m^2$ and Δ^- similar. $\Delta(x) = \Delta^+(x)$ if x is not in the negative cone and $\Delta(x) = \Delta^-(x)$ is x is not in the positive cone. Similarly, we have the Feynman propagator (and its complex conjugate) $\Delta^*(x) = \Delta^-(x)$ if x is not in the negative cone.

Now I want to introduce some slightly more complicated expressions. Formally, we have

$$\int \phi(f_1)\phi(f_2)e^{i\text{quadratic}+\lambda\phi^4}\mathcal{D}\phi = \sum \text{Feynman diagrams}$$

where the propagators on the right are Feynman propagators. Now I want to define a sort of generalization of this.

$$\phi(f_1)\phi(f_2)e^{iL_1} \Big| \phi(f_3)\phi(f_4)e^{iL_2} \Big| \cdots$$

These vertical lines mean the following. Sum over Feynman diagrams in the usual way, but any line which passes over one of these vertical lines is going to be a cut propagator instead of a Feynman propagator, unless I put a minus sign (e^{-iL_2}), in which case we use the complex conjugate of the Feynman propagator. PT: to define the Feynman diagrams, you had to make choices with Bernstein polynomials. RB: yes, to define these with just Feynman propagators, there was some ambiguity, but if you multiplying by some cut propagators is actually easy because of their wave front sets.

Now I want to define something which is more or less a scattering matrix with a perturbation by a field. $S(f_1 f_2)\overline{S}(f_3 f_4)S(f_5 f_6)\cdots$ is defined to be this sum of Feynman diagrams with $L_1, L_3, L_5, \ldots$ to be the interaction and $L_2, L_4, \ldots$ to be the complex conjugates. This S should be thought of as a scattering matrix.

The Wightman distributions $W(f_1, \ldots, f_n)$ will turn out to be $S_1(f_1)\overline{S}S(f_2)\overline{S}S(f_3)\cdots$. There is something funny going on because we've introduced these cut propagators and these inverse scattering matrices. You can think of this as scattering with a source f_1, then undo the scattering, then scatter again with source f_2, then undo, and so on. This will actually work for all $x_1, \ldots, x_n$, not just those with $x_i - x_j$ spacelike.

Now we need to check that this definition of $W(f_1, \ldots, f_n)$ satisfies the Wightman axioms. Most of them are fairly trivial (like Lorentz invariance). The main difficulty is to show that these satisfy locallity, which says that $W(f_1, f_2, \ldots, f_n) = W(f_2, f_1, \ldots, f_n)$.

Next week I'll try to explain why this choice of Wightman distributions give you a QFT.

37 NR 11-28

Last time we had a discussion of Heisenberg evolution in quantum mechanics and the Schrödinger picture of evolution. In the Heisenberg picture,

- To a point, we assign an associative $*$-algebra $A(pt) = C_h(M)$.
- To an interval $[t_1, t_2]$, we assign the LR ideal $I_{t_1,t_2} \subseteq A \otimes A$. This is the quantization of the vanishing ideal for the Lagrangian $L_{t_1,t_2} \subseteq M \times M^{op}$ (which appears in Hamilton-Jacobi). This is if there is a unique solution to the Euler-Lagrange equations. If there are serveral solutions, $L_{t_1,t_2} = \bigcup_\gamma L^\gamma_{t_1,t_2}$.

In the Schrödinger picture,

- To a point, we assign a vector space H (thought of as $C_h(M)/I_L$, where $L \subseteq M$ is the Lagrangian submanifold of boundary conditions. There are two kinds of points: in points and out points. To a point with opposite orientation, we assign the dual representation $H^\vee$.
- To the interval $[t_1, t_2]$, we assign the 1-dimensional subspace $\mathbb{C}v_{t_1,t_2} \subseteq H_1 \otimes H_2$ (the zero subspace for I_{t_1,t_2}), where H_i corresponds to $(I_{L_i}, L_i \subseteq M)$. So we are looking at Hamiltonian flows connecting the two Lagrangians L_1 and L_2. The H is a representation of $A = C_h(M)$. If A were not a $*$-algebra, we would not have the notion of a dual representation. Since A is a $*$-algebra, we have the dual representation $(\pi^\vee, H^\vee)$, where $H^\vee$ is the dual vector space to H, and $\pi^\vee(a) = \pi(a^*)^t$.

Semiclassically, this means the following. As $h \to 0$, $H \to C(L)$. First, the classical picture. We have a Hamiltonian $\mathcal{H}$. The Heisenberg picture: to a point, we assign $C(M)$, and to an interval $[t_1, t_2]$, we assign $I_{t_1,t_2} \subseteq C(M \times M^{op})$. The Schrödinger picture: to a point, we assign a Lagrangian $L \subseteq M$, and to an interval, we assign the intersection $L_{t_1,t_2} \cap (L_1 \cup L_2)$, where the L_1 and L_2 are boundary condition Lagrangians. PT: so this is an extra choice? NR: yes. PT: for geometric quantization, you choose a polarization. NR: geometric quantization gives you a hilbert space given a polarization. This is a different procedure. The problem is this. If we already know $C_h(M)$, then we can construct this H, but this is quite rare. Geometric quantization constructs the H without giving you the algebra C_h which it represents. PT: so you don't quantize the observables, you only get some of them. Are there examples where you don't get a polarization, but this $C_h(M)$ thing works. NR: yes. The simplest example is the quantum torus $\mathbb{T}^2$.

How can we connect the quantum picture with the classical picture? This is the semiclassical picture. We have a nondegenerate Lagrangian $\mathcal{L}$, and $\mathcal{H}$ is the Legendre transform of $\mathcal{L}$. First the Schrödinger picture (and only for the case where $M = T^*N$, "Lagrangian quantum mechanics"). To a point, we assign the space $H = C(L)$ for the classical $L \subseteq M$. This L is extra boundary condition data ... think of it as the zero section $N \subseteq T^*N$. We'll assume L is parallel to the zero section (i.e. $L = \{(p, q)|p = dF(q)\}$ for $F \in C^\infty N$). To the interval, we get $v_{t_1,t_2} \in C(L_1 \times L_2)$. That is, for each L_1, L_2, t_1, t_2, we get $U^{L_1,L_2}_{t_1,t_2}(q_1, q_2)$ for $q_i \in L_i$. The formula for this function is a sum of Feynman diagrams.

$$U^{L_1,L_2}_{t_1,t_2}(q_1, q_2) = \sum_\gamma \exp\bigl(iS^{L_1L_2}_\gamma(q_1, q_2)/h\bigr)(Hes?_\gamma)^{-1/2} \sum_\Gamma \frac{F_\Gamma}{|\operatorname{Aut}\Gamma|}$$

where γ varies over solutions to the Euler-Lagrange equations (we assume there are a finite number of solutions) and where $S^{L_1L_2}_\gamma(q_1, q_2) = S_\gamma(q_1, q_2) + F_1(q_1) - F_2(q_2)$. The composition (gluing) rule is that

$$\int_{L_2} U^{L_1,L_2}_{t_1,t_2}(q_1, q_2)U^{L_2,L_3}_{t_2,t_3}(q_2, q_3)\, dq_2 = U^{L_1,L_3}_{t_1,t_3}(q_1, q_3).$$

In the example where all $L_i = N$, then the F's are zero, and this is the usual composition. PT: is this a theorem or a conjecture? NR: it is a conjecture; I couldn't find it in the litarature. PT: I don't think this is true; you should have to use the path integral to define the U's. It looks like you're just taking classical solutions. NR: The question is how to define the path integral. Here is another conjecture. You can try to approximate the paths by piecewise geodesics and take a limit. What is known about this? Bruce: I'm not sure about the i in the exp. NR: [[★★★ I didn't catch all this stuff]] In the cases where you have another definition of these U's, the conjecture is that it will agree with this one.

PT: could you remind me how you compute the F_Γ? What do you put on the edges and vertices? NR: If there are n edges coming from a vertex (labelled t), you assign $\frac{\partial^n V(q)}{\partial q_{i_1}\cdots\partial q_{i_n}}\Big|_{q=\gamma(t)}$. To an edge between vertex i (with t_1) and j (with t_2), we assign K^{-1}, where $K_{ij}(t) = \left(-\frac{d^2}{dt^2}\delta_{ij} + \frac{\partial^2 V}{\partial q_i \partial q_j}\gamma(t)\right)$, $t_1 \leq t \leq t_2$ action on $L_2[t_1, t_2]$. $U_{t_1 t_2} = \exp(iH(t_2 - t_2)/h)$ acts on $L_2(N)$.

You can try to get rid of the condition $M = T^*N$. It is natural to assume that (with some corrections), this kind of procedure should work. You should expect that to ta point, we assign $H = C(L)$ for a fixed $L \subseteq M$, and to the interval, we assign $U^{L_1 L_2}_{t_1 t_2}(q_1, q_2)$. There should be some composition law, but it will be more complicated for a general symplectic manifold.

Now we have some idea of what is quantum mechanics. What is quantum field theory? It can be considered as a functor from the cobordism category of Riemannian manifolds to some other category (like Vect or SVect). Let me describe the ingredients. The Heisenberg picture is the following. To a $(d-1)$-dimensional manifold N_{d-1}, we assign an associative $*$-algebra $A(N_{d-1})$. To the same manifold with opposite orientation, we assign the opposite algebra (using the $*$ structure). To a d-dimensional manifold M_d, we assign an ideal $I(M_d) \subseteq \bigotimes_i A((\partial M)_i)$ (connected components of boundary of M). We also have a gluing axiom.

The Schrödinger picture. To N_{d-1}, we assign a vector space $H(N_{d-1})$. The opposite orientation gives the dual vector space. To M_d, we assign a 1-dimensional subspace $\mathbb{C}v(M_d) \subseteq \bigotimes_i H((\partial M)_i)$. We also have a gluing axiom.

Semiclassical (Schrödinger). Given a classical Lagrangian field theory, which means that to M_d, we assign $\Phi(M_d)$, and $\mathcal{L}$ a function on the jet space of fields. Remember I discussed the classical Bose field and I discussed how the Lagrangian picture gives the Hamiltonian picture. Roughly, the symplectic manifold we assign to N_{d-1} (think of N as part of ∂M) is $T^*\Phi(N_{d-1})$. Classically, we have to choose a Lagrangian subspace $L \subseteq S(N_{d-1})$ (e.g. the zero section). Then the picture is very similar to classical mechanics. To N_{d-1}, we assign H = functionals on L (on $\Phi(N_{d-1})$) on boundary values of fields. To M_d, we assign the 1-dimensional space $\mathbb{C}v(M_d) \subseteq \bigotimes H((\partial M)_i)$, with $v(M_d)[b] = \int \exp(i\mathcal{A}(\phi)/h)\mathcal{D}\phi$ where the integral is over fields ϕ in $\Phi(M_d)$ with given boundary values b. When $\mathcal{A}$ is non-degenerate, we can try to define this integral as a formal sum over solutions to the Euler-Lagrange equations and Feynman diagrams.

Next time I'll keep discussing this subject.

27 PT 11-29

Definition 27.1. $\mathit{TFT}_d(X) = \mathit{smBiFun}^{pb}_{\mathsf{Man}_2}(\mathsf{B}_d(x)^{fam}, \mathsf{Fr}_2^{fam})$ (symmetric monoidal bifunctors; pullback preserving)
$\mathit{EFT}_d(X) = \mathit{smBiFun}^{pb}_{\mathsf{Man}_2}(\mathsf{RB}_d(x)^{fam}, \mathsf{Fr}_2^{fam})$
$\mathit{TFT}_{d|1}(X) = \mathit{smBiFun}^{pb}_{\mathsf{SMan}_2}(\mathsf{B}_{d|1}(x)^{fam}, \mathsf{Fr}_2^{fam})$
$\mathit{EFT}_{d|1}(X) = \mathit{smBiFun}^{pb}_{\mathsf{SMan}_2}(\mathsf{RB}_{d|1}(x)^{fam}, \mathsf{Fr}_2^{fam})$ ⋄

I have to explain this family stuff. Fr_2^{fam} has objects smooth Frechét bundles $V \to S$ over manifolds S. We have the forgetful bifunctor $\mathsf{Fr}_2^{fam} \to \mathsf{Man}_2$. Recall that the 1-morphisms in Man_2 are correspondences $S_1 \leftarrow S \twoheadrightarrow S_0$. A 1-morphism in Fr_2^{fam} is a bundle map $\tilde{f}\colon p^*V_0 \to V_1$ lying over $f\colon S \to S_1$.

$$\begin{tikzcd}
V_1 \arrow[d] & p^*V_0 \arrow[l, "\tilde{f}"'] \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & V_0 \arrow[d] \\
S_1 & S \arrow[l, "f"'] \arrow[r, two heads, "p"] & S_0
\end{tikzcd}$$

2-morphisms are just morphisms $S \to S'$ as in Man_2 (i.e. all the data of the 2-morphisms is preserved under the forgetful functor to Man_2).

What is missing so far is "what are symmetric monoidal bicategories?" This is a subtle question. It is not totally obvious. What are symmetric monoidal categories? If you write it down very carefully, there are lots of natural transformations we left out (like the associator). If you're willing to go into higher categories, then a monoidal category is the same thing as a bicategory with one object. Remember that a bicategory has objects $\mathcal{C}_0$, 1-morphisms $\mathcal{C}_1$, and 2-morphisms $\mathcal{C}_2$, with maps $\mathcal{C}_2 \mathrel{\substack{\longleftarrow \\ \longrightarrow \\ \longrightarrow}} \mathcal{C}_1 \mathrel{\substack{\longleftarrow \\ \longrightarrow \\ \longrightarrow}} \mathcal{C}_0$. If $\mathcal{C}_0 = \{*\}$, then $\mathcal{C}_1(*,*)$ is a cateogry (using $\mathcal{C}_2$ as morphisms). Horizontal composition is exactly a monoidal structure on this category.

A braided monoidal category is one where there is a preffered isomorphism between $X \otimes Y$ and $Y \otimes X$ (in general, they need not be isomorphic at all). This is exactly a tricategory with one object and one 1-morphism! Again, there is no reason for the tensor structure to be symmetric. A quad-category with one object, one 1-morphism, and one 2-morphism is a symmetric monoidal category.

If we want to talk about monoidal categories, we have to shift everything by one. A symmetric monoidal category is therefore a penta-category with one object, 1-morphism, and 2-morphism. This looks like a mess, but it turns out that you can define n-categories using simplicial sets. Some people (with first names Chris and Andre) came up with an interpretation which only uses bicategories, but they have to be internal to some other category. In their language, the pullback preserving part is built in. I don't want to go into this right now. Maybe next week.

Definition 27.2. If $\mathcal{C}$ is a bicategory and $x \in \mathcal{C}$, then the *loop category at x* is $\Omega_x\mathcal{C}$ is $\mathcal{C}(x,x)$. Note that this $\Omega_x\mathcal{C}$ is a monoidal category. ⋄

Today, I want to apply the loop functor to the things in the first definition.

Example 27.3. $\Omega_\varnothing \mathsf{B}_d$ has objects closed oriented smooth d-manifolds and $\Omega_\varnothing \mathsf{B}_d(\Sigma_0, \Sigma_1) = \mathit{Diff}(\Sigma_0, \Sigma_1)$. The monoidal structure is disjoint union. We used to say that disjoint union is the monoidal structure on the bicategory B_d, but now we're just thinking of B_d as a bicategory (without any monoidal structure) and the disjoint union structure pops out (though we don't get disjoint union of $(d-1)$-manifolds)

If we loop $\mathsf{B}_d(X)$, you get closed manifolds with maps to X and diffeomorphisms respecting the map to X. ⋄

Example 27.4. $\Omega_\varnothing \mathsf{Man}_2$ is boring because there are no maps to $\varnothing$, so there aren't any non-trivial correspondances. Let's look at $\Omega_* \mathsf{Man}_2$. It has objects $* \leftarrow S \twoheadrightarrow *$ and morphisms are just morphisms of the S's. So $\Omega_* \mathsf{Man}_2 = \mathsf{Man}$. The monoidal structure is product.

In topology, you sometimes ask if one space is the loop space of another, and the answer is usually no (e.g. $\pi_1(\Omega X) = \pi_2(X)$ is abelian). Here we find that we can "de-loop" Man. ⋄

Example 27.5. $\Omega_{\mathbb{C}} \mathsf{Fr}_2$ has objects morphisms from $\mathbb{C}$ to $\mathbb{C}$, and 2-morphisms are all identities. So this is the set $\mathbb{C}$, with only identity morphisms. ⋄

Now let's start looping some stuff. A symmetric monoidal bifunctor must take the monoidal unit to the monoidal unit, so we'll loop everthing at the monoidal unit. $\mathit{TFT}_d(X) = \mathit{smBiFun}_{\mathsf{Man}_2}(\mathsf{B}_d(X)^{fam}, \mathsf{Fr}_X^{fam})$ gives

$smFun_{\mathsf{Man}}(\Omega_\varnothing \mathsf{B}_d(X)^{fam}, \Omega_{\mathbb{C}}\mathsf{Fr}_2^{fam})$. What are these things? Let's do $\Omega_{\mathbb{C}}\mathsf{Fr}_2^{fam}$ first. Our object is the vector bundle $\mathbb{C} \to pt$. The objects are pairs (S, f) where $f\colon S \to \mathbb{C}$. The morphisms from (S_0, f_0) to (S_1, f_1) are maps $g\colon S_0 \to S_1$ so that $f_1 \circ g = f_0$. This is $\mathcal{C}(\mathbb{C})$, where $\mathcal{C}(X)$ is the category with objects $(S, f\colon S \to X)$ and morphisms $g\colon S_0 \to S_1$ with $f_1 \circ g = f_0$. This is the "functor of points". Later, we'll change Man to SMan. So an element of $smFun_{\mathsf{Man}}(\Omega_\varnothing \mathsf{B}_d(X)^{fam}, \mathcal{C}(\mathbb{C}))$ is a number for every closed manifold. Restricting to connected closed manifolds (and using the fact that the functors are monoidal), we see that this is $Fun_{\mathsf{Man}}(\Omega_\varnothing \mathsf{B}_d^{conn}(X)^{fam}, \mathcal{C}_{\mathsf{Man}}(\mathbb{C}))$.

We loose information about the $(d-1)$-manifolds, but for $d = 0$, we don't loose any information by taking the loops. Moreover, for $d = 0$, there is only one connected d-manifold, a point. So in this case, we have $Fun\big(\mathcal{C}_{\mathsf{Man}}(X), \mathcal{C}_{\mathsf{Man}}(\mathbb{C})\big)$. After the break, we'll calculate this thing.

[[break]]

Arturo pointed out the following. The category $\mathcal{C}_{\mathsf{Man}}(X)$ is a well-known thing. If you have a category $\mathcal{D}$ and $x \in \mathcal{D}$, then you can define the *over category* $\mathcal{D}(x)$, whose objects are morphisms to x and whose morphisms are commutative diagrams. $\mathcal{C}_{\mathsf{Man}}(X)$ is exactly the over category $\mathsf{Man}(x)$. Note that $\mathcal{D}(x)$ has a forgetful functor to $\mathcal{D}$.

So the thing we're interested in is $Fun_{\mathsf{Man}}(\mathsf{Man}(X), \mathsf{Man}(\mathbb{C}))$.

Lemma 27.6 (2-Yoneda lemma)**.** *$\mathcal{D}(X, Y) \cong_\Phi Fun_{\mathcal{D}}\big(\mathcal{D}(X), \mathcal{D}(Y)\big)$, given by $g \mapsto (f \mapsto g \circ f)$, and the inverse is given by $F \mapsto F(\mathrm{id}_X)$.*

So We have $\mathsf{Man}(X, \mathbb{C}) = C^\infty(X)$. Thus, 0-dimensional topological field theories over X are just functions on X. This is probably the most complicated way to explain what a function is. The power of this is that we can change d to 1, 2, 3, etc.. When you take $d = 1$, you get connections; when $d = 2$, there isn't already a name for it. Since we don't want functions (we want forms), we'll use super manifolds.

If we change B_0 to RB_0, nothing changes because we're just adding a Riemannian metric to a point. This is very different if you take $d = 1$ because a circle has a length. Now let's do $TFT_{d|1}$. As before, we loop our definition (not loosing anything since $d = 0$) to get $smFun_{\mathsf{SMan}}(\Omega_\varnothing \mathsf{B}_{0|1}(X)^{fam}, \Omega_{\mathbb{C}}\mathsf{Fr}_2^{fam}) = Fun_{\mathsf{SMan}}(\Omega_\varnothing \mathsf{B}_{0|1}^{conn}(X)^{fam}, \Omega_{\mathbb{C}}\mathsf{Fr}_2^{fam})$. We have that $\Omega_\varnothing \mathsf{B}_{0|1}^{conn}(X)^{fam}$ has objects $S \xleftarrow{p_1} S \times \mathbb{R}^{0|1} \xrightarrow{f} X$, or pairs $(S, \tilde{f})$, where $\tilde{f}\colon S \to \underline{\mathsf{SMan}}(\mathbb{R}^{0|1}, X) \cong \pi TX$. So we might think that this category is $\mathsf{SMan}(\pi TX)$. This is not true. As before, $\Omega_{\mathbb{C}}\mathsf{Fr}_2^{fam} = \mathsf{SMan}(\mathbb{C})$. If we did get $\mathsf{SMan}(\pi TX)$, we would get that $TFT_{0|1}(X)$ are forms on X, and we want closed forms on X, so it would be the wrong answer anyway. What is a morphism in $\Omega_\varnothing \mathsf{B}_{0|1}^{conn}(X)^{fam}$? They are

$$\begin{tikzcd}
 & X & \\
S_0 \times \mathbb{R}^{0|1} \arrow[ur] \arrow[rr, "G"'] \arrow[d] & & S_1 \times \mathbb{R}^{0|1} \arrow[ul] \arrow[d] \\
S_0 \arrow[rr, "g"] & & S_1
\end{tikzcd}$$

where G is a fiberwise automorphism of $\mathbb{R}^{0|1}$. This is where it depends if we're doing TFT or EFT. If you don't put an inner product on $\mathbb{R}^{0|1}$, dilation is an automorphism. It turns out that TFT will only give you closed 0-forms and EFT gives you all closed forms. So topological field theories give you boring constant functions. To get de Rham cohomology, you need to use EFT.

38 NR11-30

Lagrangian QFT. I want to keep track of the quantization procedure, so this is the semiclassical view of QFT. To a compact oriented Riemannian $(d-1)$-manifold N_{d-1} we assign a vector space $H(N_{d-1})$, and to a d-manifold M_d we assign $\mathbb{C}v(M_d) \subseteq \bigotimes_i H((\partial M_d)_i)$. To the manifold $\overline{N}_{d-1}$, we assign the dual vector space $H(N_{d-1})^\vee$.

Recall classical field theory. The idea is this. In the Hamiltonian picture, to N_{d-1} we assign a symplecic manifold $S(N_{d-1})$ and to the opposite orientation, we assign the same manifold with opposite symplectic structure. To M_d, we assign a Lagrangian $L(M_d) \subseteq S(\partial M)$. We require that $S(N_1 \sqcup N_2) = S(N_1) \times S(N_2)$, and the other axioms.

An example of such a construction comes from Lagrangian mechanics. We have

- Φ, the space of fields on M_d, and
- the Lagrangian function $\mathcal{L}$, a local function on Φ (we assume it depends only on the first jet $\mathcal{L}(\phi, d\phi)$).

Then we have the action functional $\mathcal{A}[\phi] = \int_{M_d} \mathcal{L}(\phi, d\phi)$ (if the volume form on M_d is not given, we have to take the output of $\mathcal{L}$ to be a top form). Classical trajectories are solutions to the Euler-Lagrange equations $\delta\mathcal{A} = 0$. So where is the classical field theory in this? The idea is this. We can restrict Φ to $\Phi^b(\partial M)$, fields on the boundary of M (fields are usually sections of a bundle). Now let $S(N_{d-1})$ is $T^*\Phi^b(N_{d-1})$. PT: this is where collars are really useful; you don't have any trouble restricting anything to a collar because it is an open subset. In my class, we take $S(N_{d-1})$ to be the space of solutions to the Euler-Lagrange equations on the collar. In the case where fields on the boundary make sense, the two definitions agree. NR: yes, collars do the job quite nicely. Collars are absolutely natural, even in classical mechanics, but because of the lack of time, we won't do it. For me, it is easier to think about fields on the boundary because of one example: field theory on graphs. We let $L(M_d) = \{\text{solutions to the Euler-Lagrange equations}\} \subseteq S(\partial M_d)$. PT: in my language, the restriction to the collar is the inclusion of this Lagrangian into the symplectic manifold. NR: ok.

So this is the classical Hamiltonian picture coming from Lagrangian mechanics (with a non-degenerate Lagrangian). Fix the boundary conditions. For each N_{d-1}, we fix a boundary Lagrangian $L(N_{d-1}) \subseteq S(N_{d-1})$, requiring that $L(N_1 \sqcup N_2) = L(N_1) \times L(N_2)$. Then $L(M_d) \cap L(\partial M_d)$ is a collection of points. We can modify the action to

$$\mathcal{A}_L[\phi] = \int_{M_d} \mathcal{L}(\phi, d\phi) + F_L[\phi]$$

where F is the generating function for L. Solutions of $\delta\mathcal{A}_L = 0$ are then points of $L(M_d) \cap L(\partial M_d)$.

If we are on a cotangent bundle, taking $L = \{p = dF(q)\}$, then L_0 gives a representation of $[\hat{p}, \hat{q}] = ih$ on $C^\infty(\mathbb{R})$ given by $\hat{p} = ih\frac{\partial}{\partial q}$ and $\hat{q} = q$. L_F gives the representation $\hat{p} = \frac{\partial F}{\partial q} + ih\frac{\partial}{\partial q}$, $\hat{q} = q$.

The "naïve" quantization. Let $L = L_0$. We can choose $H(N_{d-1}) = C[L(N_{d-1})]$, so $v(M_d) \in C[L(\partial M_d)]$. Let's denote points of $L(\partial M_d)$ by b.

$$v(M_d)(b) = \int_{\phi|_{\partial M} = b} e^{i\mathcal{A}/h}\, \mathcal{D}\phi.$$

This is the "naïve" quantization because we don't know what this integral is. In the finite-dimensional case, we can make sense of it. If the spaces are infinite-dimensional, we can sometimes employ some functional analysis (at least in the non-oscillating case, where we don't have the i and the exponenential is rapidly decreasing). This v should be thought of as a quantization of the points in $L(M_d) \cap L(\partial M_d)$.

Let's consider the formal asymptotics of this integral as $h \to 0$. Remember that we just want to get something which satisfies all the axioms of quantum field theory. We can try to construct formal power series that do this.

$$\int_{\phi|_{\partial M} = b} e^{i\mathcal{A}/h}\, \mathcal{D}\phi = \sum_\gamma e^{i\mathcal{A}_\gamma(b)/h} \det{}'(K_\gamma(b))^{-1/2} \sum_\Gamma \frac{F(\Gamma)(b)}{|\operatorname{Aut}\Gamma|}$$

where γ is a solution to the Euler-Lagrange equations with boundary condition b. We expand $\mathcal{A}[\gamma + \psi]/h = \mathcal{A}[\gamma]/h + (K_\gamma\psi, \psi)/h +$ (higher order terms)$/h$. We scale ψ by $\sqrt{h}$, so the higher order terms have $h^{\frac{n}{2}-1}$ with $n \geq 3$, so they are small.

If we want to be consistantly naïve, we have to tensor our H with $\mathbb{C}[[h]]$. We now want to check if this satisfies gluing axiom. Consider Lagrangian mechanics on N with non-degenerate Lagrangian $\mathcal{L}(\xi, q)$. Exercising this philosophy, we get

$$U_t(q_1, q_2) = (e^{i\hat{H}t/h})(q_1, q_2)$$

where $\hat{H}$ is the quantization of the Legendre transform of $\mathcal{L}$. I want to emphasize that we don't need to choose a Riemannian structure on N. Since $\mathcal{L}$ is non-degenerate, it gives a metric on the tangent space, $d_\xi^2\mathcal{L}(\xi, q)$. I want to use this to define a natual measure on Lagrangian submanifolds. PT: in geometric quantization, you use the metaplectic structure on S and use half-forms. NR:

If we have M, and we want to glue together two parts of the boundary (b and b'; call the rest of the boundary b''). We want critical points $\delta_{b'}\mathcal{A}_\gamma^M(b', b'') = 0$.

$$\int v(M_d)(b', b'')\,\mathcal{D}b' = \sum_\gamma \cdots \sum_{\Gamma_{int}} \sum_{\Gamma_b} \quad (b'')$$

you get Feynman diagrams from the interior and Feynman diagrams from the boundary. We then would like to check the gluing rule.

Next time I'll say what to do if the Lagrangian is degenerate. We'll end up with super spaces with differentials which annihilate $v(M_d)$.

39 NR 12-03

Last time we discussed quantum field theory in terms of path integrals. Today I want to return to the degenerate systems and give a more conceptual look at Fadeev-Popov quantization.

Cohomological Field Theories

1. In the Schrödinger picture, we assign to N_{d-1} a vector space $H(N_{d-1})$ and to M_d a vector (really a 1-dimensional subspace) $\mathbb{C}v(M_d) \subseteq H(\partial M_d)$. *Cohomological field theory* simply means that we have one more operation. In addition, we have a differential $d\colon H(N_{d-1}) \to H(N_{d-1})$ (which depends on N_{d-1}) with $d^2 = 0$. $H(N)$ is a super vector space and d is odd. Furthermore, we require that $d(v(M)) = 0$.

2. We can construct QFTs on cohomologies. We can assign to N_{d-1} the cohomology $H^*(N_{d-1})$ (unfortunately, this is $H^*(H(N))$), and to M_d we assign $[v(M)] \in H^*(\partial M)$.

3. Two cohomological QFTs are *quasi-isomorphic* if the resulting cohomology QFTs are isomorphic.

If we start with a guage theory, it is impossible to use the Feynman diagram technique because the theory is degenerate. The idea is to replace the guage theory, which doesn't have any fermionic part, thought of as a cohomology QFT. Suppose G acts on a variety X (let's assume X is linear), then you can construct $C^\bullet(\mathfrak{g}, X)$, and then $C(X/G) \cong H^0(\mathfrak{g}; X)$ in many interesting cases. We know that $C^\bullet(\mathfrak{g}, X)$ is functions on $X \oplus \mathfrak{g}[1]$ (the [1] means to shift the grading, so think of the $\mathfrak{g}$ as odd). The differential is the odd vector field $Q = \sum_\alpha c^\alpha x_\alpha + \sum_{\alpha,\beta,\gamma} c_{\alpha\beta}^\gamma c^\alpha c^\beta \frac{\partial}{\partial c^\gamma}$, where $\{e_\alpha\}$ is a basis for $\mathfrak{g}$ and $\{x^i\}$ are coordinates on X (and $x_\alpha = x_\alpha^i(x)\frac{\partial}{\partial x^i}$). We can replace the guage theory on X/G by a quasi-isomorphic guage theory on $X \oplus \mathfrak{g}[1]$, which has some odd degrees. This is why there are fermionic variables in Fadeev-Popov, BRST, BV quantizations. The idea is to replace these complicated path integrals over complicated spaces by path integrals over linear spaces. PT: X is the space of fields? Do we assume it is linear? NR: yes. The fields are usually sections of some vector bundle, so it will be linear. It need not be linear in general.

The second step is to this replacement in such a way that we get a non-degenerate field theory (the action should be non-degenerate). I will return to this (BV quantization) on Wednesday, at which point I will hopefully have figured out a good way to explain it.

Example of a QFT

Today, I'll give a finite dimensional model to illustrate how these constructions can be achieved with finite dimensional approximations instead of Feynman diagrams.

We'll talk about the "discrete" Bose field. By discrete, I mean that the spacetime is a (finite) graph Γ. The spacetime category: spaces are collections of points and spacetimes are graphs Γ. To a collection of points n points N, we assign $H(N) = L^2(\mathbb{R}^n)$, and to Γ, we assign $v(\Gamma)(b) = \int_{\mathbb{R}^{v_{in}}} \exp(-A(\phi, b)/h)\, d\phi$ where v_{in} is the number of interior vertices of Γ. We assume that $A(\phi)$ is a positive polynomial in ϕ. Localilty of the action on the graph: $A(\psi) = \sum_{v\in\Gamma} A_v(\psi_v) + \sum_{e\in\Gamma} A_e(\psi_{e_+} - \psi_{e_-}) + \cdots$ where A_e is an even degree polynomial (this is the analogue of saying that the action is the integral of a Lagrangian which depends only on the jet). The natural first order action is given by

$$A(\mathfrak{p}[si) = \sum_{u,v \text{ adjacent}} A_{u,v}(\psi_v, \psi_u).$$

We want to check the gluing axiom. Suppose se have Γ, and we want to glue together parts of the boundary $\partial_+\Gamma$ and $\partial_-\Gamma$.

$$\begin{aligned} v(\Gamma)(b)|_{\text{diag}} &= \int d\psi_{\text{from } \partial_+\Gamma} \int_{\mathbb{R}^{v_{in}(\Gamma)}} d\psi_{u_0} \exp\Big(-\sum_{u,v\in\partial_\pm\Gamma} +A(\psi_u, \psi_{u_0})\Big) \\ &= \int_{\mathbb{R}^{v_{in}(\Gamma)}} \exp\Big(-\sum_{u,v \text{ adj}} A(\psi_u, \psi_v)\Big)\, d\psi. \end{aligned}$$

This graph QFT is like a universal example, thinking of the graphs as skeletons of spacetimes (and letting them grow infinite). If you want to do $d > 2$, you have to allow different things for boundaries of the graphs.

1. How to find the limits $|\Gamma| \to \infty$. 2. Compute various quantities (in the limit where $|\Gamma| \to \infty$). 3. Find functions $A(\psi, \psi)$ such that the resulting theory is a TQFT for cell complexes. This would mean that on a surface, this $v(\Gamma)$ is invariant under the standard moves of the cell decomposition (thinking of Γ as embedded in M). It is not clear that this theory exists.

So the idea is to either try to make the theory topological or to make it approximate something else as $|\Gamma| \to \infty$.

2-dimensional Yang-Mills theory

Suppose Σ is a compact oriented surface (possibly with boundary). Let Γ be a spine of a cell decomposition of Σ. Consider a trivialized G-bundle B over Γ (i.e. a copy of G over each vertex). A connection in B is a map $E(\Gamma) \to G$, which you think of as parallel transport along the edges. You can think of Γ as a subgroupoid of the fundamental groupoid $\Gamma_1(\Sigma)$. Let $g = \{B(e)\}_{e\in E(\Gamma)}$ [[★★★ let $g(e) = B(e)$]], then

$$A(g) = \sum_{\substack{f=\text{2-cell} \\ \text{of } \Sigma}} w(f)\, \mathrm{tr}(h(f))$$

where $h(f)$ is the holonomy around f (read the edges around the 2-cell f using the orientation and take the product of the $g(e_i)^{\pm 1}$) and the trace is taken in the adjoint representation.

Spacetime is $\Sigma \supset \Gamma$, space is the closed 1-dimensional cell decomposition. $H(C) = L^2(G^{E(C)})$, and $v(\Sigma) = \int_{G^{E_{in}(\Gamma)}} e^{-A(g)}\, dg$, where G is simple compact.

This action is clearly invariant under $G^{V_{in}(\Gamma)}$, and the transformation is $h\colon g(e) \to h(e_+)g(e)h(e_-)^{-1}$, corresponding to changing the trivialization.

28 PT 12-04

Next Tuesday, we'll have the mini-conference in 939.

Degree 0 cohomology groups, non-local for $d = 2$.

$$\begin{aligned}
TFT_d(X) &:= smBiFun^{pb}_{\mathsf{Man}_2}(\mathsf{B}_d(X)^{fam}, \mathsf{Fr}_2^{fam}) \\
&\xrightarrow[\text{conn. bord.}]{\Omega_e} Fun_{\mathsf{Man}}(\Omega_\varnothing \mathsf{B}_d^{conn}(X)^{fam}, \Omega_{\mathbb{R}}\mathsf{Fr}_2^{fam}) \\
&=_{d=0} Fun(\mathsf{Man}(X), \mathsf{Man}(\mathbb{R})) \\
&\cong \mathsf{Man}(X, \mathbb{R}) = C^\infty(X)
\end{aligned}$$

Similarly,

$$\begin{aligned}
TFT_{d|1}(X) &:= smBiFun^{pb}_{\mathsf{SMan}_2}(\mathsf{B}_{d|1}(X)^{fam}, \mathsf{Fr}_2^{fam}) \\
&\xrightarrow[\text{conn. bord.}]{\Omega_e} Fun_{\mathsf{SMan}}(\Omega_\varnothing \mathsf{B}_{d|1}^{conn}(X)^{fam}, \Omega_{\mathbb{R}}\mathsf{Fr}_2^{fam}) \\
&= Fun(\Omega_\varnothing \mathsf{B}_{d|1}^{conn}(X)^{fam}, \mathsf{Man}(\mathbb{R}))
\end{aligned}$$

If you weren't doing family versions, a symmetric monoidal bifunctor from $\mathsf{B}_d(X)$ to Fr_2 is the same as a functor from $\mathsf{B}_d(X)/2$-morphisms to Fr, but this doesn't work in the family version.

By the way, for $d = 0$, TFT and EFT agree (since a Riemannian metric on a point is not extra information). An object in $\Omega_\varnothing \mathsf{B}_{d|1}(X)^{fam}$ is of the form

$$\begin{tikzcd}
F^{d|1} \arrow[d] & \\
\Sigma \arrow[r, "f"] \arrow[d] & X \\
S^{m|n} &
\end{tikzcd}$$

Since $d = 0$ and we require connected, $F = \mathbb{R}^{0|1}$. There are non-trivial $\mathbb{R}^{0|1}$-bundles on S (parity reverses of non-trivial line bundles), but locally it is trivial (and since we're working over SMan, it is enough to understand stuff locally), so we may assume $\Sigma = S \times \mathbb{R}^{0|1}$. But a map from $S \times \mathbb{R}^{0|1}$ to X is the same as a map from S to $\underline{\mathsf{SMan}}(\mathbb{R}^{0|1}, X) = \pi TX$.

So the first candidate for $TFT_{0|1}(X)$ is $Fun_{\mathsf{SMan}}(\mathsf{SMan}(\pi TX), \mathsf{SMan}(\mathbb{R}))$. If this were true, this would just be differential forms on X. This isn't quite right because the morphisms in $\Omega_\varnothing \mathsf{B}_{d|1}^{conn}(X)^{fam}$ are not the same as the morphisms in $\mathsf{SMan}(\pi TX)$. A morphism in $\Omega_\varnothing \mathsf{B}_{d|1}^{conn}(X)^{fam}$ is a fiberwise diffeomorphism over a map from S_0 to S_1:

$$\begin{tikzcd} & X & \\ S_0\times\mathbb{R}^{0|1} \arrow[ur, "f_0"] \arrow[rr, "G"] \arrow[d] & & S_1\times\mathbb{R}^{0|1} \arrow[ul, "f_1"'] \arrow[d] \\ S_0 \arrow[rr, "g"] & & S_1 \end{tikzcd}$$

This G must be of the form $(g\circ p_1, \gamma\colon S_0\times\mathbb{R}^{0|1}\to\mathbb{R}^{0|1}) = (g\circ p_1, \tilde\gamma\colon S_0\to \underline{\mathrm{Aut}}(\mathbb{R}^{0|1}))$, where $\underline{\mathrm{Aut}}(\mathbb{R}^{0|1})\subseteq \underline{\mathsf{SMan}}(\mathbb{R}^{0|1},\mathbb{R}^{0|1}) = \pi T\mathbb{R}^{0|1} = \mathbb{R}^{1|1}$ is $\mathbb{R}^{0|1}\rtimes\mathbb{R}^\times$ (translations times dilations). These are the S-points of a *transport category.*

Let's talk about it for Man first. If I have a manifold Y and a Lie group G acting on Y (think $Y=\pi TX = \underline{\mathsf{SMan}}(\mathbb{R}^{0|1},X)$ and $G=\underline{\mathrm{Aut}}(\mathbb{R}^{0|1})$). The *transport category* has object set Y and morphism set $Y\times G$, where (y,g) has source and target y and $y\cdot g$, respectively. Now we're taking the S-points of this [[★★★ This transport category is a groupoid object in Man, and we take S-points of this object to get a groupoid (but we let S vary, so we're probably talking about a category fibered in groupoids)]]. Define $\mathsf{Man}(Y;G)$ to be the category with objects (S,f) where $f\colon S\to Y$ and morphisms

$$\begin{tikzcd} Y\times G \arrow[r, "\mu"] & Y \\ S_0 \arrow[u, "f_0\times\gamma"] \arrow[r, "g"] & S_1 \arrow[u, "f_1"'] \end{tikzcd}$$

The lemma is that $\Omega_\varnothing \mathsf{B}_{d|1}^{conn}(X)^{fam} = \mathsf{SMan}(\pi TX, \underline{\mathrm{Aut}}(\mathbb{R}^{0|1}))$ (to get this to come out right, we actually have to define $\tilde\gamma$ in the obvious way, but composing with the inverse map on $\underline{\mathrm{Aut}}(\mathbb{R}^{0|1})$).

Now we can finish the calculation.

Lemma 28.1. *Given Y with a G-action and any $Z\in\mathsf{SMan}$, $Fun_{\mathsf{SMan}}(\mathsf{SMan}(Y;G),\mathsf{SMan}(Z))\cong \mathsf{SMan}(Y,Z)^G := \{\alpha\in\mathsf{SMan}(Y,Z) | \alpha\circ p_1 = \alpha\circ\mu\}$ (where $\mathsf{SMan}(Y;G)$ is the transport category and $Y\times G \underset{\mu}{\overset{p_1}{\rightrightarrows}} Y \xrightarrow{\alpha} Z$).*

So the final outcome is that $TFT_{0|1}(X) = \mathsf{SMan}(\pi TX,\mathbb{R})^{\underline{\mathrm{Aut}}(\mathbb{R}^{0|1})}$. We'll see that this is exactly closed differential forms of degree zero (i.e. constant functions on X). If we put some geometry on, then we'll get all the other closed forms. It will turn out that $EFT_{d|1}(X) = \Omega_{cl}^{even}(X)$.

[[break]]

(1) Theo made the comment that when you go from Man to SMan, you have to take super Frechét spaces in place of Frechét spaces. (2) the lemma should be that $\Omega_\varnothing\cdots$ is the stackification of something.

Remember that $\mathsf{SMan}(\pi TX,\mathbb{R}) = C^\infty(\pi TX)^{ev} = \Omega^{even}(X)$.

Lemma 28.2. *$\mu\colon \pi TX\times(\mathbb{R}^{0|1}\rtimes\mathbb{R}^\times)\to\pi TX$.*

$$C^\infty(\pi TX\times\underline{\mathrm{Aut}}(\mathbb{R}^{0|1}))\cong\Omega^*(X)\otimes\bigwedge[\theta]\otimes C^\infty(\mathbb{R}^\times)\xleftarrow{\mu^*}\Omega^*(X) = C^\infty(\pi TX)$$

is given by $\omega\mapsto\omega\otimes 1\otimes s^n + d\omega\otimes\theta\otimes s^n$, where ω is of degree n. And $p_1^*\colon\omega\mapsto\omega\otimes1\otimes1$

Looking at when μ^* and p_1^* agree, we see by looking at the degree in θ and noting that $s^n\neq 0$ that $d\omega=0$ and $s^n=1$, so $n=0$. Thus, we get closed forms of degree 0.

Corollary 28.3. $\mathsf{SMan}(\pi TX,\mathbb{R})^{\underline{\mathrm{Aut}}(\mathbb{R}^{0|1})} = \Omega^0_{cl}(X)$.

Corollary 28.4. $\mathsf{SMan}(\pi TX,\mathbb{R})^{\mathbb{R}^{0|1}\rtimes\{\pm1\}} = \Omega^{ev}_{cl}(X)$.

From this we get an idea of what a super Riemannian metric on $\mathbb{R}^{0|1}$ should be.

Definition 28.5. A *super Riemannian metric* on $\mathbb{R}^{0|1}$ is something such that $\underline{Isom}(\mathbb{R}^{0|1}$,something$)\cong\mathbb{R}^{0|1})\rtimes\{\pm1\}\subseteq\underline{\mathrm{Aut}}(\mathbb{R}^{0|1})$. ⋄

The analogue of a Riemannian structure more or less works.

In the last class (Thursday), we'll talk about degree n EFTs (we've been doing degree 0). This gives us lots of different things.

$d=0$	$TFT^n_{0|1}(X)=\Omega^n_{cl}(X)$
$d=1$	introduces Cl_n-modules $KO^n(X)$
$d=2$	gives modular forms of weight $n/2$

15 RB 12-04

This is the last lecture this semester. For next semester, there is a vague plan to have a seminar on conformal field theory. I'm not really sure what conformal field theory is, and nobody else seems to know either.

Today I want to finish describing this problem of how to get from Green's functions to a quantum field theory. Recall what we did last week. We had these Green's functions given by sums of Feynman diagrams (assuming you can deal with renormalization and regularization). The quantum field theory is given by Wightman distributions. We'd like to take the Wightman distributions to be the Green's functions, but this doesn't work (e.g. Green's functions are symmetric, but Wightman distributions aren't). We take the Wightman distributions to be equal to the Green's functions at points $(x_1, \ldots, x_n)$ where $x_i - x_j$ is spacelike.

There is another way of saying this. Wightman distributions can be thought of as analytic continuations of Green's functions. What does this mean for distributions (analytic continuation only makes sense for analytic functions). What is the ananlytic continuation of a distribution? Why does analytic continuation make sense for analytic functions? It is because of the following uniqueness condition: if f is analytic in a connected open set and zero on a non-empty open subset, then f is identically zero. This tells you that analytic continuation is unique if it exists.

We want a similar condition for distributions. Obviously, we can't do this for arbitrary distributions (it isn't even true for smooth functions), so we have to add some condition to the distribution. We could require that the distribution is analytic, but that is too strong. For QFT, the correct condition turns out to be the following. Use *analytic wave front sets*, which are sort of like wave front sets (which tells you where the distribution isn't smooth and the direction in which it isn't smooth). The analytic wave front set tells you where the distribution is not analytic and in what direction it fails to be analytic (so the analytic wave front set is larger than the wave front set). Suppose we choose a proper (doesn't contain a line) closed cone C_x in the cotangent space of each $x \in M$. Then *if f is a distribution with analytic wave front set contained in C_x at each point x and $f = 0$ on some open set, then f is identically zero.* I won't prove this, but I'll give you some examples. You have to be careful because the continuation of f may depend on the choices of C_x. Also, if you start off with a real distribution, its continuation can in general be a complex distribution.

Example 15.1. Let $f = \sqrt{x}$ for $x > 0$ (on $\mathbb{R}$). We want to analytically continue it to a distribution on all of $\mathbb{R}$. There are two ways to do this.

1. Take f to be the boundary value of $\sqrt{x}$ for $Im(x) > 0$.

2. Take f to be the boundary value of $\sqrt{x}$ for $Im(x) > 0$.

The difference is that the analytic wave front sets at zero are opposite cones. ◇

Example 15.2. Suppose $M = \mathbb{R}^n$ and $C_x = C \subseteq (\mathbb{R}^n)^*$ is the same cone at each point. Then the analytic wave front set of f is in C at each point x if and only if (more or less) f is the boundary value of a holomorphic function in the cone $\mathbb{R}^n \oplus i\hat{C}$. Since the cone C is closed and proper, the dual cone $\hat{C}$ has non-empty interior. In general, M may not have a complexification, so it isn't so clear what you mean by saying that f is a boundary value of a holomorphic function on the complexification. ◇

Example 15.3. Any distribution is locally a (finite) sum of boundary values of holomorphic functions. Suppose f is a distribution of compact support on $\mathbb{R}$. Then set $g(z) = \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{1}{z-x} f(x)\,dx$ (this is well defined provided z is not in the support of f.). Then "$f(x) = g(x + i\varepsilon) - g(x - i\varepsilon)$". ◇

The point is that if you select a proper cone at each point, there is a way to make sense of analytic continuation of distributions.

We're trying to find an analytic continuation W of a Green function such that the analytic wave front set is contianed in the region $\{(z_1, \ldots, z_n) | z_i - z_j \in$cone for $i < j\}$. Last lecture I wrote down a confusing formula for what W is.

$G(x_1, \ldots, x_n)$ is a sum of Feynman diagrams that have a node of valence 1 for each x_i and nodes of valence 4 coming from the $\lambda\phi^4$ in the Lagrangian. $W(x_1, \ldots, x_n)$ will be similar sums of Feynman diagrams, but with slightly different propagators. Instead of summing over all Feynman diagrams with valence 4 nodes, you sum over diagrams like $x_1 \ \times \times | \times \times | \times \times x_2 | \times \times | x_3$, where the x_i are nodes of valence 1 and the $\times$ are nodes of valence 4. The propagators in the odd buckets are Feynman propagators, in the even buckets are conjugates of Feynman propagators, and the ones that cross buckets are cut propagators.

More generally, let's define $S(f_1, f_2, \ldots,)\overline{S}(g_1, \ldots)S(h_1, \ldots)$ to be a sum of all diagrams as in the previous paragraph, but in the first bucket we allow nodes for the f_i, in the second bucket, we allow some g_i, and so on.

Main properties:

1. $S(f_1, \ldots, f_n)\overline{S}(g, h_1, \ldots, h_m) = S(f_1, \ldots, f_n, g)\overline{S}(h_1, \ldots, h_m)$ provided no element of $\operatorname{Supp} g$ is less than or equal to any element of $\operatorname{Supp} f_i$ or $\operatorname{Supp} h_i$.
2. $S\overline{S} = 1 = \overline{S}S$ (i.e. if you don't put any functions, they cancel). This is assuming you've chosen a renormalization already.

Rather than proving these results, let me show you how to use them to prove that the Wightman distributions $W(f_1, f_2, \ldots) = S(f_1)\overline{S}S(f_2)\overline{S}\cdots$ satisfy locality.

$$\begin{aligned}
W(f_1, f_2) &= S(f_1)\overline{S}S(f_2)\overline{S} \\
&= S(f_1)\overline{S}(f_2)S\overline{S} \\
&= S\overline{S}(f_1, f_2)S\overline{S} && \text{provided } \operatorname{Supp} f_1 \not\leq \operatorname{Supp} f_2 \\
&= S(f_2)\overline{S}(f_1)S\overline{S} && \text{provided } \operatorname{Supp} f_1 \not\geq \operatorname{Supp} f_2 \\
&= W(f_2, f_1)
\end{aligned}$$

So we can switch f_1 and f_2 provided their supports are spacelike separated.

We can also handle infrared divergences. I've been saying that these divergences automatically cancel. Suppose you're working with a Lagrangian L with compact support. Add to this another Lagrangian M, also with compact support. I want to find conditions under which addition of M makes no difference to the Wightman distributions. To do this, we can make the following calculation. The new Wightman distributions are given by

$$\begin{aligned}
S(f, e^M)\overline{S}(e^{-M})S(f_2, e^M)\overline{S}(e^{-M})\cdots &= S\overline{S}\cdots S\overline{S}S(f, e^M)\overline{S}(e^{-M})S(f_2, e^M)\overline{S}(e^{-M})\cdots \\
&= S(f_1)\overline{S}S\overline{S}\cdots S(e^M)\overline{S}(e^{-M})\cdots
\end{aligned}$$

provided no element of $\operatorname{Supp} f_1$ is $\leq$ anything in $\operatorname{Supp} M$. Keep pulling the $S(f_i)$ through and cancel the left over copies of $S(e^M)\overline{S}(e^{-M})$. What is left is the Wightman distribution for the Lagrangian L. So adding M to the Lagrangian does not affect the Wightman distributions provided that no element of $\operatorname{Supp} f_i$ is less than or equal to anything in $\operatorname{Supp} M$.

A similar argiment shows that the same is true if no element of $\operatorname{Supp}\{f_i\}$ is greater than or equal to anything in $\operatorname{Supp} M$.

[[★★★ picture]]

So suppose the support of M lie in the shaded region, then adding M doesn't make a difference. So we're left with a compact region that we have to leave alone. We can mess with the Lagrangian outside of this compact region. Suppose the interaction is given by $\int \lambda(x)\phi(x)^4\,dx$, where $\lambda(x)$ has compact support. Then W are independent of λ provided that $\lambda(x) = 1$ inside a certain compact region (that depends on the supports of the f_i). This is essentially the cancellation of IR divergences (which come from λ having non-compact support).

Example 15.4. Suppose the union of the supports of the f_i is S, as in the picture.[[★★★ picture]] we need to add little bits x such that it is possible to send a message from x to S and from S to x. ⋄

We need to know that if S is compact, then this larger region is also compact. For Minkowski spacetime, this is an easy exercise, but it is not true for all spacetimes. For example, take $\mathbb{R}^{1,3} \setminus pt$ and make it so that you have to add a neighborhood of the missing point to S. But this spacetime has a hole in it, and you don't know what is coming through the hole, so you don't expect things to work very well. The compactness condition is almost equivalent to spacetime being globally hyperbolic.

So on globally hyperbolic spacetimes (which are the only ones anybody works with), the infrared divergences of quantum field theory automatically cancel.

40 NR 12-05

Last time I was explaining 2-dimensional discrete Yang-Mills theory. We started with the classical case. The objects of the spacetime category are 1-dimensional oriented closed manifolds (disjoint unione of copies of S^1) with marked points. The morphisms are cell complex decompositions Γ of 2-dimensional compact oriented manifolds. You should think of a morphism as a surface bordism between collections of circles.

Classically, the space of fields is the space of sections of the trivial principal G-bundle B over Γ (the total space is $G^{V(\Gamma)}$). A connection on this bundle B is a choice of parallel transport isomorphism for each edge. So a connection is a map $E(\Gamma) \to G$, so the space of connections is $G^{E(\Gamma)}$. Let $g(e)$ be "parallel transport along the edge e". A connection is *flat* if the ordered product $\prod_{e\in\partial f}^{\rightarrow} g(e)^{\sigma(e,f)} = h_{x_0}(f) = 1$ for each face f ($\sigma(e,f) = \pm 1$ depending on the relative orientations of f and e).

The classical Yang-Mills theory on $\Gamma \subseteq \Sigma$ has the following action.

$$A(g) = \sum_{f\subseteq(\Gamma\subseteq\Sigma)} w(f)\,\mathrm{tr}(h(f))$$

where $w(f) > 0$ is [[★★★]]. This action is gauge invariant. The gauge group can be identified with $G(\Gamma) = G^{V(\Gamma)}$. It acts on connections by $h\colon g(e) \mapsto h(e_+)g(e)h(e_-)^{-1}$, where e_- is the source vertex and e_+ is the target vertex. I don't have time to describe the Hamiltonian structure, but it is a good exercise to describe it.

What is the Quantum version? To 1-dimensional manifolds N_1, objects of the spacetime category, we should identify some space $H(N_1) = L_2(G^{E(N_1)})$ (remember that N_1 is a collection of circles with marked points, so edges are pieces of circle between marked points). To $(\Gamma \subset \Sigma)$, we assign a vector $v(\Gamma \subset \Sigma) \in L_2(G^{E(\partial(\Gamma))})$.

Our path integral philosophy tells us how to choose this vector $v(\Gamma \subset \Sigma)$.

$$v(\Gamma \subset \Sigma)(b) = \int_{G^{E_{in}(\Gamma)},\, g|_{\partial\Sigma}=b} \exp(-A(g)/h)\,\mathcal{D}_h g$$

If we have the i in the exp, it is a more realisitc theory (but still not realistic because Σ should have more structure). Removing the i is more like statistical mechanics [[★★★ I missed the explanation]].

Let's recall what we want from this quantum field theory. (1) We want locallity, which is equivalent to the gluing principle. That is, if we have two parts of the boundary of Σ and an isomorphism between the two parts (call fields b, and let b' denote fields on the rest of the boundary). We want

$$\int_{G^{E_{in}(\Gamma)}} \exp(-A(g,b,b'))\,\mathcal{D}_h g\,\mathcal{D}_h b = \int_{G^{E_{in}(\tilde\Gamma)}} \exp(-A(g,b')\,\mathcal{D}_h g$$

(2) We require Guage invariance of $v(\Gamma \subset \Sigma)(b)$. (3) We can try to find $\mathcal{D}_h g$ such that the result is a TQFT. So $v(\Gamma \subset \Sigma)(b)$ will only depend on $h(C_i)$ (C_i are the connected components of the bounary) and the genus of Σ. Since we require gauge invariance, v should only depend on the conjugacy classes $[h(C_i)]$, the genus, and the number of components of $\partial\sigma$. (4) The classical limit should recover the classical field theory.

There is no reason to expect that we should be able to find such a thing. A stronger gluing axiom: given two components of the boundary of Σ and given an isomorphism f between *part* of the components. For example, you could take two disks (each with 1 2-cell and a cell decomposition of the boundary) and try to glue together some of the 1-cells on the boundary. How about (3), can we choose the measure so that the result is topological? It turns out the answer is yes. Assume G is simple compact. Take the measure $e^{-A(g)}\mathcal{D}_h g = \prod_f w([h(f)]|A)\prod_e dg_e$, where $w([g]|A) = \sum_\lambda e^{-c_2(\lambda)A_f}\dim(V_\lambda)\chi_\lambda(g)$, where the sum is over all irreducible representations and c_2 is the second Casimir.

Then the formula for $v(\Gamma \subset \Sigma)$ is

$$v(\Gamma \subset \Sigma) = \int_{G^{E_{in}(\Gamma)}} \prod_f w([h(f)]|A_f) \prod_{e\in E_{in}(\Gamma)} dg(e).$$

Now let me explain where this measure came from. Start with disks with one 2-cell. Start with one disk, with one real number A. Then $v(\Gamma \subset D) = w([h(\partial D)]|A)$. To glue, we have to choose a measure on the boundary. We choose $\mathcal{D}_h b = \prod_{e\in\text{gluing edges}} db(e)$.

Then you can check the following identity. We want to glue two oriented disks (with real numbers A and B and opposite orientations on the gluing edges) along an edge. Say we have b assigned to the edge, the holonomy around the rest of the A-disk is g, and the holonomy around the rest of the B-disk is h. HW: check that

$$\int_G w(bg|A)w(b^{-1}h|B)\,db = w(gh|c_{A,B})$$

when $c_{A,B} = A + B$. So the natural interpretation for the real numbers A and B is area.

$$\int_G \chi_i(bg)\chi_j(b^{-1}h)\,db = \#(i,j)\chi_{\#}(gh).$$

So it is almost topological, except that we have to add up A and B when we do the gluing. There is a formula

$$v(\Gamma \subset \Sigma)(b) = \sum_{\lambda} \prod_{i \in \text{bd. comp. of } \Sigma} \chi_\lambda(h(C_i))e^{-c_2(\lambda)\sum_f A_f}(\dim V_\lambda)^{\chi(\Sigma)}.$$

This is kind of an amazing identity showing that v doesn't depend on Γ at all. This is called 2-dimensional QCD (reference is Witten, 2D Yang-Mills theory revisited).

This demonstrates that using the ideology of the path integral, one can construct 2-dimensional TQFTs. By allowing Γ to get finer and finer, Witten argues that as the size of the approximation grows, this goes to the Yang-Mills path integral and the weight goes to $e^{A_{YM}}$, where A_{YM} is the smooth Yang-Mills action. Smooth Yang-Mills theory is almost topological. You don't need a metric, just a volume form (just like we needed the area).

It turns out that 3-dimensional Yang-Mills theory is not topological, but there is a 3-dimensional TQFT, which is Chern-Simons theory. Unfortunately, it is not known how to replace the infinite-dimensional path integral by a finite-dimensional one for Chern-Simons theory. We can try to develop some formal power series which resembles what we have. The power series should (1) satisfy gluing, (2) be gauge invariant, (3) topological, and (4) classical limit reproduces the classical field theory. In Chern-Simons, it is possible to find such perturbative expansions.

Nobody has produces a gluing procedure for these invariants yet. All of these results were developed for closed 3-manifolds, but not for 3-manifolds with boundary. Even for closed manifolds, the results that exist are for "acyclic flat connections".

Based on surgery, with Turaev we developed [[★★★ didn't catch all of this]].

Next time I'll return to BV quantization.

29 PT 12-06

Tuesday's mini-conference in 939 starts at 9:00. It will probably go til 17:00. There will be 10 speakers. Bring lunch. There will be a sponsored dinner.

I was going to explain twisted field theory, but I decided that would just be more definitions. So instead, let's use the definitions we have already.

Recall that we found a very complicated way to talk about constant functions $\Omega^0_{cl}(X) = TFT_{0|1}(X) = \mathsf{SMan}\big(\pi TX/\underline{\mathrm{Aut}}(\mathbb{R}^{0|1}), \mathbb{C}\big)$. πTX is a super manifold with a Lie group acting on it, so you get a quotient stack, and we're considering functions on this stack. Being invariant under translations makes the forms closed and being invariant under dilations makes them degree 0. I wanted to do twisted stuff, which would be $\Omega^n_{cl}(X) \cong TFT^n_{0|1}(X)$.

If you put a geometry on a super point (so that $\mathrm{Aut}(\mathbb{R}^{0|1})$ is only translations), you get $EFT_{0|1}(X) = \Omega^{ev}_{cl}(X)$. Consider the map $EFT_{0|1}(X) \xrightarrow{\times S^1} EFT_{1|1}(X)$. There is a nice map (due to Florin Damitrescu), which is super parallel transport from vector bundles with connection to $EFT_{1|1}(X)$. Fei Han's thesis shows that the map from vector bundles with connection to $\Omega^{ev}_{cl}(X;\mathbb{C})$ is the Chern character form. Fei will talk about this in the student seminar next semester. Andy: what happens when you twist? PT: you get mod 2 periodicity, and if you put 1, you get odd forms.

Today I want to explain how the usual "susy cancellations" lead to modularity of the partition function ($d = 1, 2$). We'll take degree 0, and assume X is a point.

$$\begin{aligned} EFT_1 &= smBiFun_{\mathsf{Man}_2}(\mathsf{RB}_1^{fam}, \mathsf{Fr}_2^{fam}) \\ &\xrightarrow[\text{conn}]{\Omega_\varnothing} Fun_{\mathsf{Man}}(\Omega^{\text{conn}}_\varnothing \mathsf{RB}_1^{fam}, \Omega_{\mathbb{C}}\mathsf{Fr}^{fam}) \\ &= Fun_{\mathsf{Man}}\big(\Omega^{\text{conn}}_\varnothing \mathsf{RB}_1^{fam}, \mathsf{Man}(\mathbb{C})\big) \end{aligned}$$

What is $\Omega^{\text{conn}}_\varnothing \mathsf{RB}_1^{fam}$? The objects are Riemannian S^1-bundles $\Sigma \to S$ and the morphisms are bundle maps that are fiberwise isometries. This is the moduli stack of Riemannian circles. Riemannian circles are classified by their length, so this stack is $\mathbb{R}_+/SO(2)$ (the action is trivial). Thus, this functor space is just $\mathsf{Man}(\mathbb{R}_+/\mathbb{C}) = C^\infty(\mathbb{R}_+)$ (you don't see the rotations when you map to a representable stack). So given $E \in EFT_1$, you get the 1-dimensional *partition function* $Z_E(t) \in C^\infty(\mathbb{R}_+)$, given by $E(S^1_t)$. If E is the σ-model of a compact Riemannian manifold M, then $Z_E(t) = \mathrm{tr}(e^{-t\Delta})$.

An element of EFT_1 assigns to a point a Hilbert space and to intervals linear maps. When you glue the two ends of the interval together, you get a circle. The value of the theory on a circle is given by the partition function. But from the gluing law, you know it will be the trace $\mathrm{tr}(E([0,t]))$.

Theorem 29.1. *If E is susy (i.e. $E \in EFT_{1|1}$, not just EFT_1), then Z_E is a constant integer. That is, $\frac{\partial}{\partial t} Z_E = 0$ and Z_E is an integer.*

Before explaining the assumption (that E is susy), let me give an example of a supersymmetric field theory.

Example 29.2. $SE \in EFT_{1|1}$ from a compact Riemannian *spin* manifold (M, σ, g). I haven't told you exactly what the geometry on a super point or a super interval is. Remember that the objects aren't just points (they are little collars). Since the geometry on a point is a geometry on an $\mathbb{R}^{0|1}$ with some thickenning to the collar, it turns out that the isometries of the thickenned super point are just reflection (so $\{\pm 1\}$). The $\mathbb{Z}/2$-action is the same as a $\mathbb{Z}/2$-grading. So $SE(spt) = \mathbb{Z}/2$-graded Hilbert (really Frechét) space. So we define $SE(spt)$ to be $\Gamma_{L^2}(S_M)$.

Remember that $E \in EFT_1$ gives a smooth semi-group homomorphism $\mathbb{R}_+ \to B^{tc}(H)$ (bounded trace class operators) (the $\mathbb{R}_+$ comes from the moduli space of Riemannian circles), given by $t \mapsto E([0,t])$. If you have a super symmetric $SE \in EFT_{1|1}$, you get a super semi-group homomorphism $\mathbb{R}_+ \times \mathbb{R}^{0|1} \to B^{tc}(H)$. The $\mathbb{R}_+ \times \mathbb{R}^{0|1}$ is the moduli stack of Riemannian intervals, with super group structure $(t_1, \theta_1)(t_2, \theta_2) = (t_1 + t_2 + \theta_1\theta_2, \theta_1 + \theta_2)$; the Lie algebra is free on one odd generator. [[★★★ from some stuff]] you see that $E([0,t])$ must be of the form e^{-tA} and $SE(t, \theta)$ must be of the form $e^{-tD^2 + \theta D}$, where D is an odd operator on H.

For our example, we take D to be the Dirac operator. We needed compactness to get $e^{-tD^2+\theta D}$ to be trace class. ◇

So now the theorem statement makes sense, taking $A = D^2$. $Z_{SE}(t) =$

$\mathrm{str}(e^{-tD^2})$. Now we're back to what the physicists showed us a few decades ago, susy cancellation.

Let's calculate $\mathrm{str}(e^{-tD^2})$. Pretend that we can diagonalize D^2 (self-adjointness comes from symmetry if the interval when you reflect), so $H = \bigoplus_\lambda E_\lambda$, where λ are the eigenvalues of D^2.

$$\mathrm{str}(e^{-tD^2}) = \sum_\lambda e^{-t\lambda}\,\mathrm{sdim}\, E_\lambda = \mathrm{sdim}\, E_0 = \mathrm{sdim}\ker(D) = \mathrm{index}(D)$$

We have the operator $D\colon E_\lambda \to E_\lambda$ (because D commutes with D^2). The E_λ are graded, and D is an odd operator. If $\lambda \neq 0$, then D is an isomorphism of E_λ, but then the super dimension is zero.

[[break]]

Now consider

$$\begin{aligned} EFT_2 &\xrightarrow[\text{conn}]{\Omega_\varnothing} Fun_{\mathsf{Man}}\big(\Omega^{cl}_\varnothing \mathsf{RB}_2, \mathsf{Man}(\mathbb{C})\big) \\ &= C^\infty(h \times \mathbb{R}_+) \qquad (*) \end{aligned}$$

We assume that we only use *flat* surfaces because we want to do elliptic cohomology (the only compact flat manifolds are tori, which are exactly the elliptic curves). Maybe this should be called FFT. We also restrict to flat Riemannian manifolds in RB_2^{cl}. So $\Omega_\varnothing \mathsf{RB}_2$ is the moduli space of flat tori, which is $h \times \mathbb{R}_+/SL_2(\mathbb{Z})$ (where h is the upper half plane, the conformal part, and the $\mathbb{R}_+$ is the area of the torus). The $SL_2(\mathbb{Z})$ action is given by $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)(\tau,\ell) = \left(\frac{a\tau+b}{c\tau+d}, |c\tau+d|\cdot\ell\right)$. Note that you change the area of the torus when you act, so you can't just say "I'll just work with area 1 tori".

The map in $(*)$ is given by $E \mapsto E(T_{\tau,\ell}) =: Z_E(\tau,\ell)$, which is called the *partition function*. We have

$$\begin{aligned} \{\text{integral modular fuctions}\} &\subseteq_d \{\text{modular function}\} \\ &\subseteq_c Hol(h)^{SL_2\mathbb{Z}} \\ &\subseteq_b C^\infty(h)^{SL_2\mathbb{Z}} \\ &\subseteq_a C^\infty(h\times\mathbb{R}_+)^{SL_2\mathbb{Z}} \ni Z_E \end{aligned}$$

(a) says $\frac{\partial}{\partial \ell} Z_E = 0$. (b) says $\frac{\partial}{\partial \bar q} Z_E$.

There is a map $h \to D^2 \setminus 0$ (interior of the disk minus the center), given by $\tau \mapsto e^{2\pi i\tau} =: q$. $f(\tau)$ is a function of q if f is invariant under $\left(\begin{smallmatrix} 1 & k \\ 0 & 1\end{smallmatrix}\right) \subseteq SL_2\mathbb{Z}$. f is modular if $f(q)$ can be extended to 0 with a pole of finite order. (c) says Z_E is modular.

Theorem 29.3. *If $SE \in EFT_{2|1}$, then $Z_E(\tau,\ell)$ is an integral modular function (i.e. statements a,b,c, and d are true).*

For the proof for $d = 1$, we realized that

$$Z_E(t) = E(S^1_t) =_{\text{gluing}} \mathrm{str}\, E([0,1]) = \mathrm{str}(e^{-tD^2}).$$

For $d = 2$, let $A_{\tau,\ell}$ be the annulus (don't identify the top and bottom of the parallelogram).

$$\begin{aligned} Z_E(\tau,\ell) = E(T_{\tau,\ell}) &= \mathrm{str}(\underbrace{E(A_{\tau,\ell})}_{\in B^{tc}(H)}) = \mathrm{str}(q^{L_0}\bar q^{\overline{L}_0}) \\ &= \sum_{\lambda(\ell),\mu(\ell)} q^\lambda \bar q^\mu\,\mathrm{sdim}\, E_{\lambda,\mu} =_{susy} \sum_\lambda q^\lambda \underbrace{\mathrm{sdim}\, E_{\lambda,0}}_{a_n:=} \end{aligned}$$

$H_\ell = E(S^1_\ell)$. $[L_0, \overline{L}_0] = 0$. λ and μ are eigenvalues of L_0, $\overline{L}_0$.

If we fix the length ℓ, we get a 2-dimensional semigroup with 2 infinitesimal generators. Now the super symmetry tells me that $\overline{L}_0(\ell) = \overline{G}_0(\ell)^2$ for some odd operator $G_0(\ell)$ on H_ℓ ($[G_0, \overline{G}_0] = 0$). If we had two square roots, the function would have to be constant. $\lambda \in \mathbb{Z}$ because $L_0 - \overline{L}_0$ generates a circle action. Since everthing is smooth in ℓ, and things are integers, so they are independent of ℓ. From something, the sum is bounded from below (starts at some point, bigger than $-\infty$).

41 NR 12-07

Last time I talked about discrete 2-dimensional Yang-Mills theory.

In 4-dimensional Yang-Mills theory, the fields are connections on a principal G-bundle on a 4-dimensional manifold M. For physical applications, M is Minkowski (it has a non-degenerate 2-form of signature $(1,3)$). The action functional is

$$\mathcal{A}(A) = \int_M \operatorname{tr}(F(A)^2)\,dx$$

$$\overbrace{+\frac{1}{2}\int_M \langle\nabla\phi,\nabla\phi\rangle\,dx + \int_M V(\phi)\,dx}^{\text{matter}}$$

$$\underbrace{+\int_M (\psi,\mathcal{D}\psi)\,d^2x + m\int_M (\psi,\psi)[[\text{mass}]] + \int_M F((\psi,\psi))\,dx}_{\text{fermions}}$$

($\mathcal{D}$ the dirac operator) We can add matter fields (add sections of a G-bundle where fibers are some representations of G). We can add fermions (sections of a G-bundle of super vector spaces).

Can we quantize this? The machinery for (perturbative) quantizing is Feynman diagrams. The major obstruction is that the Feynman diagrams diverge. The question (for physics) is how to fix this to get something that can be tested. Mathematically, the whole problem requires redefinition. There are two kinds of divergence. Some of the divergences can be absorbed in the fact that you have infinitely many degrees of freedom. The first thing you have to do is deal with the gauge symmetry. This is what FP, BRST, and BV do. Then you have to deal with renormalization.

The standard game played in high energy physics is to start with a large symmetry group ($G = SU(N)$, $N = 4$) and try to break them.

How many orders of perturbation theory should we compute? We know the series diverges, so you should compute more than, say, 10 orders.

There are also theories that are interesting from the mathematical perspective, like Chern-Simons theory. Probably the most interesting is 3-dimensional Chern-Simons. The fields are connections on the trivial G-bundle on M_3.. The action functional is

$$\mathcal{A}(A) = \int_M \operatorname{tr}\big(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\big)$$

This is not invariant, but it is invariant under infinitesimal transformations $A \mapsto [\gamma, A] + d\gamma$, where $\gamma\colon M \to \mathfrak{g}$. $e^{2\pi i\mathcal{A}(A)}$ is invariant under global transformations because the difference is an integer (if you pick the coefficients right). One can, for any integer n, try to make sense of

$$\int_{\text{connections}} e^{2\pi i n\mathcal{A}(A)}\,\mathcal{D}A$$

If n is finite, it is hard. For $n\to\infty$, first you should find critical points. (1) it is easy to see that the critical points are flat connections, for which $F(A) = 0$. (2) the integral should be the "sum" over all guage classes of flat connections "$\sum\int \exp(2\pi i n\mathcal{A}(A+\alpha))\,\mathcal{D}\alpha$", where α is a 1-form on M with coefficients in G. α is small in the sense that that

$$n\mathcal{A}(A+\alpha) = n\mathcal{A}(A) + \int_M \operatorname{tr}(\alpha\wedge d\alpha) + n\int_M \operatorname{tr}(\alpha\wedge A\wedge\alpha) + \frac{2}{3}\int_M \operatorname{tr}(\alpha\wedge\alpha\wedge\alpha)$$

Rescale $\alpha \to \alpha/\sqrt{n}$, and consider the formal power series expansion of the integral:

$$\text{“}\sum\text{”}\, e^{2\pi i n\mathcal{A}(A)}\int e^{2\pi i\int(\alpha\wedge\mathcal{D}_A\alpha) + 2\pi i\frac{2}{3\sqrt{n}}\int\operatorname{tr}(\alpha\wedge\alpha\wedge\alpha)}\,\mathcal{D}\alpha = \sum_\Gamma \frac{F(\Gamma)}{|\operatorname{Aut}\Gamma|}$$

if such expressions existed. The propagator in the diagrams would be $(\mathcal{D}_A)^{-1}$. If we had expressions that made sense, it would overcome the problem (by introducing a metric, say) that each diagram diverges (because of the gauge invariance), then we'd have to show that the result is independent of metric or whatever. This is how finite type invariants of 3-manifolds were started.

Let's focus on the problem of how to deal with gauge theory. I promised to talk about BV, so that's what I'll do now. Next time I'll write the answer for Feynman diagrams describing Chern-Simons theory.

So what does BV do? It deals with the problem that you might have integrals

$$\int_X e^{A/h}\,dx$$

where there is a group G acting (locally freely, say) on X. We want to find a formal power series expansion for this integral. Assume that X has a volume form dx. We can try to write the integral as

$$\int_{X/G} e^{[A]/h}\,[dx]$$

The problem is that even if X is a linear space, the quotient can be quite bad. The idea is to replace the integral using the spirit of cohomological field theories by an integral over some huge superspace E:

$$\int_E e^{\tilde{A}/h} dy$$

where $d\colon F(E) \to F(E)$ is a vector field on E and $d(e^{\tilde{A}/h}) = 0$. Cattaneo and Felder define BV-style cohomological field theory, Poisson sigma model, which gives the star product for Poisson manifolds.

What are the main ingredients of BV theory?

- A manifold X with a volume form dx.
- G acting on X, preserving the volume form dx. If $\{e_\alpha\}$ is a basis for $\mathfrak{g} = Lie(G)$, let the action be given by vector fields $X_\alpha = \sum_i X^i_\alpha(x)\frac{\partial}{\partial X^i}$ in local coordinates.
- $\widetilde{X} = X \times \mathfrak{g}[1]$, the super manifold where X is the even part and $\mathfrak{g}$ is the odd part.
- because G acts on X, it induces an odd vector field Q on $\widetilde{X}$. Q acts on functions on $\widetilde{X}$, which are $Fun(X) \otimes \bigwedge^\bullet \mathfrak{g}^*[1]$. As a super vector space, it is $C^\bullet(\mathfrak{g}, Fun(X))$. It has a differential, which is Q. We assume that $H^i(\mathfrak{g}, Fun(X)) = \delta^{i,0} H^0(\mathfrak{g}, X) = Fun(X/G)$.

  In local coordinates $\{x^i\}$ on X and $\{c^\alpha\}$ on $\mathfrak{g}$, with $[e_\alpha, e_\beta] = \sum_\gamma e^\gamma_{\alpha\beta} c_\gamma$. In local coordinates, the BRST operator is

  $$Q = \sum_{\alpha,i} c^\alpha X^i_\alpha(x)\frac{\partial}{\partial x^i} + \frac{1}{2}\sum_{\alpha,\beta,\gamma} c^\gamma_{\alpha\beta} c^\alpha c^\beta \frac{\partial}{\partial c^\gamma}.$$

- The odd cotangent bundle $E = \pi T^* \widetilde{X}$. X has a volume form. [Assume $\mathfrak{g}$ has a $\mathfrak{g}$-invariant scalar product, so that $\widetilde{X}$ has a volume form.] We have the odd $\omega = \sum_i dx^i d\xi_i + \sum_\alpha dc^\alpha d\lambda_\alpha$ (ξ and λ are the cotangent directions to x and c). We have $\{F, G\}$ on $Fun(\pi T^* \widetilde{X})$.
- Q lifts of an odd vector field on E. This lift is Hamiltonian, with $h_Q = \sum_\alpha c^\alpha X^i_\alpha(x)\xi_i + \frac{1}{2}\sum_{\alpha,\beta,\gamma} c^\gamma_{\alpha\beta} c^\alpha c^\beta \lambda_\gamma$. Denote $Q_E = Q$. This is still odd, and $Q^2 = 0$. So $QF = \{h_Q, F\}$.
- If A is a G-invariant function on X, it pulls back to a function on $\widetilde{X}$ and on E. Let's denote the pull back to E also by A. What can we say about it? (i) $\{h_Q, A\} = 0$ because A is G-invariant. (ii) $\widetilde{X}$ is a Lagrangian in E, so $\{A, A\} = 0$. (iii) $\{h_Q, h_Q\} = 0$ because $Q^2 = 0$ (this is non-trivial because Q is odd). So $\{h_Q + A, h_Q + A\} = 0$.

  We have an L_∞-algebra. We have (E, Q), $\tilde{A} = A + h_Q \in Fun(E)$. This has the properties that $Q\tilde{A} = 0$ and $\{\tilde{A}, \tilde{A}\} = 0$ (this is an indication that it came from some Lagrangian submanifold).

42 NR 12-10

I'll (NR) keep working on the lecture notes. After January, they should be edited a fair amount.

Today I'll try to outline the perturbation theory of Chern-Simons.

First let me explain the concept of BV quantization. The problem whne you read the papers is that you don't see that this is equal to that. Instead, you have things that you know don't quite make sense, like $\int_{X/G} e^{A/h}[dx]$, which you have to replace by something else in order to compute it.

You can have two different quasi-isomorphic field theories. The BV concept is to replace these integrals $\int_{X/G} e^{A/h}[dx]$ by

$$\int_{L\subseteq E} e^{\tilde{A}/h}\, d\tilde{x} \qquad (*)$$

where $\widetilde{X} = X \times \mathfrak{g}[1]$ and $E = \pi T^*\widetilde{X}$. The action of G on X (which has a volume form) induces an odd vector field Q on $\widetilde{X}$. This Q lifts to a Hamiltonian vector field on the odd cotangent bundle E. E has the odd symplectic form $\omega = d\alpha$. This lift of Q is Hamiltonian, with $h_Q = \sum_{\alpha,i} c^\alpha X^i_\alpha(x)\xi_i + \frac{1}{2}\sum_{\alpha,\beta,\gamma} c^\gamma_{\alpha\beta}c^\alpha c^\beta \lambda_\gamma$, where x are coordinates on X, c^α are coordinates on $\mathfrak{g}[1]$, ξ are coordinates on the fibers of T^*X, and λ are coordinates on $\mathfrak{g}^\vee$. [[★★★ some discussion among NR, PT, and BD]] Given $Q\colon \widetilde{X} \to T\widetilde{X} \cong T^*\widetilde{X}$, $dQ\colon T\widetilde{X} \to TT\widetilde{X} \cong T(T^*\widetilde{X})$. h_Q should be a canonical construction.

We're doing all this for one reason. We want to compute $\int_X e^{A/h}\, dx$ (if G is compact). We cannot compute the perturbative expansion because the G-symmetry makes stuff degenerate. There are isolated critical points, and we only need formal neighborhoods of these critical points to compute with, and in these neighborhoods, we can use local coordinates.

The action A is G-invariant. We can pull it back to $\widetilde{X}$ and to E. Let A also denote the pullback of A to E. G-invariance says that $A|_{\widetilde{X}}$ is such that $QA = 0$. G-invariance on E means that $\{h_Q, A\} = 0$, where $\{F, G\}$ is the odd Poisson bracket on $Fun(E)$ induced by ω.

$Q^2 = 0$, which means that $\{h_Q, h_Q\} = 0$ (this is non-trivial because the bracket is odd).

$\widetilde{X} \hookrightarrow E = \pi T^*\widetilde{X}$ (the zero section) is a Lagrangian submanifold. This means that $\{A, A\} = 0$. Again, this is non-trivial because the bracket is odd.

When we combine all this, we have that $\{h_Q + A, h_Q + A\} = 0$. Let $\tilde{A} = h_Q + A$. The goal is to choose $\tilde{A}$ so that it is non-degenerate at critical points and "$\int_L e^{\tilde{A}/h}\, d\tilde{x} = \int_X e^{A/h}\, dx$". Assume $\tilde{A}$ has a critical point at zero (let X be a pointed manifold). In a neighborhood of this point, we want to look at $\tilde{A}(m) = \frac{1}{2}\big(m, q(m)\big) + S(m)$, where $S(m)$ is $O(m^3)$ and $\big(m, q(m)\big)$ is the pairing induced by ω and q is some linear operator (m is a tangent vector). Let $\hat{q}(m) = \big(m, q(m)\big)$.

Then we have $\{\tilde{A}, \tilde{A}\} = 0$. This is equivalent to $q^2 = 0$ as a linear operator, and $\{\hat{q}, S\} + \{S, S\} = 0$.

Claim. *Generically (cohomologies of $Fun(E)$ are non-zero only in degree 0), q is non-degenerate on L. That is, $\int_L e^{\tilde{A}/h}\, d\tilde{x}$ defines a formal power series that is a candidate for $(*)$.*

This handles the problem of degeneracy, but creates another problem. Such integrals, as they are written, depend on L. We should also argue why this integral has anything to do with what we started with. There are two lemmas. [[I'm very happy to have learned all this stuff, but it is still settling. I advise you to look at the lecture notes at the end of January.]] We want the integral to only depend on the cohomology class of $\tilde{A}$. We want to argue that the proposed integral is $\int_{L_0} e^{A/h}\, d\tilde{x}$ where L_0 is $X \times \mathfrak{g}[1] = \widetilde{X}$ (the zero section). Then we can argue that this is $\int_X d^{A/h}\, dx$, but not now.

X has a volume form, and G has Haar measure on it, so we get a volume form on $\widetilde{X}$. Locally, assume $\{a_i\}$ are coordinates on $\widetilde{X}$ and $\{\alpha_i\}$ are coordinates in the cotangent direction. Assume a's are even and α's are odd (this is a brave assumption; it is not true in our case). Then we have the operator (called the BV Laplacian) $\Delta = \sum_i \frac{\partial^2}{\partial a^i \partial\alpha_i}$. Locally, $Fun(\pi T^*\widetilde{X}) = \{\sum_{\{i\}} f^{i_1\dots i_k}(a)\alpha_{i_1}\cdots\alpha_{i_k}\}$, which are forms on $\widetilde{X}$. $\Delta\colon \Omega^\bullet(\widetilde{X}) \to \Omega^\bullet(\widetilde{X})$ is the Hodge dual of the differential $d*$. PT: how do you get the Hodge operator from just the volume form? I know how to do it with a metric. NR: We could equip X with a metric, but I think it is possible to get the Hodge operator just from the volume form.

Lemma 42.1 (α). *If $\Delta F = 0$, $\int_L F\,\widetilde{vol} = \int_{L_\sigma} F\,\widetilde{vol}$ where L_σ is a La-*

grangian homotopic to L. Here F is a function and we integrate against the volume form on L (and some deformaed volume form on L_σ).

The proof I know is not very satisfactory. Assume $L = L_0$ and let $L_\sigma = \{\xi = d\sigma(\tilde{x})\}$ for some function σ on $\widetilde{X}$.

Lemma 42.2 (β)**.** $\int_L \Delta F \cdot vol = 0$.

Once we have these two lemmas, we can use Lemma α to get $\int_L e^{\tilde{A}/h}\, d\tilde{x} = \int_{L_0} e^{A/h}\, d\tilde{x}$, and Lemma β to replace A by anything in the same cohomology class.

It seems like it isn't very clear how this formalism works for manifolds with boundary.

Let me say how this changes the naïve perturbation for Chern-Simons. We want to make sense of $\int e^{ikCS(A)}\mathcal{D}A$, where the integral is over connections on a G-bundle on M_3. We want to make sense of this integral as $k \to \infty$.

$$CS(A_0 + \alpha) - CS(A_0) = S_2(\alpha) + S_3(\alpha)$$

gives Feynman diagrams with trivalent vertices. Still some trouble, so we use BV. BV is too general, so use FP. The Feynman diagrams you get after applying this machinery will have the same trivalent vertices and the same propagators. The difference is that they will not be bosonic diagrams. There will be an extra factor of $(-1)^{\#}$. This happens because we used the technique of making a perturbative expansion on a theory which is a quasi-isomorphic theory. You then check that the result is independent of all the choices you made. This is what is called perturbative, or finite type, invariants of 3-manifolds. This stuff has only been done for manifolds without boundary, but for QFT we certainly need to do it for manifolds with boundary. So many things are still open.

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