Contents

How these notes are coming to exist

It is Fall 2007. QFT classes are being taught at UC Berkeley by Richard Borcherds (RB), Nicolai Reshetikhin (NR), and Peter Teichner (PT). Anton LaTeXs these notes in class and edits them later.[1] The version you're currently reading was compiled October 4, 2014. They should be available at
`http://math.berkeley.edu/~anton/index.php?m1=writings`.

- When something doesn't make sense to me, I mark it with three big, eye-catching stars [[★★★ like this]]. If you can clear any of these up for me, let me know.
- If you have notes that I'm missing or if you have a correct/clear explanation for something which is incorrect/unclear, let me know (either tell me what you'd like to modify, give me some notes to go on, or update the tex yourself and send me a copy). Real (mathematical) errors should be fixed because it would be immoral to let them propagate (er …that is, sit there), and typographical errors hardly take any time to fix, so you shouldn't be shy about telling me about them.

1 PT 08-28

There will be notes online. Office hours are Th 2-3:30 in 703. You can find the website from my site. There are also notes from a previous class there which are relevant.

Thursday, we'll be in 939. After that, we'll see.

This semester seems to be a QFT semester.

1. Kolya's class MWF 1-2
2. This class TT 11-12:30
3. Richard's class Tu 1-2
4. Hot topics course on topological conformal field theory. Tu 2-3:30. This one will be lectures by students.
5. Student seminar W 2-3
6. Topology seminar W 4-5
7. Th 3-4 QFTea

In this course, there will be homework every 2 weeks. This is to keep you honest. I think I convinced Kolya to do the same. Submit it in groups of 2-4 students. Find a group of people you like, get together with them to do the problems, and split up the writing. The homework is optional. The mini-conference will be the thing that counts for the grade. We've decided that if you give one talk, it can count for both this class and Kolya's class. The first homework will be this Thursday.

This course will have three parts.

1. Super mathematics. First super algebra, then super differential topology, then super geometry. This part should only last two weeks (this will only be a survey). There is a very good reference (Deligne-Morgan [DEF+99, Vol. 1, Notes on Supersymmetry]).
2. Fermionic field theory. Kolya is starting with bosonic field theory. We'll follow everything he does bosonically and do it fermionically. This should take four to six weeks.

---

[1] With the exception of NR22, which was done by Chris Schommer-Pries.

3. 6-8 weeks left. Kolya will go into Chern-Simons theory etc. and we'll go into the relation to algebraic topology.

In case you don't make it all the way to act 3, I'll tell you about it today.

Let Man be the category of smooth manifolds with smooth maps. Let GRing be the category of $\mathbb{Z}$-graded commutative rings. $\mathbb{Z}$-graded means that $R = \bigoplus_{i\in\mathbb{Z}} R^i$ as an abelian group, and that $R^i \cdot R^j \subseteq R^{i+j}$. A graded ring is said to be commutative if for homogeneous elements $a, b \in R$, we have

$$b \cdot a = (-1)^{|a|\cdot|b|} a \cdot b.$$

From now on, when we say ring, we mean $\mathbb{Z}$-graded commutative ring.

Note that this kind of graded commutativity makes sense for $\mathbb{Z}/2$-graded rings. A $\mathbb{Z}/2$-graded ring with this flavor of commutativity is called a *commutative super algebra*.

Definition 1.1. A *(generalized) multiplicative cohomology theory*[1] is a homotopy functor $h^*\colon \mathsf{Man}^\circ \to \mathsf{GRing}$ with the Mayer-Vietoris property: for open subsets $U, V \subseteq M$, there is a long exact sequence

$$\cdots \xrightarrow{\delta} h^i(U\cup V) \to h^i(U)\oplus h^i(V) \to h^i(U\cap V) \xrightarrow{\delta} h^{i+1}(U\cup V) \to \cdots$$

[[★★★ probably with some naturality condition]] ◇

This connecting homomorphism δ is part of the data of a cohomology theory. The Mayer-Vietoris axiom tells you that the ring $h^*(M)$ is computable (modulo knowing the cohomology of a point) since you can break a manifold up into contractible open sets with contractible intersections [[★★★ It looks like you don't even have to know what δ is! So $h^*(pt)$ really completely determines h^*?]]. $h^*(pt)$ is called the *coefficient ring* of the cohomology theory.

Example 1.2. Singular cohomology $H^*(X) = \bigoplus_{i\in\mathbb{Z}} H^i(X)$ with cup product. This H is the letter for ordinary cohomology. The coefficient ring is $\mathbb{Z}$ concentrated in degree zero. ◇

Remark 1.3. Why restrict to manifolds? Don't you usually define cohomology on arbitrary topological spaces? Somehow, the definition of a topological space is way too general; it was really defined for analysis. In topology, we usually restrict to CW-complexes by imposing the weak homotopy axiom: if $f\colon X \to Y$ induces isomorphisms on all homotopy groups with all base points (i.e. is a weak homotopy equivalence), we require that $h^*(f)\colon h^*(X) \to h^*(Y)$ be an isomorphism. For any topological space X, there is a CW-complex $X' = |S_\bullet X|$ with a map to X which is a weak homotopy equivalence. This tells you that you can't see anything with homotopy groups which doesn't show up in CW-complexes. Note, by the way, that π_* is not a cohomology theory because there is no Mayer-Vietoris sequence, so homotopy groups are not (yet) easily computable. By the weak homotopy axiom, $h^i f\colon h^i X' \xrightarrow{\sim} h^i X$ is an isomorphism, so you may as well study cohomology of CW-complexes.

There is a beautiful fact that any smooth manifold has the structure of a CW-complex. Finally, if you have a CW-complex which is finite-dimensional and countable, then you can thicken it and get a manifold. So in this class, we'll only talk about manifolds. ◇

Okay, now we have one of the definitions. The next one will take the rest of the class. But first, some examples.

1900s, Poicaré, Lefschetz. The usual H^*. I hope you've all learned how wonderful the usual cohomology is; its is a wonderful tool.

1950s. Grothendieck, Atiyah-Hirzebruch. K^*. This was the first theory which didn't satisfy the dimension axiom: $K^*(pt) = \mathbb{Z}[u, u^{-1}]$ with u of degree 2. This is very geometric: you start with vector bundles modulo stable isomorphisms. Maybe the surprising thing is that this satisfies MV. You can prove more things with this, like find the number of independent vector fields on a sphere. Division algebras over $\mathbb{R}$ have dimension 1, 2, 4, or 8. Atiyah-Singer index theorem. This got people excited about other cohomology theories.

1990s. Hopkins-Miller, Lurie. TMF^* (topological modular forms) is the universal elliptic cohomology theory. Hopkins-Miller proved that TMF^* exists and Lurie constructed it. None of it is published, but Lurie's stuff is coming out slowly. Why "elliptic"? There is a relationship between cohomology theories and formal groups (which we won't explain). Let's say R is your favorite ring (to be the coefficient ring), then if you have a

---

[1] These days, you just say "cohomology theory", leaving off the word "generalized".

formal group law on R, then you can construct a cohomology theory so that the formal group has to do with $\mathbb{CP}^\infty$. If you take the additive group law $(a+b)$, you get the usual thing; if you take ab, you get K-theory; if you take the group law of an elliptic curve, then you get some "elliptic cohomology theory". Since there are many elliptic curves, there are many elliptic cohomology theories. *TMF* is tricky to construct because there is no universal elliptic curve (so you can't just use the construction); you have to deal with stacky stuff.

What are the applications of *TMF*? There is a beautiful open question. If you have a manifold, say $\mathbb{RP}^n$, then we know that we can embed it into $\mathbb{R}^{2n}$, but what is the minimal k so that there is an embedding $\mathbb{RP}^n \hookrightarrow \mathbb{R}^k$. (Ralph Cohen did the case of an immersion instead of an embedding). For many n it is known using H^*, K^* and TMF^*. The other thing that *TMF* gives us is that it allows us to understand the homotopy groups of spheres up to 60. In particular, it allows you to detect Lie groups. The real exciting application is not yet worked out: it has to do with the index theorem on loop space. Witten has a Dirac operator on loops space, and it's index is an element in $TMF^*(pt)$. The index theorem for loop space would be nice, but we're missing the analytic side, so the theorem cannot yet be formulated.

By the way $TMF^*(pt)$ is completely known (unlike the stable homotopy groups of a point).

If the picture relating this stuff to QFTs is right, then you can use all this machinery to describe QFTs.

Definition 1.4. Let X be a manifold, $d = 0, 1, 2$ and $n \in \mathbb{Z}$. Then $QFT^n_{d|1}(X)$ are supersymmetric QFTs of dimension $d|1$ and degree n over X. ◇

We don't know how to do this for $d > 2$. Here, $d|1$ is the super-dimension of the world-sheet $\Sigma^{d|1}$.

One reason that this class will be so different from physics classes is that we'll actually give a definition, but it will be done by sucking as much intuition from the physicists and turning it into a definition.

Physicists would never look at $d = 0$, $d = 1$ is quantum mechanics, $d = 2$ is the first interesting case, and they really want to study $4|16$-dimensional world-sheets. Even through we're learning a lot from physics, we're focusing on different things (we're not trying to understand the real world, like they are).

For $d = 0$, $QFT^n_{d|1}(X)$ is a set, for $d = 1$, it's a category, and for $d = 2$, its a 2-category. $QFT^n_{d|1}$ is actually a (contravariant) functor.

Theorem 1.5 (Conjecture?)**.**

d	$QFT^n_{d\vert 1}(X)$	$QFT^n_{d\vert 1}[X]$
0	$\Omega^n_{closed}(X)$	$H^n_{dR}(X)$
1	*super vector bundles on* πTX *with Quillen connection?*	$K^n(X)$
2	*something new*	$TMF^*(X)$*?*

You might know that $[Y, X]$ are homotopy classes of maps from Y to X, so it is $Map(Y, X)/\simeq$. If you have any contravariant functor, you can make the same definition. $QFT^n_{d|1}[X]$ is $QFT^n_{d|1}(X)$/concordance. It will be an exercise that two closed forms differ by an exact form if and only if there is a closed form on $X \times I$ which restricts to the two forms at the ends. By definition of concordance, $QFT^n_{d|1}[X]$ is automatically a homotopy functor.

If you leave out supersymmetry (the $|1$), then you still get a beautiful definition, but the third column is all zeros.

The whole third act of the class is a joint project with Stephan Stolz (at Notre Dame, Indiana).

2 PT 08-30

We'll be in 87 Evans starting Tuesday. Today we'll start super mathematics. There are three books I've been reading. Dan Freed: Five lectures on supersymmetry. Darajan: Supersymmetry for mathematicians. The best notes are in "quantum fields and strings: a course for mathematicians" volume 1; this is online (in the IAS website), but it isn't well organized. The one you want is: Deligne and Morgan, Notes on Super Symmetry. `math.ias.edu/qft`

Super stuff started in physics and there are competing schools of mathematicians trying to clean it up. This sheaf-theoretic approach seems to be dominating for now.

(Physical) Motivation: you may already know that one quantum particle is represented by a vector $v \in H$ in a Hilbert space (up to phase). A 2-particle system is represented by $H \otimes H'$. If these two particles are indistinguishable, we have $H = H'$ and $v \otimes v' = \lambda v' \otimes v$ as a physical state (you can pick up some phase $\lambda \in S^1 \subseteq \mathbb{C}$). Doing this twice, we should pick up the same phase (for some reason, which doesn't work in dimension 2 for example), so we get $\lambda^2 = 1$, so $\lambda = \pm 1$. If you keep track of the world lines, you see that two switches is not the same as doing nothing. If $\lambda = 1$, you get a *boson*; if $\lambda = -1$, you get a *fermion*; if $\lambda \in S^1$, you have an *anyion* (for any phase).

Bosons live in $Sym^2 H \subseteq H \otimes H$ and fermions live in $\bigwedge^2 H \subseteq H \otimes H$. As a consequence, you get Pauli's exclusion principle (two fermions can't be in the same state because $v \otimes v = 0$. Physicists decided to write (this is a new H) $H^{new} = H^b \oplus H^f = H^e \oplus H^o$ (even and odd parts). The usual symmetries you have preserve the even and odd parts. Physicists think there should be symmetries which switch the two subspaces, called *supersymmetries* (these haven't been observed). Mathematically, this will be an odd operator on H^{new}. We'll see that $Sym^2 H^{new} = Sym^2 H^b \oplus (H^b \otimes H^f) \oplus Sym^2 H^f$ (and $Sym^2 H^f = \bigwedge^2 H^f$ because H^f is odd).

Definition 2.1. A *super vector space* (over $\mathbb{C}$ or $\mathbb{R}$) is a ($\mathbb{Z}/2$-)graded[1] vector space $V = V^e \oplus V^o$. ◇

Noah: I don't want to call this a super vector space until you have a tensor product. PT: you're right, we'll get to this later today.

If V and W are super vector spaces, then $\mathrm{Hom}(V, W)$ is a super vector space, where the even homomorphisms are the ones which preserve the grading (sending V^e and V^o to W^e and W^o respectively), and the odd ones reverse the grading. So you can think of a homomorphism as [[★★★]].

If $V = W$ is finite-dimensional and $f = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, then we have $\mathrm{str}(f) = \mathrm{tr}(A) - \mathrm{tr}(D)$ and (Berezinian superdeterminant) $\mathrm{Ber}(f) = \det(A - BD^{-1}C) \cdot \det D^{-1}$ if $\det D \neq 0$ (Lemma: this matrix is invertible if and only if A and D are invertible [[★★★ well, we'll fix this to something correct later]]. Determinant is only defined for invertible matrices?) The property you want is $\mathrm{Ber}(e^X) = e^{\mathrm{str}(X)}$ [[★★★ Homework 1]], [[★★★ Homework 2: $\mathrm{Ber}(X \cdot Y) = \mathrm{Ber}(X) \cdot \mathrm{Ber}(Y)$]]

Definition 2.2. A *superalgebra* (over $\mathbb{C}$) is a super vector space A with an even algebra structure. That is, $\mu\colon A \times A \to A$ is even in the sense that $\mu(A^i \times A^j) \subseteq A^{i+j}$. ◇

Example 2.3. If V is an ordinary vector space, then $\bigwedge^* V$ is an example of a superalgebra. It is $\mathbb{Z}$-graded a priori, so in particular it is $\mathbb{Z}/2$-graded. This is a quotient of the tensor algebra $T^*V \twoheadrightarrow \bigwedge^* V$. The tensor algebra is still graded, but it is not supercommutative (or finite-dimensional). ◇

Example 2.4. $H^*(X; \mathbb{C})$ is a superalgebra (with the cup product), so is $\Omega^*(M)$ for M a manifold. So even if you don't care about physics, super stuff shows up. ◇

These two examples are supercommutative. That is, $a \cdot b = (-1)^{|a|\cdot|b|} b \cdot a$ for homogeneous elements a and b.

Example 2.5. If V is a space with symmetric bilinear form b, then $Cl(V, b) = T(V)/(v \otimes v' + v' \otimes v - b(v, v') \cdot 1)$. If b is identically zero, you get $\bigwedge^* V$. If b is non-zero, this algebra is not supercommutative. Note that the Clifford algebra is not $\mathbb{Z}$-graded, it is only $\mathbb{Z}/2$-graded. ◇

NR: what's the difference between superalgebras and a $\mathbb{Z}/2$-graded algebra? PT: no difference yet. Noah: Well, there is a small difference: the algebra structure has to be even.

---

[1] From now on, "graded" will mean $\mathbb{Z}/2$-graded

Definition 2.6. If V and W are super vector spaces, then define $(V\otimes W)^e = V^e\otimes W^e \oplus V^o\otimes W^o$ and $(V\otimes W)^o = V^e\otimes W^o\oplus V^o\otimes W^e$. This makes $V\otimes W$ into a super vector space. ◇

This is good. For example, $\mu\colon A\otimes A\to A$ is even means exactly that it is an even map in $\mathrm{Hom}(A\otimes A, A)$. NR: still unhappy; it's still a $\mathbb{Z}/2$-graded algebra. Noah: Aren't there two homs flying around? PT: We haven't defined the category yet. I hope you'll soon understand why the experts are confused.

Remark 2.7. Associativity of $\mu\colon A\otimes A\to A$ can be written as the following commutative diagram.

$$\begin{tikzcd}
A\otimes A\otimes A \arrow[r, "\mu\otimes \mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\mu"'] & A\otimes A \arrow[d, "\mu"] \\
A\otimes A \arrow[r, "\mu"] & A
\end{tikzcd}$$

In this class, we'll assume the tensor product of vector spaces is associative. That is, we'll pretend that $(V\otimes W)\otimes X = V\otimes(W\otimes X)$. These aren't really equal, but there is a totally canonical isomorphism. I'm not sure if Kolya will need the associator. NR: at the very end.

Similarly, for sets, we'll pretend $(S_1\sqcup S_2)\sqcup S_3 = S_1\sqcup(S_2\sqcup S_3)$. ◇

Definition 2.8. A *(strict) monoidal category* is (1) a category $\mathcal{C}$ (think Set, **Vect** or **GVect** (with grading-preserving maps)), (2) an associative product functor $\mathcal{C}\times\mathcal{C}\xrightarrow{\otimes}\mathcal{C}$ ($\otimes$ or $\sqcup$), and (3) a unit object $\mathbb{1}$ such that $\mathbb{1}\otimes X = X = X\otimes\mathbb{1}$. The diagrams above define an algebra in $\mathcal{C}$. ◇

An algebra in **GVect** is a superalgebra.

To define *commutative algebras* (A,μ) in $\mathcal{C}$, we want to say that

$$\begin{tikzcd}
A\otimes A \arrow[r, "\mu"] \arrow[d, "c"'] & A \\
A\otimes A \arrow[ur, "\mu"'] &
\end{tikzcd}$$

commutes, where c is some kind of "flip map". If $\mathcal{C}=\mathsf{Vect}$, then the usual flip map is ok, but if $\mathcal{C}=\mathsf{GVect}$, then you need to involve the sign rule somehow.

Definition 2.9. A *symmetric* monidal category $(C,\otimes)$ is a monoidal category with a *braiding* natural isomorphism $c : V\otimes W\xrightarrow{\sim} W\otimes V$ satisfying

- (Yang-Baxter equation)

$$\begin{tikzcd}
U\otimes V\otimes W \arrow[r, "c_{U\otimes V,W}"] \arrow[d, "\mathrm{id}\otimes c_{V,W}"'] & W\otimes U\otimes V \\
U\otimes W\otimes V \arrow[ur, "c_{U,W}\otimes\mathrm{id}"'] &
\end{tikzcd}$$

  And the obvious symmetric diagram [[★★★]]. So far, we've defined a *braided monoidal category.*

- $c_{V,W}\circ c_{W,V} = \mathrm{id}_{V\otimes W}$ ◇

Now we have defined a commutative algebra in a symmetric monoidal category. Note that in a braided monoidal category, you have to make a choice between $c_{A,A}$ and $c^{-1}_{A,A}$.

Definition 2.10. The symmetric monoidal category SVect of super vector spaces has objects graded vector spaces, morphisms are even morphisms, monoidal structure defined as before, and braiding $V\otimes W\xrightarrow{c_{V,W}} W\otimes V$ is defined on homogeneous elements as $v\otimes w\mapsto(-1)^{|v|\cdot|w|}w\otimes v$. ◇

I invite you to check the little diagrams.

Lemma 2.11. *A commutative superalgebra is the same as a commutative algebra object in* SVect.

This categorical point of view has huge advantages because you can define all the usual objects you define in linear algebra. Once you define something categorically, you know how to "superize it". Noah: you may have seen this sign rule before with tensor products of Clifford algebras. PT: let me expand on that. If A, B are algebra objects in $(\mathcal{C},\otimes)$, then is

$A\otimes B$ an algebra object? Well, we have to find a map $A\otimes B\otimes A\otimes B \xrightarrow{\mu_{A\otimes B}} A\otimes B$

$$\begin{tikzcd}
A\otimes B\otimes A\otimes B \arrow[r, "\mu_{A\otimes B}"] \arrow[d, "\mathrm{id}\otimes c\otimes \mathrm{id}"'] & A\otimes B \\
A\otimes A\otimes B\otimes B \arrow[ur, "\mu_A\otimes\mu_B"'] &
\end{tikzcd}$$

You need the braiding to do this, and you can check that this gives you an algebra structure on $A\otimes B$. What Noah was saying is that with this tensor product, we have $Cl(V_1, b_1)\otimes Cl(V_2, b_2)\cong Cl(V_1\oplus V_2, b_1\perp b_2)$. Note that $T(V_1\oplus V_2)\cong TV_1\otimes TV_2$.

Definition 2.12. If $(\mathcal{C},\otimes,c)$ is symmetric monoidal, a *Lie algebra* in $\mathcal{C}$ is an object L together with a bracket $[,]\colon L\otimes L\to L$ such that you have (diagrammatically) (1) skew-symmetric and (2) Jacobi. ◇

Antisymmetry:

$$\begin{tikzcd}
L\otimes L \arrow[r, "{[\cdot,\cdot]}"] \arrow[d, "c_{L,L}"'] & L \\
L\otimes L \arrow[r, "{[\cdot,\cdot]}"] & L \arrow[u, "-\mathrm{id}"']
\end{tikzcd}$$

I think we need $\mathcal{C}$ to be an additive category so that $-\mathrm{id}$ makes sense. You'll need additive for Jacobi as well.

3 PT 09-04-2007

Your feedback:

- Problem 2 is wrong in the original problem set: the formula for the Berezinian is not multiplicative. The new version has the right assumptions for the formula to be true.
- However, we're still missing some assumptions for the exponential map.
- the Stokes' theorem hint is bad, you just need Stokes' theorem on the interval.

Let's extend the submission date to next week (Sept. 11).

If A is a commutative super algebra, a *free (right) module* of super dimension $(m|n)$ is a free (right) module with m (free) even generators and n (free) odd generators. Recall that $F, A\in \mathsf{SVect}$, so we have $\mu_F\colon F\otimes A\to F$ in SVect (in particular, it is an even map) with the usual commutative diagrams. There is a nice adjunction formula.

$$\mathrm{Hom}_{A\text{-}\mathsf{mod}}(F,V)\cong \mathrm{Hom}_{\mathsf{SVect}}(\mathbb{C}^{m|n}, V_{\mathrm{forget}}).$$

[[★★★ check that super dimension is unique given the module structure. No, over $\mathbb{C}$, declare odd or even. PT: no, that's not a counterexample because F should be in SVect to begin with, so you have a fixed super dimension over $\mathbb{C}$.]] In problem 2, replace the exponential formula by the following property of the Berezinian. $\mathbb{A}[e]/e^2$ with e even, and let $f\colon F\to F$ be an even morphism, then $\mathrm{Ber}(1+ef) = 1+e\,\mathrm{str}(f)$. Replace the exponential property with this.

Feedback from Barbara: I said that 87 is still too small for the class. She said that those that aren't registered shouldn't get a chair. So register, and then drop whenever you like (you can drop until the very last day of class). The only disadvantage of registering is that I might learn your name.

If you try to say something precisely, you should try to do it categorically because it will keep you honest. $X\in\mathcal{C}$ means that X is an object in $\mathcal{C}$. If $X, Y\in\mathcal{C}$, then we'll write $\mathcal{C}(X,Y)$ for $\mathrm{Hom}_{\mathcal{C}}(X,Y)$. We talked about the notion of a monoidal category. If $\mathcal{C}$ has products[1] (this is a property of $\mathcal{C}$, not extra structure).

Example 3.1. $\mathcal{C} = \mathsf{Set}, \mathsf{Top}, \mathsf{Man}$. ◇

We could define $X \otimes Y := X \times_{\mathcal{C}} Y$. We have to use choice to pick a representative of $X \times_{\mathcal{C}} Y$, which is only defined up to unique isomorphism. This $\otimes$ is only as associative as the product of sets (there is a canonical associator, which we're ignoring).

Now note that we have two different monoidal structures on Vect. In general, you have to decide which monoidal structure you use. For manifolds or topological spaces, we'll use this monoidal structure (in bordism categories, we'll use $X \otimes Y = X \sqcup Y$ instead), but for vector spaces, we'll use tensor product.

Definition 3.2. Let $(\mathcal{C}, \otimes)$ be a monoidal category. A *group object* G in $\mathcal{C}$ is an object $G \in \mathcal{C}$ together with morphisms $\mu\colon G \otimes G \to G$, $e\colon \mathbb{1} \to G$,, and $\nu\colon G \to G$ satisfying the usual axioms. ◇

Shenghao: you need a map $G \to G \times G$ for the inverse axiom. PT:

$$\begin{tikzcd}
G \times G \arrow[rr, "\mathrm{id} \times \nu"] & & G \times G \arrow[d, "\mu"] \\
G \arrow[u, "\Delta"] \arrow[r, dashed, "\varepsilon"] & \mathbb{1} \arrow[r, "e"] & G
\end{tikzcd}$$

We need $\mathbb{1}$ to be a zero object to get the dashed arrow (Noah: or just pick the map $G \to \mathbb{1}$, and then you get a Hopf algebra object. PT: yeah, let's do that, so add Δ and ε to the definition, but then you have to say what diagrams they satisfy).

If we take the monoidal structure given by product, then Δ and ε exist canonically.

Remark 3.3. For the notion of a commutative group in $\mathcal{C}$, you need a symmetric monoidal category $(\mathcal{C}, \otimes, c)$. Just as you get different notions of groups by using different monoidal structures, you get different notions of groups by using different braidings. ◇

Last time, we defined a Lie algebra object in a symmetric monoidal $\mathbb{C}$-linear category $\mathcal{C}$. Noah: I think additive is really the right thing. PT: okay, let's try to always get the right axioms, so let's just say additive instead of $\mathbb{C}$-linear. We'll also say that $\mathbb{C} \hookrightarrow \mathcal{C}(\mathbb{1}, \mathbb{1})$. In this case, we'll have $\mathcal{C}(X, Y) \otimes_{\mathbb{C}} \mathcal{C}(Y, Z)$ instead of product in Set and require composition the map to $\mathcal{C}(X, Z)$ to be $\mathbb{C}$-bilinear?[[★★★]]

Definition 3.4. We'll call this kind of thing (enriched over $\mathbb{C}$) a (symmetric) *tensor category.* ◇

Definition 3.5. A monoidal category $(\mathcal{C}, \otimes)$ is *closed* if there exist "inner Homs" $\underline{\mathcal{C}}(X, Y) \in \mathcal{C}$ for all $X, Y \in \mathcal{C}$ with natural isomorphisms $\mathcal{C}(W, \underline{\mathcal{C}}(X, Y)) \xrightarrow{\sim} \mathcal{C}(W \otimes X, Y)$. [[★★★ this is weaker than rigid because you don't get a coevaluation map]] ◇

Example 3.6. For vector spaces, you can think of the hom sets as vector spaces, so there is an inner hom which is the same as the usual hom (as a set). What is $\underline{\mathrm{Hom}}_{\mathsf{SVect}}$? It is *all* homomorphisms, not just even morphisms. This is $\mathbb{Z}/2$-graded as before. ◇

Example 3.7. $(\mathcal{C}, \otimes) = (\mathsf{Top}, \times)$. Is this closed? No; if you want to use the compact-open topology on the hom sets, you need to take the compactly generated product $\times_c$ to get the right adjunction. If you start with Top_c, the category of compactly generated topological spaces, with it's product $\times_c$, then you get a category which *is* closed monoidal. If you take $(\mathsf{Man}, \times)$, this subtlety is gone (but you have to allow infinite-dimensional manifolds for the hom sets to be manifolds). Interesting thing: $\underline{\mathrm{Hom}}_{\mathsf{SMan}}(\mathbb{R}^{0|n}, M)$ is a (finite-dimensional!) supermanifold. ◇

Note that $\mathcal{C}(\mathbb{1}, \underline{\mathcal{C}}(X, Y)) \cong \mathcal{C}(X, Y)$ canonically. For SVect, since a map from $\mathbb{1}$ to $\underline{\mathrm{Hom}}_{\mathsf{SVect}}(X, Y)$ must be even, so it is picking out an even map from X to Y.

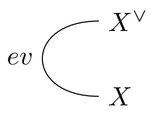
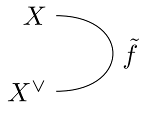
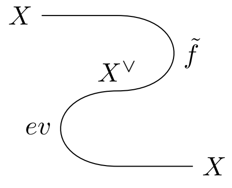
We can define the dual to X by $X^\vee := \underline{\mathcal{C}}(X, \mathbb{1})$. Deligne and Morgan get the order in this next thing wrong. There is a canonical evaluation map $X^\vee \otimes X \xrightarrow{ev} \mathbb{1}$ by taking $W = X^\vee$ and $Y = \mathbb{1}$ and taking $\mathrm{id}_{X^\vee}$ on

---

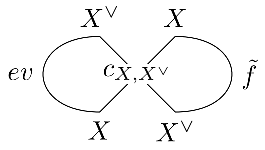
[1] If $X_1, X_2 \in \mathcal{C}$, then there is an object $X_1 \times_{\mathcal{C}} X_2 \in \mathcal{C}$ with two projection maps $p_i\colon X_1 \times_{\mathcal{C}} X_2 \to X_i$ such that the map $\mathcal{C}(Y, X_1 \times_{\mathcal{C}} X_2) \xrightarrow{(p_1 \circ -) \times (p_2 \circ -)} \mathcal{C}(Y, X_1) \times_{\mathsf{Set}} \mathcal{C}(Y, X_2)$ is a bijection.

the left hand side. You also get a canonical map $X \otimes Y^\vee \to \underline{\mathcal{C}}(Y, X)$ (this order is important!).

Now we can define a trace. Taking $X = Y$, we have $\mathbb{1} \xrightarrow{\tilde{f}} X \otimes X^\vee \to \underline{\mathcal{C}}(X, X)$ for any $f \in \mathcal{C}(X, X)$. Note that for an infinite-dimensional vector space X, there is no trace $\mathcal{C}(X, X) \to \mathcal{C}(\mathbb{1}, \mathbb{1})$. You can only define trace for $\tilde{f}$. Let's use the convention that we draw pictures with the maps going from right to left, so that it is easy to translate into symbols. We also write tensor products left-to-right into top-to-bottom.

$\mathbb{1} \xleftarrow{ev} X^\vee \otimes X$	$X \otimes X^\vee \xleftarrow{\tilde{f}} \mathbb{1}$	$X \xleftarrow{f} X$
		

Given $\tilde{f}$, you'd like to stick it onto the evaluation map, but you have to throw in a switch. The picture below is by definition the trace $\mathbb{1} \xleftarrow{\mathrm{tr}(\tilde{f})} \mathbb{1}$ or $\mathrm{tr}_{\mathcal{C}}(\tilde{f}) \in \mathcal{C}(\mathbb{1}, \mathbb{1})$.



Lemma 3.8. $\mathrm{tr}_{\mathsf{SVect}}$ *is* str.

4 PT 09-06

One bit of cleanup from Tuesday. I tried to define a category enriched over abelian groups and there was a question of whether to use $\otimes$ or $\times$ for composition. This is kind of an advertisement for monidal categories. Fix a monoidal category $(\mathcal{A}, \otimes_{\mathcal{A}})$ (think $\mathcal{A} = \mathsf{Ab}$, with monoidal structure given by either $\times$ or by $\otimes_{\mathbb{Z}}$). An $(\mathcal{A}, \otimes_{\mathcal{A}})$-enriched category $\mathcal{C}$ is a class of objects, and for any pair of objects $X, Y \in \mathcal{C}$, a hom object $\mathcal{C}(X, Y) \in \mathcal{A}$ with identity morphisms $\mathrm{id}_X \in \mathcal{A}(\mathbb{1}_{\mathcal{A}}, \mathcal{C}(X, X))$ and with associative composition morphisms $\mathcal{C}(X, Y) \otimes_{\mathcal{A}} \mathcal{C}(Y, Z) \to \mathcal{C}(X, Z)$ in $\mathcal{A}$ (we're shutting associators out of the discussion, but you could throw them in). Q: so this isn't a category? PT: no, it isn't a category. If there exists a functor $(\mathcal{A}, \otimes_{\mathcal{A}}) \to (\mathsf{Set}, \times)$, then you get a category structure on $\mathcal{C}$.

Example 4.1. $\mathcal{C} = \mathsf{Vect}$ (over $\mathbb{C}$) is enriched over $(\mathsf{Vect}, \otimes)$. $f \circ (g + h) = fg + fh$, so the composition is actually bilinear. A category enriched over Vect is called a *linear category.* ◇

Today I want to get to supermanifolds, so I need to take some shortcuts. Let A be a commutative super algebra (i.e. a commutative algebra object in $(\mathsf{SVect}, \otimes, c)$; in particular, the multiplication is even and commutativity means that $a \cdot b = (-1)^{|a|\cdot|b|} b \cdot a$).

Example 4.2. $A = \bigwedge^* \mathbb{C}^n$. Remember that the Clifford algebra is not super commutative. ◇

The category cSAlg of commutative super algebra is itself a monoidal category via

$$\begin{tikzcd}
(A \otimes B) \otimes (A \otimes B) \arrow[r, "\mu_{A \otimes B}"] \arrow[d, "\mathrm{id}_A \otimes c_{B,A} \otimes \mathrm{id}_B"'] & A \otimes B \\
(A \otimes A) \otimes (B \otimes B) \arrow[ur, "\mu_A \otimes \mu_B"'] &
\end{tikzcd}$$

Fix such an A. Then the category $\mathsf{mod}\text{-}A$ is the category of right A-modules with even A-module homomorphisms.

Lemma 4.3. $\mathsf{mod}\text{-}A$ *is a closed monoidal category.*

Next we will introduce supermanifolds, which can be thought of as a commutative super algebra, and modules will be like sheaves of modules on the supermanifold.

"Proof". We have to define the tensor product and inner hom and verify the adjunction formula. In general, $M \otimes_A N$ makes sense if M is a right module and N is a left module (which it isn't). To define the monoidal structure, we'll turn N into a left module via $\mu_N^\ell := \mu_N \circ c_{A,N}$.

$$\begin{tikzcd} A \otimes N \arrow[r, "c_{A,N}"'] \arrow[rr, bend left, "\mu_N^\ell"] & N \otimes A \arrow[r, "\mu_N"'] & N \end{tikzcd}$$

[[★★★ To check this, you must use that A is commutative]] Now we have two maps $\mathrm{id}_M \otimes \mu_N^\ell, \mu_M \otimes \mathrm{id}_N \colon M \otimes A \otimes N \to M \otimes N$. We define the monoidal structure $M \otimes_A N$ to be the coequalizer in SVect of these two: $M \otimes A \otimes N \rightrightarrows M \otimes N \to M \otimes_A N$.

Now let's define the inner hom. Recall that it should come with isomorphisms $\mathcal{C}(X, \underline{\mathcal{C}}(Y, Z)) \cong \mathcal{C}(X \otimes Y, Z)$.

Lemma 4.4. *If $M, N \in \mathsf{mod}\text{-}A$, the A-module $\mathrm{Hom}_A(M, N)$ (this includes even and odd morphisms) is an inner hom.*

I have to give you a right A-action on $\mathrm{Hom}_A(M, N)$. $\mathrm{Hom}_A(M, N)$ is a left module via $(a \cdot \phi)(m) = a \cdot \phi(m) = \mu_N^\ell(a \otimes \phi(m))$. Now make this into a right A-module as above. □

Remark 4.5. We used that the two actions (left and right action on N) commute, which follows from commutativity of A. ◇

Q: does Yoneda's lemma work for enriched categories? PT: let's check during the break.

Office hours are moving today; this week, they'll be Friday at 2.

Super-manifolds

These are defined via sheaves.

Definition 4.6. A *super-manifold* $M = (|M|, \mathcal{O}_M)$ *of dimension* $(m|n)$ is a sheaf $\mathcal{O}_M$ of commutative super algebras over a (Hausdorff, second countable) topological space $|M|$ which is locally isomorphic (as a ringed space) to $(U, C^\infty(U) \otimes \bigwedge^* \mathbb{R}^n)$ where $U \subseteq \mathbb{R}^m$ is an open subset. ◇

Remark 4.7. We'll see that $|M|$ will be come a smooth manifold of dimension n. ◇

Definition 4.8. If X is a topological space and $\mathcal{C}$ is a category. A ($\mathcal{C}$-valued) presheaf on X is a functor $\mathcal{F} \colon \mathit{Open}(X)^\circ \to \mathcal{C}$, where $\mathit{Open}(X)$ has objects open subsets of X and morphisms inclusions (i.e. $\mathrm{Hom}_{\mathit{Open}(X)}(U, V) = *$ if $U \subseteq V$ and $\varnothing$ otherwise). ◇

Example 4.9. $\mathcal{F}(U) = C^0(U)$ (we could take $\mathcal{C}$ to be commutative algebras). If X is a smooth manifold, we could define $\mathcal{F}(U) = C^\infty(U)$. ◇

Definition 4.10. Assume $\mathcal{C}$ has all products. A presheaf $\mathcal{F}$ is a *sheaf* if the gluing property is satisfied: for any open covering $U = \bigcup_i U_i$, the sequence

$$\begin{tikzcd} F \arrow[r, "\prod F(\iota_i)"] & \prod_i \mathcal{F}(U_i) \arrow[r, shift left, "\prod(\mathcal{F}\iota_{ij} \circ p_i)"] \arrow[r, shift right, "\prod(p_j \circ \mathcal{F}\iota_{ij})"'] & \prod_{i,j} \mathcal{F}(U_i \cap U_j) \end{tikzcd}$$

is an equalizer in $\mathcal{C}$. ◇

Remark 4.11. If follows (by taking the empty cover of $\varnothing$) that $\mathcal{F}(\varnothing)$ is the terminal object in $\mathcal{C}$. ◇

Definition 4.12. A smooth structure on a topological manifold X is a sheaf $C^\infty(X) \subseteq C^0(X)$ so that $(X, C^\infty(X))$ is locally isomorphic to $(U, C^\infty(U))$ for some $U \subseteq \mathbb{R}^n$ open. ◇

Morphisms of sheaves. (a) Say $\mathcal{F}, \mathcal{G} \colon \mathit{Open}(X)^\circ \to \mathcal{C}$ are two sheaves on the same space X. Then a morphisms of sheaves ($\mathcal{F} \to \mathcal{G}$ is a natural transformation between them. That is, it is a T such that the following diagram commutes for every inclusion $\iota\colon V \hookrightarrow U$.

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "\mathcal{F}\iota"] \arrow[d, "T(U)"'] & \mathcal{F}(V) \arrow[d, "T(V)"] \\
\mathcal{G}(U) \arrow[r, "\mathcal{G}\iota"] & \mathcal{G}(V)
\end{tikzcd}$$

(b) Let $\mathcal{F}$ be a sheaf on X and $\mathcal{G}$ a sheaf on Y (both valued in $\mathcal{C}$. A morphism from $(X, \mathcal{F})$ to $(Y, \mathcal{G})$ is a continuous map $f\colon X \to Y$ and a natural transformation from $\mathcal{G}$ to $\mathcal{F} \circ f^{-1}$ [[★★★ The other way?!? Ok, this is supposed to be a morphisms of "$\mathcal{C}$-sheaved spaces". Clean up.]].

$$\begin{tikzcd}
\mathit{Open}(X) \arrow[r, "\mathcal{F}", ""{name=A, below}] & \mathcal{C} \\
\mathit{Open}(Y) \arrow[u, "f^{-1}"] \arrow[ur, "\mathcal{G}"', ""{name=B}] &
\arrow[Rightarrow, from=A, to=B, "T"']
\end{tikzcd}$$

Another way to think of it:

$$\begin{tikzcd}
\mathcal{F} \arrow[r] \arrow[d] & f_*\mathcal{F} \arrow[d] & \mathcal{G} \arrow[l] \arrow[dl] \\
X \arrow[r, "f"] & Y &
\end{tikzcd}$$

5 PT 09-11

Definition 5.1. A *supermanifold* $M = (|M|, \mathcal{O}_M)$ of dimension $(m|n)$ is a (second countable, Hausdorff) topological space $|M|$, together with a sheaf $\mathcal{O}_M$ of commutative super algebras, which is locally isomorphic to $\mathbb{R}^{m|n} := (\mathbb{R}^m, \mathcal{O}_{\mathbb{R}^{m|n}})$, where $\mathcal{O}_{\mathbb{R}^{m|n}} = C^\infty(\mathbb{R}^m) \otimes \bigwedge^* \mathbb{R}^n$. ("$m$ even variables, n odd variables") ◇

If you like sheaves, you should use them, but we'll see that you can avoid sheaves if you really hate them.

Remark 5.2. Depending on the ground field ($\mathbb{R}$ or $\mathbb{C}$), you get a different notion because $C^\infty\mathbb{R}^m$ could be smooth $\mathbb{R}$-valued or $\mathbb{C}$-valued functions. This gives us real or complex super manifolds. The usual convention is that real super manifolds are called *super manifolds* and complex super manifolds are called *CS manifolds*. You don't say "complex supermanifolds" because it gets confused with "super complex manifolds". You know that complex and real manifolds are very different, but this distinction between supermanifolds and CS manifolds is new . . . we didn't have two different notions before. In the super world, these are very different beasts. ◇

Definition 5.3. A *super complex manifold* is something locally isomorphic to $\mathcal{O}^{an}_{\mathbb{C}^{m|n}} := C^{an}(\mathbb{C}^m) \otimes \bigwedge^* \mathbb{C}^n$. ◇

NR: you can draw an analogy with representation theory; real and complex representations are very different. PT: yes, we'll see that it is basically the same reason that supermanifolds and CS manifolds are different.

Let $J(U)$ (for $U \subseteq |M|$ open) be the ideal in $\mathcal{O}_M(U)$ generated by the odd elements. Note that any section of J is nilpotent because a high enough power of a (finite) linear combination of odd elements eventually has a square of an odd element in each term, and odd elements square to zero.

Claim. *$M_{red} := (|M|, \mathcal{O}_M/J)$ is a smooth structure on $|M|$.*

Locally, we have $\big(C^\infty(\mathbb{R}^m) \otimes \bigwedge^* \mathbb{R}^n\big)/J \cong C^\infty(\mathbb{R}^m)$. I won't explain why this quotient is actually a sheaf (you don't need to sheafify); it will be in the homework. A smooth structure is actually an embedding of the sheaf of smooth functions into the sheaf of continuous functions. So given $\phi \in \mathcal{O}_M/J$, we construct the function $(x \mapsto \lambda(x))$ where λ is uniquely determined by the condition that $\phi - \lambda(x) \cdot 1 \in \mathcal{O}_M/J$ is *non*-invertible in any neighborhood of x.

The procedure above gives you a way to take any section of $\mathcal{O}_M$ and produce an honest function on $|M|$. The projection $\mathcal{O}_M \to \mathcal{O}_M/J$ induces an embedding of supermanifolds $M_{red}^{m|0} \hookrightarrow M^{m|n}$ (interpreting a manifold as a supermanifold). The map on underlying spaces is the identity, and the projection $\mathcal{O}_M \to \mathcal{O}_M/J$ is the morphism of sheaves. Sometimes we'll confuse $M_{red}^{m|0}$ with $|M|$, just like we do all the time (even though the first one has an extra structure: a sheaf of smooth functions).

Now let's look at the main examples. We've already seen one example: the local model $(\mathbb{R}^m, \mathcal{O}_{\mathbb{R}^{m|n}})$. We can take any smooth manifold and tensor with $\bigwedge^* \mathbb{R}^n$, throwing in a constant odd fiber. But you can also twist the odd fiber around.

Example 5.4. Let $E^n \to X^m$ (n is the dimension of the fiber, so E is $(m+n)$-dimensional) be a smooth vector bundle over a manifold X. Define the super manifold πE as the pair $(X, \mathcal{O}_{\pi E})$ via $\mathcal{O}_{\pi E}(U) = C^\infty(U, \bigwedge^* E) = \{\text{smooth sections of the bundle } \bigwedge^* E \to X\}$. This is a super manifold of dimension $m|n$. ◇

Let's discuss the different notions from the remark [[★★★]]. We have two different notions of vector bundle: real and complex. Depending on which one we use, you get a different notion. Take your favorite example of a complex vector bundle which is not a real vector bundle tensor $\mathbb{C}$ and you get the following corollary.

Corollary 5.5. *There are "more" CS manifolds than (real) supermanifolds.*

Take the canonical line bundle on a complex manifold. Say $\mathcal{O}(-1)$ on $\mathbb{CP}^1 = S^2$. This is a complex bundle, but every real line bundle is trivial on S^2 is trivial because S^2 is simply connected. But $\mathcal{O}(-1)$ has non-trivial Chern class, so it is non-trivial. There could still be some weird real manifold so that when you tensor up with $\mathbb{C}$ you get this $\pi\mathcal{O}(-1)$. To rule that out, you need the following theorem.

Theorem 5.6 (Batchelor)**.** *Any ($\mathbb{R}$ or $\mathbb{C}$) super manifold M is isomorphic to πE for some smooth vector bundle E over M_{red}.*

Noah: do these isomorphisms play well with the morphisms? PT: No. if the answer were yes, then we wouldn't have introduced super manifolds. Santiago: you want to think of differential forms as sections of the odd tangent bundle, so shouldn't you be taking sections of the dual of $\bigwedge^* E$? PT: that's true if you're thinking of a quotient of a tensor power, but I'm thinking of multilinear forms (which maybe I should denote $\bigwedge^*(E^*)$).

Proof. I'll write $\mathcal{O}_M \to |M|$ to mean a sheaf over $|M|$. We can form $\mathcal{O}_M/J \to |M|$, which is the smooth manifold M_{red}. We have $J/J^2 \to |M|$, a sheaf of $\mathcal{O}_M/J$-modules over M_{red}. Locally, $J/J^2 \cong C^\infty(\mathbb{R}^m) \otimes \bigwedge^1 \mathbb{R}^n \cong C^\infty(\mathbb{R}^m) \otimes \mathbb{R}^n$ (since $J = C^\infty(\mathbb{R}^m) \otimes \bigwedge^{\geq 1} \mathbb{R}^n$ and $J^2 = C^\infty(\mathbb{R}^m) \otimes \bigwedge^{\geq 2} \mathbb{R}^n$. This is a vector bundle over M_{red}.[1]

[[★★★ HW to finish the proof. In particular, prove that all these things are sheaves. Show that if E is the vector bundle J/J^2, $\pi E \cong M^{m|n}$. The dimensions are right and the underlying spaces are the same. In the proof, you have to use partitions of unity (we're in the smooth category); in the analytic context, this is not true. It may look obvious, but you have to check. If you take a filtered vector space, take the associated graded, and think of it as a filtered vector space, you don't get the original thing back. You really have to check that the two things are the same.]] □

Thanks for the homework. I'm saying the homework in class today, and waiting for feedback before I put it on the website.

I put the first project up: Let $(\mathcal{A}, \otimes)$ be a closed monoidal category. Formulate a proof of the Yoneda lemma for $\mathcal{A}$-enriched categories.

Kolya and I discussed this business. We agreed that you can do the projects in groups of one, two, or three people. If there is just one of you, you can write a paper or give a talk. If there are three of you, you should do both. If there are two, it will be something in between. If you

---

[1] By the way, a vector bundle can be though of as a sheaf. If you have a vector bundle $E \to X$, then the sheaf of smooth sections of $E \to X$ is a sheaf of modules over $\mathcal{O}_X = C^\infty(X)$. Locally, this is just smooth fiber-valued (so $\mathbb{R}^n$-valued) functions on X, i.e. $C^\infty(\mathbb{R}^m) \otimes \mathbb{R}^n$. This means that the sheaf is locally free of rank n. Now the statement is that if you start with a sheaf of modules which is locally free of rank n, then you can construct a vector bundle. I invite you to check this [[★★★]]

want this problem, you should take it, but if you think there will be more interesting things, then you should wait. It will be first come first served.

Next we'll do a theorem which will get rid of sheaves. We've already gotten rid of sheaves on objects (the theorem that specifying a vector bundle is enough). Now let's get rid of sheaves on morphisms.

We'll define $C^\infty(M) := \mathcal{O}_M(|M|)$ to be the commutative super algebra of functions. Note that this is not $C^\infty(M_{red})$.

Theorem 5.7. *The map* $\mathsf{SMan}(M,N) \to \mathsf{SAlg}\big(C^\infty(N), C^\infty(M)\big)$ *is bijective. That is, a morphism of supermanifolds is completely determined by the induced morphism on global sections of the structure sheaves.*

I'm going to skip the theorem. We could make it HW, but let's not. Again, you need partitions of unity and it fails in the analytic case.

Corollary 5.8. *There is an equivalence of categories* $\bigwedge\text{-}\mathsf{Vect} \to \mathsf{SMan}$, *where the objects of* $\bigwedge\text{-}\mathsf{Vect}$ *are smooth vector bundles and the morphisms are given by the theorem: a morphism* $(E \to X) \to (E' \to X')$ *are pairs* (f,ϕ), *where* $f\colon X \to X'$ *and a* $\phi \in \mathsf{SAlg}(C^\infty(X', \bigwedge^*(f^*E')), C^\infty(X, \bigwedge^* E))$ *(these are very different from usual bundle maps; this is a morphism from* $C^\infty(\pi f^* E') \to C^\infty(\pi E)$*).*

Example 5.9. $\mathbb{R}^{0|2} \xrightarrow{\alpha} \mathbb{R}^{1|0}$ given by "$(\theta_1,\theta_2) \mapsto \theta_1 \cdot \theta_2$", is described via the algebra homomorphism $\Lambda^*[\theta_1,\theta_2] = C^\infty(\mathbb{R}^{0|2}) \leftarrow C^\infty(\mathbb{R}^1) \ni x$ given by $x \mapsto \lambda\theta_1\theta_2$ for some $\lambda \in \mathbb{R}$. Why is this well-defined? It looks like it's only defined on the polynomial algebra $\mathbb{R}[x]$. If this were well-defined, we could take any map $f\colon \mathbb{R} \to \mathbb{R}$ and get $x \mapsto f(\theta_1\theta_2)$. The point is that we have Taylor series, and $\theta_1\theta_2$ is nilpotent, so the Taylor series automatically converges (normally, it only converges for analytic functions).

A more interesting example might have been $\mathbb{R}^{1|2} \to \mathbb{R}$, given by $(y,\theta_1,\theta_2) \mapsto y + \theta_1\theta_2$. Here you'd need to find

$$f(y+\theta_1\theta_2) = \sum_{n=0}^{\infty} \frac{f^{(n)}(y)}{n!}(\theta_1\theta_2)^n$$

which will converge. ◇

Remark 5.10. There is no (good) theory of C^k super manifolds because you don't get Taylor series. ◇

[[★★★ HW 2 (this even holds in the analytic setting, no partition of unity needed): $\mathsf{SMan}(M, \mathbb{R}^{p|q}) \cong (C^\infty(M)^{ev})^p \times (C^\infty(M)^{odd})^q$. That is, specifying a map to $\mathbb{R}^{m|n}$ is just specifying m even functions and n odd functions. If x_i, θ_j are coordinates on $\mathbb{R}^{m|n}$, the even functions are F^*x_i and the odd functions are $F^*\theta_j$, where $F \in \mathsf{SMan}(M, \mathbb{R}^{m|n})$. If you like, you can think about $\mathsf{SAlg}(C^\infty(\mathbb{R}^{p|q}), C^\infty(M))$ instead of $\mathsf{SMan}(M, \mathbb{R}^{m|n})$]]

6 PT 09-13

Three main theorems on super manifolds.

(a) $M \cong \pi E$ for some smooth vector bundle $E \to M_{red}$. Recall that $C^\infty(\pi E) = C^\infty(M_{red}, \bigwedge^* E^*)$. [[★★★ HW1]]

(b) $\mathsf{SMan}(M,N) \cong \mathsf{SAlg}(C^\infty N, C^\infty M)$. Super manifolds are in some sense "affine".

(c) If $U \subseteq \mathbb{R}^{p|q}$ is a *domain* (the restriction of the super manifold structure on $\mathbb{R}^{p|q}$ to an open subset U), then $\mathsf{SMan}(M,U) \cong \{f_1,\dots,f_p \in C^\infty(M)^{ev}, \phi_1,\dots,\phi_q \in C^\infty(M)^{odd} | (f_1(x),\dots,f_p(x)) \in |U| \text{ for all } x \in M\}$.[[★★★ HW2]]

The first two use partitions of unity, so you must be in the smooth category, not the analytic category.

Combining, $C^\infty(U) = C^\infty(U_{red})[\xi_1,\dots,\xi_q] \supseteq \mathbb{R}[x_1,\dots,x_p] \otimes \bigwedge^*[\xi_1,\dots,\xi_q]$, which is a free commutative super algebra on x_i and ξ_j, so it is really easy to specify a morphism from this subalgebra (it is determined by choosing an even guy for each x and an odd guy for each ξ). To extend this to the whole algebra, you use Taylor expansion.

Last time we saw that there are maps

$$\mathsf{SMan}(M,U) \to \mathsf{Man}(M_{red}, U_{red}) \to C^0(|M|,|U|)$$

[[★★★]].

Reimundo: do you have a map $M \to M_{red}$. PT: there is a map which comes from the zero section using $M \cong \pi E$, but since this isomorphism is not canonical, they map is not canonical.

Remark 6.1 (Historical digression?)**.** Once you have this third result, you can throw the other stuff away and define super manifolds locally.

If $S, U \subseteq \mathbb{R}^{p|q}$ are domains, then $\mathsf{SMan}(S,U) \cong \{f_i, \phi_j \text{ as above}\}$, with $f_i \in C^\infty(S)^{ev} = \big(C^\infty(S_{red}) \otimes \bigwedge^*[\xi_i]\big)^{ev}$, so

$$f_i = \sum_{I \text{ even}} f_I^i(x) \cdot \xi_I = \underbrace{f^i_\varnothing(x)}_{\text{body}} + \underbrace{\quad\cdots\quad}_{\text{soul}}$$

I varies over index sets (sequences of 0's and 1's) $(i_1,\dots,i_r)$ where $1 \le i_a \le q$ and $f_I \in C^\infty(S_{red})$. Similarly, the ϕ_j have the same, but for odd. The soul is nilpotent. The ϕ_j don't have a body, just a soul.

Historically, people defined morphisms of super manifolds this way (before defining objects). Then you have to figure out how to compose these guys. I'm going to skip this. You have to use the Taylor expansion on all of these f_i's around the body. If you do the second homework, you have to deal with these Taylor expansions.

Then you know how to deal with maps between domains, so you can talk about isomorphisms. Now you can glue domains together just like you'd glue ordinary manifolds, using these isomorphisms to glue.

What we've done is equivalent, but is more elegant in my opinion. To do computations, you should use the local language, but to formulate precise statements, use the global language. ◇

Example 6.2. $f\colon U \to \mathbb{R}^{1|1}$ given by $\sum_I f_I(x) \cdot \xi_I$. The body is $f_\varnothing(x)$ and the rest, $f_s(x)$, is the soul. The soul has some even stuff in it, but it is nilpotent because it has odd factors. The body is (like Reimundo was saying) an ordinary function in $C^\infty(M_{red})$. You cannot write is as this body/soul as a direct sum if U is not a domain. You can always pick out the body, but the soul is not canonical.[1]. Q: isn't the soul just f minus the body? PT: no, the body is a function on the reduced guy.

[[From (c), by the way, you get $C^\infty(M) \cong \mathsf{SMan}(M,\mathbb{R}^{1|1}) \cong \mathsf{SMan}(M,\mathbb{R}^{1|0}) \times \mathsf{SMan}(M,\mathbb{R}^{0|1})$. This gets into this inner Hom stuff. This is saying that any function breaks up uniquely as an even and an odd function.]]

Now we have $g\colon \mathbb{R}^{1|1} \to \mathbb{R}^{1|0}$. Ah, there is trouble, you can think of g as a function (not necessarily even) on $\mathbb{R}^{1|1}$ or as a morphism in the category, so even. I mean that $g(y,\eta) = g_0(y)$, so I mean a morphism in the category. That is, g is an even function. Reimundo: write $\mathbb{R}^{1|0}$ on the target; then there is no confusion. PT: yes, that's a good idea.

Now let's try to calculate $g \circ f(x,\xi)$. I'll write in terms of body and soul because I have to Taylor expand around the body. g is equal to its

---

[1] "The body is canonical. Everything else you have to search for."

body.

$$\begin{aligned} g\big(f_0(x)+f_s(x,\xi)\big) &= \sum_{n=0}^{\infty} \frac{g^{(n)}(f_0(x))}{n!}(f_x(x,\xi))^n \\ &= gf_0(x) + g'(f_0(x))f_s(x,\xi) + g''(f_0(x))??? + \cdots \end{aligned}$$

And we know that this will be a finite sum because $f_s(x,\xi)$ is nilpotent. ◇

Notation: Let $A \in \mathsf{SAlg}$ and $M \in \mathsf{mod}\text{-}A$. Then we get the *parity reversed module* πM, given by defining $(\pi M)^{ev} = M^{odd}$ and $(\pi M)^{odd} = M^{ev}$. For example, A is a free module on one even generator, and πA is a free module of dimension $(0|1)$ with free generator $1_A \in A^{ev} = (\pi A)^{odd}$.

Thus, a free module of dimension $(p|q)$ is (by definition, if you want) isomorphic to $A^{\oplus p} \oplus (\pi A)^{\oplus q} =: A^{p|q}$. If the algebra is commutative (super), then p and q are determined by the module, otherwise they are not.

Super objects in differential geometry.

Philosophy: Anything one can formulate in terms of functions (more generally, sections of bundles) has a super analogue.

Definition 6.3. M a super manifold. Then a *vector bundle* (of dimension $(p|q)$) E over M is a locally free sheaf of $\mathcal{O}_M$-modules over $|M|$ (of rank $(p|q)$). ◇

Q: Are these bundles completely determined by their global sections. PT: that is an excellent question. I think they are, but I haven't checked it. It should follow from the same techniques we used to prove result (b).

Definition 6.4. If A is a commutative super algebra, $Der(A) = \{D\colon A \to A | D(ab) = D(a)b + (-1)^{|D||a|}aD(b)\}$ is an A-module. ◇

Example 6.5. $TM \to M^{m|n}$ is a vector bundle of dimension $(m|n)$ given by $\mathcal{O}_{TM}(U) = Der(\mathcal{O}_M(U))$.

"Proof". in a coordinate chart (x_i, ξ_j), basis is given by ∂_{x_i} and ∂_{ξ_j}, which act like you'd think on the coordinate functions. ◇

[[break]]

Consider $U^{p|q} \xrightarrow{f} \mathbb{R}^{1|1} \xrightarrow{g} \mathbb{R}^{1|1}$ as before. Body and soul decomposition only makes sense because we're working in $\mathbb{R}^{p|q}$. The even part of f is $f_0(x) + f_x^{ev}(x,\xi)$, which is like your y and the odd part is $f_x^{odd}(x)$, which is like your η. So we have

$$g_0(f_0(x) + f_x^{ev}(x,\xi)) + g_1(f_0(x) + f_s^{ev}(x,\xi)) \cdot f_x^{odd}(x,\xi)$$

You do the Taylor expansion around the body, with variable the *even part* of the soul.

Q: in the $\mathbb{R}^{1|1}$ case, the soul only has an odd part. PT: yes, this is a bad example.

Let's get back to these vector bundles. Jonah asked a good question over the break: we defined $TM \to M$ as a sheaf, but is TM a super manifold? The answer is yes. there is an alternative way to define vector bundles where you have a total space and you say it's locally trivial.

Example 6.6. $M = \mathbb{R}^{1|1}$ with coordinates (t,θ), and consider the odd vector field $D = \partial_\theta + \theta\partial_t$. What is $D^2 = \frac{1}{2}[D,D]$? It is not a vector field in general, but remember that $\frac{1}{2}[D,D]$ is not zero because D is odd. Given $f \in C^\infty(\mathbb{R}^{1|1})$, we can write it as $f_0(t) + f_1(t)\theta$. So we have

$$\begin{aligned} D(f) &= (\partial_\theta + \theta\partial_t)(f_0(t) + f_1(t)\theta) \\ &= f_1(t)\underbrace{\partial_\theta(\theta)}_{1} + \theta(f_0'(t) + f_1'(t)\theta) \\ &= f_1(t) + f_0'(t)\theta \\ D^2(f) &= f_0'(t) + f_1'(t)\theta \\ &= \partial_t(f) \end{aligned}$$

So $D^2 = \partial_t$. ◇

The notation is bad. I should have denoted the tangent bundle as $\chi(M)$ (vector fields). $\Omega^1 M$ are sections of the "cotangent bundle", given by $\Omega^1 M(U) = \mathrm{Hom}_{\mathcal{O}_M(U)}(\chi M(U), \mathcal{O}_M(U))$.

Then there is a beautiful operator $d\colon \mathcal{O}_M \to \Omega^1 M$, the de Rham differential. It is determined by $\langle D, df\rangle = D(f)$ for $D \in \chi M$. This is an even map. d extends to a graded derivation of square zero $\Omega^\bullet M = \bigwedge^\bullet_{\mathcal{O}_M}(\Omega^1 M)$, which is $\mathbb{Z}$-graded as usual; this is what I mean by *graded derivation* . . . there are no other signs because d is even. Reimundo: you're making a choice of what is the even and odd parts. PT: there is a whole chapter in Deligne and Morgan about two very natural choices. Reimundo: I think you're making a choice by saying that d is even. PT: I don't know about that.

Theorem: $H^*(\Omega^* M, d) \cong H^*_{dR}(M_{red})$.

You should think of the reduced guy as having all the real features ... the odd part is like a nilpotent cloud.

7 PT 09-18

Ways of defining:

	super manifolds	vector bundles
1	sheaf $\mathcal{O}_M$ in SAlg	locally free $\mathcal{O}_M$-modules
2	gluing domains $U \subseteq \mathbb{R}^{m\|n}$	gluing $U \times \mathbb{R}^{m\|n}$, fiberwise linear
3	$C^\infty(M) \in \mathsf{SAlg}$	projective modules[1] over $C^\infty(M)$
4	functor of points (Yoneda)	

[[★★★ Project 2: characterize $C^\infty(M)$ for a supermanifold M, *algebraically*, among all commutative super algebras. It turns out there is a beautiful characterization: if you give me a commutative super algebra, there is a beautiful way to decide if it is the functions on a supermanifold. This is known for ordinary manifolds: you prove that points in the manifold correspond to maximal ideals with real residue field, then use the Zariski topology, then work some more to get the sheaf of functions. This is all explained in some book; I'll put a pdf on my web site.]]

Once you have various points of view, you jump around and always use the most convenient interpretation.

Q: does 3 work for non-compact things? PT: yes, this is because manifolds are paracompact, so any vector bundle lies in a trivial bundle.

<u>Pullbacks of vector bundles</u>. You know the construction of pullbacks for ordinary manifolds.

$$\begin{tikzcd}
f^*E \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & E \arrow[d] \\
M \arrow[r, "f"] & N
\end{tikzcd}$$

Point of view 3 is most convenient here. Given $f\colon M \to N$, we have $f^*\colon C^\infty(N) \to C^\infty(M)$. If P_E is a projective module, we can define $f^*(P_E) := P_E \otimes_{C^\infty(N)} C^\infty(M)$.

Remark 7.1. If P_E is a $C^\infty(M)$-module, it can also be viewed (via our algebra map f^*) as a $C^\infty(N)$-module. We'll call this push-forward. Note that this operation doesn't preserve "projective". $\diamond$

To check that we preserve projective, it is enough to observe that we preserve direct sum and freeness. It is clear that the first construction (pull-back) does this, but the second (push-forward) does not.

Example 7.2. Let's pull back to a point. What is a map from a point to N? If X is an ordinary manifold, then $\mathsf{SMan}(X,N) \cong \mathsf{SAlg}(C^\infty N, C^\infty X)$. I claim this is the same as $\mathsf{SAlg}(C^\infty N_{red}, C^\infty X)$; this is because $C^\infty X$ has no odd part, so the ideal in $C^\infty N$ generated by the odd functions must be killed. But this is the same as $\mathsf{Man}(X, N_{red})$. [[★★★ This is just saying that ${}_{red}$ is adjoint to the forgetful functor from manifolds to super manifolds.]]

When you pull back,

$$\begin{tikzcd}
f^*E \arrow[d] & & E^{p|q} \arrow[d] \\
pt \arrow[r, "f"] & N_{red} \arrow[r, phantom, "\subseteq"] & N
\end{tikzcd}$$

This fiber f^*E is the fiber, but remembering that it is $(p|q)$-dimensional. ◇

<u>Products in SMan</u>. I want to define these so I can talk about super Lie groups.

Definition 7.3. A super Lie group is a super manifold with $G \times G \xrightarrow{\mu} G$, $G \xrightarrow{\nu}$, and $\mathbb{R}^0 = \mathbb{R}^{0|0} = pt \xrightarrow{e} G$. ◇

In an arbitrary monoidal category, you need to define a Hopf algebra object, but if you have products, you don't need the diagonal and counit maps.

The easiest way to define products is using description 2 (there was a homework which told you how to get maps to $\mathbb{R}^{p|q}$, from which you can prove that products are what you think), but let's use 3. Question: what is the coproduct in Alg. Answer: it is tensor product (over whatever the base is). Two maps $A_1 \to A$ and $A_2 \to A$ gives you a map $A_1 \otimes A_2 \to A$ (for this, you need to be in commutative rings), and you can go the other way by restriction. Question: is this ok for supermanifold (or just regular manifolds)? For ordinary manifolds, is $C^\infty(X_1 \times X_2) \cong C^\infty X_1 \otimes_{alg} C^\infty X_2$? No! There is a map $C^\infty X_1 \otimes_{alg} C^\infty X_2 \to C^\infty(X_1 \times X_2)$, but this map is only onto if one of them is discrete. The point is that any function which depends interestingly on both variables isn't a finite linear combination of simple tensors.

Theorem 7.4. *If one completes $\otimes_{alg}$ to the projective tensor product $\otimes$ of Frechét spaces, the above map becomes an isomorphism.*

This Frechét spaces stuff will be relevent later when we do quantum field theories, so let's explain a little of this now.

We'll have to prove that $C^\infty M$ is a Frechét space. It turns out that there is a unique structure of a Frechét algebra on $C^\infty(M)$ (check the reference for Project 2). So $C^\infty M \in \mathsf{FSAlg}$.

To define a super Lie group, you turn around all the arrows to see that $C^\infty G$ has a Hopf algebra structure. The only subtlety is that this is a Hopf algebra in the category FSAlg (commutative as an algebra, but possibly non-cocommutative).

[[break]]

Let me explain how these projects should work. It's first come first served, and Matthias grabbed this one and somebody is already interested in Project 1. But since there can be more than one person per project, it's ok to say that you'd also like to work on that project.

Frechét spaces

We'll call them F-spaces. Banach spaces are B-spaces. The difference is like compact and non-compact spaces, or C^k versus C^∞. For supermanifolds, even if we restrict to compact super manifolds, you still have this trouble with C^k. (Andy: if you limit the odd dimensions, you can do C^k.)

Start with a topological space X. Then what structure do we have on $C^0(X)$? We have the compact-open topology. We only have a norm if X is compact (the supp-norm). Matt: you could take functions vanishing at infinity if it isn't compact. PT: yes, but I don't want to; this amounts to looking at functions on the one point compactification. If X is not compact, we might still want to give a good description than the compact-open topology. This is done by the notion of uniform convergence on compact sets. If X is any topological space, you get a semi-norm for each compact $K \subseteq X$, $\rho_K(f) := \max_{x \in K} |f(x)|$. This family of semi-norms leads to the topology of uniform convergence on compact sets. The nice thing about manifolds is that they are second countable, so we just need a countable sequence of these semi-norms.

On a (topological) manifold, we may pick a countable sequence $K_1 \subseteq K_2 \subseteq \cdots$ such that $X = \bigcup K_i$. Then we get $\rho_{K_1} \leq \rho_{K_2} \leq \cdots$ defining the topology on $C^0(X)$. If you want, you can even define a metric now by some formula. This implies that $C^0(X)$ is metrizable.

Theorem 7.5. *$C^0(X)$ is complete with respect to this topology.*

Completeness of a topological vector space means that Cauchy sequences converge (this is ok so long as the space has a countable basis; otherwise you need to use nets). [[★★★ btw, how does this work?]]

Definition 7.6. A *Frechét space* (or *F-space*) is a complete topological vector space whose topology is given by an increasing countable sequence of semi-norms. ◇

You can get around picking the norms; here is an equivalent definition.

Definition 7.7. A *Frechét space* (or *F-space*) is a complete locally convex[2] topological vector space that is metrizable. ◇

Remark 7.8. Locally convex is important because the Hahn-Banach theorem tells you that linear continuous maps V' can detect points (there will always be a continuous linear map which doesn't vanish at a given point). ◇

Example 7.9. $L^p[0,1]$ $(p > 0)$ is locally convex if and only if $p \geq 1$ (in which case they are actually Banach spaces). ◇

Example 7.10. Any Banach space. In this case, you have just one norm which defines the topology. As we saw (by taking X non-compact), there are other F-spaces. ◇

Example 7.11. Let X be a compact smooth manifold. Then $C^k(X)$ is a B-space for all $0 \leq k < \infty$ using the C^k semi-norms [[★★★ a sequence converges if derivatives up to order k converge pointwise?]]. The norm will depend on some choices (like a choice of some charts or a choice of a Riemannian metric), but the induced topology does not depend on these choices. An F-space only has the topological structure, not on the sequence of semi-norms. Similarly, C^k has lots of norms, but they all define the same topology, so it is more naturally an F-space than a B-space.

If X is not compact, then $C^k(X)$ is only an F-space.

Finally, if X is not compact, you can make $C^\infty(X)$ into an F-space via controlling more and more derivatives on larger and larger compact sets. It is enough to check completeness. [[★★★ We're taking an inverse limit of Frechét spaces $\cdots \to C^{k+1}(X) \hookrightarrow C^k(X)$. Shouldn't completeness follow immediately from the fact that all the C^kX are complete and $C^\infty X = \bigcap C^k X$?]] ◇

If V and W are topological vector spaces, then $V \otimes_{alg} W$ has lots of possible topologies. This is one of the problems with topological vector spaces. We'll use the projective topology, characterized by: given a continuous bilinear $V \times W \to Z$, it factors uniquely through $V \otimes_{proj} W$ by a continuous linear map (through the usual bilinear map $\varepsilon\colon V \times W \to V \otimes_{proj} W$).

Definition 7.12. If V and W are F-spaces, then define $V \otimes W$ to be the completion of $V \otimes_{proj} W$. ◇

The result is that if we take complete topological vector spaces Z, the property given above characterizes $\otimes$ (because a map to a complete thing extends uniquely to the completion).

Lemma 7.13. *If X_1 and X_2 are smooth manifolds, then $C^\infty(X_1) \times C^\infty(X_2) \to C^\infty(X_1 \times X_2)$ given by $(f_1, f_2) \mapsto \big((x_1, x_2) \mapsto f_1(x_1) \cdot f_2(x_2)\big)$ is bilinear* and continuous.

Theorem 7.14. *The induced map $C^\infty X_1 \otimes C^\infty X_2 \to C^\infty(X_1 \times X_2)$ is an isomorphism.*

For any super manifold, $C^\infty(M)$ is a Frechét algebra. In fact, the structure sheaf $\mathcal{O}_M$ is a sheaf of Frechét algebras. Locally, that sheaf is $C^\infty(\mathbb{R}^m) \otimes \bigwedge^* \mathbb{R}^n$, and this exterior algebra is finite dimensional. We know that $C^\infty(\mathbb{R}^m)$ has a unique Frechét structure. $\otimes$ defines a product on super manifolds which is compatible with the product on manifolds.

---

[2] A subset is convex if the line between any two points in the set is in the set. Locally convex means that any neighborhood of 0 contains a convex neighborhood.

8 PT 09-20 Super Lie algebras (over ℂ)

Classical super Lie algebras (see Kac):

$$\begin{tikzcd}
\mathfrak{gl}(p|q) \arrow[r, phantom, "\supseteq"] & \mathfrak{sl}(p|q) \\
\mathfrak{osp}(p|q) \arrow[u, phantom, "\cup|"] &
\end{tikzcd}$$

$\mathfrak{osp}$ has several versions over $\mathbb{R}$.

$$P(n) = \left\{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \,\middle|\, \mathrm{tr}(A) = 0, B \text{ symm}, C \text{ skew symm} \right\}$$

$$Q(n) = \left\{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} \,\middle|\, \mathrm{tr}(B) = 0 \right\}$$

$\mathfrak{g}$	F_4	G_3	$D_{2\|1}(\alpha)$
sdim	24|16	17|14	9|8
$(\mathfrak{g}_0, \mathfrak{g}_1)$	$\cdots$	$(G_2 \times \mathfrak{sl}_2, \mathbb{C}^7 \otimes \mathbb{C}^2)$	$((\mathfrak{sl}_2)^3, \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2)$

Note that $\mathfrak{g}_1$ is a representation of $\mathfrak{g}_0$. The table doesn't give you the information of what brackets of $\mathfrak{g}_1$ with $\mathfrak{g}_1$ are.

A super Lie algebra is a super vector space $\mathfrak{g}$ together with a Lie bracket $[,]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that is skew symmetric and satisfies the Jacobi identity (in SVect, so $[b,a] = -(-1)^{|a|\cdot|b|}[a,b]$).

Example 8.1. If $\mathrm{sdim}(V) = (p|q)$, then $\mathfrak{gl}(p|q) = \mathrm{Hom}(V,V) = \mathsf{SVect}(V,V)$. This contains $\mathfrak{sl}(p|q) := \{\alpha \in \mathrm{Hom}(V,V) | \, \mathrm{str}(\alpha) = 0\}$. Note that if $p = q$, then the identity has supertrace zero. If you want a simple Lie algebra, you'd have to quotient out by the multiples of the identity. The bracket here is $[\alpha, \beta] := \alpha\beta - (-1)^{|\alpha|\cdot|\beta|}\beta\alpha$. $\mathfrak{sl}(p|q)$ inherits this bracket. ◇

Example 8.2. If ϕ is a non-degenerate symmetric bilinear form on $V = V^e \oplus V^o$ (i.e. it is symmetric on V^e, skew on V^o, and vanishes on pairing even with odd), we have $\mathfrak{osp}(p|q) = \{\alpha \in \mathrm{Hom}(V,V) | \phi(\alpha v, w) + (-1)^{|\alpha|\cdot|v|}\phi(v, \alpha w) = 0\}$. Over $\mathbb{R}$ there are many non-isomorphic forms like this, so you should really write $\mathfrak{osp}(p_1, p_2|q)$, where (p_1, p_2) is the signature. Over $\mathbb{C}$ there is only one such form up to isomorphism. The Lie bracket is inherited from $\mathfrak{gl}(p|q)$. ◇

Example 8.3. $P(n)$ and $Q(n)$ ◇

Example 8.4. Finally, there are a few exceptional super Lie algebra. The last one has a continuous parameter α. This completes the list of *simple* super Lie algebras (actually, not all of these examples are simple, because you have to quotient $\mathfrak{sl}(p|p)$ by identities to get simple, and erase $\mathfrak{gl}(p|q)$. Then you have all the simples. This is a theorem of Kac [[★★★ ref]]). You have to form extensions between these things ◇

Andy: why don't you also have one for skew-symmetric bilinear forms? PT: It turns out to be isomorphic to $\mathfrak{osp}$ (as a super Lie algebras), so it is already on the list. Reimundo: It's not the same vector space though; $\mathfrak{spo}(p|q)$ should be isomorphic to $\mathfrak{osp}(q|p)$. What happens if you take a form which pairs the even and the odd part non-trivially? [[★★★ something I didn't catch]]

Super Lie groups

Definition 8.5. A super Lie group is a super manifold G with $\mu\colon G \times G \to G$, $\nu\colon G \to G$, and $e\colon pt \to G$ satisfying .... Here we had to use existence of products in SMan. ◇

We were asking if SMan has products. We thought of it as some category of sheaves over ordinary manifolds, together with some Frechét structure. Then you can take products by taking these completed tensor products. Another way of proving this was that we had an equivalence of categories $\bigwedge$-Vect to SMan, given by taking $(E^q \to X^p) \mapsto C^\infty(\pi E) = C^\infty(\bigwedge^* E^*)$, where the morphisms in $\bigwedge$-Vect are induced by the morphisms in SMan. Now it suffices to show that $\bigwedge$-Vect has products. Define $(E_1 \to X_1) \times (E_2 \to X_2) = (E_1 \oplus E_2 \to X_1 \times X_2)$ (since $\bigwedge^*(E_1^* \oplus E_2^*) \cong \bigwedge^* E_1^* \otimes \bigwedge^* E_2^*$). You can use whatever model for the product you like.

Definition 8.6. A *(left) G-action* on a super manifold M is a morphism $\ell\colon G\times M\to M$ satisfying the associativity and unit diagrams. ◇

Example 8.7. $G = \mathbb{R}^{1|1}$, with $\mathbb{R}^{1|1}\times\mathbb{R}^{1|1}\xrightarrow{+}\mathbb{R}^{1|1}$ given by $\big((t_1,\theta_1),(t_2,\theta_2)\big)\mapsto(t_1+t_2,\theta_1+\theta_2)$ (this actually works on any $\mathbb{R}^{p|q}$). ◇

Example 8.8 (Super Heisenberg group)**.** $\mathbb{R}^{1|1}\times\mathbb{R}^{1|1}\xrightarrow{\mu}\mathbb{R}^{1|1}$ given by $\big((t_1,\theta_1),(t_2,\theta_2)\big)\mapsto(t_1+t_2+\theta_1\theta_2,\theta_1+\theta_2)$. Note that this group is not commutative (not even super commutative), but you can prove that this is associative. You can check that $\nu(t,\theta)=(-t,-\theta)$ and $e=(0,0)$ (this is just the point 0 in the underlying $\mathbb{R}^1$. You don't actually have a choice of θ because choosing e amounts to choosing an even and an odd function on $\mathbb{R}^{0|0}$, and there aren't any odd functions. ◇

We want to go from Lie groups to Lie algebra, so I have to define left invariant vector fields. What are left invariant vector fields on a G-manifold M? The usual formula: $\xi\in Vect(M)$ is left invariant if $\ell_g^*\xi=\xi\ell_g^*$ for all $g\in G$ ($\ell_g\colon M\to M$ is multiplication by g). This is problematic in our case because this is defined pointwise; you have to use arrows. Let's check if the formula itself is a problem. Given $\xi\in Vect(M)=Der(C^\infty M)$, we have $C^\infty M\xrightarrow{\xi}C^\infty M\xrightarrow{\ell_g^*}C^\infty M$ is a derivation and $C^\infty M\xrightarrow{\ell_g^*}C^\infty M\xrightarrow{\xi}C^\infty M$ is also, so the equation makes sense.

Define $L:=(\pi_1\times\ell)\colon G\times M\to G\times M$. Note that $C^\infty(G\times M)=C^\infty G\otimes C^\infty M$ (using our definition of $\otimes$). $\mathrm{id}\otimes\xi$ is a derivation on this thing. You have to check that this is continuous on the usual tensor (then there is a unique continuous extension to the completion). You would call this "the vertical vector field on $G\times M$ corresponding to ξ".

Definition 8.9. $\xi\in Vect(M)$ is *G-invariant* if $L^*(\mathrm{id}\otimes\xi)=(\mathrm{id}\otimes\xi)L^*$. ◇

If you think about it just a bit, you'll see that this L has all group elements built in, and this is equivalent to the usual definition.

The left invariant vector fields form a Lie subalgebra of all vector fields (the derivations form a Lie algebra, but it is infinite-dimensional): $Vect(M)$ is a super Lie algebra under the bracket and $Vect(M)^G$ forms a Lie subalgebra.

Theorem 8.10. *Let G be a super Lie group and $\mathfrak{g}$ be the super Lie algebra of left invariant vector fields on G (so $M=G$ and the action ℓ is the multiplication μ). Then the restriction map $\mathrm{res}_e\colon\mathfrak{g}\to T_eG$ is an isomorphism of super vector spaces.*

If our group G had dimension $(m|n)$, then T_eG has dimension $(m|n)$. [[break]]

There is a little confusion about what this restriction map. Let's start with a vector bundle "$E\to M$", which is by definition a locally free $\mathcal{O}_M$-module $C^\infty E$ (this is the "sheaf of sections"). Then we have

$$\begin{tikzcd}
\text{“} f^*E \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & E \text{ ”} \arrow[d] \\
N \arrow[r, "f"] & M
\end{tikzcd}$$

We specify $C^\infty(f^*E):=C^\infty N\otimes_{(C^\infty M,f^*)}C^\infty E$. Now if $N=\mathbb{R}^{0|0}$ is a point, then $C^\infty N=\mathbb{R}$ or $\mathbb{C}$, so the module $C^\infty(f^*E)$ is just a finite-dimensional super vector space, and this is what I meant by $T_eG=C^\infty(e^*TG)$ (as a $C^\infty(pt)$-module). What is this restriction? Remember that $\mathfrak{g}\subseteq Vect(G):=Der(C^\infty(G))=:C^\infty(TG)$. I claim there is a map $C^\infty(TG)\to C^\infty(e^*TG)$. This is just saying that there is a map $C^\infty(TM)\to C^\infty(pt)\otimes C^\infty(TM)$ given by $1\mapsto 1\otimes s$ for $s\in C^\infty(TM)$.

$$\begin{tikzcd}
\text{“} & f^*E \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & E \arrow[d] & \text{ ”} \\
& N \arrow[r, "f"] \arrow[u, bend left, dashed, "f|"] & M \arrow[u, bend right, "s"'] &
\end{tikzcd}$$

Remark 8.11. By the way, the notation is $C^\infty(E)=\Gamma_{C^\infty}(X;E)$, which is confusing because this is not the same as functions on E (thought of as a manifold). ◇

Proof of 8.10. We construct the inverse map $T_eG\to\mathfrak{g}$. What is T_eG? An element $v\in T_eG$ is a derivation $v\colon C^\infty G\to\mathbb{R}$ in the sense that $v(f\cdot g)=v(f)\cdot g(e)\pm f(e)\cdot v(g)$. Remember that we can evaluate functions at points by composition: $pt\xrightarrow{e}G\xrightarrow{f}\mathbb{R}$. If you had a global vector field (derivation) $V\colon C^\infty G \to C^\infty G$, then you could compose with evaluation at the point e. So whenever you have a global vector field, you can evaluate at any point you want to get the value of the vector field at the point.

Now we construct $T_eG \to \mathfrak{g}$, given by $v \mapsto \xi_v$. We define $\xi_v\colon C^\infty G \xrightarrow{\mu^*} C^\infty(G\times G) = C^\infty G \otimes C^\infty G \xrightarrow{\mathrm{id}\otimes v} C^\infty G \otimes \mathbb{R} = C^\infty G$ [[★★★]]. Symbolically, $f(x) \mapsto f(x\cdot y) \mapsto v\big(y \mapsto f(xy)\big)|_{y=e}$.

Claim. *ξ_v is left invariant and $v \mapsto \xi_v$ is the inverse to restriction to the identity.*

□

Example 8.12. $(\mathbb{R}^{1|1}, \mu)$ and $v = \partial_{\theta_2}$. Then $\xi_v f = \partial_{\theta_2}\big(f(t_1 + t_2 + \theta_1\theta_2, \theta_1 + \theta_2)\big)|_{(t_2,\theta_2)=(0,0)}$

$$\begin{aligned} f(t,\theta) &= f_0(t) + f_1(t)\theta \\ \xi_v f &= \partial_{\theta_2}\Big(f_0(t_1+t_2) + f_0'(t_1+t_2)\theta_1\theta_2 + f_1(t_1+t_2)(\theta_1+\theta_2)\Big) \\ &= -f_0'(t_1)\theta_1 + f_1(t_1) \end{aligned}$$

So $\xi_v = \partial_\theta - \theta\partial_t$ on $\mathbb{R}^{1|1}$ which is that vector field for which $\frac{1}{2}[\xi,\xi] = -\partial_t$. Then since the dimension of the Lie algebra is $1|1$, we have that it is generated by ξ and $[\xi,\xi]$. ◇

9 PT 09-25

About the first homework: nobody got the characterization of the Berezinian because I didn't put enough conditions. You need that the Berezinian is natural. If F is free over A of finite dimension, then $\mathrm{Ber}\colon \mathrm{Hom}_A(F,F)^\times \to GL_1(A^e)$ is multiplicative. If $\phi\colon A \to B$ is a morphism of commutative super algebras, then the diagram

$$\begin{tikzcd} \mathrm{Hom}_A(F,F) \arrow[r, "\mathrm{Ber}"] \arrow[d, "\alpha\otimes\mathrm{id}_B"'] & GL_1(A) \arrow[d, "GL_1(\phi)"] \\ \mathrm{Hom}_B(F\otimes_A B, F\otimes_A B) \arrow[r, "\mathrm{Ber}"] & GL_1(B) \end{tikzcd}$$

Recall that we proved the following theorem.

Theorem 9.1. *If G is a super Lie group with $\mathfrak{g} = Lie(G)$ the left invariant vector fields on G (a super Lie algebra), then $res_e\colon \mathfrak{g} \to T_eG$ is an isomorphism in SVect.*

We proved this by constructing an explicit inverse.

If M is a super manifold and $m \in M$, what is T_mM? It is a super vector space (it is a vector bundle over a point). I want to say that $T_mM \cong Der(C^\infty M, \mathbb{R}_m)$ (as a super vector space) [[★★★ HW1. this is an easy one]]. Where is m? $\mathbb{R}$ is a bimodule over $C^\infty M$ via evaluation at m. By the way, a points of M are $\mathsf{SMan}(\mathbb{R}^{0|0}, M) \cong \mathsf{Man}(\mathbb{R}^0, M_{red})$, so a point really is a point of the underlying manifold. You get evaluation at a point by $C^\infty M \to C^\infty M_{red} \xrightarrow{ev_m} \mathbb{R}$.

Ok, so what was this explicit inverse? Remember that $Vect(M) = Der(C^\infty M, C^\infty M)$ is a $C^\infty M$-module. Given $v \in T_eM$, $v\colon C^\infty G \to \mathbb{R}$ a derivation,we produced $\xi_v\colon C^\infty G \xrightarrow{\mu^*} C^\infty(G\times G) \cong C^\infty G \otimes C^\infty G \xrightarrow{\mathrm{id}\otimes v} C^\infty G \otimes \mathbb{R} = C^\infty G$. [[★★★ HW2. fill in the gaps. In particular, check that ξ_v is left invariant]] Explicitly, this ξ_v is given by $f(x) \mapsto f(x\cdot y) \mapsto v(y \mapsto f(x\cdot y))$. This will be left invariant because we are using right multiplication by y. If you wanted right invariant, you'd use $v\otimes\mathrm{id}$ instead of $\mathrm{id}\otimes v$. Part of this homework is to prove the following lemma.

Lemma 9.2 (Inverse Function Theorem [[★★★ this is the right name?]])**.** *If $f\colon M \to N$ induces an isomorphism $df_m\colon T_mM \to T_{f(m)}N$, then f is an isomorphism in some neighborhood of $m \in M$.*

What is this df_m? It is given by $T_mM \cong Der(C^\infty M, \mathbb{R}_m) \xrightarrow{-\circ f} Der(C^\infty N, \mathbb{R}_{f(m)}) \cong T_{f(m)}N$.

Warning 9.3. We do not have $df\colon Vect(M) \to Vect(N)$. ⌟

Examples of super Lie groups and their Lie algebras

Example 9.4. Let $V = V_0 \times V_1$ be a super vector space. Then V is a super Lie group under addition. We have to produce a super manifold and a multiplication, so we need to define $C^\infty(V)$. It is $C^\infty V := C^\infty(V_0) \otimes \bigwedge^*(V_1^*) \supseteq V^*$ (the linear functions on V_0 are smooth and $\bigwedge^1 V_1^* = V_1^*$).

Now we need to describe the super Lie group structure. We have morphisms of super manifolds $V \times V \xrightarrow{+} V$, $V \xrightarrow{-} V$, and $pt \xrightarrow{0} V$. I hope it is clear what exactly these maps are. Note that this is a commutative super Lie group. ◇

Super Heisenberg groups are really key.

Example 9.5 (Super Heisenberg groups)**.** Let $V \in \mathsf{SVect}$, and let b an even skew form on V. So $b\colon V \otimes V \to \mathbb{R}$ is a skew symmetric morphism in SVect, so it is skew on the even part, symmetric on the odd part, and the cross terms are zero because $\mathbb{R}$ is even. Then we can construct $H(V, b)$. As a super manifold, it is $\mathbb{R} \times V$ (or $\mathbb{C} \times V$) (the $\mathbb{R}$ will be the center). But as a group, it is not the product (I hope it is clear that there are products of super Lie groups; you have to be a little careful because you get a sign). The group structure is $(\mathbb{R} \times V) \times (\mathbb{R} \times V) \to \mathbb{R} \times V$, given by $\big((t_1, v_1), (t_2, v_2)\big) \mapsto \big(t_1 + t_2 + b(v_1, v_2), v_1 + v_2\big)$. This is an extension of V by $\mathbb{R}$:

$$1 \to \mathbb{R} \to H(V, b) \to (V, +) \to 1$$

Taking $v_1 = 0$, we see that this $\mathbb{R}$ is central. This is not a semi-direct product because there is no splitting (as groups) if $b \neq 0$. Explicitly, in terms of functions, we have $C^\infty(\mathbb{R} \times V \times \mathbb{R} \times V) \leftarrow C^\infty(\mathbb{R} \times V)$, given by sending $t \mapsto t_1 + t_2 + b$, where $b\colon V \times V \to \mathbb{R}$ pulled back to $\mathbb{R} \times V \times \mathbb{R} \times V$, and $V^* \ni \phi \mapsto \phi_1 + \phi_2$, the pull back of the dual addition map $V \times V \xrightarrow{+} V$, $V^* \times V^* \xleftarrow{+} V^*$. Now I've told you where the linear maps go, and this determines the whole algebra map.

[[break]]

Now let's compute $h(V, b) := Lie(H(V, b))$. As a super vector space, it is $\mathbb{R} \oplus V \in \mathsf{SVect}$. The Lie bracket is given by

$$\frac{1}{2}[v, w] = b(v, w) \cdot c_{\mathbb{R}}$$

where $c = 1 \in \mathbb{R}$ is the central element.

How do we prove this? If $v \in V$, we want a derivation $\psi_v \in h(V, b)$ by the procedure in the proof of Theorem 8.10. Note that $v \in V$ extends to a global vector field $\partial_v \in Vect(V)$. $\xi_v\colon (t, \phi) \overset{\mu^*}{\longmapsto} (t_1 + t_2 + b\ [[b(v_1, v_2)]], \phi_1 + \phi_2\ [[\phi_1(v_1) + \phi_2(v_2)]]) \xmapsto{\mathrm{id} \otimes v\ \ [[1 \otimes \partial_{v_2}]]} \big((-1)^{|v_1| \cdot |v_2|} b(v_1, -)\ [[= v_1^*]], \partial_{v_2}\phi_2\big)$. The conclusion is that $\xi_v = \partial_v \pm v^*\partial_t$. I should have done the right invariant case so that I don't pick up the sign. So $\xi_v\colon (t, \phi) \overset{\mu^*}{\longmapsto} (t_1 + t_2 + b, \phi_1 + \phi_2) \xmapsto{v_1 \otimes \mathrm{id}} \big(v_2^*, \partial_{v_1}\phi_1\big)$, so $\xi_v = \partial_v + v^*\partial_t \in Vect(\mathbb{R} \times V)$.

Now the claim is that $[\xi_v, \xi_w]$, the Lie bracket of derivations, is $2b(v, w)\partial_t$.

$$\begin{aligned}
[\xi_v, \xi_w] &= \cdots \\
&= \underbrace{[\partial_v, \partial_w]}_{0} + \underbrace{[v^*\partial_t, w^*\partial_t]}_{0} + \partial_v(w^*\partial_t) \pm \partial_w(v^*\partial_t) + v^*\partial t\partial_w \pm w^*\partial_t\partial_v \\
&= \partial_v(w^*)\partial_t \pm \underbrace{\partial_w(v^*)}_{b(w,v)=v^*(w)} \partial_t \\
&= 2b(v, w)\partial_t
\end{aligned}$$

The ∂_v terms commute so they fall out and the ∂_t stuff commutes. If you believe the signs are good, then some more stuff cancels out. If you think about this some more, you get the right result.

Now you can take the universal enveloping algebra $U(h(V, b))$ defined as usual $(T^*(h(V, b))/(\alpha\beta \pm \beta\alpha - [\alpha, \beta]))$. Inside of this tensor algebra, you have this central part $\mathbb{R}c_{\mathbb{R}}$ (which is different from the unit of the tensor algebra). You can quotient further by this central piece. If V is completely odd, then the quotient $U(h(V, b))/(c_{\mathbb{R}} - 1)$ is the Clifford algebra $Cl(V, b)$. All of this stuff comes from the super Lie group. When you quantize, you use representations of these Clifford algebra, which really come from representations of the super Lie groups. ◇

10 PT 09-27

More examples of super Lie groups

Problem: If $G \not\cong \mathbb{R}^{p|q}$ (as a super manifold), how does one *write down* $\mu\colon G \times G \to G$? Use the following trick.

Trick: For any super manifold $S \in \mathsf{SMan}$, you can study the *S-points of* G, $G(S) = \mathsf{SMan}(S, G)$. For any S, $G(S)$ is an ordinary group. I hope the group structure is clear:

$$G(S) \times G(S) = \mathsf{SMan}(S,G) \times \mathsf{SMan}(S,G) = \mathsf{SMan}(S, G \times G) \xrightarrow{\mu_*} G(S).$$

Similarly, you get the inverse and identity.

Yoneda's lemma tells us that this determines G. If you want to describe G, it is enough to give a bunch of ordinary groups (with some naturality). To determine $GL_{p|q}$, we just need to write down the (ordinary) groups $GL_{p|q}(S)$.

Definition 10.1. $GL_{p|q}(S) := GL_{p|q}(C^\infty S)$. That is, thinking of $C^\infty S$ as a commutative super algebra, this is the set of invertible even endomorphisms of $(C^\infty S)^{p|q}$. ◇

Now I claim that this defines the super Lie group $GL_{p|q}$. We have to check naturality.

$$\begin{tikzcd}
S \arrow[d, "f"'] & G(S) \arrow[r, phantom, "="] & \mathsf{SMan}(S,G) \\
S' & G(S') \arrow[u, "f^*=G(f)"'] \arrow[r, phantom, "="] & \mathsf{SMan}(S',G)
\end{tikzcd}$$

This is clear because we get $C^\infty S' \xrightarrow{f^*} C^\infty S$.

Given such a collection $\{G(S), G(f)\}$, you get a functor $G\colon \mathsf{SMan}^\circ \to \mathsf{Gp}$, but we don't know that the induced $G\colon \mathsf{SMan}^\circ \to \mathsf{Set}$ is representable.

Example 10.2. We have the functor $GL_{p|q}\colon \mathsf{SMan}^\circ \to \mathsf{Gp}$. Is $GL_{p|q}(S) \cong \mathsf{SMan}(S, GL_{p|q})$ for some super manifold $GL_{p|q}$? This looks like we're back to where we started, but the point is that now we're just looking for a super manifold (because the group structure is determined). ◇

General setting: Yoneda embedding. If we have a category $\mathcal{C}$ (SMan in our case), then we get a functor $Y\colon \mathcal{C} \to \mathsf{Fun}(\mathcal{C}^\circ, \mathsf{Set}) =: \hat{\mathcal{C}}$ given by $M \mapsto (S \mapsto \mathcal{C}(S,M))$ and $(g\colon M \to M') \mapsto \big(\mathcal{C}(S,M) \xrightarrow{g_*} \mathcal{C}(S,M')\big)$. $\hat{\mathcal{C}}$ is a kind of completion of $\mathcal{C}$; it has all limits.

Lemma 10.3. $\mathcal{C}(M,M') \xrightarrow[\sim]{Y} \hat{\mathcal{C}}(Y(M), Y(M'))$ *(i.e. Y is fully faithful).*

But it is not true that every object in $\hat{\mathcal{C}}$ is in the image of Y.

Definition 10.4. An object in $\hat{\mathcal{C}}$ is *representable* if it is isomorphic (in $\hat{\mathcal{C}}$) to something in the image of Y. ◇

If $\mathcal{C}$ is the category of finite-dimensional vector spaces, then an infinite-dimensional vector space will give you a functor which will not be representable.

Claim. *$S \mapsto GL_{p|q}(S)$ is representable as a domain in $\mathbb{R}^{p^2+q^2|2pq}$ (the space of all even endomorphisms of $(C^\infty S)^{p|q}$).*

We get a functor $S_{red} \mapsto GL_{p|q}(S_{red}) \cong \mathsf{Man}(S_{red}, GL_p\mathbb{R} \times GL_q\mathbb{R})$ induced by $GL_{p|q}$, and $GL_p\mathbb{R} \times GL_q\mathbb{R} \subseteq \mathbb{R}^{p^2+q^2}$ is an open subset.

We define the domain by $U_{red} := \big\{(a,d) | \det(a_{red}), \det(d_{red}) \neq 0\big\}$. This gives us a manifold representing our functor [[★★★ hmmm …flesh out the hidden effective descent statement and topology]].

Remark 10.5. Given a super manifold M, as soon as $Y(M)$ factors through Gp, you have the group structure.

$$\begin{tikzcd}
\mathsf{SMan} \arrow[r, "Y(M)"] \arrow[dr, dashed, "\widetilde{Y(M)}"'] & \mathsf{Set} \\
& \mathsf{Gp} \arrow[u, "\text{forget}"']
\end{tikzcd}$$

Need $\mu\colon M \times M \to M$, and you have $M(S) \times M(S) \to M(S)$ for each S. Taking $S = M \times M$ and looking at the image of the identity, you get μ. ◇

Definition 10.6. $SL_{p|q}(S) = \{\alpha \in GL_{p|q} | \operatorname{Ber}(\alpha) = 1\}$. ◇

You can then check that $SL_{p|q}$ is representable.

[[★★★ There is Project 3 on the website: The K-theory of a super manifold is the same as the K-theory of the underlying manifold]]

[[★★★ Project 4: describe super Lie groups corresponding to Kac's list of simple super Lie algebras over $\mathbb{C}$. After the break, we'll see that there always is some super Lie group giving you the super Lie algebra.]]

[[break]]

There is a theorem that any super Lie group embeds into $GL_{p|q}$, so why do need all this machinery if we're just multiplying matrices? Well, why do you need conceptual mathematics? If you really want to define something precisely (physicists don't define what a field theory is, they just know what it is), you need some conceptual stuff. There is no natural embedding into $GL_{p|q}$.

Theorem 10.7. *There is an equivalence of categories* $\mathsf{SMan} \ni G \mapsto (G_{red}, \mathfrak{g}, \mathfrak{g}_{red} \cong Lie(G_{red}), G_{red} \times \mathfrak{g} \xrightarrow{a} \mathfrak{g}$ *extending the adjoint action on* $\mathfrak{g}^e)$.

Theorem 10.8 (Stated in [DEF+99, Deligne-Morgan])**.** *Fix a super Lie group G and a super manifold M. Then there is a natural bijection* $\{$*actions* $M \times G \xrightarrow{r} M\} \leftrightarrow \Big\{$*actions* $M_{red} \times G_{red} \xrightarrow{\rho} M_{red}$, *Lie homo* $\mathfrak{g} \xrightarrow{\phi} Vect(M)$ *such that* $\mathfrak{g}^e \xrightarrow{\phi|_{\mathfrak{g}^e}} Vect(M)^e \to Vect(M_{red})$ *is* $d\rho\Big\}$

$dr\colon \mathfrak{g} \to Vect(M)$.

The idea of the proof: Frobenius theorem [[★★★ Project 5: prove the two theorems using the Frobenius theorem. You have to read between the lines in Deligne and Morgan]]

If X is an ordinary manifold and $\xi \in Vect(X)$, then you get a local $\mathbb{R}$ action (the flow of ξ). This action is global if X is compact. So if $G = \mathbb{R}$, then the reduced action is actually redundant.

Corollary 10.9. *If M is a compact supermanifold, then odd vector fields are in bijective correspondence with actions $(\mathbb{R}^{1|1}) \times M \to M$. (The $\mathbb{R}^{1|1}$ has the Heisenberg group $H(\mathbb{R}^{0|1}, b)$ where b is a non-zero bilinear form. $(t_1, \theta_1) \cdot (t_2, \theta_2) = (t_1 + t_2 + \theta_1\theta_2, \theta_1 + \theta_2)$).*

Proof. An odd vector field $\xi \in Vect(M)$ is a Lie homomorphism from $B(1)$ (free Lie algebra on one odd generator, which is $(1|1)$-dimensional) to $Vect(M)$. $[\xi, \xi]$ is an even vector field on M, so it induces a vector field on M_{red}. Integrating this in the usual way, we get an action of $\mathbb{R}$. ☐

This is a funny way to do it. You should really solve some ODE with even and odd elements. For project 5, I think you'll have to prove this corollary directly.

Reimundo: how do you know $[\xi, \xi] \neq 0$? PT: it could be zero. This is a good case. Given $\xi \in Vect(M)^o$ such that $[\xi, \xi] = 0$ leads to an action $\mathbb{R}^{0|1} \times M \to M$ and vice versa.

So odd vector fields correspond to $\mathbb{R}^{1|1}$ actions and odd vector fields whose bracket with themselves is zero correspond to $\mathbb{R}^{0|1}$ actions. Later, we'll get $d \in Vect(\pi TX)$ (the de Rham d) from the obvious $\mathbb{R}^{0|1}$ action on $\pi TX = \mathsf{SMan}(\mathbb{R}^{0|1}, X)$ induced by the action of $\mathbb{R}^{0|1}$ on itself.

11 PT 10-02

Project 5 (on super group actions) use the following Fröbenius theorem: Every involutive distribution (on a super manifold) is integrable (to a foliation). A *distribution* is a sub-bundle of the tangent bundle. On a super manifold, we have $Der(\mathcal{O}_M)$, which is a locally free $\mathcal{O}_M$-module, and a distribution $\mathcal{D}$ is just a locally free submodule. If you have such a thing, there is the Fröbenius map $\mathcal{D}\otimes_{\mathcal{O}_M}\mathcal{D}\xrightarrow{[,]} Der(\mathcal{O}_M)/\mathcal{D}$, $[X, fY] = X(f)Y + (-1)^{|X||f|}f\cdot[X,Y]$. The first term is an "error term" which is in $\mathcal{D}$, so modulo $\mathcal{D}$, this map is well defined. A distribution is *involutive* if the Fröbenius map is zero. A distribution $\mathcal{D}$ on a manifold M is *integrable* if locally $M \cong M_1 \times M_2$ where $\mathcal{D} \cong Der(\mathcal{O}_{M_1})$. This is exactly analogous to the classical version. You can find the proof in [DEF+99, Deligne-Morgan].

Project 6: Formulate a theory of G-principal bundles (where G is a super Lie group) and their connections. This is really a joint project with Kolya's class. A connection picks out a horizontal distribution in $P \to M$. A connection is *flat* exactly when the corresponding horizontal distribution is involutive. The curvature is exactly the Fröbenius map. This should be done in such a way that a representation $G \to GL_{p|q}$ takes a G-principal bundle P to a vector bundle $\mathcal{E}$ and a connection on P to a connection on $\mathcal{E}$ in the following sense. A vector bundle $\mathcal{E}$ is a locally free sheaf of $\mathcal{O}_M$-modules (you should think of this as sections of a total space ... making this precise will be your homework). A connection on $\mathcal{E}$ is an $\mathbb{R}$-linear map $\nabla\colon \mathcal{E} \to \Omega^1 M \otimes \mathcal{E}$ (remember that $\Omega^1 M = \mathrm{Hom}_{\mathcal{O}_M}(Der(\mathcal{O}_M), \mathcal{O}_M)$) such that for a section $s \in \mathcal{E}$ and $f \in \mathcal{O}_M$,

$$\nabla(f\cdot s) = df \otimes s + f\cdot\nabla(s).$$

Usually there would be a sign, but we require ∇ to be even. In particular, if you have a vector field, you can use ∇ to differentiate a section of $\mathcal{E}$ along the vector field:

$$\nabla_X(s) = \langle X, \nabla(s)\rangle \in \mathcal{E}.$$

In my first class, I used the word "Quillen connection". Let me explain that. Given a connection ∇, you can extend it uniquely to a derivation $\widetilde{\nabla}\colon \Omega^* M \otimes_{\mathbb{R}} \mathcal{E} \to \Omega^* M \otimes_{\mathbb{R}} \mathcal{E}$ such that

$$\widetilde{\nabla}(\alpha\otimes s) = d\alpha\otimes s + (-1)^{\text{c-deg}\,\alpha}\alpha\cdot\nabla(s).$$

The sign comes from cohomological degree (which is different from the sign coming from parity). In the ordinary setting, you need the sign so that the curvature $\widetilde{\nabla}^2$ is $\mathcal{O}_M$-linear.

Warning 11.1. $\Omega^* M$ has two gradings: (1) cohomological degree "c-deg"$\in \mathbb{N}_0$, and (2) parity $|\alpha| \in \mathbb{Z}/2$. The de Rham d has odd cohomological degree, but even parity. Deligne and Morgan have a good sign convention which doesn't mix the two gradings: $\Omega^* M$ is a $\mathbb{Z}$-graded object in the category of super algebras. ⌟

Definition 11.2. A *Quillen connection* on a super vector bundle $\mathcal{E}$ is a $\widetilde{\nabla}$ satisfying $\widetilde{\nabla}(\alpha\otimes s) = d\alpha\otimes s + (-1)^{\text{c-deg}\,\alpha}\alpha\cdot\widetilde{\nabla}(s)$. ⋄

Today I want to explain zero dimensional quantum field theories, but you won't know it yet because we haven't said what a quantum field theory is. Recall that if you have a category $\mathcal{C}$ (which will be SMan soon) and if you have $Y, Z \in \mathcal{C}$, then you'd sometimes like to have an inner home $\underline{\mathcal{C}}(Y,Z) \in \mathcal{C}$ so that you get natural isomorphisms

$$\mathcal{C}(X\times Y, Z) \cong \mathcal{C}(X, \underline{\mathcal{C}}(Y,Z)).$$

(here we assume $\mathcal{C}$ has products; in general you can use some other monoidal structure).

Remark 11.3. The adjunction defines a functor $\mathcal{C}^\circ \to \mathsf{Set}$ for a given Y and Z. Then you can ask if the functor is representable. ⋄

Take $\mathcal{C} = \mathsf{Man}$. Is there an inner hom? Only when one of the manifolds has dimension 0. If you extend your category to include infinite-dimensional manifolds, then you get an inner hom.

Theorem 11.4. *If $M \in \mathsf{SMan}$, then $\underline{\mathsf{SMan}}(\mathbb{R}^{0|1}, M)$ exists and is isomorphic to πTM (by the way, it exists for $\mathbb{R}^{0|q}$ for any q). That is, there are natural isomorphisms*

$$\mathsf{SMan}(S\times\mathbb{R}^{0|1}, M) \cong \mathsf{SMan}(S, \pi TM).$$

We'll call $\mathbb{R}^{0|1}$ a "super point", and we'll call πTM the "odd tangent bundle". The caveat is that you need to know what πTM is as a super manifold. [[★★★ current HW2 covers this. If $\mathcal{E}$ is an $\mathcal{O}_M$-module, you'll figure out how this leads to a super manifold E with a morphism $E \to M$ so that E is locally isomorphic to $p_1 \colon U \times \mathbb{R}^{p|q} \to U$.]]

[[break]]

Statement of HW2: $\mathcal{E}$ a locally free sheaf of dimension $(p|q)$ over $\mathcal{O}_M$ (think $\mathcal{E} = \Gamma(\pi TM)$). Define a morphism $E \xrightarrow{p} M$ which locally looks like $p_1 \colon U \times \mathbb{R}^{p|q} \to U$. There are two approaches:

(Approach a) $\mathcal{O}_E$ is a completion of $\mathrm{Sym}^*_{\mathcal{O}_M} \mathcal{E}$. If M is an ordinary manifold and $\mathcal{E}$ is an ordinary vector bundle, then we did something which produced a super manifold whose sections were $\mathrm{Sym}^* \mathcal{E}$.

Definition 11.5. If $\mathcal{E}$ is an A-module, then you can form $\mathrm{Sym}_A \mathcal{E}$, the free (commutative) A-algebra on $\mathcal{E}$. That is,

$$\mathsf{SAlg}(\mathrm{Sym}_A \mathcal{E}, B) \cong A\text{-}\mathsf{mod}(\mathcal{E}, B_{\mathrm{forget}}). \qquad \diamond$$

This gives me things which are polynomial on the fibers. $\mathrm{Sym}_{\mathbb{R}}(V^*)$ is polynomials on V. This is why you need to complete: the smooth functions are the completion of polynomial functions (in the Frechét topology).

If F and M are ordinary things, then we defined a super manifold πF. We said that $C^\infty(\pi F) := \bigwedge^*_{\mathcal{O}_M}(F)$.

Definition 11.6. $\bigwedge^*_A \mathcal{E} := \mathrm{Sym}_A(\pi\mathcal{E})$. $\diamond$

Q: Something is funny .... PT: I'm using the sign convention where there is only one grading, which mixes the cohomological degree and the parity.

Sign convention for today: add the parity and cohomological degree to get a single sign. For example, $\bigwedge^*_A \mathcal{E}$ is just a commutative super algebra.

Now the two π's are consistent. What we did a month ago is the special case where M and F are ordinary things, and $F := \pi E$. For the projection, I just have to remember the $\mathcal{O}_M$-module structure.

(Approach b) Functor of points approach. Define

$$\mathsf{SMan}(S, E) := \{(f, s) | f \in \mathsf{SMan}(S, M), s \in f^*(\mathcal{E})^e\}.$$

$$\begin{tikzcd}
f^*E \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & E \arrow[d, "p"] \\
S \arrow[u, bend left, "s"] \arrow[r, "f"'] \arrow[ur, dashed] & M
\end{tikzcd}$$

Proof of Theorem 11.4. Pick $S \in \mathsf{SMan}$.

$$\begin{aligned}
\mathsf{SMan}(S \times \mathbb{R}^{0|1}, M) &\cong \mathsf{SAlg}(C^\infty M, C^\infty S \otimes C^\infty \mathbb{R}^{0|1}) \qquad \text{(for convenience)} \\
&\cong \{(f, s) | f \in \mathsf{SAlg}(C^\infty M, C^\infty S), \\
&\qquad s \colon C^\infty M \to C^\infty S \text{ odd derivation w.r.t } f\}
\end{aligned}$$

Since $C^\infty \mathbb{R}^{0|1} \cong \mathbb{R}[\theta]/\theta^2$, the tensor in the first line is an algebraic tensor product. Given $\phi \in \mathsf{SMan}(S \times \mathbb{R}^{0|1}, M)$, I can write $\phi = f + \theta s$. The condition that ϕ is an algebra map: $\phi(ab) = f(ab) + \theta s(ab)$ for functions a, b, so I have $\phi(a)\phi(b) = (f(a) + \theta s(a))(f(b) + \theta s(b)) = f(a)f(b) + \theta(s(a)f(b) + (-1)^{|a|} f(a)s(b)$. So f must be an algebra map and s is a derivation as desired.

We'll have to finish the last step on Thursday. □

12 PT 10-05

Theorem 12.1. *If M is a super manifold, then $\underline{\mathsf{SMan}}(\mathbb{R}^{0|1}, M) \cong \pi TM$. That is, there is a natural isomorphism $\mathsf{SMan}(S, \pi TM) \cong \mathsf{SMan}(S \times \mathbb{R}^{0|1}, M)$.*

This theorem will explain zero dimensional quantum field theory. I haven't quite explained what πTM is, but the homework shows that for any locally free sheaf of modules $\mathcal{E}$, there is a total space $E \xrightarrow{p} M$ such that $\mathsf{SMan}(S, E) = \{(f, g) | f \in \mathsf{SMan}(S, M), g \in (f^*\mathcal{E})^e\}$ for any super manifold S. In particular, taking $\mathcal{E} = \pi Der(C^\infty M)$, we get $E = \pi TM$. In particular, the homework shows that these functor are actually representable by a super manifold. We did part of the proof last time, but let's start over.

Proof. Pick S.

$$\begin{aligned}
\mathsf{SMan}(S \times \mathbb{R}^{0|1}, M) &\cong \mathsf{SAlg}(C^\infty M, C^\infty S \otimes C^\infty \mathbb{R}^{0|1}) \\
&\cong \{(f, g) | f \in \mathsf{SAlg}(C^\infty M, C^\infty S), \\
&\qquad\qquad g\colon C^\infty M \to C^\infty S \text{ odd derivation } \} \\
&\cong \{(f, g) | f \in \mathsf{SMan}(S, M), \\
&\qquad\qquad g \in \Gamma(f^*TM)^{odd} = \Gamma(f^*\pi TM)^{ev}\} \\
\cong \mathsf{SMan}(S, \pi TM) &
\end{aligned}$$

Let $\phi \in \mathsf{SAlg}(C^\infty M, C^\infty S \otimes C^\infty \mathbb{R}^{0|1})$, then $\phi = f + \theta g$ for $f, g\colon C^\infty M \to C^\infty S$. Saying that ϕ is a super algebra homomorphism says that

$$\begin{aligned}
f(ab) + \theta g(ab) &= \phi(ab) \\
&= \phi(a)\phi(b) \\
&= \big(f(a) + \theta g(a)\big)\big(f(b) + \theta g(b)\big) \\
&= f(a)f(b) + \theta\big(g(a)f(b) + (-1)^{|a|} f(a)g(b)\big)
\end{aligned}$$

Comparing coefficients, we get that $f \in \mathsf{SAlg}(C^\infty M, C^\infty S)$, and that $g(ab) = g(a)f(b) + (-1)^{|a|} f(a)g(b)$, which is equivalent to saying that $g\colon C^\infty M \to {}_{f^*}C^\infty S_{f^*}$ is an odd derivation (we're thinking of $C^\infty S$ as a $C^\infty M$-bimodule via f).

For the third isomorphism, we're using the fact that

$$\begin{tikzcd}
f^*TM \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\ulcorner"] & TM \arrow[d] \\
S \arrow[r, "f"] & M
\end{tikzcd}$$

$\Gamma(f^*TM) = \Gamma(TM) \otimes_{(C^\infty M, f^*)} C^\infty S$. This works for any bundle, but there is a lemma that $\Gamma(TM) \otimes_{(C^\infty M, f^*)} C^\infty S \cong Der(C^\infty M, {}_{f^*}C^\infty S_{f^*})$.

The last isomorphism follows from the homework. I'm not going to check the naturality, but everything we did was totally obvious. ☐

πTM is the odd tangent bundle. Writing it as in the theorem, we get an action of $\mathbb{R}^{0|1}$ on πTM. Today we'll try to understand the action of $\underline{\mathrm{Aut}}(\mathbb{R}^{0|1})$ on πTM. Let's first understand the endomorphisms group.

Note that by the theorem $\underline{\mathsf{SMan}}(\mathbb{R}^{0|1}, \mathbb{R}^{0|1}) \cong \pi T\mathbb{R}^{0|1} = \mathbb{R}^{1|1}$ (over $\mathbb{R}^{0|1}$ there are no twisted bundles; one of the projects is to show that bundles over a super manifold are the same as bundles over the reduced manifold). Now the super Lie group $\underline{\mathrm{Aut}}(\mathbb{R}^{0|1})$ is $\mathbb{R}^{0|1} \rtimes \mathbb{R}^\times$ (odd translations times even nonzero dilations).

Theorem 12.2. *For any super manifold M, there is an embedding $\Omega^*M \hookrightarrow C^\infty(\pi TM)$ with image the fiberwise polynomial functions such that*

- *(a) the infinitesimal generator of the odd translations extends the de Rham d (using the convention that adds the two gradings). This is the first time that this super stuff has given us a better understanding of something classical.*
- *(b) the (even) $\mathbb{R}^\times$ action restricts to the grading operator on Ω^*M. That is, for $\lambda \in \mathbb{R}^\times$ and $\omega \in \Omega^k M$, we have that $\lambda_*(\omega) = \lambda^k \omega$.*

If M is an ordinary manifold, this is the definition [[★★★ I missed the explanation]]. In general, $C^\infty(\pi TM)$ is the completion of $\bigwedge^*_{\mathcal{O}_M}(\pi TM)$ (the theorem is that this is isomorphic to Ω^*M).

If you don't know anything about the de Rham d, this defines it for you. The best property it has is that $d^2 = 0$. This means that $[d, d] = 0$, which just says that translation commutes with translation. From (b), we can see that d has cohomological degree 1 (this is equivalent to saying that dilations and translations commute the way they do).

[[break]]

Now we'll prove the theorem using local coordinates, and leave it as an exercise that the proof works globally.

Proof. Consider the case $M = U \subseteq \mathbb{R}^{p|q}$ a domain, with coordinates x_i and η_j on $\mathbb{R}^{p|q}$. Then [[★★★ PT: this is actually a third way to understand the total space; by gluing]] $\pi TU \cong U \times \mathbb{R}^{q|p}$, with coordinates $\hat{\eta}_j$ and $\hat{x}_i$ on $\mathbb{R}^{q|p}$ (these are just totally different coordinates, not operators or anything like that). Note that x_i and $\hat{\eta}_j$ are even and $\hat{x}_i$ and η_j are odd. If we don't need to separate the xs and ηs, then we'll write y_k to mean x_k or η_{k-p} (if $k > p$). Now I'll write the isomorphism from the first theorem in local coordinates.

$$\begin{aligned}
\mathsf{SMan}(S, \pi TU) &\overset{HW}{\cong} \{(X_i, H_j, \hat{H}_j, \hat{X}_i) | X_i, \hat{H}_j \in (C^\infty S)^e \text{ such that} \\
&\qquad\qquad X_i(s) \in |U|, \hat{X}_i, H_j \in (C^\infty S)^o\} \\
\mathsf{SMan}(S \times \mathbb{R}^{0|1}, U) &\overset{HW}{\cong} \{X_1 + \theta\hat{X}_1, \dots, X_p + \theta\hat{X}_p \in C^\infty(S \times \mathbb{R}^{0|1})^e \\
&\quad H_1 + \theta\hat{H}_1, \dots, H_q + \theta\hat{H}_q \in C^\infty(S \times \mathbb{R}^{0|1})^o \text{ s.t. } \dots\}
\end{aligned}$$

The notation presents the isomorphism of the first theorem for you.

(a) (Right) translation action of $\mathbb{R}^{0|1} \subseteq \underline{\mathrm{Aut}}(\mathbb{R}^{0|1})$ (coord η, the coordinate on the $\mathbb{R}^{0|1}$ in $S \times \mathbb{R}^{0|1}$ is θ). The group law $\mathbb{R}^{0|1} \times \mathbb{R}^{0|1} \to \mathbb{R}^{0|1}$ is given by $(\theta, \eta) \mapsto \theta + \eta$. On the coordinates of πTU, you get $\pi TU \times \mathbb{R}^{0|1} \to \pi TU$, given by $(y_k, \hat{y}_k, \eta) \mapsto (y_k + \eta\hat{y}_k, \hat{y}_k)$. $X_i + \theta\hat{X}_i \mapsto X_i + (\theta + \eta)\hat{X}_i = X_i + \theta X_i + \eta\hat{X}_i$. y_k is just X_i, which goes to $X_i + \theta 0 \mapsto X_i + (\theta + \eta)0 \mapsto X_i$. if y_k is $\hat{X}_i$, then it goes to $\theta\hat{X}_i \mapsto (\theta + \eta)\hat{X}_i \mapsto$[[★★★]].

We have to differentiate this action with respect to η. This gives a vector field D on πTU, which is given by $D(y_k, \hat{y}_k) = (\hat{y}_k, 0)$. So in local coordinates, we get $D = \sum_k \hat{y}_k \partial_{y_k}$.

Claim. *The embedding $i \colon \Omega^* M \hookrightarrow C^\infty(\pi TM)$ is given by taking $y_k \mapsto y_k$, and $dy_k \mapsto \hat{y}_k$. These two determine the map i.*

From this we can see that D comes from $\sum_{k=1}^{p+q} dy_k \partial_{y_k}$, which is just the de Rham d. □

Maybe I shouldn't start this again on Tuesday. Let's call this done. Part (b) is way easier, so I'll leave it as an exercise (I don't want to make it a homework).

13 PT 10-09

Last week we proved that $\underline{\mathsf{SMan}}(\mathbb{R}^{0|1}, M) \cong \pi TM$. This implies that $\underline{\mathrm{Aut}}(\mathbb{R}^{0|1})$ ats on πTM. This leads to a $\mathbb{Z}$-grading on $C^\infty(\pi TM) \supseteq \Omega^* M$, with $d^2 = 0$ and d of degree 1. So the easiest super point leads to the de Rham differential.

Project 7: $\underline{\mathsf{SMan}}(\mathbb{R}^{0|2}, M)$ leads to the structure of differential gorms on M. There is a paper to read. These gorms have more structure because the automorphism group of $\mathbb{R}^{0|2}$ is bigger. (Research project: keep going ... structure on $\underline{\mathsf{SMan}}(\mathbb{R}^{0|q}, M)$)

On the homework: nobody proved Batchelor's theorem. You have a super manifold $M^{m|n}$, and you functorially construct $Gr(M) := \pi E_M$, where $E_M \to M_{red}$ is an ordinary vector bundle over M_{red}, given by $\Gamma(E_M) = J_M/J_M^2$, where J_M is the ideal of nilpotents (the ideal generated by the odd elements). Notice that $C^\infty(Gr(M)) = \Gamma(\bigwedge^*(J_M/J_M^2)) = \bigoplus_{k=0}^n \bigwedge^k(J_M/J_M^2)$, but $\bigwedge^k(J_M/J_M^2) \cong J_M^k/J_M^{k+1}$ canonically, given by $a_1 \wedge \cdots a_k \mapsto asym(a_1 \cdots a_k)$. A theorem: there is an isomorphism $M \xrightarrow{\phi} Gr(M)$ such that $Gr(M) \xrightarrow{Gr(\phi)} Gr(Gr(M)) = Gr(M)$ is the identity map. These will be much easier to glue (actually, extend).

We were tied up for the last weeks trying to make super manifolds precise. Now let's loosen up and try to describe where we are and how it is related to Kolya's class. Today will be a kind of a survey of the rest of the class.

Classical Field Theory

Data:

(1) a spacetime M (of dimension d, if this is a d-dimensional field theory), which is usually a Lorentz manifold (has a non-degenerate metric with one negative). This is often too hard, so you do a "Wick rotation" to turn it in to a Riemannian metric (this is called "Euclidean field theory").

(2) Fields $\Phi(M)$. One example is scalar fields (spin 0), in which case $\Phi(M) = C^\infty(M; V)$ (linear case: V a vector space), or $C^\infty(M; X)$ (the nonlinear case, X is the "target"). Another example is guage fields (spin 1), where a field is a G-principal bundle with a connection. In the easiest case (where you have a line bundle, like in electrodynamics), you have $\Omega^1 M/d\Omega^0 M$. There are p-form fields where the fields are p-forms. Another research project could be "noncommutative p-form fields" for which you'd have to go into gerbes and their connections. Another example of fields is gravitational fields (spin 2), where fields are metrics on M. These integral spin fields will correspond to bosons. There are also spinor fields (spin 1/2), which are sections of the spinor bundle. The last thing is spin 3/2.

(3) Classical action $A \colon \Phi(M) \to \mathbb{R}$ (or $\mathbb{C}$), which describes the physics on the fields.

Remark 13.1. The structure on M (and on the bundle over it) is chosen such that A can be defined. ◇

The examples Kolya studied:

- Classical mechanics ($d = 1$), where $M = \mathbb{R}$, and $\Phi(M) = C^\infty(M; N)$ ($N = X$ is configuration space). M is also sometimes called the world-line. AJ: world-line usually refers to the embedding, so a field would be a world line. If you take $N = V$ a vector space with the simplest possible action (the action from classical mechanics), you call this the *linear σ-model*; if N is something else, it is the *non-linear σ-model.* $A = \int_M L(\gamma)$, where $L(\gamma) = |\dot\gamma|_g^2$, so you need a metric g on N; you also need a measure on M so you can integrate. You could add a potential. This is special because the lagrangian only depends on the 1-jet.

- Classical string theory ($d = 2$), where M is a surface (the world-sheet)

- Electromagnetism (Yang-Mills).

Classically, you get a space of classical solutions $\mathcal{M}$, which is a symplectic manifold, defines as solutions to the Euler-Lagrange equations for the given action A (with respect to variations vanishing near boundary).

If you have $M = [a, b]$, and $A_{ab} \colon \Phi([a, b]) \to \mathbb{R}$ is the 1-form $dA_{ab}(\phi, \delta\phi) = \int_a^b \big(\frac{\partial L}{\partial \phi}\delta\phi dt\big) + \alpha(\phi_b, \delta\phi_b) - \alpha(\phi_a, \delta\phi_a)$ (the "bulk term" and the "boundary terms"; here we do not require any vanishing on $\delta\phi$). Classical solutions are the ones for which the bulk term vanishes. If we restrict to classical solutions $\phi \in \mathcal{M}$, we get that $dA_{ab} = \alpha_b - \alpha_a$. This implies that $\omega = d\alpha_b = d\alpha_a \in \Omega^2\mathcal{M}$ is well defined. In good cases, this is a symplectic form.

Now I want to get to what data are associated to the boundary itself. If we choose a boundary point (a or b in this case), it gives in addition this 1-form α_a (or α_b). This is the kind of input you need if you want to do geometric quantization (here you take the trivial line bundle, and $d + \alpha_a$ is the connection; the non-trivial line bundles come from the case where your action is $\mathbb{R}/\mathbb{Z}$-valued).

[[break]]

Example 13.2. Kinetic energy only. $L(\phi)(t) = |\dot{\phi}(t)|_g^2$ so $A(\phi) = \int_{\mathbb{R}} |\dot{\phi}(t)|^2 dt$ for $\phi\colon \mathbb{R} \to (N, g)$. Then $\mathcal{M}$ is the space of geodesics in (N, g). You could identify these with TN. You could also use the metric to identify this with T^*N. Then this 1-form α_0 becomes the canonical 1-form α on T^*N (and α_t is given by pushing forward by the flow). ◇

In general, the dimension of $\mathcal{M}$ is $2r \cdot \dim N$ if L depends only on the r-jets of the fields. In general, you won't get the cotangent bundle, but you will get a symplectic manifold under good conditions.

Time translation gives a vector field ξ on $\mathcal{M}$. This leads to the Hamiltonian function $h\colon \mathcal{M} \to \mathbb{R}$ by $h(\phi) := i_\xi \alpha_t(\phi) - L_t(\phi)$. Independence of t: $i_\xi \alpha_a - i_\xi \alpha_b = i_\xi A_{ab} = \xi(A_{ab}) = L_b - L_a$ by the fundamental theorem of calculus. What we're using is the extra structure that once we fix a time t, we get his 1-form, and for different ts, the difference of the two 1-forms is d of a function.

Qunatization

We have $(M, \Phi(M), A)$. There are different opinions about what a quantization should be. In Kolya's class, we quantize the classical observables (the algebra $C^\infty(M)$) to some $\mathcal{A}_h$ (sometimes you can't set h to Planck's constant). Then we study representations $\mathcal{A}_h \to B(H)$.

I want to go a slightly different route, where we get the state space H directly. The states (positive linear operators on H) only see the projective space on H, and physically, you only expect this much (since scaling a state doesn't change it). We get H by choosing a *polarization.* If you make this extra choice, then you actually get a Hilbert space, and you have to discuss how the Hilbert space changes when you change your choice. In the cases we'll do, it will only change by scaling, so in the end of the day, we'll get a projective space. Good choices of polarization usually lead to isomorphic "Hilbert spaces" (they'll actually be Frechét spaces) and the isomorphism will be canonical up to phase. Moreover, we want operators on H (e.g. we want to quantize functions, like the Hamiltonian h). One way to get these is to go through the deformation quantization (where the operators come first, and then you take a representation). The other way is to choose a polarization and use the path integral approach, but I don't want to start this today. We'll do it Thursday.

If you start with a classical field theory $(M, \Phi(M), A)$, then you can take the classical solutions to pick up $(\mathcal{M}, \omega, \alpha_t, A_{ab})$. Kolya explained how to think of these data (as a "Hamiltonian field theory"): if $\dim M = d$, then you associate to ∂M a symplectic manifold $\mathcal{M}$ (with $\omega = d\alpha$) and to M you associate a Lagrangian submanifold $L \subseteq \mathcal{M}$ (so that $\alpha|_L = dA$). Then you can do geometric quantization to get a Hilbert space and a Hamiltonian operator. Or you could go through deformation quantization and pick a representation to get this stuff. It is a little silly to only look at the classical solutions and then try to quantize. It would be better to look at the whole field theory and then try to quantize. This was Feynman's idea: particles don't travel along the classical solution, but there is some probability which peaks at the classical solution (but is non-trivial outside of the classical solutions).

Supersymmetric (Susy) classical field theories

We're trying to change the data to something that makes sense for super manifolds. We need

- super spacetime $M^{p|q}$ (maximal susy in physics literature is $M^4|16$; here we'll only go up to $M^{2|1}$, which is all you need for elliptic cohomology). There will be something like a metric g, which we'll get by trying to define a classical action.
- space of fields $\Phi(M)$, which you can do by adding the word super a few times to everything we did. We'll concentrate on the scalar field case where $\Phi(M) = \mathsf{SMan}(M, X)$ "Susy σ-model with target X".
- classical action $A\colon \Phi(M) \to \mathbb{R}$. Here we run into trouble. The integral is ok (you integrate sections of the Berezinian line bundle).

> To get dA_{ab}, we differentiated the classical action. For this, we needed some kind of smooth structure on $\mathsf{Man}([a,b],N)$. What we really want, therefore, is $\Phi(M) = \underline{\mathsf{SMan}}(M,X)$, which is not a finite-dimensional super manifold (unless M or X is zero dimensional). We understand this in terms of the functor of points. For every super manifold S, we have $A_S\colon \underline{\mathsf{SMan}}(M,X)(S) = \mathsf{SMan}(S\times M, X) \to \mathbb{R}(S) = C^\infty(S)^e$, and this should be natural in S.

These are the theories that we'll quantize via path integrals.

14 PT 10-11

Classical field theory consists of the data

- space time Σ^d,
- Fields $\Phi(\Sigma)$, and
- a classical action $A\colon \Phi(\Sigma) \to \mathbb{R}$.

The easiest case (so far), which we saw in NR's class, is $d = 1$, with $\Sigma = [a,b]$, $\Phi(\Sigma) = \mathcal{C}^\infty(\Sigma, N)$ for some configuration space N, and action $A(\phi) = \int_\Sigma |\dot\phi(t)|^2\, dt$ (using some Riemannin metrics g on N and something on Σ). Take $N = \mathbb{R}$. We want to quantize thisby taking the hilbert space $\mathcal{H} = L^2(N, vol_g)$. The Hamiltonian operator is $H = -\Delta_g$. In the case $N = \mathbb{R}$, $H = -\partial_x^2$. I want to explain how to get this from path integrals instead of deformation quantization. This case is actually very precise; the measure in question actually do exist (it is called the Weiner(?) measure).

Quantum mechanical evolution on $\mathcal{H}$ is given by e^{itH}. In other words, if you know H, you can apply this operator to a state to determine how the state evolves in time. This is one way to solve the Schrödinger equation. That is, solutions of the Schrödinger equation are $e^{itH}(\psi)$.

I want to do the Euclidean version of this, which is the heat equation. I want to study the operator e^{-tH}. We see that this is the heat equation because $\frac{d}{dt}e^{-tH}(\phi) = -H(e^{-tH}\phi) = \partial_x^2(e^{-tH}\phi)$, so $e^{-tH}\phi$ is a solution to the heat equation. If ϕ gives the distribution of heat on $\mathbb{R}$ at time 0, then $e^{-tH}\phi$ tells you the distribution of heat at time t. This is the easiest way to solve the heat equation if you happen to know how to write down the operator e^{-tH}. In particular, if we put a unit of heat at $y \in \mathbb{R}$ (roughly, "$\phi = \delta_y$", in quotes because $\delta_y \notin L^2$), then we get the "integral kernel" of e^{-tH}.

Because H is self-adjoint, you can write an eigenspace decomposition of $\mathcal{H}$. e^{itH} will have the same eigenspaces, but with eigenvalues $e^{it\lambda}$. Even though the operator H is unbounded, you can check that e^{-tH} is bounded (because of the minus sign).

Let me explain the notion of integral kernels. Let's say I want an operator $O_k\colon L^2N_1 \to L^2N_2$. I claim it can be described by its integral kernels (if it has them) $k \in C^0(N_1 \times N_2)$. Think of the

k as matrix coefficients and O as a linear operator. We have that $(O_k f)(n_2) := \int_{N_1} k(n_2, n_1) f(n_1)\, dn_1$. It turns out that if k is continuous, then this will always be a compact operator (assuming N_1 and N_2 are compact). In the non-compact setting, if $k \in L^2$, you get Hilbert-Schmidt operators, and some other things in general. Can you imaging how these things compose? What is $O_{k_2} \circ O_{k_1}$, where $k_1 \in C^0(N_1 \times N_2)$, $k_2 \in C^0(N_2 \times N_3)$. This is just like matrix multiplication: $(O_{k_2} \circ O_{k_1})(f)(n_3) = \int_{N_1 \times N_2} k_2(n_3, n_2) k_1(n_2, n_1) f(n_1)\, dn_1\, dn_2 = (O_{k_3} f)(n_3)$, where $k_3(n_3, n_1) = \int_{N_2} k_2(n_3, n_2) k_1(n_2, n_1)\, dn_2$ (by Fubini's theorem).

These integral kernels are quite convenient. I claim that any good enough compact operator has a kernel. For example, $(e^{-tH})(\phi)(x) = \int_{\mathbb{R}} k_t(x, y) \phi(y)\, dy$ (here, all $N_i = \mathbb{R}$), where k_t is the *heat kernel*. What is the interpretation of k_t? If we plug in $\phi = \delta_{y_0}$ (start with a unit of heat at y_0), then we just get $k_t(x, y_0)$, which is supposed to tell us the amount of heat at time t at the point x. It turns out that $k_t(x, y) = \frac{1}{\sqrt{2\pi t}} e^{\frac{1}{2t} d_g(x,y)^2}$ (for the derivation, take a PDE class).

Now I want to derive why k_t is given by path integrals. Let's write this in terms of a path integral. This is the one case where we can actually do this. Note that we could be doing this on any Riemannian manifold N; we're just using $\mathbb{R}$ to be explicit. We want to write out the matrix coefficients

$$\begin{aligned}
e^{-tH}(x, y) &:= k_t(x, y) \\
&= (e^{-\frac{t}{n}H} \cdots e^{-\frac{t}{n}H})(x, y) \\
&= \overbrace{\int_{\mathbb{R}} \cdots \int_{\mathbb{R}}}^{n-1} k_{t/n}(x, x_1) \cdots k_{t/n}(x_{n-1}, y)\, dx_1 \ldots dx_{n-1} \\
&= \int_{\mathbb{R}^{n-1}} dx_1 \ldots dx_{n-1} \underbrace{\frac{1}{(2\pi t/n)^{n/2}}}_{Z_n(t)} \left(e^{-\frac{1}{2} \sum_{i=1}^{n} \frac{d(x_i, x_{i-1})^2}{t/n}} \right) \\
&= \int \frac{dx_1 \ldots dx_{n-1}}{Z_n(t)} \underbrace{e^{-\frac{1}{2} \int_0^t |\ldots\sigma(t)|^2\, dt}}_{e^{-A(\sigma)}} \\
&= \int_{\substack{\text{pcws linear } \sigma \\ \text{w/ crnrs at } i \cdot t/n}} \frac{d\sigma}{Z_n(t)} e^{-A(\sigma)}
\end{aligned}$$

where $\sigma\colon [0, t] \to \mathbb{R}$ is a piecewise geodesic path connecting $x_0 = x$ and $x_n = y$ (you go by a geodesic from x_i at time it/n to x_{i+1} at time $(i+1)t/n$; this is ok because $\mathbb{R}$ has unique geodesics). Our space time is $\Sigma = [0, t]$ and the fields are $\phi\colon [0, t] \to \mathbb{R}$.

Now we let n go to infinity, and this is always the heat kernel, so the limit exists. We can write this as

$$O_{[0,t]}(x, y) = \text{“} \int_{\substack{\sigma\colon [0,t] \to \mathbb{R} \\ \sigma(0) = x,\ \sigma(t) = y}} \underbrace{\frac{d\sigma}{Z(t)} e^{-A(\sigma)}}_{\text{Wiener measure}} \text{”}$$

You can't just take C^1 paths because they have Wiener measure zero. If you take all continuous paths, then the individual parts don't make sense, but somehow this Wiener measure makese sense. You compute this integral by taking the limit as n goes to infinity (this is not the way Wiener got it, this is Bruce Driver's and somebody's reinterpretation).

Now we can change the minus signs in the exponents to i's, and you get Feynman's interpretation of the Schrödinger equation.

$$e^{itH}(x, y) = \text{“} \int_{\substack{\sigma\colon [0,t] \to \mathbb{R} \\ \sigma(0) = x,\ \sigma(t) = y}} \frac{d\sigma}{Z(t)} e^{iA(\sigma)} \text{”}$$

The idea is that because of the i, you get oscillatory behavior which cancels almost everything out in the integral except for the extrema of the action. Now we got the quantum mechanical evolution without using deformation quantization. We put in the classical action and the magic of the path integral. The data are the classical field theory and the Wiener measure.

[[break]]

Theorem 14.1 (Driver-Anderson)**.** *If (M,g) is a compact Riemannian manifold, then $e^{-t\Delta_g}(x,y) = \lim_{n\to\infty}\int \frac{d\sigma}{Z_n(t)}e^{-A(\sigma)}$, where the integral is over piecewise minimal geodesic paths $\sigma\colon [0,t]\to M$ with $\sigma(0)=x$ and $\sigma(t)=y$ with corners at $i\cdot t/n$.*

Note that both sides make perfect sense. The quotation mark come in when we try to write the right hand side as "$\int \frac{d\sigma}{Z(t)}e^{-A(\sigma)}$", where the integral is over (some) paths $\sigma\colon [0,t]\to M$ with $\sigma(0)=x$ and $\sigma(t)=y$. In the $d=2$ case, nobody can make sense of either side of the theorem. If you know what Wiener measure is, you can remove the quotation marks, but $\mathbb{R}$ is the only case where that works. Before you take the limit, the right hand side of the theorem makes sense, but it is unknown in general if the limit exists.

Now we'll try to formalize this expression in quotation marks.

Path integral quantization of classical field theories

Let Σ be our d-dimensional space time, with $\partial_{in}\Sigma \hookrightarrow \Sigma$ and $\partial_{out}\Sigma \hookrightarrow \Sigma$. We have our classical action $A\colon \Phi(\Sigma)\to\mathbb{R}$, which we can restrict to $\Phi(\partial_{in}\Sigma)$ and $\Phi(\partial_{out}\Sigma)$. When you try to make sense of this restriction, it becomes really important what your spacetime category is. Now we want to quantize.

If Y^{d-1} is one of these $(d-1)$-manifolds (like one of the boundary components), we want to get a Hilbert space $\mathcal{H}_Y$. Before, we associated to the endpoints L^2N, and to the interval, we associated the operator $e^{itH}\colon L^2N\to L^2N$. We also want an operator $O_\Sigma\colon \mathcal{H}_{\partial_{in}\Sigma}\to\mathcal{H}_{\partial_{out}\Sigma}$ for each compact Riemannian manifold Σ.

We define "$\mathcal{H}_Y = L^2(\Phi(Y))$" (doesn't make much sense because $\Phi(Y)$ will not be finite dimensional unless Y is zero dimensional). Now for $\phi_{in}\in\Phi(\partial_{in}\Sigma)$ and $\phi_{out}\in\Phi(\partial_{out}\Sigma)$, we define "$\mathcal{O}_\Sigma(\phi_{in},\phi_{out})$" to be $\int\frac{\mathcal{D}\phi}{Z(\phi)}e^{iA(\phi)}$, where the integral is over all $\phi\in\Phi(\Sigma)$ such that the restriction to $\partial_{in}\Sigma$ is ϕ_{in} and the resetriction to $\partial_{out}\Sigma$ is ϕ_{out}. This "normalized Lebesgue measure" $\mathcal{D}\phi/Z(\phi)$ doesn't really exist.

[[★★★ "HW: check that the functor Q from Riemannian category in dimension d to the category of Hilbert spaces given by $Q(Y)=\mathcal{H}_Y$ and $Q(\Sigma)=O_\Sigma$ is a symmetric monoidal functor."]] [[★★★ it isn't completely clear to me that this is even a functor]]

15 PT 10-16

Be sure to do the second problem on the homework (the one in quotes), because it will be the motivation for most of what we will do in this class. This was the problem of showing that the path integral, if it made sense, would define a symmetric monoidal functor. By the way, the symmetric monoidal structure on Hilbert spaces is given by taking the usual algebraic tensor product, with the pairing $\langle v_1 \otimes v_2, w_1 \otimes w_2 \rangle = \langle v_1, w_1 \rangle \langle v_2, w_2 \rangle$. What is the difference between this tensor product and the projective tensor product (if the hilbert space is a Frechét space)? All of these lie in the space of continuous linear operators from H_1 to H_2. The relationship is $H_1^* \otimes_{Hilb} H_2 \cong HS(H_1, H_2)$ (Hilbert-Schmidt operators) which contains $H_1^* \otimes_{proj} H_2 \cong TC(H_1, H_2)$ (trace class operators) which contains $H_1^* \otimes_{alg} H_2 = FR(H_1, H_2)$ (finite rank operators).

Kolya talked about trace class operators. A product of two Hilbert-Schmidt operators is a trace class operator. $HS(H_1, H_2)$ is supposed to be a Hilbert space, so we have to define $\langle A, B \rangle_{HS}$, which we define as $\operatorname{tr}(A^* B)$ (trace of an endomorphism of H_2).

What about the d-dimensional bordism category? The Atiyah-Segal definition of a d-dimensional quantum field theory is exactly a symmetric monoidal functor from the bordism category to the category of Hilbert spaces. The various flavors, like TQFTs, CFTs, etc. come from differences in the precise definition of the bordism category (smooth, conformal, Riemannian manifolds make up the bordism category). Kolya did the special case of TQFTs where you only use topological manifolds.

Easiest version of a d-dimensional bordism category (as in Kolya's class): let objects be closed (topological) $(d-1)$-manifolds[1] (note that the objects don't form a set) and let $\operatorname{Hom}(Y_1, Y_2)$ be the set of triples $(Y_2 \overset{i_2}{\hookrightarrow} \Sigma^d \overset{i_1}{\hookleftarrow} Y_1)$ where $\partial\Sigma = i_1 Y_1 \sqcup i_2 Y_2$ and Σ is a compact manifold, up to homeomorphism relative boundary. This makes the morphisms into a set, and the composition is what you think (just gluing works because we are working with topological manifolds). What should we do about the identity? Note that the identities are already in there. This will be the wrong answer in the Riemannian category, where we'll basically have to throw in identities by hand. Andy: so all the different choices of i_1 and i_2 give you different morphisms? Chris: if i_1 and i_1' are pseudo-isotopic (in the same concordence class), then they will give you the same morphism set. Another Chris: well, you really want Σ-pseudo-isotopic; that is, pseudo-isotopic using a cylinder. [[★★★ I don't think I understand what pseudo-isotopic means]] PT: so to answer Andy's question, if you have different parameterizations, then they're different if they're different using the given equivalence relation.

If $d = 2$, then we're interested in $Homeo(S^1)/\text{isotopy}$, which is $\mathbb{Z}/2$. If $d = 3$, then we're interested in $Homeo(\Sigma^2)/\text{isotopy}$, which is the mapping class group of Σ. If $\Sigma = \mathbb{T}^2$, then this group is $GL_2(\mathbb{Z})$. The action on H_1 for an arbitrary Σ gives you a map to $Sp(2g, \mathbb{Z})$ (g is the genus of Σ).

If you try to put a smooth structure on top of everything, do you get a problem with gluing along the boundary? You have to use collars around the boundary (you prove that the boundary has a collar by integrating an inward-pointing vector field). Q: Actually, you needed collars on topological manifolds to get identities in the topological bordism category. PT: that's right; good point. You can glue the interiors of the collars to get a smooth structure. Note that we had to make some choices, but different choices are in the same equivalence class, so the composition is well defined (using diffeomorphisms instead of homeomorphisms). This defines the smooth bordism category.

We want a Riemannian bordism category. We could try the same trick. We could assume there is a collar where the metric is a product metric, and people sometimes do, but we don't want to do that (it throws out a lot of manifolds). The categories we've seen so far have a weird symmetry to them (you can reverse a bordism).

[[break]]

We can start with a Riemannian manifold M without boundary (but possibly non-complete), and metrically complete it (adding a boundary) to get $\widehat{M}$. An isometry $\phi\colon M_1 \to M_2$ induces $\hat{\phi}\colon \widehat{M_1} \to \widehat{M_2}$.

Definition 15.1 (Stolz, Teichner)**.** Riem_d is the category with objects Riemannian d-manifolds without boundary (possibly non-complete, non-compact) together with a decomposition $\widehat{M} \smallsetminus M = \partial_{out} M \sqcup \partial_{in} M$ (as *sets*), and morphisms $\mathsf{Riem}_d(M_1, M_2) = \{\text{isometries } \phi\colon M_1 \to M_2 | \phi(\partial_{in} M_1) = \partial_{in} M_2, \phi(\partial_{out} M_1) = \partial_{out} M_2\}$. ◇

---

[1] With an orientation, spin structure, graph drawn on the manifolds, or whatever extra structure you like.

Note that this is a symmetric monoidal category under disjoint union (rather *distant union*).

Example 15.2 ($d = 1$)**.** We have manifolds that look like (a, b) (with $\partial_{out} = a$, $\partial_{in} = b$), (a, ∞) (with $\partial_{out} = a$, $\partial_in = \varnothing$), $(-\infty, b)$, and $(-\infty, \infty)$. Note that the inclusion $(a, b) \hookrightarrow (a, \infty)$ is not a morphism. By the way, if I decided that $\partial_{in}(a, b) = \{a, b\}$, then I would draw it as . We could take the manifold $M = (a, b) \smallsetminus \{c\}$ with some choices of ∂_{in} and ∂_{out}. No, this is bad because the metric completion of this has two extra points in the middle, not one. As a manifold with metric tensor, it is the disjoint union of two open intervals.

Chris: do you require the in and out parts to consist of whole connected components? PT: not, it is just a disjoint union as sets. Chris: so you could have an open disk, with the boundary broken up in a really nasty way, like a Cantor set and the complement. PT: yes, I guess I'm allowing that for now. ◇

Definition 15.3. The *Riemannian bordiam category* RB_d has objects $\{Y \in \mathsf{Riem}_d | \partial_{out} Y \text{ is a closed } (d-1)\text{-manifold}\}/(\text{germs towards } \partial_{in})$, where germs toward ∂_{in} is the equivalence relation generated by saying that $Y_1 \sim Y_2$ if $Y_1 \overset{i}{\hookrightarrow} Y_2$ (isometrically) with $\hat{\imath}\colon \partial_{out} Y_1 \to \partial_{out} Y_2$ an isomorphism. ◇

I'll do the morphisms on Thursday. The punchline is that we get an asymmetry of the bordism category (which we want) because there is a little germ hanging off one end of the morphism, so you can't reverse it.

16 PT 10-18

Homework question: what is the "Hilbert space"? You have a classical field theory $\Phi(\Sigma^d) \xrightarrow{A} \mathbb{R}$, with restriction maps $\Phi(\Sigma) \to \Phi(\partial_{in/out}\Sigma)$. A quantum field theory would be a functor Q from the bordism category to hilbert spaces. You take $Q(Y^{d-1}) =$"$L^2(\Phi(Y))$".

I'm going to start over again with the Riemannian category Riem_d of Riemannian d-manifolds.

Definition 16.1. The objects are Riemannian d-manifolds (M^d, g) without boundary with finitely many connected components M_i and finitely many ends[1] such that $\widehat{M} := \sqcup \widehat{M_i}$, the metric completion of M (completing the connected components), is compact. And together with the decomposition $\delta M := \widehat{M} \smallsetminus M = \delta_{perm} M \sqcup \delta_{germ} M$ (perm for permanent) so that $\delta_p M$ is closed. Note that the boundary of $\widehat{M}$ will be contained in δM. Note also that $\widehat{M}$ need not be a manifold (for example, take M to be the open cone on a torus; then $\widehat{M}$ adds in the point of the cone, which doesn't have a neighborhood that looks like $\mathbb{R}^n$). The morphisms are $\mathsf{Riem}_d(M_1, M_2) = Isom(M_1, M_2)$, isometric embeddings (these don't have to preserve the decomposition of δM_1). ◇

Remark 16.2. Such an embedding induces a map $\widehat{M_1} \to \widehat{M_2}$, but this map doesn't have to have any nice properties (like injectivity). ◇

Definition 16.3. The objects of RB_d are objects Y of Riem_d such that $Y \cup \partial_p Y$ is a (topological) manifold with boundary $\delta_p Y$ (note that $\delta_p Y$ is a closed $(d-1)$-manifold, topologically), modulo a germ equivalence relation generated by $\sim$ saying $Y_1 \sim Y_2$ if $Y_1 \subseteq Y_2$ (open isometric inclusion) such that $\delta_p Y_1 = \delta_p Y_2$. We denote the equaivalence class by $[Y]$.. $\mathsf{RB}_d([Y_1], [Y_2]) = \{(\Sigma, [i_1], [i_2]) | \Sigma \in \mathsf{Riem}_d, i_k \in \mathsf{Riem}_d(Y_k, \Sigma)^2$ such that i_2 induces homeomorphism $\delta_p Y_2 \cong \delta_p \Sigma$ and i_1 induces a homeomorphism $\hat{i}_1\colon \delta_g Y_1 \to \delta_g \Sigma$ for some representative $Y_1\}$/germ equivalence of

---

[1] The *ends* of a space is $Ends(X) = \varprojlim_{K \subseteq X} \pi_0(X \smallsetminus K)$ where the limit is over compact subsets of X. For example, if you have $\mathbb{R}^2$ minus some points, each deleted point is an end (and one end "at infinity").

[2] For some choice of representative Y_k in $[Y_k]$.

($Y_1 \overset{i_1}{\hookrightarrow} \Sigma$). You also mod out by isometries rel boundary to get a category (actually, I want to make a bicategory where the 2-morphisms are isometries). ◇

Chris will lecture next Wednesday at 2 on bicategories.

Maybe next time I'll give you an equivalent definition without germs.

17 PT 10-23

What Kolya called spacetime categories, I'll call bordism categories. Let B_d be the category whose objects are closed $(d-1)$-manifolds, with $\mathsf{B}_d(Y_{in}, Y_{out}) = \{Y_{out} \xhookrightarrow{j_{out}} \Sigma \xhookleftarrow{j_{in}} Y_{in} | \Sigma \text{ compact}, \partial\Sigma = \operatorname{im} j_{in} \sqcup \operatorname{im} j_{out}\}/\text{equivalence}$. Really, you shouldn't mod out by equivalence. $\mathsf{B}_d(Y_{in}, Y_{out})$ is really a category with objects $(Y_{out} \xhookrightarrow{j_{out}} \Sigma \xhookleftarrow{j_{in}} Y_{in})$ and morphisms are isomorphisms $\Sigma \to \Sigma'$ respecting the embeddings $j_{in/out}$. Thus, B_d should really be thought of as a bicategory. A bicategory $\mathcal{C}$ has a class of objects, hom *categories* $\mathcal{C}(x, y)$ for each pair of objects $x, y \in \mathcal{C}$, and composition functors $\mathcal{C}(y, z) \times \mathcal{C}(x, y) \to \mathcal{C}(x, z)$ (called *horizontal composition*). Furthermore, we reqire an associator natural transformation satisfying the pentagon identity. Note that the composition in $\mathcal{C}(x, y)$ is associative (because it is a category); this composition is called the *vertical composition.* Furthermore, there is some $\mathrm{id}_x \colon pt \to \mathcal{C}(x, x)$, which are *weak identities.*

Definition 17.1. A *2-category* is a category enriched over Cat. That is, a 2-category is where the composition functors are associative on the nose. ◇

Mac Lane proved that any bicategory is equivalent (as a bicategory) to a 2-category, but this rigidification is unnatural (there are more functors between bicategories than between 2-categories).

Let me tell you the horizontal composition $\mathsf{B}(Y_2, Y_3) \times \mathsf{B}(Y_1, Y_2) \to \mathsf{B}(Y_1, Y_3)$. If $Y_3 \xleftarrow{\Sigma'} Y_2 \xleftarrow{\Sigma} Y_1$, then the horizontal composition is the pushout $\Sigma \cup_{Y_2} \Sigma' \colon Y_1 \to Y_3$. Since union of sets is not associative, this composition is not associative. Note that this gluing doesn't work in the Riemannian category. Even in the smooth category, you need to use collars. This is ok in the topological category.

Claim. *This pushout defines a composition in* RB_d.

Let me first give an equivalent version of RB_d, with no germs. Recall that Riem_d has objects Riemannian d-manifolds without boundary Y^d with three tameness conditions:

- finitely many ends,

- finitely many components, [[★★★ follows from next condition]] and
- $\widehat{Y}$ is compact,

together with the decomposition $\widehat{Y} \smallsetminus Y = \delta_p Y \sqcup \delta_g Y$. Then $\mathsf{Riem}(Y_1, Y_2)$ consists of isometric embeddings $Y_1 \hookrightarrow Y_2$.

RB_d has objects $Y \in \mathsf{Riem}_d$ such that $Y \cup \delta_p Y$ is a topological d-manifold with boundary $\delta_p Y$. Last time we used a germy equivalence relation, but we aren't doing that this time. $\mathsf{RB}_d(Y_{in}, Y_{out}) = \{Y^1_{out} \xhookrightarrow{j_{out}} \Sigma \xhookleftarrow{j_{in}} Y^1_{in} | \Sigma \in \mathsf{Riem}_d, Y^1_{in/out}$ an open subset of $Y_{in/out}$ such that

- $\delta_p Y^1_{out} \cong \delta_p Y_{out}$,
- $\hat{j}_{out}\colon \delta_p Y^1_{out} \xrightarrow{\sim} \delta_p \Sigma$, $\hat{j}_{in}\colon \delta_g Y^1_{in} \xrightarrow{\sim} \delta_g \Sigma\}$.

Note that the germ boundary of Y^1_{in} doesn't have to match the germ boundary of Y_{in}. Given $Y \in \mathsf{RB}_d$, $\mathrm{id}_Y \in \mathsf{RB}_d(Y,Y)$ is $Y \hookrightarrow Y \hookleftarrow Y$. Given another morphism $Y^2_{out} \hookrightarrow \Sigma' \hookleftarrow Y^2_{in}$, a 2-morphism $\Sigma \to \Sigma'$ is an isometry $\Sigma \to \Sigma'$ such that *there exist* $Y^3_{in/out} \subseteq Y^1_{in/out}, Y^2_{in/out}$ so that $\Sigma \to \Sigma'$ respects the inclusions of $Y^3_{in/out}$. Composition is just the usual composition.

I still have to define the (horizontal) composition. Say we have $Y_3 \hookrightarrow \Sigma \hookleftarrow Y_2 \hookrightarrow \Sigma' \hookleftarrow Y_1$, then we can glue Σ and Σ' along Y^3_2 (intersecting the representatives Y^1_2 and Y^2_2).

Variations on the theme: You can't just choose Y^3_2, you have to take it ot be the intersection $Y^1_2 \cap Y^2_2$. Another point: for 2-morphisms, you have to use $\Sigma' \smallsetminus (Y^2_{in} \smallsetminus Y^3_{in})$ instead of Σ'.

There is a bit of a subtlety with the gluing. Consider two copies of $\mathbb{R}^1$ glued together along $(0,1)$. This is still locally $\mathbb{R}^1$, but the result is non-Hausdorff (the two 0's and the two 1's are indistinguishable). We can remove $(-\infty, 0)$ from the first $\mathbb{R}$ and $[1,\infty)$ from the other $\mathbb{R}$, and then the problem is gone.

More generally, if you have embeddings $U \xhookrightarrow{i_1} M_1$ and $U \xhookrightarrow{i_2} M_2$ such that $i_1 \times i_2 \colon U \to M_1 \times M_2$ is proper, then $M_1 \cup_U M_2$ is Hausdorff. [[★★★ HW. You probably only need that U, M_1, and M_2 are Hausdorff.]]

Now we have to check properness in the horizontal composition in RB_d, which I claim follows from our conditions (this is where you use that $\widehat{Y}$ is compact). [[★★★ HW: show that if $Y^1 \subseteq Y$ open, then $Y_1 \cong Y$ in RB_d]]

18 PT 10-25

Definition 18.1 (Reminder Chris' talk)**.**

(a) A *(strict) 2-category* is a category enriched over $(\mathsf{Cat}, \times)$.

(b) A *bicategory* is a "weak" version of a 2-category where one only has canonical "assiciators" and "identitators". ◇

This means that in a 2-category $\mathcal{C}$, we have

- objects $x, y \in obj(\mathcal{C})$,
- morphism categories $\mathcal{C}(x,y)$,
- composition functors $\mathcal{C}(y,z) \times \mathcal{C}(x,y) \to \mathcal{C}(x,z)$, and
- identity functors $pt \to \mathcal{C}(x,x)$ for each $x \in \mathcal{C}$

such that the identity is an identity and the composition is associative. In a bicategory, instead of imposing these conditions on the identity and the composition, we add the extra data of the associators and identitators. The associator must satisfy the pentagon identity, which (by Mac Lane's theorem) tells you that any two ways you associate are canonically isomorphic. There is a similar thing for the identitators; they satisfy the triangle identity (which also tells you that anything you want to do is canonical).

Example 18.2. Alg is a bicategory, where the objects, 1-morphisms, and 2-morphisms are algebras, bimodules and intertwiners, respectively. Tensor product gives you the horizontal composition, and the universal property of tensor product gives you the associators and identitators.

There are also more subtle versions of this. For example, you could take Frechét algebras and Frechét bimodules. ◇

A QFT will be a bifunctor from a geometric bicategory to this algebraic bicategory.

Example 18.3. B_d, whose objects, 1-morphisms, and 2-morphisms are closed (topological) $(d-1)$-manifolds, compact d-manifolds (with a decomposition of the boundary into inclusions of incoming and outgoing

$(d-1)$-manifolds), and homeomorphisms of compact d-manifolds respecting the inclusions.

RB_d has a forgetful bifunctor F to B_d. The objects, 1-morphisms, and 2-morphisms of RB_d are germs of Riemannian structures around $(d-1)$-manifolds Y, agreeable bordisms Σ between them (with some germiness at the in-boundary), and isometries. $F(Y) = \delta_{perm}Y$, $F(\Sigma) = \Sigma_{core}$ (where you've removed the germiness) and F of an isometry is the isometry restricted to the cores.

Is this F a 2-functor? Well, Chris hasn't discussed 2-functors yet, so we won't get into it too much. It is weaker than you might think it is. A 2-functor doesn't have to respect (horizontal) composition on the nose, but only up to natural isomorphism.

Theo: This functor doesn't send identities anywhere. PT: If you add homeomorphisms to the bordisms in B_d, then it becomes ok. You have to make sure that you still have a bicategory; you have to know how to compose these homeomorphisms with bordisms. You do this by composing the embedding with the homeomorphism (or its inverse, depending on which side you're composing). It turns out that this bigger B_d is equivalent. Chris S-P: no, now you have some more automorphisms . . . [[★★★ some stuff about the cylinder not playing like it should]]. ◇

I want to give you some examples of 1-morphisms in RB_d.

Assume we have a geometric bi-collar Σ of $Y = \delta_p\Sigma$ [[★★★ Σ is a cylinder, with the whole boundary declared to be germ boundary]]. Let's cut it in half (along Y) and call the (open) two sides Y_R and Y_L. We have that $F(Y_R) = F(Y_L) = Y$. I claim that we can make a nice 1-morphism out of this. You can think of it as living in $\mathsf{RB}_d(Y_L \sqcup Y_R, \varnothing)$ by letting $j_{in}\colon Y_L \sqcup Y_R \hookrightarrow \Sigma$ be the inclusion.

By the way, I'm going to give up the right-to-left notation because it gets us confused; you just have to label the boundary components as in or out. In RB_d, we'll draw collars on the outside for the in boundary and collars on the inside for the out boundary.

Can you think of this as an element of $\mathsf{RB}_d(Y_L, Y_R)$? Can you think of this as an element of $\mathsf{RB}_d(\varnothing Y_L \sqcup Y_R)$? No. $\delta_p\Sigma$ is empty, so the target must be the empty set.

Analogy: in Frechét spaces, it is much easier to get bilinear maps $V \otimes W \to \mathbb{C}$ than to get vectors in $V \otimes W$. For example, take $W = V'$; you get a canonical pairing, but you only get a canonical vector if V happens to be finite dimensional. This kind of corresponds to the fact that in B_d, you can flip bordisms around to get a "reversed morphism".

Given a monoidal bifunctor $Q\colon \mathsf{RB}_d \to \mathsf{Frechét}$, we get two Frechét spaces $Q(Y_L) = V_L$ and $Q(Y_R) = V_R$, and a pairing $Q(\Sigma)\colon V_L \otimes V_R \to \mathbb{C}$. In TQFTs, this morphism doesn't exist, so it is sometimes added as an axiom that the two guys are dual.

[[break]]

A couple of remarks. If you apply F to this Σ, you're going to get nonsense: $F(\Sigma) = Y$, which is supposed to have an inclusion from $Y \sqcup Y$. This doesn't make sense, but I invitet you to make a bigger category where this works. Chris convinced me that the enlarged B_d is not equivalent to the smaller B_d because the cylinder is not the identity on the homeomorphisms. Another thing you can do is use the axiom of choice to choose slightly larger cores for all bordisms (so in our case, the core of Σ would be a cylinder on Y, not Y). Then you don't have to throw in homeomorphisms either.

The other thing I should announce is that Bruce was volunteered to talk about how path integrals connect the three different classes next week.

I think of a quantum field theory as a representation of the category RB_d. A representation of a group is exactly the same thing as a functor from G (thought of as a 1-object category) to the category of vector spaces.

About HW2: You can enrich B_d by adding a space X (which you think of as the target of some classical field theory) to get $\mathsf{B}_d(X)$, where the objects are continuous maps $f\colon Y^{d-1} \to X$, bordisms have maps to X, and homeomorphisms must respect the maps to X. If you do this for RB_d (you probably want to take smooth maps to X). Another thing you could do is equip Y with a bundle and a section (a crazy way to think of a map $Y \to X$ is to think of it as a section of the trivial bundle $X \times Y \to Y$, but there is no reason to take the trivial bundle). When I wrote $\Phi(\Sigma^d)$ in the homework, this is what I had in mind; Σ was equipped with a bundle $P \to \Sigma$, and the fields $\Phi(\Sigma)$ is the space of sections of this bundle. This part was precise, the imprecise part was that the Hilbert space associated to this Σ was supposed to be $L^2(\Phi(\Sigma))$.

Consider $d = 0$ (this is below mechanics, where $d = 1$. This is called instanton theory because there is no time). What are symmetric monoidal functors $\mathsf{B}_0(X) \to \mathsf{Vect}$. The objects are (-1)-manifolds mapping to X. There is only one (-1)-manifold, which is $\varnothing$.

If you have a functor between two monoidal categories $F\colon (\mathcal{C}, \otimes) \to (\mathcal{D}, \otimes)$, you can require a map $F(x \otimes_{\mathcal{C}} y) \to Fx \otimes_{\mathcal{D}} Fy$, and you usually require it to be an isomorphism (or quasi-isomorphism), but I'll require it to be an equality for this class.

So there is one object of $\mathsf{B}_0(X)$, which is $Q(\varnothing) = Q(\mathbb{1}) = \mathbb{C}$. $\mathsf{B}_0(\varnothing, \varnothing)$ is sets of circles, so $x \in \mathsf{B}_0(X)(\varnothing, \varnothing)$ is a map from a bunch of circles to X, and x get's mapped to some $Q(x) \in \mathbb{C}$.

We need to add requirements on the Q's such that

- Q is smooth. Recall that $H^n_{dR}(X) \overset{HW1}{=} \Omega^n_{cl}(X)/\text{concordence}$. $\Omega^n_{cl}(X) =$ 0-dimensional susy QFTs. $\pi TX =$ super points in X. $C^\infty(X)$, $C^\infty(\pi TX) = \Omega^*(X)$.
- susy Q's leading to functions on super points, which is $\Omega^*(X)$.
- understand closedness and degree.

19 PT 10-30

Today's lecture was given by Bruce Driver.

Today is a bosonic day. I'll start with some finite-dimensional calculation and hopefully get to the point where you'll see the connection with Borcherds' class.

Let $A > 0$ be an $N \times N$ real matrix. The invariant way to do this is not to introduce a matrix at all; just use an inner product, but we've always been using this A. Consider the partition function $Z_A = \int_{\mathbb{R}^N} \exp\bigl(-\frac{1}{2}Ax \cdot x\bigr)\, dx = \sqrt{\det(2\pi A^{-1})}$. Let $d\mu_A(x) = \frac{1}{Z_A}\exp\bigl(-\frac{1}{2}Ax \cdot x\bigr)dx$ be the associated Guassian probability measure. The measure you've been seeing in Borcherds' class is

$$d\mu(\phi) = \frac{1}{Z_A}\exp\Bigl(-\frac{1}{2}\int \bigl((\nabla\phi(x))^2 + m^2\phi(x)^2\bigr)\, dx \qquad (*)$$

which we've been writing as $\mathcal{D}\phi$. To get the operator A, you do integration by parts on this integral. You can rewrite the thing in the exp as $\bigl(-\Delta + m^2)\phi, \phi\bigr)_{L^2(\mathbb{R}^d, dx)}$, so our A is $-\Delta + m^2$.

<u>Integration formulas</u>. Now I'll give you some ways to integrate against this measure. The first one is

$$\int_{\mathbb{R}^N} e^{\lambda \cdot x}\, d\mu_A(x) = e^{\frac{1}{2}A^{-1}\lambda\cdot\lambda} \qquad (1)$$

You do this by completing the square. Define the operator $L = L^A := \sum_{i,j=1}^N A^{-1}_{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}$. If A is the identity matrix, this is the Laplacian. Next, we have, for a positive number t,

$$\int f(x - \sqrt{t}y)\, d\mu_A(y) = (e^{tL/2}f)(x) = u(t, x). \qquad (2)$$

By definition, u solves the heat equation: $\frac{\partial u}{\partial t} = Lu/2$, $u(0, , x) = f(x)$. This measure is completely determined by this, by the way. Again, I won't prove this; you could do it with Fourier transforms. Let me at least do an example.

Example 19.1. Take $f(x) = e^{\lambda \cdot x}$. Compute $Lf = \bigl(\sum_{i,j=1}^N A^{-1}_{ij}\lambda_i\lambda_j\bigr)f = (A^{-1}\lambda \cdot \lambda)f$. Since f is an eigenfunction for L, it is easy to write down $e^{tL/2}f = e^{t(A^{-1}\lambda\cdot\lambda)/2}f$.

On the other side, we have (recall that $d\mu_A(y)$ is invariant under $y \mapsto -y$)

$$\begin{aligned}\int f(x-\sqrt{t}y)d\mu_A(y) &= \int e^{\lambda(x-\sqrt{t}y)}d\mu_A(y)\\ &= \overbrace{e^{\lambda\cdot x}}^{=f}\overbrace{\int e^{\sqrt{t}\lambda)\cdot y}d\mu_A(y)}^{e^{\frac{1}{2}A^{-1}(\sqrt{t}\lambda)\cdot\sqrt{t}\lambda}}\\ &= e^{\frac{t}{2}A^{-1}\lambda\cdot\lambda}f \qquad\diamond\end{aligned}$$

Next, by setting $x = 0$ and $t = 1$ or $\log(2)$, we get

$$\int_{\mathbb{R}^N} f(y)d\mu_A(y) = (e^{L/2}f)(0)$$

This is a good way to compute these integrals on polynomials.

Example 19.2. $\int(\lambda\cdot x)^2 d\mu_A(x) = (e^{L/2}(\lambda\cdot x)^2|_{x=0}$. Since L is nilpotent here, you can just use the power series to get $(I + L/2 + (L/2)^2/2! + \cdots)(\lambda\cdot x)^2$, which at $x = 0$, you get $\frac{L}{2}(\lambda\cdot x)^2|_{x=0} = (A^{-1}\lambda\cdot\lambda)$. $\diamond$

You can do this for any polynomial. The other way to do this is to use formula (1) and differentiating with respect to λ to get new formulas.

If you want to see Feynman diagrams coming out, you can do integration by parts. Suppose we have $\int \partial_v f(x)d\mu_A(x)$, where $v \in \mathbb{R}^N$ and ∂_v is the directional derivative. You can compute this as

$$\begin{aligned}\int \partial_v f(x)\frac{1}{Z_A}e^{-\frac{1}{2}Ax\cdot x}dx &= \int f(x)(Av\cdot x)\frac{1}{Z_A}e^{-\frac{1}{2}Ax\cdot x}dx\\ &= \int (Av\cdot x)f(x)d\mu_A(x)\end{aligned}$$

You're using that some things go to zero fast enough. If you replace f by fg, then we have that the adjoint ∂_v^* with respect to the inner product given by the integral is $-\partial_v + M_{(Av\cdot x)}$. You can write this as

$$M_{(v,x)} = \partial^*_{A^{-1}v} + \partial_{A^{-1}v}.$$

It is this sum of creation and annihilation operators which is interesting.

Example 19.3. Apply the formula to the function 1, giving $\int(v_1\cdot x)\cdots(v_4\cdot x)d\mu_A(x)$, where $v_1\cdot x = \partial^*_{A^{-1}v_1}1$, which is

$$\begin{aligned}&\int \partial_{A^{-1}v_1}[(v_2\cdot x)(v_3\cdot x)(v_4\cdot x)]d\mu_A(x)\\ &\qquad = \int (v_2\cdot A^{-1}v_1)(v_3\cdot x)(v_4\cdot x)d\mu_A(x) + \cdots (2\text{ more terms})\end{aligned}$$

The way to compute this is by drawing a dot for each pairing of four dots, and for each such pairing you have to assiciate the weight $(A^{-1}v_i, v_j)$ when i and j are paired, then multiply the weights together. This simply comes from integration by parts. $\diamond$

Suppose $c\mu(\phi)$ as before, and let $f, g \in L^2(\mathbb{R}^d)$ or $C_c^\infty(\mathbb{R}^d)$. Then, by analogy with the two-vertex formula, we get

$$\int(\phi, f)_{L^2}(\phi, g)_{L^2}d\mu(\phi) = (A^{-1}f, g)_{L^2}.$$

The point of Gaussian measure is the once you know it for two, you can get everything else as products. I really want to stick in delta functions for f and g to compute $\int\phi(x)\phi(y)d\mu(\phi)$, but there are problems.

$(A^{-1}f)(x) = \int\Delta_m(x-y)f(y)dy$. Here, Δ_m is the propagator, which you can write as a function of one variable (usually you need a Greens function). Here are some of the properties of the function Δ_m:

1. $\Delta_m(x) \sim \begin{cases}|x| & d = 1\\ -\log|x| & d = 2\\ |x|^{2-d} & d > 2\end{cases}$ for $x \sim 0$.
2. $\Delta_m(x) \sim e^{-m|x|}$ for $x \gg 1$ independent of dimension.

If we take $f = \delta_x$ and $g = \delta_y$, then we get $\int\phi(x)\phi(y)d\mu(\phi) = \Delta_m(x-y)$.

The only reasonable interpretation of $\int\phi(x)^2 d\mu(\phi)$ is $+\infty$ for $d \geq 2$. If you thought the measure lived on functions, you'd expect this integral to be finite. Fact: there exists a measure μ on some space of distributions (depending on dimension) that does deserve to be thought of as $(*)$.

Let me go over one more theorem in preparation for Richard's class today.

Theorem 19.4. *Let f and g be polynomials on $\mathbb{R}^N$.* $\mu_A(f \cdot g) := \int_{\mathbb{R}^N} f \cdot g \, d\mu_A = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\substack{i_1,\dots i_n=1 \\ j_1,\dots j_n=1}}^{N} A^{-1}_{i_1 j_1} \cdots A^{-1}_{i_n j_n} \mu_A(\partial_{i_1} \cdots \partial_{i_n} f) \cdot \mu_A(\partial_{j_1} \cdots \partial_{j_n} g)$

This is the formula that Richard said a Lagrangian has to satisfy, which he said was hard to describe. think of the i's as x's and the j's as being y's. If you assume the i's and j's have disjoint support, you can make sense of all the $A^{-1}_{i_k j_k}$.

[[break, and back to PT talking]]

We don't have that much time. It would be good to somehow see how the different classes are related. Path integrals are at the heart of all three classes, which is why I asked Bruce to give the analytic side of the story. Are there any questions about how the classes are related? Let me try to say what happens in all three classes.

We're trying to "quantize" a classical field theory. Remember that a classical field theory consists of the data

- Space-time M. In RB's class, he works in flat Minkowski space $\mathbb{R}^{d-1,1}$. In this class, we've been trying to keep this generic; the space-time is a bordism (which we usually draw as surfaces). One of the differences between the classes is that we think of the boundary as very important; that's where you get your Hilbert space. RB keeps looking at compact support things, and you have a distinguished time direction, so you can kind of think of it as a bordism from $\mathbb{R}^{d-1}$ to $\mathbb{R}^{d-1}$. NR talked about mechanics, where the space-time is $[0,t]$, and quantum mechanical evolution is e^{itH}.
- Fields $\Phi(M)$. In RB's class, we think of it as $C_c^{\infty}(\mathbb{R}^d, \mathbb{R})$. These are the things we're going to integrate over. We just learned from Bruce that the Gaussian measure makes these things measure zero, so you have to go into distributions ... you have to enlarge your space of fields. In this class, we haven't specified this yet, but we've been thinking of fields as sections of some bundle over space-time. The easiest case is the σ-model, where the bundle is trivial, so $\Phi(M) = C^{\infty}(M, X)$ for some target space X. In mechanics (NR's class), the fiber in this bundle is a specification of a configuration space N.
- Action $A\colon \Phi(M) \to \mathbb{R}$. In RB, $\Phi(M) = C_c^{\infty}(\mathbb{R}^d, \mathbb{R})$, and the action is given by $\int_{\mathbb{R}^d} (-\Delta + m^2)\phi(x)\, dx$. This $-\Delta + m^2$, which Bruce called A, is a positive operator on $C_c^{\infty}(\mathbb{R}^d, \mathbb{R})$. In this class, we haven't specified any action. We'll do the supersymmetric version of the classical action later. To do that, we'll have to explain the integral over super manifolds using the Berezinian. In NR's class and in mechanics, a field is a path in configuration space, and $A[\gamma] = \int_0^t \big(\dot\gamma(\tau)^2 + V(\gamma(\tau))\big)\, d\tau$.

To get quantum mechanics, you want $Q\colon \mathsf{RB}_1 \to \mathsf{Vect}$. It is enough to evaluate Q on a point to get it on objects; we say $Q(pt)$ is the Hilbert space of states. An interval $[0,t]$ goes to the operator e^{itH}, where H is the Hamiltonian operator. This is the quantum mechanical evolution (the fact that it is an evolution is built into the assumption that Q is a functor).

If $V = 0$, you can quantize by taking $Q(pt) = L^2(N)$, and $H = -\Delta$. Remember that N must have a Riemannian metric, so we have a Laplacian Δ. In this class, we'll study space-times of dimensions $d|1$, with $d = 0, 1, 2$.

If N is a spin manifold, we'll see that if we take space-time to be $M = [0,t] \times \mathbb{R}^{0|1}$, then geometric quantization will give you $Q(pt \times \mathbb{R}^{0|1})$ will be the L^2-spinors on M, and H will be the square of the Dirac operator $\not{\nabla}_N$.

Let's say that you believe that the Hilbert space is right (there is the subtle issue of polarization, telling you why you can just consider functions of position and not momentum). You'd like to know how the system evolves, so you want to understand the operator $e^{it\Delta}$. This operator has a kernel $e^{it\Delta}(x,y)$, which you think of as matrix coefficients. We get that

$$e^{it\Delta}(x,y) = \int_{\substack{\gamma\colon [0,t] \to N \\ \gamma(0)=x, \gamma(t)=y}} \frac{e^{iA(\gamma)}}{Z_\gamma} \mathcal{D}\gamma$$

This is the Feynman-Kac formula. In all the classes, we're trying to compute this integral. The i in the exponent makes this highly oscillatory. If we get rid of it and assume A is positive, then you get rapid decay, and these integrals are quite computable, as Bruce showed us today.

In RB's class, we're struggling with integrating $e^{\lambda \phi(x)^4}$ because it is not Gaussian, which doesn't make sense, so we expand as a power series in λ so that we get terms which are polynomials multiplying the Guassian part, which we know how to do. So when the thing in the exponent is not purely quadratic, you have to do perturbative stuff. If you have such a term in the classical action, you'll run into this problem.

20 PT 11-01

Bruce talks again.

Notation:

- $d\mu_A(x) = \frac{1}{Z_A} e^{-\frac{1}{2}Ax\cdot x}\,dx$, $Z_A = \int_{\mathbb{R}^N} \exp\bigl(-\frac{1}{2}Ax\cdot x\bigr)\,dx$.
- $\int e^{\lambda\cdot x} d\mu_Z(x)$
- $\int f(x) d\mu_A(x) = (e^{L/2}f)(0)$

For a while, we'll take $A = \mathrm{id}$. In general, you define a new inner product $(v,w)_A = (v, Aw)$. The thing that tells you independence of A is that $\sum_{i,j=1}^N A_{ij}^{-1} e_i \otimes e_j = \sum_{k=1}^N u_k \otimes u_k$, where $\{u_k\}$ is an orthonormal basis with respect to $(,)_A$. In general, you interpret $Ax\cdot x$ by $(x,x)_A$. We'll set $\mu = \mu_I$.

Before I go to the infinite-dimensional case, let me go back to the theorem from last time.

Theorem 20.1. *If p and q are polynomials, then $\mu(p\cdot q) := \int_{\mathbb{R}^N} pq\,d\mu = \sum_{n=0}^\infty \frac{1}{n!}\sum_{i_1,\ldots,i_n=1}^N \mu(\partial_{i_1}\cdots\partial_{i_n}p)\cdot\mu(\partial_{i_1}\cdots\partial_{i_n}q)$, where $\partial_i = \frac{\partial}{\partial x_i}$.*

There are no j_k's because we took A to be the identity.

This is more or less what RB is using as his definition of a Feynman measure. Actually, RB was a little misleading by saying that you can't make sense out of these measures. The thing to have in mind that at the end of the day you want to apply this stuff to $d\mu(\phi) = \frac{1}{Z}\exp\bigl(-\frac{1}{2}\int(|\nabla\phi|^2 + m\phi^2)dx\bigr)\mathcal{D}\phi$, where $\phi\colon \mathbb{R}^d \to \mathbb{R}$. Polynomials in $\tilde{\phi}(f) = \int \phi(x)f(x)\,dx$ where $f \in C_c^\infty(\mathbb{R}^d)$. PT: are there two different ϕ's? BD: we're identifying the function ϕ with the distribution it gives you. The f's are like the i's. We've basically already worked out that $\int \phi(f)\phi(g)\,d\mu(\phi) = \bigl((-\Delta+m^2)^{-1}f, g\bigr)$. What RB wants to do is integrate things like "$\phi \mapsto \int f(x)\phi^4(x)\,dx$". If ϕ were defined, then this would be a polynomial and everything would be fine. But this would amount to setting f and g to be delta functions. We want to find an extension of this μ, but in a nice way. PT: how does the notion of polynomials translate? BD: the analogue of $p(x)$ would be $p(\phi)$ =polynomial$(\int f(x)\phi^4(x)\,dx, \cdots)$. A polynomial before was something of the form $p(x) = \sum c_\alpha x^\alpha$. Now we think of c_α as $f(\alpha)$.

Proof of Theorem. Let $P(t,x) = (e^{-tL/2}p)(x)$. Expand this as a Taylor series in t, then we get $\sum_{n=0}^\infty \frac{1}{n!}\bigl((tL/2)^n p\bigr)(x)$. This is a finite sum because p is of finite degree, so it will eventually be killed by the derivatives. In the litirature, you see the notation $:p: = e^{-1/2}p$, the Wick ordering of p. I'm going to drop the x from the notation, then we want to compute

$$\begin{aligned}\frac{d}{dt}e^{tL/2}[P(t)\cdot Q(t)] &= \text{chain rule computation}\\ &= \sum_i e^{tL/2}\partial_i P\cdot\partial_i Q\end{aligned}$$

This essentially completes the proof. Just repeat the process. You see that $\frac{d^n}{dt^n}e^{tL/2}[P\cdot Q] = \sum_{i_1,\ldots,i_n} e^{tL/2}\partial_{i_1}\cdots\partial_{i_n}P\cdot\partial_{i_1}\cdots\partial_{i_n}Q$. Since these series always truncate, we get polynomials in t and x, so there is no problem in applying Taylor's theorem. We get

$$e^{L/2}[e^{-L/2}p\cdot e^{-L/2}q] = \sum_{n=0}^\infty \frac{1}{n!} \underbrace{t^n\Bigl(\frac{\partial}{\partial t}\Bigr)^n e^{tL/2}[P\cdot Q]\Big|_{t=1}}_{=\sum_{i_1,\ldots,i_n} e^{L/2}(\partial_{i_1}\cdots\partial_{i_n}e^{-L/2}p)\cdot(\partial_{i_1}\cdots\partial_{i_n}e^{-L/2}q)}$$

Now let $P \to e^{L/2}p$, $Q \to e^{L/2}q$ then evaluate at $x = 0$. Something got screwed up. If Peter gives me five minutes, I'll fix it next time. ☐

One of the corollaries of this computation is this.

Corollary 20.2. *The mapping $L^2(\mu) \ni p \mapsto (e^{L/2}p) \in \mathrm{Sym}^*(\mathbb{R}^N)$ extends to a unitary map after completion. The right-hand side is usually called the Fock space. The norm on $\mathrm{Sym}^*\mathbb{R}^N$ (with respect to which you complete) is $\|q\|^2 = \sum_{n=0}^\infty \frac{1}{n!}\sum|\partial_{i_1}\cdots\partial_{i_n}q|^2(0)$.*

The way you're supposed to think about these infinite-dimensional wave equations is $\ddot{\phi}(t) = -\nabla(\phi(t))$, where $\phi(t,\cdot) \in L^2(\mathbb{R}^{\text{space}})$, $\partial_t^2\phi - \Delta_{\text{space}}\phi + \cdots = 0$. In the description of taking L^2 of L^2, you run into trouble… the Fock space description is better.

<u>Question</u>: suppose we have a Hilbert space H. Can you make sense out of $d\mu(x) = \frac{1}{Z}e^{-\frac{1}{2}(x,x)}\mathcal{D}x$? That is, can you find a measure μ on H such that for each $\lambda \in H$, $\int_H e^{(\lambda,x)}\,d\mu(x) = e^{\frac{1}{2}(\lambda,\lambda)}$? We know this is supposed to be the answer in finite dimensions. You should be thinking

of H as $\{f \in L^2 \mid \int(|\nabla f|^2 + m^2 f^2)dx < \infty\}$. The thing that causes all the problems (and all the good things) is that the answer is NO.

The problem is that H is too small. A good analogue is this. If you just have $\mathbb{Q}$ and you want a Lebesgue measure on it. Since a measure must be countably additive, you run into trouble because $\mathbb{Q}$ is countable. This is similar to what is happenning here. Let me state the theorem.

Theorem 20.3. *Suppose H and K are real separable Hilbert spaces such that $H \overset{i}{\hookrightarrow} K$ is a continuous embedding (i.e. $\|i(h)\|_K \le C\|h\|_H$ for all $h \in H$) and so that $\overline{i(H)}^K = K$. If (and only if) i is Hilbert-Schmidt, then there exists a unique (Gaussian) measure μ on K such that for $\lambda \in K^*$, $\int_K e^{\lambda(x)}\,d\mu(x) = e^{\frac{1}{2}(\lambda|_H,\lambda|_H)_{H^*}}$.*

This tells you what you need to do to enlarge your space. $A\colon H \to K$ is *Hilbert-Schmidt* if $\sum_{j=1}^\infty \|Ae)i\|_K^2 = \|A\|_{HS}^2 < \infty$ for any orthonormal basis $\{e_i\}$ of H. This is also $\|A\|_{HS}^2 = \mathrm{tr}(A^*A)$.

Note that $\mu(H) = 0$.

[[break]]

For lack of time, let me stick to the following example.

Example 20.4 (Wiener measure)**.** $d\mu(\omega) = \frac{1}{Z}\exp\big(-\frac{1}{2}\overbrace{\int_0^T \dot\omega(t)^2\,dt}^{(\omega,\omega)_H}\big)\mathcal{D}\omega$, where $\omega(0) = 0$, $\omega(t) \in \mathbb{R}^d$. When you see a $\mathcal{D}$, it means that this is an informal expression which doesn't make sense. Where can you construct this measure? $H = \{\omega\colon [0,T] \to \mathbb{R}^n | (\omega,\omega)_H < \infty\}$. Note that Z is usually 0 or ∞ …no one term in this expression makes sense.

Claim. *$H \overset{i}{\hookrightarrow} L^2([0,T]^1,\mathbb{R}^n)$ is Hilbert-Schmidt.*

Where is the Hamiltonian operator coming from? In other setups, $\mathbb{R}^n$ is replaced by an infinite-dimensional thing. In our language, $\mathbb{R}^N$ is $L^2([0,T]^1,\mathbb{R}^n)$. It turns out that $\|i\|_{HS}^2 = T^2/2$. You should compute $i^*\colon L^2 \to H$.

Claim. *$(i^*f)(s) = \int_0^T s\wedge t\, f(t)\,dt$, where $s \wedge t = \min(s,t)$.*

This is not hard to check. The Hilbert-Schmidt norm $\|i\|_{HS}^2$ is $\mathrm{tr}(ii^*)$, and ii^* is basically i^* since the inclusion doesn't do anything. So we should evaluate $\int_0^T s\wedge s\,ds = T^2/2$.

Now you do another little exercise. You can ask what is $\int_K (x,k_1)_K(x,k_2)_K\,d\mu(x) = \sum_{n=1}^\infty (e_n,k_1)_K(e_n,k_2)_K$, where $\{e_n\}$ is an orthonormal basis for H. This is really $\big((k_1,\cdot)_K,(k_2,\cdot)_K\big)_{H^*}$. Now you may as well replace e_n by $i(e_n)$, and then $(i(e_n),k_j)_K = (e_n,i^*k_j)_H$, so we see that the result is $(i^*k_1,i^*k_2)_H = (k_1,ii^*k_2)_K$.

Now we compute $\int_K (f,\omega)_K^2\,d\mu(\omega) = (f,ii^*f) = \iint_{[0,T]} s\wedge t\, f(s)f(t)\,ds\,dt$. Formally, you can take f to be a delta function at t_0. The left hand side is $\int \omega(t_0)^2 d\mu(\omega)$ and the right hand side is $t_0 \wedge t_0 = t_0$. More generally, you get $\int \omega(s)\omega(t)d\mu(\omega) = s\wedge t$. We haven't shown that this makes sense. Weiner showed that something something making the total measure of continuous functions 1, so you can just work with continuous functions. ◇

Where is the operator theory? This is supposed to have to do with Quantum mechanics, which has to do with operators. In particular, where is the Hamiltonian? Define

$$(T_t^\vee f)(x) = \text{“}\frac{1}{Z_t}\int_{\omega\colon [0,1]\to\mathbb{R}^n,\omega(0)=x} \exp\Big(-\int_0^t \big(\tfrac{1}{2}\dot\omega(\tau)^2 + V(\omega,\tau)\big)\,d\tau\Big) f(\omega(t))\mathcal{D}\omega\text{”}$$

If you didn't have the V, this would be the Weiner measure. The rigorous definition of T_t would be

$$T_t^\vee f(x) = \int_{C([0,t],\mathbb{R}^n),\omega(0)=0} f\big(x + t\omega(t)\big)\exp\Big(-\int_0^t V(\omega(\tau+x)d\tau\Big)d\mu(\omega)$$

On of the homework problems is the gluing axiom. From the informal formula, you should check that $T_s^\vee T_t^\vee = T_{s+t}^\vee$ and $T_0^\vee = \mathrm{id}$.

$$\frac{d}{dt}\Big|_{t=0} T_t f = -\hat{H}f \qquad \hat{H} = -\frac{1}{2}\Delta + M_V$$

When you do this calculation, you get an $\dot\omega$, which you know doesn't make sense (because the paths aren't smooth), but you pretend that it does make sense. If the paths were smooth, you wouldn't get the $-\Delta$.

You have to be a little more careful, you can't just pull the derivative under the integral. The key is

$$\int \omega(t)^2 d\mu(\omega) = t \qquad \int \omega(t) d\mu(\omega) = 0$$

(these paths are rough). When Ito was developing stochastic calucus, this had to do with the fact that

$$df(\omega(t)) = f'(\omega(t))d\omega(t) + \frac{1}{2}f''(\omega(t)d\omega(t)^2 dt$$

21 PT 11-06

Homework 5 is due Nov. 20 (before Thanksgiving).Projects:

1. Yoneda for $\mathcal{A}$-enriched categories. [Theo, Dan]
2. Super manifolds via algebra. [Matthias, Dan]
3. K-theory for super manifolds. [Manuel]
4. Simple super Lie algebras. [Andre, Jonah]
5. G-actions on super manifolds.
6. Super principal bundles and connections. [Alan, Dan B.]
7. Differential gorms. [Kevin]

I'll give you at least three more projects in the next weeks.

We're way behind in this class because we spent so much time on super manifolds. We'll go very quickly through geometric quantization. I have to do this very quickly so that we can actually get to cohomology theories.

Start with a symplectic manaifold (M, ω). I explained at some point (as did Kolya) how to go from a classical field theory to such a setup (look at classical solutions). For today, I'll assume M is finite-dimensional, though it could be infinite-dimensional in general. We want to somehow quantize this classical system. We want a vector space V (I won't discuss the inner product for now), which is like the state space of a quantum system, and Lie homomorphism $C^\infty(M, \omega) \to End(V)$. You should think of $C^\infty(M, \omega)$ as classical observables and $End(V)$ as quantum observables. You want to quantize the observables in such a way that Poisson bracket goes to Lie bracket. Furthermore, you want a map that takes the constant function 1 to the identity id_V (this doesn't follow from Lie homomorphism).

Warning 21.1. I will ignore factors of $\hbar$, π, and i (I won't "set π or i to 1"). How do you make sure an operator has real eigenvalues? You require that it is self-adjoint. However, unitary operators are skew-adjoint. This i converts between skew-adjoint and self-adjoint operators. ⌟

So far this is *pre-quantization.* We require some more properties to call it *geometric quantization.* These other properties come from physics motivation. To do the geometric quantization, you have to give up part of $C^\infty M$ (the map will not be a Lie homomorphism).

This pre-quantization always exists if ω is *integral*, i.e. $\int_{\Sigma^2} f^*(\omega) \in \mathbb{Z}$ for all $f\colon \Sigma^2 \to M$, where Σ^2 is a closed oriented surface. Using some algebraic topology, this is the same as saying that $[\omega] \in H^2_{dR}(M)$ is actually lies in the image of $H^2(M;\mathbb{Z})$.

Construction of pre-quantization

Step 1: If ω is integral, there exists a hermitian line bundle $L \to M$ with connection such that the curvature of the connection is ω. The reason you need integrality for this is that if $[\omega] = [\text{curvature}(\nabla)]$, then it come from the first Chern class $c_1(L) \in H^2(M;\mathbb{Z})$.

You can discuss how many such line bundles there are up to isomorphism. If you have two line bundles, you can complex conjugate and tensor, so there is a notion of a "difference of line bundles". Any two such bundles will differ by a flat bundle. Thus, isomorphism classes of such (L,∇) forms a torsor under the group of isomorphism classes of flat line bundles. Flat line bundles, up to isomorphism, are classified by their holonomy, and because we assume unitary and assume M connected, the holonomy is just $\mathrm{Hom}(\pi_1 M, S^1) = H^1(M;S^1)$ (by the universal coefficient theorem, thinking of the circle S^1 as a discrete abelian group). So there is this torus which acts simply transitively on the choice of line bundle.

I will skip the construction of the bundle. The name of this line bundle with connection is called the *pre-quantum line bundle.* Choose one such (L,∇).

Remark 21.2. $H^2(M;\mathbb{Z}) \cong H^1(M, \underline{S}^1)$ ($\underline{S}^1$ is sheaf of S^1-valued functions). ◇

The main diagram of this story is the following. There is an exact sequence of Lie algebras (we get this for any symplectic manifold)

$$\begin{tikzcd}
0 \arrow[r] & H^0(M;\mathbb{R}) \arrow[r] \arrow[d, "\exp"'] & C^\infty(M,\omega) \arrow[r, "d"] \arrow[d, "E"'] & sp(M,\omega) \arrow[r, "\text{cohom class}", "x\mapsto[i_x\omega]"'] \arrow[d, "\exp"'] & H^1(M,\mathbb{R}) \arrow[r] \arrow[d, "\exp"'] & 0 \\
1 \arrow[r] & H^0(M;S^1) \arrow[r, "\text{rot.}", "\text{fibers}"'] & \mathrm{Aut}(L,\nabla) \arrow[r, "\text{diffeo}", "\text{on base}"'] & Sp(M,\omega) \arrow[r, "\text{diff}", "\text{holonomy}"'] & H^n(M;S^1) \arrow[r] & 1
\end{tikzcd}$$

where $sp(M,\omega)$ are *symplectic vector fields*, vector fields X so that $\mathcal{L}_X(\omega) = 0$. If you use the Cartan formula, you see that $\mathcal{L}_X(\omega) = d(i_x\omega)$. Each of these are Lie algebras, all the maps are Lie homomorphisms, and the sequence is exact

I want to exponentiate to an exact sequence of Lie Groups, which is the bottom row (using our choice of (L,∇)). If $\tilde{f}\colon (L,\nabla) \to (L,\nabla)$ (respecting connection), then $f\colon M \to M$ must be a symplectomorphism. The map $E\colon C^\infty(M,\omega) \to \mathrm{Aut}(L,\nabla)$ is given as follows, you can flow along the function in M and simultaneously rotate the fibers.

Step 2: Define $V := \Gamma(L)$, the sections of L. This is a complex vector space. Let $C^\infty(M)$ act on V via the map E (remember we wanted a Lie homomorphism $p\colon C^\infty M \to End(V)$). The diagram is

$$\begin{tikzcd}
C^\infty(M) \arrow[r, "p"] \arrow[d, "E"'] & End(V) \arrow[d, "\exp"] \\
\mathrm{Aut}(L,\nabla) \arrow[r] & GL(V)
\end{tikzcd}$$

More concretely, for $f \in C^\infty M$ and $s \in \Gamma(L)$, $(p(f))(s) = \nabla_{X_f}(s) + f\cdot s$. Remember that we wanted $p(1) = \mathrm{id}_V$. The reason this is true is because of the big diagram above commuting (you get a constant rotation of the fibers, where $1 \in \mathbb{R}$ exponentiates to $1 \in S^1$). You can also see this from the formula.

The other formula I wanted to give you is this. If we have $\omega = d\alpha$ (i.e. if we have a *symplectic potential* $\alpha \in \Omega^1 M$), then we can take $L = \mathbb{C} \times M$ and $\nabla = d + m_\alpha$ (m_α is multiplication by α). We add the 1-form α to get curvature $d\alpha$, which we want. Now we get $p(f)(s) = X_f(s) + (\alpha(X_f) + f)\cdot s$. In the cases that lead to quantum mechanics, $M = T^*N$ (N is the configuration space), so we have the symplectic potential α.

After the break, I'll do the even more special case where M is a vector space.

[[break]]

Linear case (i.e. M is a finite dimensional vector space and $\tilde{\omega}\colon M\times M\to\mathbb{R}$ is a non-degenerate skew pairing). We have $TM\cong M\times M$, and $\omega_m(v_1,v_2)=\tilde{\omega}(v_1,v_2)$ defines ω. Then $\omega=d\alpha$, where $\alpha_m(v)=\tilde{\omega}(m,w)$ is in $\Omega^1(M)$. Now we can evaluate these formulas. I want to evaluate them on linear functions $M\overset{\phi}{\cong}M^*\subseteq C^\infty(M)$. We have the pre-quantization $p(v)=\partial_v+m_{\phi(v)}$. Because I defined the symplectic form to be constant, this symplectic manifold has translational symmetry. That is, M acts on itself by translations, which are symplectomorphisms (since the form is constant).

Remark 21.3. Translations do *not* preserve the 1-form α. ◇

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & \mathbb{R} \arrow[r] \arrow[d, equal] & heis(M,\omega) \arrow[r] \arrow[d, hook] & (M,[,]=0) \arrow[rr] \arrow[d, hook] & & 0 \arrow[d, equal] \\
0 \arrow[r] & H^0(M;\mathbb{R}) \arrow[r] \arrow[d, "\exp"'] & C^\infty(M,\omega) \arrow[r, "d"] \arrow[d, "E"'] & sp(M,\omega) \arrow[rr, "\text{cohom class}", "x\mapsto[i_x\omega]"'] \arrow[d, "\exp"'] & & H^1(M,\mathbb{R}) \arrow[r] \arrow[d, "\exp"'] & 0 \\
1 \arrow[r] & H^0(M;S^1) \arrow[r, "\text{rot.}", "\text{fibers}"'] & \mathrm{Aut}(L,\nabla) \arrow[r, "\text{diffeo}", "\text{on base}"'] & Sp(M,\omega) \arrow[rr, "\text{diff}", "\text{holonomy}"'] & & H^n(M;S^1) \arrow[r] & 1 \\
1 \arrow[r] \arrow[u, equal] & S^1 \arrow[r] \arrow[u, equal] & Heis(M,\omega) \arrow[r] \arrow[u, hook] & (M,+) \arrow[rr] \arrow[u, hook] & & 0 \arrow[u, equal]
\end{tikzcd}$$

Since translations preserve the de Rham d, but don't preserve α, so they don't preserve the connection, but when we do the extension to get the Heisenberg group, which does respect the connection. This formula leads to a representation of $heis(M,\omega)$ on $V=C^\infty(M;\mathbb{C})$.

We're sort of going back between symmetries an observalbes, so we're really hiding Noether's theorem in this diagram.

You don't expect symmetries to quantize well, you have to get a phase. This corresponds to the fact that you had to take the extension to the $Heis(M,\omega)$.

Fact: This representation is highly reducible! This is bad for quantization from some point of view. [[PT: I'm using some notes I wrote a couple of years ago, which are probably on my web site.]] To get an irreducible representation, we choose a *polarization*.

In the linear case, this is a decomposition $M\cong N\oplus N^*$, where N is a Lagrangian subspace, with $\tilde{\omega}$ corresponding to $\left(\begin{smallmatrix}0 & -ev\\ ev & 0\end{smallmatrix}\right)$. In the linear case, this decomposition always exists.

Geometric quantization in this case takes $V:=C^\infty(N)$. Then I'd still like to write down an action of all functions, but as I said, you can't always do that. I certainly want to quantize the Heisenberg, so I'll just do that. $heis(M,\omega)$ acts on V via $N\oplus N^*\to End(V)$ given by $(n,\phi)\mapsto\partial_n+m_\phi$. The ∂_n are *annihilators* and the m_ϕ are *creators*. You have to check it, but the commutation relations are exactly the commutation relation in the Heisenberg algebra, so this is really a Lie homomorphism $M\to End(V)$, with the property that the central element (corresponding to the constant function) goes to id_V. It turns out that this V is *the* irreducible representation of (M,ω) for which the central element acts as the identity (Stone-von Neumann theorem tells us that the center either acts trivially, or the irrep is characterized by the action of the center).

22 PT 11-08

Referecnes for geometric quantization:

- Bates-Weinstein (Berkeley MLN)
- Kirillov (Springer)
- Guilleman-Sternberg "Geometric Asymptotics"

Last time we had (M,ω) (integral) symplectic manifold. Our pre-quantization is $p\colon C^\infty M \to End(V)$ a Lie algebra homomorphism with $p(f) = \nabla_{X_f} + m_f$, where $V := \Gamma_{C^\infty}(L)$ for a particular line bundle with connection (L,∇). The easiest case is the example where M is a vector space and ω is constant $\tilde{\omega}\colon M\times M \to \mathbb{R}$. In that case, we have $M \cong_{\tilde{\omega}} M* \subseteq C^\infty M$, and on that subspace, we have $p(v) = \partial_v + m_v$. Notice that if you write down the commutation relations, then the vectors do not commute. We really want to take the constants as well: $\mathbb{R}\cdot c\oplus M =: heis(M,\omega) \subseteq C^\infty M$, and $p\colon heis(M,\omega)\to End(V)$ is a Lie homomorphism.

I'm purposly allowing ∂_v to act on $C^\infty M$. I'm not completing to a Hilbert space. Remember that $C^\infty M$ has a Frechét topology, and these ∂_v and m_v are continuous, but the spectrum is unbounded. I'm avoiding the problem by taking $C^\infty M$ and now worrying about what happens on the completion. Once you have an inner product, you can ask about adjoints. I'm ignoring some i's; ∂_v is skew-adjoint and m_v is self-adjoint. If you us im_v, then this actually leads to a unitary representation of $Heis(M,\omega)$ on $L^2(M,\omega^n/n!)$. Physicists like self-adjoint rather than skew-adjoint operators, so they would use $i\partial_v + m_v$ instead.

The problem is that this representation is not irreducible, and we expect it to be because it comes from a single particle. Before we cut it down to make it irreducible, let's do the odd version of this. [[★★★ Project 8: geometric quantization for symplectic super manifolds. Everything goes through beautifully. There are some notes for pre-quantization by Kostant]]

Odd (linear) analogue of pre-quantization

Take W a finite dimensional vector space and $b\colon W\times W\to\mathbb{R}$ a non-degenerate symmetric bilinear form. This means that $(\pi W, b)$ is a symplectic super manifold (remember that $(\pi W)_{red} = pt$). What is the analogue of pre-quantization? We have to look at $C^\infty(\pi W) = \bigwedge^*(W^*) \supseteq \mathbb{R}\cdot c\oplus W^* \cong_b \mathbb{R}c\oplus W =: heis(W,b)$ (this is a super Lie algebra). $C^\infty(\pi W)$ is also a super Poisson algebra. The Lie algebra structure and the Poisson structure are compatible, and this compatibility tells us that the Lie algebra structure on $C^\infty(\pi W)$ is completely determined by its behavior on the linear functions because $\{f,gh\} = \{f,g\}h + (-1)^{|f||g|}g\{f,h\}$.

Super pre-quantization is a super Lie homomorphism $C^\infty(\pi W) \to End(C^\infty(\pi W))$. On $heis(W,b)\subseteq C^\infty(\pi W)$, the c goes to $\mathrm{id}_{C^\infty(\pi W)}$ and $w\mapsto \partial_w + m_w$ (again, we're ignoring some i's if you want to work with inner products). So we have a super Lie algebra represented on an associative algebra, and an associative algebra is always a super Lie algebra, so we get a unique extension to a super algebra homomorphism $U(heis(W,b)) \to End(C^\infty(\pi W))$. Our criterion is that c goes to the identity, so we get an algebra homomorphism from $U(heis(W,b))/(c=1)$ =:Weyl(W,b). All this works in the even case by the way. $p\colon U(heis(W,b))/(c=1)\to End(\bigwedge^*(W^*))$.

Lemma 22.1. *$U(heis(W,b))/(c = 1_U) \cong Cl(W,b)$, with defining relations $w_1w_2 + w_2w_1 = b(w_1,w_2)$.*

Since the w_i are odd, $[w_1,w_2] = w_1w_2 + w_2w_1 = b(w_1,w_2)$. So the way to prove the lemma is to observe that the two algebras have the same defining relations.

So what we've constructed is a representation of the Clifford algebra on the exterior algebra, given by the formula $w\mapsto \partial_w + m_w$. You can get $Cl(W,b)\to\bigwedge^*(W^*)$ by taking $a\mapsto p(a)\cdot 1_{\bigwedge^*}$. This turns out to be an isomorphism of $Cl(W,b)$-modules. This is something you might have seen before.

Corollary 22.2. *$\bigwedge^*(W^*)$ is* not *irreducible as a $Cl(W,b)$-module.*

Example 22.3. If you take $End(V)$, as a representation over itself, it is isomorphic to $V\otimes V^*$ (with the action on the left side, so the V^* is the multiplicity space).

If you take a finite group G, then $k[G] = \bigoplus_\lambda V_\lambda\otimes V_\lambda^*$ (with the action on the left again). ◇

We see that we kind of have to take some sort of "square root" to extract the irreducible representations. This is what the polarization does

Back to (M, ω)

A polarization of $(M, \tilde{\omega})$ is a decomposition $M \cong N \oplus N^*$ such that ω corresponds to $\left(\begin{smallmatrix} 0 & ev \\ -ev & 0 \end{smallmatrix}\right)$.

Theorem 22.4. *$heis(M, \tilde{\omega}) \to End(C^\infty N)$, with $c \mapsto \mathrm{id}$ and $(v, \phi) \mapsto \partial_v + im_\phi$, is an irreducible representation.*

The proof is not hard, but we won't do it. It is proved in all of the references.

Theorem 22.5 (Stone-von Neumann)**.** *This representation exponentiates to an irreducible unitary representation $\rho\colon Hesi(M, \omega) \to U(L^2N)$, with $c \mapsto \mathrm{id}$. Moreover, this is the* unique *irreducible unitary representation of $Heis(M, \omega)$ on a Hilbert space sending c to* id.

This theorem is also not that hard.

I'm very interested in how unique the representation $heis(M, \tilde{\omega}) \to End(C^\infty N)$ is. Bruce: it isn't always unique. PT: ok.

Corollary 22.6. *$Sp(M, \omega)$ acts projectively on L^2N.*

Note that Sp must fix the origin; earlier we were looking at translations as well.

Proof of Corollary. For $g \in Sp(M, \omega)$, the key formula is $\rho(g(h)) = U_g \circ \rho(h) \circ U_g^*$ for all $h \in Heis(M, \omega)$, where $g(h)$ is the action of $Sp(M, \omega)$ on $Heis(M, \omega)$. That is, we've precomposed the representation with the action of $Sp(M, \omega)$. But by uniqueness of the representation, there must be some unitary operator U_g making the key formula work. The U_g is not quite unique. Two such U_g could differ by a phase (by Schur's lemma). That means that $g \mapsto U_g$ must be multiplicative up to phase, which is what it means to have a projective representation: $Sp(M, \tilde{\omega}) \to U(L^2N)/S^1$. □

If you differentiate this action, it leads to $\mathfrak{sp}(M, \omega) \to End(C^\infty N)/S^1$. But $\mathfrak{sp}(M, \omega) \subseteq C^\infty(M)$ are exactly the "quadratic" functions (given by $A \mapsto (m \mapsto \tilde{\omega}(Am, m))$). In fact, quantization gives a Lie homomorphism $\dots heis(M, \omega) \rtimes \mathfrak{sp}(M, \omega) = \mathbb{R}c \oplus M^* \oplus \mathrm{Sym}^2(M^*) \subseteq C^\infty(M)$. If you're careful, you'll see that this representation extends to a Lie homomorphism $heis(M, \omega) \rtimes \mathfrak{sp}(M, \omega) \to End(C^\infty(N))$ (this is quantization of observables). It *does not* extend to $C^\infty(M)$. This is the down side of geometric quantization; you can only quantize some classical observables, not all of them. This Lie homomorphism actually integrates to $Heis(M, \omega) \rtimes Sp(M, \omega) \to U(L^2N)$. This is quantization of symmetries.

[[break]]

Corollary 22.7. *Geometric quantization of $(M, \tilde{\omega})$ does* not *depend on the polarization (up to phase).*

This follows from the uniqueness part of Stone-von Neumann. This "up to phase" is good because physically, changing the phase doesn't change anything.

Example 22.8. $(M, \omega) = (\mathbb{R}^2, dx \wedge dy)$. I choose the polarization $\mathbb{R}^2 \cong \mathbb{R}_x \oplus \mathbb{R}_y = N \oplus N^*$. The two operators are ∂_x and m_x. Let $P = i\partial_x$ and $Q = m_x$ so that everything is self-adjoint. Question: how does $SO(2) \subseteq Sp(2)$ act on $L^2\mathbb{R}$? The answer is that the infinitesimal generator of the circle action is the energy $E = \frac{1}{2}(P^2 + Q^2)$ (remember that geometric quantization can only quantize quadratic observables). $L^2\mathbb{R}$ has an orthonormal basis of eigenvectors for this operator and you can get between them with these creation and annihilation operators. The spectrum of E is $\frac{1}{2} + \mathbb{N}_0$. The lowest eigenvector if $\frac{1}{2}$ and the eigenvector is $e^{-x^2/2}$.

When we integrate, we get $e^{2\pi i E} = -1$, so we see that we only get a projectiv action. In fact, you can do this for any symplectic group and you see that it is actually a double cover that acts.

$$\begin{tikzcd}
& S^1 \arrow[r, phantom, "="] \arrow[d] & S^1 \arrow[d] \\
Mp \arrow[rr, bend left] \arrow[dr, "2"'] & \widetilde{Sp} \arrow[d] & U(L^2N) \\
& Sp & U(L^2N)/S^1
\end{tikzcd}$$

[[★★★ This is the Weil representation of Mp, yes?]] ◇

The fermionic case is way easier than the bosonic case because the Clifford algebra is finite-dimensional. In the bosonic case, you get this infinite-dimensional universal enveloping algebra of the Heisenberg. The analytic subtleties come from the fact that $Heis(M, \omega)$ is not compact. In the fermionic case, the underlying space is a point which is compact, and the rest is just linear.

Let me tell you the punch line, and I'll do the polarization for fermions on Tuesday.

Let $(\pi W, b)$ be an odd symplectic vector space, as before. We want a polarization $W = N \oplus N^*$, with b corresponding to $\left(\begin{smallmatrix} 0 & ev \\ ev & 0 \end{smallmatrix}\right)$.

Problem: This only exists if the signature of b is zero. If we study Riemannian manifolds, then we want b to be positive definite, so it looks like we're in trouble, but I'll show you some tricks which will allow us to polarize.

If the signature is zero, then $Cl(W, b) \to End(C^\infty(\pi N)) = End(\bigwedge^*(N^*))$. Counting dimensions (say $\dim N = n$), we see that this has dimension $(2^n)^2 = 2^{2n}$ on the right, and $\dim Cl(W, b) = 2^{2n}$. It turns out that this map is an isomorphism in the signature zero case. So in this case, the clifford algebra is a matrix algebra.

Corollary 22.9. *$Cl(W, b)$ has the irreducible representation $\bigwedge^*(N^*)$ and it is unique.*

23 PT 11-13

Let (M, ω) be a symplectic vector space. Then we have the Lie algebra $h(M, \omega) \subseteq C^\infty(M)$, and a polarization $(M, \omega) \cong (N \oplus N^*, \left(\begin{smallmatrix} 0 & ev \\ -ev & 0 \end{smallmatrix}\right))$ and an irreducible representation $h(M, \omega) \to End(C^\infty N)$, where $n \mapsto \partial_n$ and $n^* \mapsto m_{n^*}$. By Stone-von Neumann, we get uniqueness (up to phase) of the unitary representation of $H(M, \omega)$ on $L^2 N$. So we could extend the representation to quadratic functions, but not to all of $C^\infty M$. The outcome is a projective representation of $Sp(M, \omega)$, which is independent of the polarization.

Definition 23.1. This is called the *metaplectic* representation of the *metaplectic group* $Mp(M, \omega) \xrightarrow{2} Sp(M, \omega)$. ◇

By the way, how do you classify covers of $Sp(M, \omega)$? We have to compute $\pi_1(Sp(M^{2n}, \omega))$. A maximal compact inside of $Sp(M, \omega)$ is $U(n)$, which must then have the same π_1, which is $\mathbb{Z}$. We all know that $\pi_1(U(1)) \cong \mathbb{Z}$. And $U(n-1) \subseteq U(n)$, and $U(n) \to U(n)/U(n-1) \cong S^{2n-1}$ is a fibration. So there is one non-trivial connected double cover of $Sp(M, \omega)$.

Now for the fermionic side. If (W, b) is symmetric bilinear non-degenerate, then we get a super Lie algebra $h(W, b) \subseteq C^\infty(\pi W) = \bigwedge^*(W^*)$, a polarization $(W, b) \cong (N \oplus N, \left(\begin{smallmatrix} 0 & ev \\ ev & 0 \end{smallmatrix}\right))$ (this exists if and only if the signature of b is zero), and an irreducible representation $h(W, b) \to End(C^\infty(\pi N))$. The uniqueness here is obvious because it leads to a representation of the associative algebra $Cl(W, b) \to End(C^\infty N) = End(\bigwedge^* N^*)$. Counting dimensions, we see that this is an isomorphism, so $Cl(W, b)$ is the algebra of endomorphisms of a vector space, which has a unique irreducible representation. The outcome here is a projective representations of the orthogonal group $O(W, b)$ on $\bigwedge^*(N^*)$. I want to get to $SO(W, b)$. Given $g \in Cl(W, b)$, we act on $\bigwedge^*(N^*)$. This action is even if and only if $g \in SO(W, b)$.

Definition 23.2. This is the *spinor representation* of $\mathrm{Spin}(W, b) \xrightarrow{2} SO(W, b)$. ◇

Since we're under the assumption that the signature of b is zero, we should take care of that. It should be clear that we should not have done two different cases. We should have just started with a symplectic super vector space. The claim is that you can do it in this generality [[★★★ Project 8: do this geometric quantization story in this case. There are now 10 projects online.]][[★★★ Another project (10): in the bosonic case, we said that once you have an irrep of the Heisenberg, you can't extend it to all of C^∞. In the odd case, if you have $h(W,b) \rtimes O(b) \cong \bigwedge^{\leq 2} W^*$, which acts on $End(\bigwedge^* N^*)$. But $\bigwedge^{\leq 2} W^* \subseteq \bigwedge^*(W^*) = C^\infty(\pi W^*)$. Can we extend the representation to $C^\infty(\pi W^*) \to End(\bigwedge^* N^*)$. There is a filtration preserving iso $Cl^{\leq 2}(W) \cong \bigwedge^{\leq 2} W^*$. We have a map of associative algebras $Cl(W,b) \to End(\bigwedge^* N^*)$. If we had an iso of super Lie algebras $Cl(W,b) \cong C^\infty(\pi W^*)$, you could extend the representation.]]

What if $sign(b) \neq 0$? There are two answers.

1. (this is the usual one in physics books) Assume $\dim W = 2n$. Then remove the signature by complexifying: $(W \otimes_{\mathbb{R}} \mathbb{C}, b \otimes \mathbb{C})$. Because we assumed even dimension, we can still write this as $(W \otimes_{\mathbb{R}} \mathbb{C}, b \otimes \mathbb{C}) \cong (N_{\mathbb{C}}, N^*_{\mathbb{C}}, \left(\begin{smallmatrix} 0 & ev \\ ev & 0 \end{smallmatrix}\right))$, which we call a *complex polarization*. The story continues just as above, but now you have a complex vector space. You're now dealing with CS super manifolds. Then $Cl(W,b) \otimes_{\mathbb{R}} \mathbb{C} \cong Cl(W \otimes \mathbb{C}, b \otimes \mathbb{C}) \to End(C^\infty(\pi N_{\mathbb{C}})) \cong \bigwedge^*_{\mathbb{C}}(N^*_{\mathbb{C}})$. The dimension count argument still works, so this is still an isomorphism. The outcome is that we get a *complex* representation of the real group $SO(W,b) \subseteq SO(W \otimes \mathbb{C}, b \otimes \mathbb{C})$. In particular, we can now talk about $\mathrm{Spin}(2n)$ and $SO(2n)$.

   If you take this representation, you can just pull back

$$\begin{tikzcd}
 & S^1 \arrow[r, equal] \arrow[d] & S^1 \arrow[d] \\
\mathrm{Spin}(W,b) \times_{\{\pm 1\}} S^1 \arrow[r, phantom, "\cong"] & \mathrm{Spin}^c(W,b) \arrow[r] \arrow[d] & U(\bigwedge^* N^*) \arrow[d] \\
 & SO(W,b) \arrow[r] & PU(\bigwedge^* N^*)
\end{tikzcd}$$

   It is a little work to show that the maps from Spin^c factor through Spin. But what do we do if W is not even dimensional?

2. If $b \cong \left(\begin{smallmatrix} I_{n_+} & 0 \\ 0 & -I_{n_-} \end{smallmatrix}\right)$, so $sign(b) = n_+ - n_-$ and $rk(b) = n_+ + n_- = n$. Then take the orthogonal direct sum $W \perp \mathbb{R}^n = (W,b) \oplus (\mathbb{R}^n, \left(\begin{smallmatrix} I_{n_-} & 0 \\ 0 & -I_{n_+} \end{smallmatrix}\right)) \cong (N \oplus N^*, \left(\begin{smallmatrix} 0 & ev \\ ev & 0 \end{smallmatrix}\right))$, where $\dim N = n = \dim W$. By this trick, we get $Cl(W \perp \mathbb{R}^n) \xrightarrow{\sim} End(\bigwedge^* N^*)$. But $Cl(W \perp \mathbb{R}^n) \cong Cl(W) \otimes Cl_{n_-,n_+}$, where $Cl_{m,n} = Cl(\mathbb{R}^{n+m}, \left(\begin{smallmatrix} I_n & 0 \\ 0 & -I_m \end{smallmatrix}\right))$. Since these b's are diagonalizable, these are all the clifford algebras. Watch out, this is a graded tensor product of *super algebras*. Now you can think about it like this. $\bigwedge^* N^*$ is a graded irreducible $Cl(W)$-Cl_{n_-,n_+}-bimodule (the signs you pick up are exactly cancelled by the signs from the tensor product).

[[break]]

Outcome: $\bigwedge^* N^*$ is graded irreducible $Cl(W,b)$-Cl_{n_-,n_+}-bimodule and $\mathrm{Spin}(W,b) \xrightarrow{2} SO(W,b)$ acts on it, *commuting with the Cl_{n_-,n_+} action.* As a bimodule, this guy is unique, as before (if you preserve gradings at least), then you play the same game to get a projective representation of $SO(W,b)$ which lifts to $\mathrm{Spin}(W,b)$. In this case, the lifting is easy to check because everything is real, so you can only get ± 1 for phases. That's also the way to show it for the metaplectic group.

Now I want to put these things on bundles. There are two generalizations of these linear discussions. (a) Discuss the vector bundle case. This is the "family version" of what we've been doing. (b) Curved case of symplectic super manifolds. If you work purely fermionically, it's all linear. The intersection of these two is the tangent bundle of a symplectic manifold. I want to say a few things about (b), but basically we'll follow (a).

(b) Let's do the bosonic case first. Let (M,ω) be a symplectic manifold [[★★★ Project 9 is to do all this for super symplectic manifolds.]]. Prequantization worked very nicely to give us a Lie algebra homomorphism $C^\infty M \to End(\Gamma_{C^\infty}(L \to M))$ $(f \mapsto \nabla_{X_f} + m_f)$ by writing ω as the curvature of a connection on the line bundle $L \to M$. For geometric quantization, we need a polarization $P \subseteq TM$, a Lagrangian subbundle (which, it will turn out, has to be integrable/involutive). If you have this, then M^{2n} looks locally like a cotangent bundle $\mathbb{R}^n \times \mathbb{R}^n$. In this case, you can half the space $\Gamma_{C^\infty}(L \to M)$ by taking the sections that are covariantly constant with respect to P, i.e. if $X \in \Gamma(P) \subseteq Vect(M)$, then $V_P := \{s \in \Gamma(L \to M) | \nabla_X = 0 \; \forall x \in \Gamma(P)\}$.

Now we will get a representation of some subalgebra of $C^\infty(M)$ on V_P. What is this subalgebra? Which functions act? V_P has a caonical action by $C^\infty(M)_P = \{f \in C^\infty M | X_(f) = 0 \; \forall X \in \Gamma(P)\}$.

Example 23.3. If $(M, \omega) \cong (T^*N, d\alpha)$, then the typical P is $P_{vert} \subseteq T(T^*N)$. If N had a metric, then we would get a canonical connection, but in general, the vertical subbundle is the only canonical thing around. Now the claim is that $V_P \cong C^\infty N$ (canonically). We are taking sections which are covariantly constant in the vertical direction, i.e. functions which are constant along fibers. If you pick different isomorphisms $(M, \omega) \cong (T^*N, d\alpha)$, then you get different polarizations, which lead to different geometric quantizations. The subalgebra $C^\infty(T^*N)_{P_{vert}} \cong C^\infty N$ acts on $C^\infty N$ by multiplication operators. This is terribly boring.

In our linear example, we took $M \cong N \oplus N^* = T^*N$, which is a special case of this example. If T^*N happens to be linear, then we got the differentiation operators as well. So if you try harder, in some cases, you can extend the action. But in general, the machine doesn't give you much. This is why people say quantization is an art, not a functor.

In fact, on can quantize more functions on $C^\infty(T^*N)$. In particular, those that are linear on the fibers (these are the p's, momenta) and some that are quadratic in fibers (e.g. Riemannian metrics). Remember that $V_{P_{vert}} \cong C^\infty N$. A Riemannian metric g on N acts by the corresponding Laplacian Δ_g. One way to get this is to do the path integral (I'm thinking of only potential energy). ◇

24 PT 11-15

Today we'll put linear quantizations on bundles. Let $\pi\colon E \to X$ be a real vector bundle of fiber dimension k on a topological space X. Let $P(E) \to X$ be the corresponding bundle of frames (this is actually a $GL(k)$-principal bundle).

(1) We want to reduce the structure group from $GL(k, \mathbb{R})$ to a subgroup G. (2) Given some extension $A \hookrightarrow \widetilde{G} \twoheadrightarrow G$, we want to lift G-bundles to $\widetilde{G}$-bundles (this is sometimes called "reducing").

For (1), we can formulate it in terms of (0) putting extra structure on E (a) transition functions, (b) classifying spaces.

Definition 24.1. *Reducing the structure group to G* means finding a G-principal bundle $P \to X$ and an isomorphism of bundles $P \times_G GL(k) \cong P(E)$. ◇

In terms of transition functions, you take an open covering $\{U_i\}$ of X with trivializations of $P(E)$ (or E if you like). Then the transition functions $\phi_{ij}\colon U_i \cap U_j \to GL(k)$ measure the difference between the two trivializations. An automorphism of the trivial bundle $(U_i \cap U_j) \times GL(k) \to U_i \cap U_j$ is the same as a transition function. Because the bundle is a bundle, the transition functions satisfy the cocycle condition $\phi_{kj} \circ \phi_{ji} = \phi_{ki}$ on $U_i \cap U_j \cap U_k$. If you order the indices and only consider $i < j$, then you don't have to have $\phi_{ii} = \mathrm{id}$ and $\phi_{ij} = \phi_{ji}^{-1}$. On the other hand, these data (covering with transition functions satisfying the cocycle condition) exactly recover the data of the bundle. Restricting is just factoring ϕ_{ij} trough $G \subseteq GL(k)$ (you're allowed to change the ϕ_{ij} by a coboundary, which doesn't change the bundle). This happens when the cohomology class lies in $H^1(X; G)$.

Theorem 24.2. *$H^1(X; G)$ (actually $\check{H}^1(X; G)$) naturally parameterizes isomorphism classes of principal G-bundles over X, which are also parameterized by homotopy classes of maps to BG.*

There is a contractible principal G-bundle $EG \to BG$ so that for a principal G-bundle $P \to X$, there is a unique homotopy class of maps $X \to BG$ so that P is the pullback of EG. Fact: $BGL(k)$ is the Grassmanian of k-planes in $\mathbb{R}^\infty$.

Let $EGL(k)$ be the total space of the universal $GL(k)$-bundle. Given $G \subseteq GL(k)$, I claim that $EGL(k)/G \cong BG$. This is because $EGL(k) \to EGL(k)/G$ is a universal bundle. We can divide out further:

$$\begin{tikzcd}
EGL(k) \arrow[d] & & GL(k)/G \arrow[d, "\text{fiber}"] & \\
EGL(k)/G \arrow[d, two heads] & \cong & BG \arrow[d] & \\
EGL(k)/GL(k) & = & BGL(k) & X \arrow[l] \arrow[ul, dashed]
\end{tikzcd}$$

If $GL(k)/G \cong *$, then $BG \cong BGL(k)$ and therefore $[X, BG] \cong [X, BGL(k)]$. If you wanted to use the transition function point of view, you'd use partitions of unity.

$$\begin{tikzcd}
 & & GL(2n, \mathbb{R}) \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash, "\simeq"] & \\
 & Sp(n) \arrow[dl, dash] \arrow[dr, dash, "\simeq"'] & GL(n, \mathbb{C}) \arrow[d, dash, "\simeq"] & O(2n) \arrow[dl, dash] \\
GL(n, \mathbb{R}) \arrow[dr, dash, "\simeq"'] & & U(n) & \\
 & O(n) \arrow[ur, dash] & &
\end{tikzcd}$$

If $k = 2n$, then $GL(2n, \mathbb{R})$ has three nice subgroups (corresponding to putting symplectic, complex, or inner product structure on a real bundle). Which of these quotients are contractable? The first obstruction to $GL(2n, \mathbb{R})/GL(n, \mathbb{C})$ being contractible is that complex vector bundles have natural orientations; the same problem shows up for Sp. $GL(n, \mathbb{R})$ has two components, but $GL(n, \mathbb{C})$ and $Sp(n)$ are connected. The obstructions to lifting lie in $H^{i+1}\big(X, \pi_i(GL(k)/G)\big)$. So if you have some π_0, you get some obstruction (called the first Steifel-Whitney class) in H^1. In general, the obstructions can be identified with certain characteristic classes of the bundle.

What is the inclusion $GL(n, \mathbb{R}) \to Sp(n)$? It is given by $A \mapsto \left(\begin{smallmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{smallmatrix}\right)$, the symplecomorphisms that preserve the polarization $\mathbb{R}^{2n} \cong N \oplus N^*$, with ω corresponding to $\left(\begin{smallmatrix} 0 & ev \\ -ev & 0 \end{smallmatrix}\right)$. This is not a homotopy equivalence, so there is some obstruction to finding a polarization on a symplectic vector bundle.

Now let's discuss separately the two cases of $Sp(n)$ and $O(k)$ (k need not be even). I have a bundle $E \to X$. $Sp(n)$ structure would give me a non-degenerate skew form $\omega \in \Gamma(\bigwedge^2 E^*)$ on E. $O(k)$ structure would give me a fiberwise positive definite for $b \in \Gamma(\operatorname{Sym}^2 E^*)$ on E. By the way, I don't assume any integrability condition. Usually a symplectic (or complex) structure has the additional condition that the form be closed, and we're ignoring this for now.

Let's say we got a symplectic structure ω. Next, we look for a polarization (a $GL(n, \mathbb{R})$ structure). Again, we get obstructions (and again we ignore the integrability condition on the polarization).

Claim. *We have the following diagram.*

$$\begin{tikzcd}
\mathbb{R}^\times & ML(n) \arrow[l] \arrow[d, two heads, "2"] & \subseteq & Mp(n) \arrow[d, two heads, "2"] \\
\mathbb{R}_{>0} \arrow[from=u, two heads, "square"', "2"] & GL^+(n) \arrow[l, "\det"'] & \subseteq & Sp(n)
\end{tikzcd}$$

[[★★★ PT: no, this is not right. Neither square is right, but that might cancel out.]] In particular, if $E \to X$ is TX, X is a symplectic manifold, the polarization $P \subseteq TX$ is integrable (i.e. we have a foliation $\mathcal{F}$ of X whose tangent bundle is P), and a metaplectic structure on TX gives a metalinear structure on each leaf of $\mathcal{F}$ (i.e. $\bigwedge^{-top/2} \mathcal{F}$ exists).

The reason I mention this is, as I pointed out on Tuesday, if you have T^*N, you get functions on N ($= X/\mathcal{F}$, the space of leaves), but you're missing an inner product. These things give us "half-forms" and resolve the inner product problem. Instead of looking at functions, we look at these half-forms. Then you can multiply two of them to get a volume form, which you can integrate.

[[break]]

Let's go back to this problem. wWhat does it mean to lift the structure group from G to $\widetilde{G}$. It means finding factoring the ϕ_{ij} through $\widetilde{G} \to G$ in such a way that the lifts still satisfy the cocycle condition. In general, $c_{ijk} = \tilde{\phi}_{kj} \tilde{\phi}_{ji} \tilde{\phi}_{ij}^{-1}$ will be a map $U_i \cap U_j \cap U_k \to \ker(\widetilde{G} \to G) =: A$.

Assume A is abelian (in our case it will be $\mathbb{Z}/2$). This will define a class in $\check{H}^2(X;A)$ (you have to check that c_{ijk} is a cocycle). If this c_{ijk} is a coboundary, then you can change the $\widetilde{\phi}_{ij}$ by whatever c_{ijk} is the coboundary of to get it to be zero. The punch line is that

$$\begin{tikzcd}
H^1(X;A) \arrow[r] & H^1(X,\widetilde{G}) \arrow[r] & H^1(X;G) \arrow[r] & H^2(X;A) \\
 & {[\widetilde{P}]} \arrow[r, mapsto] & {[P]} \arrow[r, mapsto] & {[c]}
\end{tikzcd}$$

is an exact sequence. You can see this from the fact that

$$BA \to B\widetilde{G} \to BG$$

is a fibration, so

$$H^1(X;A) = [X, BA] \to [X, B\widetilde{G}] \to [X, BG] \to [X, K(A,2)] = H^2(X;A)$$

is exact (this can be extended to the left by taking loop spaces). But it turns out that $BA = \Omega K(A,2)$. $\pi_1(\Omega Y) \cong \pi_2(Y)$, so $BA \cong K(A,1)$.

So if you want to lift a G-bundle to a $\widetilde{G}$-bundle, the obstruction lies in $H^2(X;A)$, and the non-uniqueness is given by $H^1(X,A)$.

Given any H-principal bundle $P_H \to X$ and $\rho\colon G \to H$, we say that P_H has a "reduction" to G (over ρ) if we can find a G-principal bundle and an isomorphism of H-principal bundles $P_G \times_G H \cong P_H$.

Example 24.3. $\mathbb{Z}/2 \to Mp(n) \to Sp(n)$ and $\mathbb{Z}/2 \to \mathrm{Spin}(n) \to SO(n)$. In both cases, it turns out that the obstruction is $w_2(E) \in H^2(X;\mathbb{Z}/2)$, the second Steifle-Whitney class.

The first Steifle-Whitney class $w_1 E \in H^1(X;\mathbb{Z}/2)$ gives you the obstruction to reducing to $O(2n)$.

To get the symplectic structure, you need more things to vanish, but the obstruction to lifting to $Mp(n)$ is the same as the obstruction of lifting from $O(2n)$ to $\mathrm{Spin}(2n)$. ◇

Definition 24.4. Let $E^k \to X$ be an oriented vector bundle with inner product (i.e. frame bundle has $SO(k)$ structure). Then a *spin structure* on E is a lift of the structure group to $\mathrm{Spin}(k)$. ◇

Geometric quantization leads to a more concrete picture. We'll rewind the definition on Tuesday and answer the following question: What is a spin structure on a (single) vector space?

25 PT 11-20

Projects 5 and 10 are still open. Please let me know your preferences on

- talk and/or paper version. The papers will be due Dec. 6 so that we can read them.
- date of mini-conference (Fr/Mo afternoons Dec. 7 and 10 or all day Tuesday Dec. 11)

Clean up of the mess. This is about the symplectic group $Sp(n)$ and the metaplectic group $Mp(n)$, the unique connected double cover of $Sp(n)$. Inside $Sp(n)$, there is $GL(n)$, the subgroup of matrices of the form $\left(\begin{smallmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{smallmatrix}\right)$. Pulling back, we get the metalinear group $ML(n)$.

$$\begin{tikzcd}
\mathbb{Z}/2 \arrow[r, equal] \arrow[d] & \mathbb{Z}/2 \arrow[d] \\
ML(n) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & Mp(n) \arrow[d, "2"] \\
GL(n) \arrow[r, hook] & Sp(n)
\end{tikzcd}$$

I thought this was trouble before because $\pi_1(GL_n^+) = \mathbb{Z}/2$, so the map to $\pi_1(Sp(n)) = \mathbb{Z}$ must be the zero map. So the extension $ML(n) \to GL(n)$ is *topologically* trivial, though not group-theoretically trivial. Another example: consider

$$\begin{tikzcd}
\mathbb{Z}/4 \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & S^1 \arrow[d, "2"] \\
\mathbb{Z}/2 \arrow[r, hook] & S^1
\end{tikzcd}$$

As a topological space, $\mathbb{Z}/4 \cong \mathbb{Z}/2 \times \mathbb{Z}/2$, but not group-theoretically.

Conclusion: There is an interesting map $\mathrm{Hom}(\pi_1 G, \mathbb{Z}/2) \to H^2(G_\delta; \mathbb{Z}/2)$ (this H^2 parameterizes group extensions; the δ means discrete topology). If you want to learn more about this, come to 253 next semester. One of the things I'll do is the relation between group cohomology and extensions.

The other thing I did, which was look at $\det\colon GL(n)^+ \to \mathbb{R}_{>0}$, was not right. As a Lie group, $\mathbb{R}_> \times \mathbb{Z}/2 \cong \mathbb{R}^\times$. So we keep all of $GL(n)$:

$$\begin{tikzcd}
\mathbb{Z}/2 \arrow[d] & \mathbb{Z}/2 \arrow[d] & \mathbb{Z}/2 \arrow[r, equal] \arrow[d] & \mathbb{Z}/2 \arrow[d] \\
\mathbb{C}^\times \arrow[d, "2"'] & \{\text{x-axis}\cup\text{y-axis}\setminus 0\} \arrow[l] \arrow[d] & ML(n) \arrow[l] \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Mp(n) \arrow[d, "2"] \\
\mathbb{C}^\times & \mathbb{R}^\times \arrow[l] & GL(n) \arrow[l] \arrow[r, hook] & Sp(n)
\end{tikzcd}$$

So we get the non-trivial extension $\mathbb{R}_> \times \mathbb{Z}/4$ of $\mathbb{R}^\times \cong \mathbb{R}_> \times \mathbb{Z}/2$.

If $E^n \to X$ is a vector bundle with metalinear structure, then $\bigwedge^n(E^*) \otimes_\mathbb{R} \mathbb{C}$ has a canonical square root: half forms (if $E = TX$). $TX \to X$, then $L \to X$ is a line bundle such that $L^{\otimes 2} \cong_\rho \bigwedge^n(T^*X)$. Then we get a pairing $\Gamma(L) \times \Gamma(L) \to \mathbb{C}$, given by $(s_1, s_2) \mapsto \int_X \rho(s_1 \otimes s_2)$ (if X is oriented).

Spin structures on inner product spaces

Let (W, b) be a real *inner product* space (i.e. finite dimensional with positive definite symmetric bilinear form). We know

- $O\big((W_1, b_1), (W_2, b_2)\big) = \{f\colon W_1 \xrightarrow{\sim} W_2 \text{ linear} | f^*b_2 = b_1\}$,
- orientations o_1 and o_2 on W_1 and W_2 are equivalence classes of orthonormal bases, so we know what $SO\big((W_1, b_1, o_1), (W_2, b_2, o_2)\big) \subseteq O\big((W_1, b_1), (W_2, b_2)\big)$.

Question: define a notion of "spin structure" σ on (W, b) such that $\mathrm{Spin}\big((\mathbb{R}^n, \mathrm{std}), (\mathbb{R}^n, \mathrm{std})\big) \cong \mathrm{Spin}(n)$ canonically. Bruce: left ideal in $Cl(W, b)$? PT: maybe.

Definition 25.1. A *spin structure* on (W, b) is a graded irreducible $Cl(W, b)$-Cl_n-bimodule (where $Cl_n = Cl(\mathbb{R}^n, \mathrm{std})$, where $e_i^2 = -1$). The $Cl(W, b)$ are $*$-algebras, and modules should come equipped with an inner product such that $Cl(W, b) \to End(M)$ is a morphism of $*$-algebras. ⋄

[[★★★ HW 6: there are exactly two isomorphism classses of such bimodules, and they correspond canonically to orientations on W. So an orientation is a choice of isomorphism class of such a bimodule, but a spin structure is an actual choice of bimodule. Since you only care about isoclasses for this homework, it is enough to do it for $\mathbb{R}^n$ with the standard spin structure.]]

Example 25.2. Let $(W, b) = (\mathbb{R}^n, \mathrm{std})$. If you ignore the grading, there is only one irreducible Cl_n-Cl_n-bimodule (if you don't take grading into account). ChrisSP: A is only irreducible as an A-A-bimodule if A is simple. PT: yes, ok. If A if finite dimensional simple, then it is $End(D)$ for some division ring D, and then A is the only irreducible bimodule [[★★★]]. Cl_n is simple as a *super algebra*, so therefore a spin structure on $\mathbb{R}^n$, called the *standard* spin structure on $\mathbb{R}^n$. The isomorphism class is given by πCl_n, the parity reversed bimodule. ⋄

Theo: where is the orientation? Andy: it comes from the right action of Cl_n. The right action by $e_1 \cdots e_n$ (note that we make this one choice once and for all) is equivalent to the left action of some orientation.

Definition 25.3. $\mathrm{Spin}\big((W_1, b_1, \sigma_1), (W_2, b_2, \sigma_2)\big) := \{(f, F) | f \in O(W_1, W_2), F\colon {}_{Cl(W_1,b_1)}\sigma_1{}_{Cl_n} \xrightarrow{\sim} {}_{Cl(W_1,b_1)}Cl(f)_*\sigma_2{}_{Cl_n}$ even bimodule iso$\}$ ⋄

Theorem 25.4. *The map* $\mathrm{Spin}\big((W_1, b_1, \sigma_1), (W_2, b_2, \sigma_2)\big) \to SO\big((W_1, b_1, o_{\sigma_1}), (W_2, b_2, o_{\sigma_2})\big)$*, given by* $(f, F) \mapsto f$ *is a connected double cover.*

[[break]]

Proof. [[★★★ HW 2]] □

The idea is Schur's lemma. In the easiest case, where $W_1 = W_2$, then let's check that over $f = \mathrm{id}$, there are two points. By Schur's lemma, an isomorphism must be a multiple of the identity, and it must be $\pm\,\mathrm{id}$ because it has to respect the inner product on the module.

Classical (Lagrangian) field theory $\mathcal{L}$:

- space-time Σ^d

- classical fields $\Phi(\Sigma)$
- classical action $\mathcal{A}\colon \Phi(\Sigma) \to \mathbb{R}$ (could depend on a finite number of deriviatives at a point)

(Step 1) From this we can get a classical Hamiltonian field theory $H_{\mathcal{L}}$ using the Euler-Lagrange equations. this is a functor from RB_d^Σ (where all the d-manifolds are submanifolds of Σ) to Symp, the category of symplectic manifolds (with a potential) with morphisms Lagrangians (with a function) (not quite a category because to compose Lagrangians, you need them to intersect cleanly). It is give by taking $H_{\mathcal{L}}(Y)$ to be the set of solutions of the Euler-Lagrange equations on Y for variations with compact support. Notice that this is a subset of $\Phi(Y)$ (sections of the pull-back bundle giving $\Phi(\Sigma)$). $H_{\mathcal{L}}(Y)$ comes equipped with a 1-form α, coming from variations on $\delta_{perm}Y$, as NR discussed. In good cases, $d\alpha$ is a symplectic structure. Let's assume that we're in a good case.

Given a morphism M from Y_0 to Y_1, then $H_{\mathcal{L}}(M)$ (solutions to the Euler-Lagrange equations on M) is a Lagrangian in $H_{\mathcal{L}}(\partial M)$. The punch line is that $\alpha|_{H_{\mathcal{L}}(M)} = d\mathcal{A}|_{\Phi(M)}$.

(Step 2) Quantization. Given any functor $H\colon \mathsf{RB}_d \to \mathsf{Symp}$, you want to quantize to a functor $Q_H\colon \mathsf{RB}_d \to \mathsf{ProjHilb}$ (projective Hilbert spaces). Note that we're throwing out the dependence on Σ. On objects, (M,α), you choose a polarization P (this is why it's an art, not a functor). NR picked one or two Lagrangians (L_1 and L_2); I'm picking a whole foliation by Lagrangians. Then you get $V = C^\infty(M/P)$ (where M/P is the space of leaves). If you assume existence of a polarization, then we want any other one will be connected to it (using the Hichen connection, for example).

On morphisms, Q uses the functional integral (path integral, but we're in higher dimensions). It isn't good enough to just compose with $H\colon \mathsf{Symp} \to \mathsf{ProjHilb}$ by taking solutions because you have to subdivide your morphism arbitrarily and then use the functor H. NR: there is functoriality in the other direction; semi-classical limit is functorial.

26 PT 11-27

Projects 11, 12: Stuff about spin structures.

Recall that if you have an inner product space W of dimension n, then a *spin structure* is a graded irreducible $Cl(W)$-Cl_n-$*$-bimodule. There are exactly two isomorphism classes of these bimodules. If you pick a spin structure, it makes sense to talk about a two-fold covering. In the case $W = \mathbb{R}^n$, you get the usual $\mathrm{Spin}(n)$. The reason this is good is that you can now define spin strucutures on vector bundles. If $E \to X$ is a vector bundle with fiberwise inner product, then a spin structure on E is an irreducible bimodule bundle $S \to X$ over the algebra bundles $Cl(E)$ and Cl_n (this one is trivial) which gives spin structures S_x on the fibers E_x. The good thing about this is that this bimodule bundle can be equipped with a natural connection as soon as you have a connection on E. Project 11 is to show that these are good definitions, and Project 12 is to show that if ∇ is a metric-preserving connection on E, then there is a unique connections ∇^S on any spin structure S on E (with some compatibility conditions, spelled out in the statement of the project).

The consequence is that it is very easy to write the Dirac operator on a spin manifold. Say X is a Riemannian spin manifold (i.e. we have a spin structure on the tangent bundle TX) and corresponding connection (coming from the Levi-Civita connection) ∇^S. Then the Dirac operator on X $\not{D}_X\colon \Gamma(S) \xrightarrow{\nabla^S} \Omega^1 X \otimes \Gamma(S) \cong_g Vect(X) \otimes \Gamma(S) \xrightarrow{\text{left } Cl(TX) \text{ action}} \Gamma(S)$. This is the Cl_n-linear Dirac operator (also called the Atiyah-Singer operator). The nice thing about having a Cl_n-linear operator is that $\ker \not{D}_X$ is a finite dimensional graded Cl_n-module. The Dirac operator is an odd operator, so the kernel is actually graded. Finite dimensional comes from ellipticity. Usually, you define the index to be the dimension of this kernel. Here we take the dimension of the even part of the kernel minus the dimension of the odd part. This is not a good definition, because it totally ignores the Cl_n action. This is the right invariant if you use the usual Dirac operator. The real thing to do is take the kernel itself, and consider it as $\ker \not{D}_X \in \{\text{isoclasses of finite dimensional graded } Cl_n\text{-modules}\}$. This will depend on the metric. What if we change the metric on the same topological manifold? Because you impose this Cl_n action, the isomorphism class of $\ker \not{D}_X$ will only change by restrictions of finite dimensional Cl_{n+1}-modules. Thus, you can think of $[\ker \not{D}_X] \in$

{f.d. Cl_n-mods}/{f.d. Cl_{n+1}-mods} $\cong KO_n(pt) \cong KO^{-n}(pt)$, which is $\mathbb{Z}$ if $n|4$, $\mathbb{Z}/2$ if $n \equiv 1, 2\text{-}\mathsf{mod}4$, and 0 otherwise.

Theorem 26.1. *If the scalar curvature of g is positive, then* $[\ker \not{D}_X] = 0$.

So we get some obstruction to having positive scalar curvature. For example, there are 9-dimensional Riemannian manifolds which are homeomorphic to S^9, but do not have positive scalar curvature.

Now I'd like to go back to the first class, where I told you that this stuff is related to cohomology theories. I'll start in arbitrary dimension (I think we'll eventually figure out how to do everything in arbitrary dimension). Plan for the next lectures.

Explain $(d|1)$-dimensional (if you change the 1 to a 0, you get boring zeros for everything) Euclidean (Euclidean signature) field theories over X of degree n. We'll call this $EFT^n_{d|1}(X)$. Then $EFT^n_{d|1}[X] =$
$$EFT^n_{d|1}(X)/\text{concordance} \cong \begin{cases} H^n_{dR}(X) & d=0 \\ KO^n(X) & d=1 \\ TMF^n(X)? & d=2 \end{cases}.$$
Today, we'll do $n = 0$ and $(d|0)$-dimensional stuff (that is, show that it's boring). We'll do $TFT_d(X)$, topological field theory.

Reminder: there is a symmetric monoidal bicategory B_d, whose objects are closed oriented smooth $(n-1)$-manifolds Y with germy collars (but no metric). The 1-morphisms $\mathsf{B}_d(Y_0, Y_1)$ are bordisms with the usual embeddings. The 2-morphisms are diffeomorphisms relative boundary. You can mod out by 2-morphisms to get a category, but I don't want to do that. The symmetric monoidal structure is disjoint union.

There is a very easy variation of the theme. Given a manifold X (which you think of as the target thing that you're trying to compute the cohomology of), we can define $\mathsf{B}_d(X)$ by adding smooth maps to X to everthing in sight (i.e. an object is a manifold with a smooth map to X, a morphism is a bordism with a smooth map to X, and a 2-morphism is a diffeomorphism commuting with the maps to X).

Remark 26.2. A smooth map $\phi\colon X_1 \to X_2$ induces a symmetric monoidal bifunctor $\mathsf{B}_d(X_1) \to \mathsf{B}_d(X_2)$ by composing everything with ϕ. So roughly speaking, a field theory is a representation of this category. Recall that given a group G (or a monoid or something), then a representation is the same thing as a functor from $\mathcal{C}_G \to \mathsf{Vect}$, where $\mathcal{C}_G$ is the category with one object $*$ and $\mathcal{C}(*, *) = G$. Bordism categories are covariant, but we're going to take representations, so we'll get something contravariant. ◇

Instead of Vect, the target bicategory is Fr_2, whose objects are separable Frechét spaces, morphisms are continuous linear maps, and the only 2-morphisms are identities.

Remark 26.3. bifun($\mathsf{B}_d(X), \mathsf{Fr}_2$) =fun($\mathsf{B}_d(X)$/2-morphisms, Fr) (actually an isomorphism of categories), so we're not changing the definition of a QFT, just making things more complicated. But this complication is important. ◇

[[break]]

Over the break, we decided there is an adjunction between taking a category and thinking of it as a bicateogry and collapsing all 2-morphisms. The functors forms a category, and the bifunctors form a bicategory (which turns out just to be a category), and you get an isomorphism of categories.

Now we have to do the major additional step, which is to formulate the smoothness condition on such a bifunctor from $\mathsf{B}_d(X)$ to Fr_2.

Example 26.4. Let's do $d = 0$ right now. $\mathsf{B}_0(X)/2\text{-mor} \to \mathsf{Fr}$. The unique object in $\mathsf{B}_0(X)$ is $\varnothing$. $\mathsf{B}_0(X)(\varnothing, \varnothing) = \{\Sigma^0 \to X\}/\text{diffeo} \supseteq X$, where Σ^0 is a compact 0-manifold (a bunch of points). We have that $\varnothing \mapsto \mathbb{C}$ because the functor must be monoidal. If Σ^0 is a single point, then we get a copy of X in the hom set. Since the functor must be monoidal, the whole functor is determined by where the points in X go (they go to some scalar in $\mathbb{C}$). Thus, symmetric monoidal functors from $\mathsf{B}_0(X)$/2-mor to Fr is the same thing as $Map(X, \mathbb{C})$.

We want to do two things. First, we really want to restrict to smooth functors, so we want to restrict to $C^\infty(X)$. Secondly, we really want closed forms on X, and this will come from the supersymmetry. $\Omega^*(X) \cong C^\infty(\pi TX) \cong C^\infty(\underline{\mathsf{SMan}}(\mathbb{R}^{0|1}, X))$ (C^∞ on the *super points* of X). Just the smoothness gives us TFT. ◇

So what does it mean that a functor is smooth? We have to put more structure on these categories. The idea is that a smooth map is something which takes smooth functions to smooth functions. If we enrich the categories a bit to say that some of the morphisms are smooth, then we can use this as a definition. First, we need to make family versions of these (bi)categories $\mathsf{B}_d(X)$ and Fr_2 over Man_2. This is like in algebraic geometry, where you always work over some base scheme.

First we define a Grothendieck site Man_2, and put smooth structures (fiber functors) $\mathsf{B}_d(X)^{fam} \to \mathsf{Man}_2$ and $\mathsf{Fr}_2^{fam} \to \mathsf{Man}_2$. Then a smooth functor is one which respects these extra maps to Man_2. If we look at the fiber over the point in Man_2, we get the old story.

To define Man_2, we have to be a little more clever than the stupid adding of identity 2-morphisms. Man_2 is the bicategory of correspondences between smooth manifolds. The objects are smooth manifolds S (because there will be obvious generalizations to super manifolds). The 1-morphisms from S_0 to S_1 are correspondences between S_0 and S_1. A correspondence is a third manifold S, with maps $f\colon S \to S_1$ and $p\colon S \twoheadrightarrow S_0$ (we require p to be a submersion). The 2-morphisms from $(S_1 \xleftarrow{f} S \xrightarrow{p} S_0)$ to $(S_1 \xleftarrow{f'} S' \xrightarrow{p'} S_0)$ is a morphism $\phi\colon S \to S'$ making the two triangles commute. The vertical composition is easy. The horizontal composition of 1-morphisms is more interesting; you take pull-back:

$$\begin{tikzcd}[column sep=small]
 & & S_{20} \arrow[dl] \arrow[dr] & & \\
 & S_{21} \arrow[dl] \arrow[dr] & & S_{10} \arrow[dl] \arrow[dr] & \\
S_2 & & S_1 & & S_0
\end{tikzcd}$$

There is a sub-bicategory where you require $p = \text{id}$. That is the category of smooth manifolds!

27 PT 11-29

Definition 27.1. $TFT_d(X) = smBiFun^{pb}_{\mathsf{Man}_2}(\mathsf{B}_d(x)^{fam}, \mathsf{Fr}_2^{fam})$ (symmetric monoidal bifunctors; pullback preserving)
$EFT_d(X) = smBiFun^{pb}_{\mathsf{Man}_2}(\mathsf{RB}_d(x)^{fam}, \mathsf{Fr}_2^{fam})$
$TFT_{d|1}(X) = smBiFun^{pb}_{\mathsf{SMan}_2}(\mathsf{B}_{d|1}(x)^{fam}, \mathsf{Fr}_2^{fam})$
$EFT_{d|1}(X) = smBiFun^{pb}_{\mathsf{SMan}_2}(\mathsf{RB}_{d|1}(x)^{fam}, \mathsf{Fr}_2^{fam})$ ◇

I have to explain this family stuff. Fr_2^{fam} has objects smooth Frechét bundles $V \to S$ over manifolds S. We have the forgetful bifunctor $\mathsf{Fr}_2^{fam} \to \mathsf{Man}_2$. Recall that the 1-morphisms in Man_2 are correspondences $S_1 \leftarrow S \twoheadrightarrow S_0$. A 1-morphism in Fr_2^{fam} is a bundle map $\tilde{f}\colon p^*V_0 \to V_1$ lying over $f\colon S \to S_1$.

$$\begin{tikzcd}
V_1 \arrow[d] & p^*V_0 \arrow[l, "\tilde{f}"'] \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & V_0 \arrow[d] \\
S_1 & S \arrow[l, "f"'] \arrow[r, two heads, "p"] & S_0
\end{tikzcd}$$

2-morphisms are just morphisms $S \to S'$ as in Man_2 (i.e. all the data of the 2-morphisms is preserved under the forgetful functor to Man_2).

What is missing so far is "what are symmetric monoidal bicategories?" This is a subtle question. It is not totally obvious. What are symmetric monoidal categories? If you write it down very carefully, there are lots of natural transformations we left out (like the associator). If you're willing to go into higher categories, then a monoidal category is the same thing as a bicategory with one object. Remember that a bicategory has objects $\mathcal{C}_0$, 1-morphisms $\mathcal{C}_1$, and 2-morphisms $\mathcal{C}_2$, with maps $\begin{tikzcd} \mathcal{C}_2 \arrow[r, shift left] \arrow[r, shift right] & \mathcal{C}_1 \arrow[l, bend right, hook'] \arrow[r, shift left] \arrow[r, shift right] & \mathcal{C}_0 \arrow[l, bend right, hook'] \end{tikzcd}$. If $\mathcal{C}_0 = \{*\}$, then $\mathcal{C}_1(*, *)$ is a cateogry (using $\mathcal{C}_2$ as morphisms). Horizontal composition is exactly a monoidal structure on this category.

A braided monoidal category is one where there is a preffered isomorphism between $X \otimes Y$ and $Y \otimes X$ (in general, they need not be isomorphic at all). This is exactly a tricategory with one object and one 1-morphism! Again, there is no reason for the tensor structure to be symmetric. A quad-category with one object, one 1-morphism, and one 2-morphism is a symmetric monoidal category.

If we want to talk about monoidal categories, we have to shift everything by one. A symmetric monoidal category is therefore a pentacategory with one object, 1-morphism, and 2-morphism. This looks like a mess, but it turns out that you can define n-categories using simplicial sets. Some people (with first names Chris and Andre) came up with an interpretation which only uses bicategories, but they have to be internal to some other category. In their language, the pullback preserving part is built in. I don't want to go into this right now. Maybe next week.

Definition 27.2. If $\mathcal{C}$ is a bicategory and $x \in \mathcal{C}$, then the *loop category at* x is $\Omega_x\mathcal{C}$ is $\mathcal{C}(x, x)$. Note that this $\Omega_x\mathcal{C}$ is a monoidal category. ◇

Today, I want to apply the loop functor to the things in the first definition.

Example 27.3. $\Omega_\varnothing\mathsf{B}_d$ has objects closed oriented smooth d-manifolds and $\Omega_\varnothing\mathsf{B}_d(\Sigma_0, \Sigma_1) = \mathit{Diff}(\Sigma_0, \Sigma_1)$. The monoidal structure is disjoint union. We used to say that disjoint union is the monoidal structure on the bicategory B_d, but now we're just thinking of B_d as a bicategory (without any monoidal structure) and the disjoint union structure pops out (though we don't get disjoint union of $(d-1)$-manifolds)

If we loop $\mathsf{B}_d(X)$, you get closed manifolds with maps to X and diffeomorphisms respecting the map to X. ◇

Example 27.4. $\Omega_\varnothing\mathsf{Man}_2$ is boring because there are no maps to $\varnothing$, so there aren't any non-trivial correspondances. Let's look at $\Omega_*\mathsf{Man}_2$. It has objects $* \leftarrow S \twoheadrightarrow *$ and morphisms are just morphisms of the S's. So $\Omega_*\mathsf{Man}_2 = \mathsf{Man}$. The monoidal structure is product.

In topology, you sometimes ask if one space is the loop space of another, and the answer is usually no (e.g. $\pi_1(\Omega X) = \pi_2(X)$ is abelian). Here we find that we can "de-loop" Man. ◇

Example 27.5. $\Omega_\mathbb{C}\mathsf{Fr}_2$ has objects morphisms from $\mathbb{C}$ to $\mathbb{C}$, and 2-morphisms are all identities. So this is the set $\mathbb{C}$, with only identity morphisms. ◇

Now let's start looping some stuff. A symmetric monoidal bifunctor must take the monoidal unit to the monoidal unit, so we'll loop everthing at the monoidal unit. $\mathit{TFT}_d(X) = \mathit{smBiFun}_{\mathsf{Man}_2}(\mathsf{B}_d(X)^{fam}, \mathsf{Fr}_X^{fam})$ gives $\mathit{smFun}_{\mathsf{Man}}(\Omega_\varnothing\mathsf{B}_d(X)^{fam}, \Omega_\mathbb{C}\mathsf{Fr}_2^{fam})$. What are these things? Let's do $\Omega_\mathbb{C}\mathsf{Fr}_2^{fam}$ first. Our object is the vector bundle $\mathbb{C} \to pt$. The objects are pairs (S, f) where $f\colon S \to \mathbb{C}$. The morphisms from (S_0, f_0) to (S_1, f_1) are maps $g\colon S_0 \to S_1$ so that $f_1 \circ g = f_0$. This is $\mathcal{C}(\mathbb{C})$, where $\mathcal{C}(X)$ is the category with objects $(S, f\colon S \to X)$ and morphisms $g\colon S_0 \to S_1$ with $f_1 \circ g = f_0$. This is the "functor of points". Later, we'll change Man to SMan. So an element of $\mathit{smFun}_{\mathsf{Man}}(\Omega_\varnothing\mathsf{B}_d(X)^{fam}, \mathcal{C}(\mathbb{C}))$ is a number for every closed manifold. Restricting to connected closed manifolds (and using the fact that the functors are monoidal), we see that this is $\mathit{Fun}_{\mathsf{Man}}(\Omega_\varnothing\mathsf{B}_d^{conn}(X)^{fam}, \mathcal{C}_{\mathsf{Man}}(\mathbb{C}))$.

We loose information about the $(d-1)$-manifolds, but for $d = 0$, we don't loose any information by taking the loops. Moreover, for $d = 0$, there is only one connected d-manifold, a point. So in this case, we have $\mathit{Fun}\big(\mathcal{C}_{\mathsf{Man}}(X), \mathcal{C}_{\mathsf{Man}}(\mathbb{C})\big)$. After the break, we'll calculate this thing.

[[break]]

Arturo pointed out the following. The category $\mathcal{C}_{\mathsf{Man}}(X)$ is a well-known thing. If you have a category $\mathcal{D}$ and $x \in \mathcal{D}$, then you can define the *over category* $\mathcal{D}(x)$, whose objects are morphisms to x and whose morphisms are commutative diagrams. $\mathcal{C}_{\mathsf{Man}}(X)$ is exactly the over category $\mathsf{Man}(x)$. Note that $\mathcal{D}(x)$ has a forgetful functor to $\mathcal{D}$.

So the thing we're interested in is $\mathit{Fun}_{\mathsf{Man}}(\mathsf{Man}(X), \mathsf{Man}(\mathbb{C}))$.

Lemma 27.6 (2-Yoneda lemma)**.** $\mathcal{D}(X, Y) \cong_\Phi \mathit{Fun}_\mathcal{D}\big(\mathcal{D}(X), \mathcal{D}(Y)\big)$, *given by* $g \mapsto (f \mapsto g \circ f)$*, and the inverse is given by* $F \mapsto F(\mathrm{id}_X)$*.*

So We have $\mathsf{Man}(X, \mathbb{C}) = C^\infty(X)$. Thus, 0-dimensional topological field theories over X are just functions on X. This is probably the most complicated way to explain what a function is. The power of this is that we can change d to 1, 2, 3, etc.. When you take $d = 1$, you get connections; when $d = 2$, there isn't already a name for it. Since we don't want functions (we want forms), we'll use super manifolds.

If we change B_0 to RB_0, nothing changes because we're just adding a Riemannian metric to a point. This is very different if you take $d = 1$ because a circle has a length. Now let's do $\mathit{TFT}_{d|1}$. As before, we loop our definition (not loosing anything since $d = 0$) to get $\mathit{smFun}_{\mathsf{SMan}}(\Omega_\varnothing\mathsf{B}_{0|1}(X)^{fam}, \Omega_\mathbb{C}\mathsf{Fr}_2^{fam}) = \mathit{Fun}_{\mathsf{SMan}}(\Omega_\varnothing\mathsf{B}_{0|1}^{conn}(X)^{fam}, \Omega_\mathbb{C}\mathsf{Fr}_2^{fam})$. We have that $\Omega_\varnothing\mathsf{B}_{0|1}^{conn}(X)^{fam}$

has objects $S \xleftarrow{p_1} S \times \mathbb{R}^{0|1} \xrightarrow{f} X$, or pairs $(S, \tilde{f})$, where $\tilde{f}\colon S \to \underline{\mathsf{SMan}}(\mathbb{R}^{0|1}, X) \cong \pi T X$. So we might think that this category is $\mathsf{SMan}(\pi T X)$. This is not true. As before, $\Omega_{\mathbb{C}}\mathsf{Fr}_2^{fam} = \mathsf{SMan}(\mathbb{C})$. If we did get $\mathsf{SMan}(\pi T X)$, we would get that $TFT_{0|1}(X)$ are forms on X, and we want closed forms on X, so it would be the wrong answer anyway. What is a morphism in $\Omega_{\varnothing}\mathsf{B}_{0|1}^{conn}(X)^{fam}$? They are

$$\begin{tikzcd}
 & X & \\
S_0 \times \mathbb{R}^{0|1} \arrow[ur] \arrow[rr, "G"'] \arrow[d] & & S_1 \times \mathbb{R}^{0|1} \arrow[ul] \arrow[d] \\
S_0 \arrow[rr, "g"] & & S_1
\end{tikzcd}$$

where G is a fiberwise automorphism of $\mathbb{R}^{0|1}$. This is where it depends if we're doing TFT or EFT. If you don't put an inner product on $\mathbb{R}^{0|1}$, dilation is an automorphism. It turns out that TFT will only give you closed 0-forms and EFT gives you all closed forms. So topological field theories give you boring constant functions. To get de Rham cohomology, you need to use EFT.

28 PT 12-04

Next Tuesday, we'll have the mini-conference in 939.

Degree 0 cohomology groups, non-local for $d = 2$.

$$\begin{aligned} \mathit{TFT}_d(X) &:= \mathit{smBiFun}^{pb}_{\mathsf{Man}_2}(\mathsf{B}_d(X)^{fam}, \mathsf{Fr}_2^{fam}) \\ &\xrightarrow[\text{conn. bord.}]{\Omega_e} \mathit{Fun}_{\mathsf{Man}}(\Omega_\varnothing \mathsf{B}_d^{conn}(X)^{fam}, \Omega_\mathbb{R}\mathsf{Fr}_2^{fam}) \\ &=_{d=0} \mathit{Fun}(\mathsf{Man}(X), \mathsf{Man}(\mathbb{R})) \\ &\cong \mathsf{Man}(X, \mathbb{R}) = C^\infty(X) \end{aligned}$$

Similarly,

$$\begin{aligned} \mathit{TFT}_{d|1}(X) &:= \mathit{smBiFun}^{pb}_{\mathsf{SMan}_2}(\mathsf{B}_{d|1}(X)^{fam}, \mathsf{Fr}_2^{fam}) \\ &\xrightarrow[\text{conn. bord.}]{\Omega_e} \mathit{Fun}_{\mathsf{SMan}}(\Omega_\varnothing \mathsf{B}_{d|1}^{conn}(X)^{fam}, \Omega_\mathbb{R}\mathsf{Fr}_2^{fam}) \\ &= \mathit{Fun}(\Omega_\varnothing \mathsf{B}_{d|1}^{conn}(X)^{fam}, \mathsf{Man}(\mathbb{R})) \end{aligned}$$

If you weren't doing family versions, a symmetric monoidal bifunctor from $\mathsf{B}_d(X)$ to Fr_2 is the same as a functor from $\mathsf{B}_d(X)/2$-morphisms to Fr, but this doesn't work in the family version.

By the way, for $d = 0$, *TFT* and *EFT* agree (since a Riemannian metric on a point is not extra information). An object in $\Omega_\varnothing \mathsf{B}_{d|1}(X)^{fam}$ is of the form

$$\begin{tikzcd} F^{d|1} \arrow[d] & \\ \Sigma \arrow[r, "f"] \arrow[d] & X \\ S^{m|n} & \end{tikzcd}$$

Since $d = 0$ and we require connected, $F = \mathbb{R}^{0|1}$. There are non-trivial $\mathbb{R}^{0|1}$-bundles on S (parity reverses of non-trivial line bundles), but locally it is trivial (and since we're working over SMan, it is enough to understand stuff locally), so we may assume $\Sigma = S \times \mathbb{R}^{0|1}$. But a map from $S \times \mathbb{R}^{0|1}$ to X is the same as a map from S to $\underline{\mathsf{SMan}}(\mathbb{R}^{0|1}, X) = \pi T X$.

So the first candidate for $\mathit{TFT}_{0|1}(X)$ is $\mathit{Fun}_{\mathsf{SMan}}(\mathsf{SMan}(\pi T X), \mathsf{SMan}(\mathbb{R}))$. If this were true, this would just be differential forms on X. This isn't quite right because the morphisms in $\Omega_\varnothing \mathsf{B}_{d|1}^{conn}(X)^{fam}$ are not the same as the morphisms in $\mathsf{SMan}(\pi T X)$. A morphism in $\Omega_\varnothing \mathsf{B}_{d|1}^{conn}(X)^{fam}$ is a fiberwise diffeomorphism over a map from S_0 to S_1:

$$\begin{tikzcd} & X & \\ S_0 \times \mathbb{R}^{0|1} \arrow[ur, "f_0"] \arrow[rr, "G"] \arrow[d] & & S_1 \times \mathbb{R}^{0|1} \arrow[ul, "f_1"'] \arrow[d] \\ S_0 \arrow[rr, "g"] & & S_1 \end{tikzcd}$$

This G must be of the form $(g \circ p_1, \gamma \colon S_0 \times \mathbb{R}^{0|1} \to \mathbb{R}^{0|1}) = (g \circ p_1, \tilde{\gamma} \colon S_0 \to \underline{\mathrm{Aut}}(\mathbb{R}^{0|1}))$, where $\underline{\mathrm{Aut}}(\mathbb{R}^{0|1}) \subseteq \underline{\mathsf{SMan}}(\mathbb{R}^{0|1}, \mathbb{R}^{0|1}) = \pi T \mathbb{R}^{0|1} = \mathbb{R}^{1|1}$ is $\mathbb{R}^{0|1} \rtimes \mathbb{R}^\times$ (translations times dilations). These are the S-points of a *transport category.*

Let's talk about it for Man first. If I have a manifold Y and a Lie group G acting on Y (think $Y = \pi T X = \underline{\mathsf{SMan}}(\mathbb{R}^{0|1}, X)$ and $G = \underline{\mathrm{Aut}}(\mathbb{R}^{0|1})$). The *transport category* has object set Y and morphism set $Y \times G$, where (y, g) has source and target y and $y \cdot g$, respectively. Now we're taking the S-points of this [[★★★ This transport category is a groupoid object in Man, and we take S-points of this object to get a groupoid (but we let S vary, so we're probably talking about a category fibered in groupoids)]]. Define $\mathsf{Man}(Y; G)$ to be the category with objects (S, f) where $f \colon S \to Y$ and morphisms

$$\begin{tikzcd} Y \times G \arrow[r, "\mu"] & Y \\ S_0 \arrow[u, "f_0 \times \gamma"] \arrow[r, "g"] & S_1 \arrow[u, "f_1"'] \end{tikzcd}$$

The lemma is that $\Omega_\varnothing \mathsf{B}_{d|1}^{conn}(X)^{fam} = \mathsf{SMan}(\pi T X, \underline{\mathrm{Aut}}(\mathbb{R}^{0|1}))$ (to get this to come out right, we actually have to define $\tilde{\gamma}$ in the obvious way, but composing with the inverse map on $\underline{\mathrm{Aut}}(\mathbb{R}^{0|1})$).

Now we can finish the calculation.

Lemma 28.1. *Given Y with a G-action and any $Z \in \mathsf{SMan}$, $\mathit{Fun}_{\mathsf{SMan}}(\mathsf{SMan}(Y; G), \mathsf{SMan}(Z)) \cong \mathsf{SMan}(Y, Z)^G := \{\alpha \in \mathsf{SMan}(Y, Z) | \alpha \circ$*

$p_1 = \alpha \circ \mu\}$ *(where* $\mathsf{SMan}(Y;G)$ *is the transport category and* $Y \times G \underset{\mu}{\overset{p_1}{\rightrightarrows}} Y \xrightarrow{\alpha} Z$ *).*

So the final outcome is that $TFT_{0|1}(X) = \mathsf{SMan}(\pi TX, \mathbb{R})^{\underline{\mathrm{Aut}}(\mathbb{R}^{0|1})}$. We'll see that this is exactly closed differential forms of degree zero (i.e. constant functions on X). If we put some geometry on, then we'll get all the other closed forms. It will turn out that $EFT_{d|1}(X) = \Omega^{even}_{cl}(X)$.

[[break]]

(1) Theo made the comment that when you go from Man to SMan, you have to take super Frechét spaces in place of Frechét spaces. (2) the lemma should be that $\Omega_\varnothing \cdots$ is the stackification of something.

Remember that $\mathsf{SMan}(\pi TX, \mathbb{R}) = C^\infty(\pi TX)^{ev} = \Omega^{even}(X)$.

Lemma 28.2. *$\mu \colon \pi TX \times (\mathbb{R}^{0|1} \rtimes \mathbb{R}^\times) \to \pi TX$.*

$$C^\infty(\pi TX \times \underline{\mathrm{Aut}}(\mathbb{R}^{0|1})) \cong \Omega^*(X) \otimes {\textstyle\bigwedge}[\theta] \otimes C^\infty(\mathbb{R}^\times) \xleftarrow{\mu^*} \Omega^*(X) = C^\infty(\pi TX)$$

is given by $\omega \mapsto \omega \otimes 1 \otimes s^n + d\omega \otimes \theta \otimes s^n$*, where* ω *is of degree* n*. And* $p_1^* \colon \omega \mapsto \omega \otimes 1 \otimes 1$

Looking at when μ^* and p_1^* agree, we see by looking at the degree in θ and noting that $s^n \neq 0$ that $d\omega = 0$ and $s^n = 1$, so $n = 0$. Thus, we get closed forms of degree 0.

Corollary 28.3. $\mathsf{SMan}(\pi TX, \mathbb{R})^{\underline{\mathrm{Aut}}(\mathbb{R}^{0|1})} = \Omega^0_{cl}(X)$*.*

Corollary 28.4. $\mathsf{SMan}(\pi TX, \mathbb{R})^{\mathbb{R}^{0|1} \rtimes \{\pm 1\}} = \Omega^{ev}_{cl}(X)$*.*

From this we get an idea of what a super Riemannian metric on $\mathbb{R}^{0|1}$ should be.

Definition 28.5. A *super Riemannian metric* on $\mathbb{R}^{0|1}$ is something such that $\underline{Isom}(\mathbb{R}^{0|1}$,something$) \cong \mathbb{R}^{0|1}) \rtimes \{\pm 1\} \subseteq \underline{\mathrm{Aut}}(\mathbb{R}^{0|1})$. ◇

The analogue of a Riemannian structure more or less works.

In the last class (Thursday), we'll talk about degree n EFTs (we've been doing degree 0). This gives us lots of different things.

$d = 0$ $TFT^n_{0|1}(X) = \Omega^n_{cl}(X)$
$d = 1$ introduces Cl_n-modules $KO^n(X)$
$d = 2$ gives modular forms of weight $n/2$

29 PT 12-06

Tuesday's mini-conference in 939 starts at 9:00. It will probably go til 17:00. There will be 10 speakers. Bring lunch. There will be a sponsored dinner.

I was going to explain twisted field theory, but I decided that would just be more definitions. So instead, let's use the definitions we have already.

Recall that we found a very complicated way to talk about constant functions $\Omega^0_{cl}(X) = TFT_{0|1}(X) = \mathsf{SMan}\big(\pi TX/\underline{\mathrm{Aut}}(\mathbb{R}^{0|1}), \mathbb{C}\big)$. πTX is a super manifold with a Lie group acting on it, so you get a quotient stack, and we're considering functions on this stack. Being invariant under translations makes the forms closed and being invariant under dilations makes them degree 0. I wanted to do twisted stuff, which would be $\Omega^n_{cl}(X) \cong TFT^n_{0|1}(X)$.

If you put a geometry on a super point (so that $\mathrm{Aut}(\mathbb{R}^{0|1})$ is only translations), you get $EFT_{0|1}(X) = \Omega^{ev}_{cl}(X)$. Consider the map $EFT_{0|1}(X) \xrightarrow{\times S^1} EFT_{1|1}(X)$. There is a nice map (due to Florin Damitrescu), which is super parallel transport from vector bundles with connection to $EFT_{1|1}(X)$. Fei Han's thesis shows that the map from vector bundles with connection to $\Omega^{ev}_{cl}(X;\mathbb{C})$ is the Chern character form. Fei will talk about this in the student seminar next semester. Andy: what happens when you twist? PT: you get mod 2 periodicity, and if you put 1, you get odd forms.

Today I want to explain how the usual "susy cancellations" lead to modularity of the partition function ($d = 1, 2$). We'll take degree 0, and assume X is a point.

$$\begin{aligned} EFT_1 &= smBiFun_{\mathsf{Man}_2}(\mathsf{RB}^{fam}_1, \mathsf{Fr}^{fam}_2) \\ &\xrightarrow[\text{conn}]{\Omega_\varnothing} Fun_{\mathsf{Man}}(\Omega^{\text{conn}}_\varnothing \mathsf{RB}^{fam}_1, \Omega_{\mathbb{C}} \mathsf{Fr}^{fam}) \\ &= Fun_{\mathsf{Man}}\big(\Omega^{\text{conn}}_\varnothing \mathsf{RB}^{fam}_1, \mathsf{Man}(\mathbb{C})\big) \end{aligned}$$

What is $\Omega^{\text{conn}}_\varnothing \mathsf{RB}^{fam}_1$? The objects are Riemannian S^1-bundles $\Sigma \to S$ and the morphisms are bundle maps that are fiberwise isometries. This is the moduli stack of Riemannian circles. Riemannian circles are classified by their length, so this stack is $\mathbb{R}_+/SO(2)$ (the action is trivial). Thus, this functor space is just $\mathsf{Man}(\mathbb{R}_+/\mathbb{C}) = C^\infty(\mathbb{R}_+)$ (you don't see the rotations when you map to a representable stack). So given $E \in EFT_1$, you get the 1-dimensional *partition function* $Z_E(t) \in C^\infty(\mathbb{R}_+)$, given by $E(S^1_t)$. If E is the σ-model of a compact Riemannian manifold M, then $Z_E(t) = \mathrm{tr}(e^{-t\Delta})$.

An element of EFT_1 assigns to a point a Hilbert space and to intervals linear maps. When you glue the two ends of the interval together, you get a circle. The value of the theory on a circle is given by the partition function. But from the gluing law, you know it will be the trace $\mathrm{tr}(E([0,t]))$.

Theorem 29.1. *If E is susy (i.e. $E \in EFT_{1|1}$, not just EFT_1), then Z_E is a constant integer. That is, $\frac{\partial}{\partial t} Z_E = 0$ and Z_E is an integer.*

Before explaining the assumption (that E is susy), let me give an example of a supersymmetric field theory.

Example 29.2. $SE \in EFT_{1|1}$ from a compact Riemannian *spin* manifold (M, σ, g). I haven't told you exactly what the geometry on a super point or a super interval is. Remember that the objects aren't just points (they are little collars). Since the geometry on a point is a geometry on an $\mathbb{R}^{0|1}$ with some thickenning to the collar, it turns out that the isometries of the thickenned super point are just reflection (so $\{\pm 1\}$). The $\mathbb{Z}/2$-action is the same as a $\mathbb{Z}/2$-grading. So $SE(spt) = \mathbb{Z}/2$-graded Hilbert (really Frechét) space. So we define $SE(spt)$ to be $\Gamma_{L^2}(S_M)$.

Remember that $E \in EFT_1$ gives a smooth semi-group homomorphism $\mathbb{R}_+ \to B^{tc}(H)$ (bounded trace class operators) (the $\mathbb{R}_+$ comes from the moduli space of Riemannian circles), given by $t \mapsto E([0,t])$. If you have a super symmetric $SE \in EFT_{1|1}$, you get a super semi-group homomorphism $\mathbb{R}_+ \times \mathbb{R}^{0|1} \to B^{tc}(H)$. The $\mathbb{R}_+ \times \mathbb{R}^{0|1}$ is the moduli stack of Riemannian intervals, with super group structure $(t_1, \theta_1)(t_2, \theta_2) = (t_1 + t_2 + \theta_1\theta_2, \theta_1 + \theta_2)$; the Lie algebra is free on one odd generator. [[★★★ from some stuff]] you see that $E([0,t])$ must be of the form e^{-tA} and $SE(t, \theta)$ must be of the form $e^{-tD^2 + \theta D}$, where D is an odd operator on H.

For our example, we take D to be the Dirac operator. We needed compactness to get $e^{-tD^2+\theta D}$ to be trace class. ◇

So now the theorem statement makes sense, taking $A = D^2$. $Z_{SE}(t) = \mathrm{str}(e^{-tD^2})$. Now we're back to what the physicists showed us a few decades ago, susy cancellation.

Let's calculate $\mathrm{str}(e^{-tD^2})$. Pretend that we can diagonalize D^2 (self-adjointness comes from symmetry if the interval when you reflect), so $H = \bigoplus_\lambda E_\lambda$, where λ are the eigenvalues of D^2.

$$\mathrm{str}(e^{-tD^2}) = \sum_\lambda e^{-t\lambda}\,\mathrm{sdim}\, E_\lambda = \mathrm{sdim}\, E_0 = \mathrm{sdim}\ker(D) = \mathrm{index}(D)$$

We have the operator $D\colon E_\lambda \to E_\lambda$ (because D commutes with D^2). The E_λ are graded, and D is an odd operator. If $\lambda \neq 0$, then D is an isomorphism of E_λ, but then the super dimension is zero.

[[break]]

Now consider

$$\begin{aligned} EFT_2 \xrightarrow[\text{conn}]{\Omega_\varnothing} &\ Fun_{\mathsf{Man}}\big(\Omega^{cl}_\varnothing \mathsf{RB}_2, \mathsf{Man}(\mathbb{C})\big) \\ &= C^\infty(h \times \mathbb{R}_+) \end{aligned} \qquad (*)$$

We assume that we only use *flat* surfaces because we want to do elliptic cohomology (the only compact flat manifolds are tori, which are exactly the elliptic curves). Maybe this should be called FFT. We also restrict to flat Riemannian manifolds in RB^{cl}_2. So $\Omega_\varnothing \mathsf{RB}_2$ is the moduli space of flat tori, which is $h \times \mathbb{R}_+ / SL_2(\mathbb{Z})$ (where h is the upper half plane, the conformal part, and the $\mathbb{R}_+$ is the area of the torus). The $SL_2(\mathbb{Z})$ action is given by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)(\tau, \ell) = \left(\frac{a\tau+b}{c\tau+d}, |c\tau + d| \cdot \ell\right)$. Note that you change the area of the torus when you act, so you can't just say "I'll just work with area 1 tori".

The map in $(*)$ is given by $E \mapsto E(T_{\tau,\ell}) =: Z_E(\tau, \ell)$, which is called the *partition function.* We have

$$\begin{aligned} \{\text{integral modular fuctions}\} &\subseteq_d \{\text{modular function}\} \\ &\subseteq_c Hol(h)^{SL_2\mathbb{Z}} \\ &\subseteq_b C^\infty(h)^{SL_2\mathbb{Z}} \\ &\subseteq_a C^\infty(h \times \mathbb{R}_+)^{SL_2\mathbb{Z}} \ni Z_E \end{aligned}$$

(a) says $\frac{\partial}{\partial \ell} Z_E = 0$. (b) says $\frac{\partial}{\partial \bar{q}} Z_E$.

There is a map $h \to D^2 \smallsetminus 0$ (interior of the disk minus the center), given by $\tau \mapsto e^{2\pi i \tau} =: q$. $f(\tau)$ is a function of q if f is invariant under $\left(\begin{smallmatrix} 1 & k \\ 0 & 1 \end{smallmatrix}\right) \subseteq SL_2\mathbb{Z}$. f is modular if $f(q)$ can be extended to 0 with a pole of finite order. (c) says Z_E is modular.

Theorem 29.3. *If $SE \in EFT_{2|1}$, then $Z_E(\tau, \ell)$ is an integral modular function (i.e. statements a,b,c, and d are true).*

For the proof for $d = 1$, we realized that

$$Z_E(t) = E(S^1_t) =_{\text{gluing}} \operatorname{str} E([0,1]) = \operatorname{str}(e^{-tD^2}).$$

For $d = 2$, let $A_{\tau,\ell}$ be the annulus (don't identify the top and bottom of the parallelogram).

$$\begin{aligned} Z_E(\tau,\ell) = E(T_{\tau,\ell}) &= \operatorname{str}(\underbrace{E(A_{\tau,\ell})}_{\in B^{tc}(H)}) = \operatorname{str}(q^{L_0}\bar{q}^{\overline{L}_0}) \\ &= \sum_{\lambda(\ell),\mu(\ell)} q^\lambda \bar{q}^\mu \operatorname{sdim} E_{\lambda,\mu} =_{susy} \sum_\lambda q^\lambda \underbrace{\operatorname{sdim} E_{\lambda,0}}_{a_n:=} \end{aligned}$$

$H_\ell = E(S^1_\ell)$. $[L_0, \overline{L}_0] = 0$. λ and μ are eigenvalues of L_0, $\overline{L}_0$.

If we fix the length ℓ, we get a 2-dimensional semigroup with 2 infinitesimal generators. Now the super symmetry tells me that $\overline{L}_0(\ell) = \overline{G}_0(\ell)^2$ for some odd operator $G_0(\ell)$ on H_ℓ ($[G_0, \overline{G}_0] = 0$). If we had two square roots, the function would have to be constant. $\lambda \in \mathbb{Z}$ because $L_0 - \overline{L}_0$ generates a circle action. Since everthing is smooth in ℓ, and things are integers, so they are independent of ℓ. From something, the sum is bounded from below (starts at some point, bigger than $-\infty$).

References

[CdS03] Ana Cannas da Silva. *Introduction to symplectic and Hamiltonian geometry.* Publicações Matemáticas do IMPA. [IMPA Mathematical Publications]. Instituto de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, 2003.

[DEF+99] Pierre Deligne, Pavel Etingof, Daniel S. Freed, Lisa C. Jeffrey, David Kazhdan, John W. Morgan, David R. Morrison, and Edward Witten, editors. *Quantum fields and strings: a course for mathematicians. Vol. 1, 2.* American Mathematical Society, Providence, RI, 1999. Material from the Special Year on Quantum Field Theory held at the Institute for Advanced Study, Princeton, NJ, 1996–1997. 2, 24, 25