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How these notes are coming to exist

It is Fall 2007. QFT classes are being taught at UC Berkeley by Richard Borcherds (RB), Nicolai Reshetikhin (NR), and Peter Teichner (PT). Anton LaTeXs these notes in class and edits them later.[1] The version you're currently reading was compiled October 4, 2014. They should be available at
http://math.berkeley.edu/~anton/index.php?m1=writings.

– When something doesn't make sense to me, I mark it with three big, eye-catching stars [[★★★ like this]]. If you can clear any of these up for me, let me know.

– If you have notes that I'm missing or if you have a correct/clear explanation for something which is incorrect/unclear, let me know (either tell me what you'd like to modify, give me some notes to go on, or update the tex yourself and send me a copy). Real (mathematical) errors should be fixed because it would be immoral to let them propagate (er . . . that is, sit there), and typographical errors hardly take any time to fix, so you shouldn't be shy about telling me about them.

[1] With the exception of NR22, which was done by Chris Schommer-Pries.

1 NR 08-27

Quantum field theory is a very big subject (in both physics and math), even though it is relatively new (late 50s and 60s). It was designed to describe the interactions of particles and the structure of the micro-world. From the beginning, there were some formidable mathematical (and intrinsic) problems:

1. renormalization problem
2. perturbation theory

On Tuesdays, Richard Borcherds will have a seminar which will be focused on these problems, so these things won't be in this course.

By the 60s and 70s, there were well-developed ways to get around these things, but there were more and more particles showing up, and they needed explanation. The main outcome of this was the Standard Model and Gauge theory. This stuff is very interesting, but we won't talk about it.

The goals of this course: Give a mathematical summary of the basic ideas in classical and quantum field theory. You can't really talk about these theories unless you start from classical and quantum mechanics, so this will be the subject of the first few lectures. Newton did stuff ... 2nd order ODE; variational principals and Lagrangians; Hamiltonian mechanics (reduce to first order ODEs); Symplectic geometry (M,ω), $C^\infty(M)$. The symplectic form induce a Poisson structure $\{\cdot,\cdot\}$. So you get $C^\infty(M)$, which is (1) a commutative algebra, (2) a Lie algebra with $\{f,g\} = (df\wedge dg, \omega^{-1})$, where $\omega\in\bigwedge_x^2 T^*M$ and $\omega^{-1}\in\bigwedge_x^2 TM$, and we have that

$$\{f,gh\} = \{f,g\}h + \{f,h\}g.$$

Such an object is called a Poisson algebra.

How to go from 2nd order to first order, well we have $\ddot q = F(q)$, which can be turned into the system $p=\dot q$ and $\dot p = F(q)$, so we have twice as many variables.

Physicists would not approve of this approach because it isn't very physical.

We have $C^\infty(M)$ with $\{\cdot,\cdot\}$. We want to deform this algebra in the category of associative algebras. That is, we want to find some (associative) multiplication

$$f *_h g = fg + \frac{h}{2}\{f,g\} + O(h^2)$$

This is called deformation quantization. There is a powerful technique called geometric quantization which Peter will talk about.

This does the classical mechanics to quantum mechanics. Hopefully we'll do this in the first two weeks.

What is a classical field theory? It is the same sort of thing, but where the n-dimensional manifold N $(M=T^*N)$ is replaced by $\Gamma(E\to M)$ for some bundle E (which we can't even say is a manifold).

The real goal of this course is to explain invariants of 3-manifolds and corresponding conformal field theories (we'll just focus on Chern-Simons theory). (Something about affine lie algebras. Lately, there is the theory of SLE processes which relates this stuff to probability theory. If you have Brownian motion, you get a random curve you'd like to describe. There is an analytic technique)

When I started thinking about this course, I realized that Peter is teaching a seminar course, so we'll try to coordinate (for the first half of the semester). There will be a certain division of labor. I'll focus on bosonic field theories, and Peter will be focusing (at least in the beginning) on fermionic field theories, which are really important for bosonic field theories (for Chern-Simons theory). There is an infinite-dimensional group which makes the Lagrangian something. The way people deal with this in QFT and perturbation theory is known as the theory of Faddeev-Popov ghosts. These are fields which you don't see, but they play a certain role. It is important to have these objects because they give you finite type invariants of 3-manifolds. One of the goals of this course is to explain that there are different points of view on the same thing, and they produce different kinds of results. Fermionic fields are essential for this part. For this reason, it is a good idea to go to both classes (at least for the first half of the semester).

There is a syllabus on my website.

Q: What are the prerequisites? NR: I assume you know differential geometry and some symplectic geometry. If you don't know this stuff, you can look in a textbook.

We'll start next lecture with Lagrangian mechanics and then Hamiltonian mechanics (that will be this week). Next week, we'll spend some time on Hamilton-Jacobi theory. After that, we'll move on to quantization and semi-classical analysis. Then we'll go to classical field theories. Then we'll talk about quantum mechanics and quantum field theory. Then symmetries (action of the gauge group) with some examples. In about a month (or a month and a half), we'll start focusing on Chern-Simons theory.

PT: if you decide to take this class, then register so that we can get a bigger room.

There will be homework, but it won't be graded. The office hours are on Tuesdays by appointment. At the end of the class, we'll have a mini-conference. Everybody registered will give a short presentation, and this will be instead of the final exam.

2 NR 08-29 Lagrangian Mechanics

Today we'll start with classical mechanics. Recall some basic facts.

Recall that Newton's equations say that a trajectory γ in $\mathbb{R}^n$ should satisfy the following equation.

$$m\ddot{\gamma}(t) = -\frac{\partial U}{\partial q}\big(\gamma(t)\big)$$

People often write $F := -\frac{\partial U}{\partial q}$ and $a := \ddot{\gamma}$, in which case this is written as $F = ma$. Thinking of a solid as a collection of constrained points, you can understand the motion of solids. (Euler)

Let's reformulate these differential equations as a variational problem (Lagrange) on $T\mathbb{R}^n$, or more generally, on TN for a smooth manifold N (which we'll refer to as *configuration space*). Choose a *Lagrangian* $\mathcal{L} \in C^\infty(TN)$. If N is equipped with a Riemannian metric (as in our example $N = \mathbb{R}^n$), then we take the Lagrangian

$$\mathcal{L}(\xi, q) = \frac{(\xi, \xi)}{2} - U(q) \tag{2.1}$$

for $(\xi, q) \in TN$, where $(\cdot, \cdot)$ is the Riemannian metric and U is some potential function. For a parameterized path $\gamma = \{\gamma(\tau)\}_{\tau=0}^t$ in N, define the *action functional*

$$\mathcal{A}[\gamma] = \int_0^t \mathcal{L}\big(\dot{\gamma}(\tau), \gamma(\tau)\big)\, d\tau.$$

As we'll see later, solutions to Newton's equations are parameterized paths γ_{cl} in N on which $\mathcal{A}[\gamma_{cl}]$ is extremal (i.e. the variation vanishes).

Heuristically the (first) variation is the infinitesimal change in action when γ is changed infinitesimally.

$$\delta\mathcal{A}[\gamma] = \text{“}\mathcal{A}[\gamma + \delta\gamma] - \mathcal{A}[\gamma]\text{”}$$

More precisely, if f is a function on paths, let γ_s be a family of parametrized paths such that $\gamma_0 = \gamma$ and define $\delta f(\gamma) = \frac{df(\gamma_s)}{ds}\big|_{s=0}$. Note that this depends on the choice of the family $\{\gamma_s\}$. In particular, $\delta\gamma = \frac{d\gamma_s}{ds}|_{s=0} = \big\{\big(\delta\dot{\gamma}(\tau), \delta\gamma(\tau)\big) \in T_{\dot{\gamma}(\tau)}(T_{\gamma(\tau)}N)\big\}_{\tau=0}^t$ is a vector field along the path $\{\dot\gamma(\tau),\gamma(\tau)\}_{\tau=0}^t$ which descibes how we are wiggling γ [[★★★ It looks like $\delta\gamma$ is more naturally a vector field along γ, not along $(\dot\gamma,\gamma)$, but this does induce a vector field along $(\dot\gamma,\gamma)$.]].

The variation of $\mathcal{A}$ is

$$\begin{aligned}\delta\mathcal{A}[\gamma] &= \frac{d\mathcal{A}[\gamma_s]}{ds}\Big|_{s=0}\\ &= \sum_i \int_0^t \Big(\frac{\partial \mathcal{L}}{\partial \xi^i}(\dot\gamma,\gamma)\delta\dot\gamma^i(\tau) + \frac{\partial\mathcal{L}}{\partial q^i}(\dot\gamma,\gamma)\delta\gamma^i(\tau)\Big)\,d\tau\\ &= \underbrace{\sum_i \int_0^t \Big(-\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \xi^i}(\dot\gamma,\gamma) + \frac{\partial\mathcal{L}}{\partial q^i}(\dot\gamma,\gamma)\Big)\delta\gamma^i(\tau)\,d\tau}_{\text{bulk term}} \qquad \text{(integrating by parts)}\\ &\quad + \underbrace{\sum_i \Big(\frac{\partial\mathcal{L}}{\partial\xi^i}\big(\dot\gamma(t),\gamma(t)\big)\delta\gamma^i(t) - \frac{\partial\mathcal{L}}{\partial\xi^i}\big(\dot\gamma(0),\gamma(0)\big)\delta\gamma^i(0)\Big)}_{\text{boundary terms } = \frac{\partial\mathcal{L}}{\partial\xi}(\delta\gamma(t)) - \frac{\partial\mathcal{L}}{\partial\xi}(\delta\gamma(0))}\end{aligned}$$

We call the first term a "bulk term" and the last terms "boundary terms".

If γ is a solution to the Euler-Lagrange equations

$$-\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\xi^i}(\dot\gamma,\gamma) + \frac{\partial\mathcal{L}}{\partial q^i}(\dot\gamma,\gamma) = 0 \qquad \text{(Euler-Lagrange equations)}$$

then the bulk term vanishes, so $\delta\mathcal{A}[\gamma]$ is given by boundary terms. Note that this is the *only* way to get the bulk term to vanish because if this factor doesn't vanish, we can change our choice of the family $\{\gamma_s\}$ (thus changing $\delta\gamma$) to get the integral to not vanish. The Euler-Lagrange equations for (2.1) are Newton's equations in $\mathbb{R}^n$. Classical trajectories are those that satisfy the Euler-Lagrange equations.

Boundary Problems

A *boundary problem* is a submanifold $B \subseteq TN\times TN$. A solution to the boundary problem is a classical trajectory γ_{cl} (one that solves the Euler-Lagrange equations) such that $\big(\dot\gamma(t),\gamma(t),\dot\gamma(0),\gamma(0)\big)\in B$. Here are three examples of boundary problems (conditions we can impose on our paths).

(i) Boundary value problems: $\gamma(0)=Q$ and $\gamma(t)=Q'$. In this case, $B = T_{Q'}N\times T_QN$.

(ii) $\gamma(0)=Q$ and $\dot\gamma(t)=V$. In this case, $B=V(N)\times T_QN$, where $V(N)$ is the image of the vector field $V\colon N\to TN$ in TN.

(iii) Initial value problems: $\gamma(0)=Q$ and $\dot\gamma(0)=V$. In this case, $B = TN\times\{(V,Q)\}$. This means we fixed a point on TN at $\tau=0$.

Consider the first problem. If we restrict to families $\{\gamma_s\}$ for which $\big(\dot\gamma_s(t),\gamma_s(t),\dot\gamma_s(0),\gamma_s(0)\big)\in B$, then we see that $\delta\gamma(t)$ and $\delta\gamma(0)$ are zero, so the boundary terms in $\delta\mathcal{A}[\gamma]$ vanish. Thus, the extrema of $\mathcal{A}[\gamma]$ (subject to the boundary conditions) are precisely solutions to this boundary problem.

In general, if we restrict to families $\{\gamma_s\}$ satisfying the boundary conditions it is not true that a solution to the Euler-Lagrange equations is an extremum of the action (because the boundary terms of $\delta\mathcal{A}[\gamma]$ may not vanish). To remedy this, we can try to change the action functional so that solutions to the boundary problem are extrema of the new action functional.

Consider the second boundary problem. We have that $\delta\gamma(0)=0$ (because we fixed $\gamma(0)$), so the remaining boundary terms are (in the Riemannian case, using the Lagrangian in (2.1))

$$\begin{aligned}d\mathcal{L}\big(\delta\gamma(t)\big) &= \sum_i \frac{\partial\mathcal{L}}{\partial\xi^i}\big(\dot\gamma(t),\gamma(t)\big)\delta\gamma^i(t) && \text{(in coordinates)}\\ &= \sum_i \dot\gamma^i(t)\delta\gamma^i(t) = \sum_i V^i_{\gamma(t)}\delta\gamma^i(t) && \Big(\mathcal{L}(\xi,q) = \tfrac{(\xi,\xi)}{2} - U(q)\Big)\\ &= \big(V_{\gamma(t)},\delta\gamma(t)\big) && \text{(the pairing from the metric)}\end{aligned}$$

So extrema of our action functional are not solutions to the boundary problem. If $N=\mathbb{R}^n$, consider the modified action functional

$$\mathcal{A}_V[\gamma] = \mathcal{A}[\gamma] - \sum_i V^i_{\gamma(t)}\gamma^i(t).$$

Then it is easy to see that extrema of $\mathcal{A}_V$ are exactly the solutions to the boundary problem. [[★★★ For $N\neq\mathbb{R}^n$, what is the analogue of this weird term $\sum V^i_{\gamma(t)}\gamma^i(t)$?]]

General strategy: If we want solutions to a boundary problem $B\subseteq TN\times TN$ to be extrema of an action functional, we should consider the modified action functional

$$\mathcal{A}_{F,B}[\gamma] = \mathcal{A}[\gamma] + F$$

for some function F on B. We want solutions to the equations of motion to be exactly the extrema of the action $\mathcal{A}_{F,B}$, so we want

$$\delta\mathcal{A}_{F,B}[\gamma_{cl}] = \text{boundary terms}|_B + \delta F = 0$$

where γ_{cl} is a solution to the Euler-Lagrange equations and the family $\{\gamma_s\}$ is contstrained by the boundary conditions (in particular, the vector $\big(\delta\dot\gamma(t), \delta\gamma(t), \delta\dot\gamma(0), \delta\gamma(0)\big)$ is tangent to B). If we use coordinates (V, Q, v, q) on $TN \times TN$,[1] this is equivalent to F satisfying the differential equations

$$dF = -d\mathcal{L} \circ dQ + d\mathcal{L} \circ dq.$$

If we choose local coordinates $\{x^a\}$ on B, these become

$$\frac{\partial F}{\partial x^a} = -\sum_i \frac{\partial \mathcal{L}}{\partial \xi^i}\big(V(x), Q(x)\big)\frac{\partial Q^i(x)}{\partial x^a} + \sum_i \frac{\partial \mathcal{L}}{\partial \xi^i}\big(v(x), q(x)\big)\frac{\partial q^i(0)}{\partial x^a}.$$

The question is whether we can always find such an F. If you try to do this for the third boundary problem (iii), you'll discover it's impossible. It is clear that a necessary condition is that the form $-d\mathcal{L} \circ dQ + d\mathcal{L} \circ dq$ is closed. In local coordinates, this means that

$$\Big[\Big(\frac{\partial}{\partial x^b}\sum_i \frac{\partial \mathcal{L}}{\partial \xi^i}\big(V(x), Q(x)\big)\frac{\partial Q^i(t)}{\partial x^a}\Big) - \big((V,Q) \leftrightarrow (v,q)\big)\Big] - \big[a \leftrightarrow b\big] = 0$$

Once this condition is satisfied, the obstruction to finding such an F is an element of $H^1_{dR}(B)$. [[★★★ that is, $\frac{\partial^2 F}{\partial x^a \partial x^b} = \frac{\partial^2 F}{\partial x^b \partial x^a}$]]

Example 2.2. If N is a Riemannian manifold with Lagrangian $\mathcal{L}(\xi, q) = \frac{1}{2}(\xi, \xi)$, then the classical trajectories are geodesics in N. [[★★★ HW1]]

If $N = \mathbb{R}^n$ with the usual metric, the Euler-Lagrange equations say that $0 = -\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \xi}(\dot\gamma, \gamma) + \frac{\partial \mathcal{L}}{\partial q}(\dot\gamma, \gamma) = \ddot\gamma$, so the trajectories are $\gamma(t) = A + Bt$.

If $N = S^2$, with the metric induced by the standard embedding in $\mathbb{R}^3$, then [[★★★]]. On S^2, there are generically two geodesics connecting a pair of points, demonstrating that there can be more than one trajectory connecting two points. ◇

The physical systems we've descibed so far are called *conservative*, meaning that $\mathcal{L}$ is independent of time. More generally, we could take $\mathcal{L}$ to be a smooth function on $TN \times \mathbb{R}$. In this case, the action functional is

$$\mathcal{A}[\gamma] = \int_{t_1}^{t_2} \mathcal{L}\big(\dot\gamma(\tau), \gamma(\tau), \tau\big)\, d\tau.$$

[[★★★ how much of the previous analysis works?]]

[1] [[★★★ Q and q are well-defined globally (and these are the only coordinates we care about), but V and v don't make sense. I'd like to fix this without making the meaning unclear.]]

3 NR 08-31 Legendre transform, Hamiltonian formulation

The Legendre transform

Definition 3.1. A smooth function $\mathcal{L} \in C^\infty(TN)$ is *non-degenerate* if $\det\left(\frac{\partial^2 \mathcal{L}}{\partial \xi_i \partial \xi_j}\right) \neq 0$ for all (ξ, q). ◇

Definition 3.2. $\mathcal{L}$ is *strongly non-degenerate* if $\left(\frac{\partial^2 \mathcal{L}}{\partial \xi_i \partial \xi_j}\right)$ is positive definite for all (ξ, q). ◇

For a function $\mathcal{L} \in C^\infty(TN)$, define $\mathcal{H} \in C^\infty(T^*N)$ by

$$\mathcal{H}(p, q) = \max_{\bar\xi \in T_qN} \big(p(\bar\xi) - \mathcal{L}(\bar\xi, q)\big).$$

To find the $\bar\xi$ for which $p(\bar\xi) - \mathcal{L}(\bar\xi, q)$ is maximum, we differentiate with respec to $\bar\xi$, and we see that we are trying to solve for $\bar\xi$ in

$$p(\bar\xi) = \frac{\partial \mathcal{L}}{\partial \xi}(\bar\xi, q). \tag{3.3}$$

Definition 3.4. The *Legendre transfrom* is the map $TN \to T^*N$ given by $(\xi, q) \mapsto \left(\frac{\partial \mathcal{L}}{\partial \xi}(\xi, q), q\right)$. [[★★★ notation . . . two different ξ's.]] ◇

This definition of the Legendre transform also works for $\mathcal{L}$ a non-smooth convex function (some higher than first derivatives may be discontinuous).

Proposition 3.5. *If $\mathcal{L}$ is strongly non-degenerate (resp. non-degenerate), then the Legendre transform is an isomorphisms (resp. local isomorphism). In particular, equation (3.3) has a unique solution $\bar\xi$ for a given (p, q).*

Thus, if $\mathcal{L}$ is strongly non-degenerate, there is a unique solution to (3.3), so we can define $\mathcal{H}$.

The "inverse transformation" is

$$\tilde{\mathcal{L}}(\xi, q) = \max_{\bar p \in T_q^*N} \big(\bar p(\xi) - \mathcal{H}(\bar p, q)\big).$$

This $\tilde{\mathcal{L}}$ is the convex hull of $\mathcal{L}$ (see, e.g. [CdS03]).

Theorem 3.6. *If $\mathcal{L}$ is fiberwise convex (for all ξ, each q), then $\tilde{\mathcal{L}} = \mathcal{L}$.*

[[★★★ how is convexity used . . . wikipedia picture]] If we assume $\mathcal{L}$ is smooth, convexity is equivalent being strongly non-degenerate.

In our applications, $\mathcal{L}$ will either be non-degenerate, which insures fiberwise convex, or the Hessian $\left(\frac{\partial^2 \mathcal{L}}{\partial \xi_i \partial \xi_j}\right)$ will be identitcally zero.

Theorem 3.7. *The image of classical trajectories on TN in T^*N (with respect to the Legendre transform) are solutions to the following first order system.*

$$\dot p_i = \frac{\partial \mathcal{H}}{\partial q^i} \qquad \dot q^i = -\frac{\partial \mathcal{H}}{\partial p^i}$$

Where $\mathcal{H}$ is the Hamiltonian of the system (given by the Legendre transform of the Lagrangian).

Proof. [[★★★ HW4. It's easy from the ingredients.]] □

We'll see a coordinate-free formulation of this theorem later (Corollary 3.11).

Elements of symplectic geometry 1

Recall that a symplectic manifold is a pair (M, ω) where ω is a closed non-degenerate 2-form on M. Nondegeneracy means that when you think of ω_x as a map $T_xM \to T_x^*M$, it is an isomorphism. Thus, we have an inverse map ω_x^{-1}, and we can think of ω^{-1} as a section of $\bigwedge^2 TM$ (i.e. as a bivector field). If we choose local coordinates x^i on M, then we can write ω_x as

$$\omega_x = \sum_{i,j} (\omega_x)_{ij} dx^i \wedge dx^j$$

$$\omega_x^{-1} = \sum_{i,j} (\omega_x)^{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}$$

Example 3.8 ($M = T^*N$)**.** For a smooth manifold N, the projection $\pi \colon T^*N \to N$ induces a 1-form $T^*T^*N \overset{\alpha}{\leftarrow} T^*N$ on M. This 1-form

α is called the *canincal 1-form* on T^*N. Explicitly, if $x \in T_{(p,q)}T^*N$ is a tangent vector,

$$\alpha_{(p,q)}(x) = p\big(d\pi(x)\big)$$

where $p(\beta)$ is the natural pairing of $p \in T_q^*N$ with $\beta \in T_qN$. and $d\pi\colon TT^*N \to TN$ is the differential of the canonical projection $\pi\colon T^*N \to N$.

In local coordinates $\{q^i\}$ and corresponding coordinates $\{p_i\}$ on T_q^*N, one can check that

$$\alpha = \sum_i p_i dq^i.$$

The 2-form $\omega = d\alpha$ is a sypmlectic form on $M = T^*N$. In local coordinates, $\omega = \sum_i dp_i \wedge dq^i$. ◇

Definition 3.9. *A* a commutative algebra over $\mathbb{C}$ is a *Poisson algebra* if it is a Lie algebra with some bracket $\{\,,\}$ and the Lie algebra structure acts by derivations on the commutative algebra structure:$\{ab, c\} = a\{b, c\} + b\{a, c\}$. ◇

[[★★★ HW2 formulate the notion of super Poisson algebras. c.f. PT's lectures for commutative super algebras and Lie super algebras]]

Theorem 3.10. *For a syplectic manifold (M, ω), $C^\infty(M)$ with point-wise multiplication and the Poisson bracket given by*

$$\{f, g\} = \omega^{-1}(df \wedge dg) = \sum_{ij} (\omega^{-1})^{ij}\Big(\frac{\partial f}{\partial x^i}\frac{\partial g}{\partial x^j} - \frac{\partial g}{\partial x^i}\frac{\partial f}{\partial x^j}\Big)$$

is a Poisson algebra.

Proof. It is clear that it is a commutative associative algebra. The operation $\{\,,\}$ is a first order bidifferential operator, it will satisfy the Liebniz rule, so we need only check the Jacobi identity, which follows from $d\omega = 0$ [[★★★ HW3, it's very easy]]. □

A function $f \in C^\infty(M)$ induces a vector field $v_f := \omega^{-1}(df)$ on M.

Corollary 3.11 (to Theorem 3.7)**.** *The image $x(\tau)$ in T^*N of a classical trajectory $\big(\dot\gamma(\tau), \gamma(\tau)\big)$ on TN (with respect to the Legendre transform) is a flow line of the Hamiltonian vector field $v_{\mathcal{H}} = \omega^{-1}(d\mathcal{H})$. In other words, $\dot x(\tau) = \omega^{-1}(d\mathcal{H})(x(\tau))$ [[★★★ can this be simplified]].*

The main moral of this transformation is that under the Legendre transform, the trajectories become flow lines of the Hamiltonian vector field.

Variational principle in Hamiltonian mechanics

Assume $\mathcal{L}$ is strongly non-degenerate, so it gives an isomorphism between tangent and cotangent bundles. Define $p(\tau) := \frac{\partial\mathcal{L}}{\partial\xi}\big(\dot\gamma(\tau), \gamma(\tau)\big)$. Recall that the Hamiltonian is a function on T^*N given by $\mathcal{H}\big(p(\tau), \gamma(\tau)\big) = p(\bar\xi) - \mathcal{L}\big(\bar\xi, \gamma(\tau)\big)$, where $\bar\xi$ satisfies $p = \frac{\partial\mathcal{L}}{\partial\xi}(\bar\xi, q)$ (that is, $\bar\xi = \dot\gamma(\tau)$). So we have that

$$\mathcal{L}\big(\dot\gamma(\tau), \gamma(\tau)\big) = p\big(\dot\gamma(\tau)\big) - \mathcal{H}\big(p(\tau), \gamma(\tau)\big).$$

We can write the action as

$$\begin{aligned}\mathcal{A}[\gamma_*] := \mathcal{A}[\gamma] &= \int_0^t \Big(p(\tau)\dot\gamma(\tau) - \mathcal{H}\big(p(\tau), q(\tau)\big)\Big)\, d\tau \\ &= \int_{\gamma^*} \alpha - \int_0^t \mathcal{H}\big(p(\tau), q(\tau)\big)\, d\tau\end{aligned}$$

where $\gamma_* = \{p(\tau), \gamma(\tau)\}_{\tau=0}^t$ is the image of $\gamma = \{\dot\gamma(t), \gamma(t)\}$ in T^*N under the Legendre transformation. The first term universal (in the sense that it only depends on γ_* and the manifold N, but not on the Lagrangian), and the other term really describes the dynamics.

The variation of the action on a classical trajectory can now be written in terms of γ^*_{cl}.

$$\delta\mathcal{A}[\gamma^*_{cl}] = \sum_i \frac{\partial\mathcal{L}}{\partial\xi^i}\big(\dot q(t), q(t)\big)\delta q^i(t) - (t \leftrightarrow 0) = \alpha\big(di(\delta q(t))\big) - \alpha\big(di(\delta q(0))\big)$$

where $\alpha(\delta q)$ is the value of the form α on the vector field $di(\delta q)$ and $i\colon N \hookrightarrow T^*N$ is the zero section.

The goal now is to show that for $B \subseteq TN \times TN$, the image $B_* \subseteq T^*N \times T^*N$ is Lagrangian if we take the difference of the form on the two cotangent bundles. $\Omega = \omega_1 - \omega_2$. There is some L_t some family of Lagrangians with the following property. Solutions to the E-L equations with given boundary conditions (L_*) are $L_* \cap L_t$.

Let γ^*_{cl} be a classical trajectory in the phase space originating at $q \in N$ at time t_1 and ending at time t_2 at $Q \in N$. We can evaluate the action on such a trajectory, giving a function on $N \times N \times \mathbb{R} \times \mathbb{R}$

$$\mathcal{A}[\gamma^*_{cl}] = \mathcal{A}(q, Q, t_1, t_2).$$

Theorem 3.12 (Hamilton-Jacobi)**.**

$$\frac{\partial \mathcal{A}}{\partial t_2} + H(d_Q\mathcal{A}, Q) = 0 \qquad \frac{\partial \mathcal{A}}{\partial t_1} - H(-d_q\mathcal{A}, q) = 0.$$

4 NR 09-05

Recall from lecture 2 that given a boundary problem $B \subseteq TN \times TN$, we try to find a function F on B satisfying the condition

$$dF(V, Q, v, q) = \frac{\partial \mathcal{L}}{\partial \xi}(v, q)dq - \frac{\partial \mathcal{L}}{\partial \xi}(V, Q)dQ = p\,dq - P\,dQ.$$

where $p = \frac{\partial \mathcal{L}}{\partial \xi}(v, q)$ and $P = \frac{\partial \mathcal{L}}{\partial \xi}(V, Q)$. If the form on the right is closed (i.e. if $dp \wedge dq - dP \wedge dQ = 0$), then such an F exists locally. [[★★★ I'm not sure this coordinate-free description is quite right.]] In coordinates $\{x^a\}$ on B, the closedness condition is

$$\sum_i \Big(\frac{\partial P_i}{\partial x^a}\frac{\partial Q^i}{\partial x^b} - \frac{\partial q_i}{\partial x^a}\frac{\partial q^i}{\partial x^b}\Big) - [a \leftrightarrow b] = 0$$

In other words, a necessary condition for existence of such an F is that

$$\Omega|_{B_*} = 0$$

where B_* is the Legendre transform of B and $\Omega = \pi_1^*\omega - \pi_2^*\omega_2$ is the symplectic form on $T^*N \times \overline{T^*N}$.[1] Recall that such a submanifold is called *isotropic*.

Elements of symplectic geometry 2

Definition 4.1. B_* is an *isotropic submanifold* of a symplectic manifold if the restriction of the symplectic form to B_* is zero. ⋄

Definition 4.2. An isotropic submanifold of maximal dimension is called a Lagrangian submanifold. ⋄

It is clear that if the dimension of the symplectic manifold is $2n$, then the maximal dimension of an isotropic submanifold is n (this follows from non-degeneracy of the symplectic form).

Now let's look at the E-L solutions. From the theory of ODEs, we know that we generically need to fix $2n$ coordinates to get finitely many

[1] Note that $\lambda_1\pi_1^*\omega + \lambda_2\pi_2^*\omega$ gives a symplectic form on $T^*N \times T^*N$ for any (non-zero) λ_1 and λ_2.

solutions. Thus, if you want extrema, this boundary manifold B_* should be Lagrangian.

Thus, a boundary condition B is variational and the number of solutions is generically finite if and only if B_* is a Lagrangian submanifold in $T^*N \times T^*N$ with the symplectic form Ω.

For each extremum of $\mathcal{A}$ with fixed end points (Q, q), we have the function $\mathcal{A}[\gamma^*_{cl}]$ on a neighborhood of $(Q, q) \in N \times N$. Assume that each of these functions extends to a function on $N \times N$. In free motion on a Riemannian manifold N these will be the geodesics connecting two generic points. Now we can take the differential of this function with respect to the first or second argument gives the submanifold

$$L_t^{(\gamma_{cl})} := \{(P, Q, p, q) \in T^*N \times T^*N | P = d_Q\mathcal{A}[\gamma_{cl}], p = -d_q\mathcal{A}[\gamma_{cl}]\}$$

[[★★★ the condition that $(P, Q, p, q) \in L_t$ is exactly "if you start at (p, q) and flow for time t along the Hamiltonian vector field v_H, you will end up at the point (P, Q)]]

Proposition 4.3. *L_t is a Lagrangian submanifold.*

Proof. [[★★★ HW]] [[★★★ This follows from the fact that flowing for time t, $F_t: T^*N \to T^*N$ is a symplectomorphism, and L_t is the graph of this symplectomorphism. In general, the graph of a symplectomorphism $M \to M$ is a Lagrangian in $M \times \overline{M}$.]] □

Now solutions to the E-L equations with boundary conditions B can be identified with points in $L_t \cap B_*$. Note that solutions to the E-L equations can be regarded as points on L_t, and imposing the boundary conditions on them intersects with B_*.

5 NR 09-07

Last time I gave the Hamiltonian interpretation of these variational problems. Let's move forward now, and we'll come back to it when it becomes more relevant to the goals of the class. I want to talk about Chern-Simons theory, which is an infinite-dimensional version of something. How you should look at the previous sections: $S(A) = \int_M \operatorname{tr}(A \wedge dA + \frac{2}{3} \wedge^3 A)$ is the Chern-Simons action, and $\delta S(A) = 0$. I gave the impression that there is a unique trajectory which connects two points, which was an (untrue) assumption. The condition that $\delta S(A) = 0$ says that $F(A) = 0$ (A is flat connection). $\mathfrak{g} = Lie(G)$ (compact Lie group). Gauge classes of flat connections correspond to classical trajectories. The counterpart to these classical trajectories are γ_{cl}, solutions of the E-L equations. There could be several such γ_{cl}, so all sentences where I said "the classical solution" should be replaced with "a classical solution". For example, if we fix $\gamma_{cl}(0) = q$, $\gamma_{cl}(t) = Q$, then we get $\mathcal{A}[\gamma_{cl}] = \mathcal{A}(q, Q, t)$. $\mathcal{A}_{\gamma_{cl}}$ is then a function on $N \times N \times \mathbb{R}$, but there is a value for each solution γ_{cl}.

I explained that boundary conditions B that have a variational interpretation are, by the Legendre transform, the Lagrangian submanifolds $B_* \subseteq T^*N \times T^*N$ (under the form $\Omega = \omega_1 - \omega_2$). I also described for each γ_{cl} a Lagrangian submanifold $L_t = \{(P, Q, p, q) \in T^*N \times T^*N | P = d_Q\mathcal{A}_{\gamma_{cl}}, p = -d_q\mathcal{A}_{cl}\}$.

Proposition 5.1. *For every γ_{cl}, L_t is a Lagrangian submanifold (I'm a little loose with the term "submanifold"; it's generically smooth).*

Then we have the following intrepretations for solutions to the E-L equations. A solution is a system of Lagrangian submanifolds in $T^*N \times \overline{T^*N}$. Solutions to E-L equations with given boundary conditions B_* are intersection points in $B_* \cap L_t^{\gamma_{cl}}$. If there is only one trajectory connecting two points, then there is only one such L_t, and if B_* and L_t are in generic position, they'll only intersect once, so you'll have a unique trajectory. You can already see the benifit of the Hamiltonian point of view: it gives this geometric interpretation of solutions to the E-L equations.

Now let's move on.

Hamiltonian dynamics on a symplectic manifold

So far, we've had $M = T^*N$. Now let (M, ω) be any symplectic manifold. Fix the Hamiltonian function $H \in C^\infty(M)$. It defines a vector field $v_H = \omega^{-1}(dH)$. The Hamiltonian dynamical system generated by H has trajectories which are flow lines of this vector field v_H (you can take this as the definition of a Hamiltonian dynamical system); i.e. they are solution to the (generally nonlinear) sytem of differential equations

$$\dot{x}(t) = v_H(x(t)). \qquad (*)$$

We can think of these flow lines as the motion of points. Define $F_t(x) = x(t)$ (assuming the existence and uniqueness of solutions t0 $(*)$, at least for some small interval), where $x(t)$ is the flow line passing through x at $t = 0$. You can think of this as the local action of $\mathbb{R}$ on M, given by shifting points along flow lines. Suppose $f \in C^\infty(M)$, then define $f_t = F_t^*(f)$ (i.e. $f_t(x) = f\big(x(t)\big)$). This gives an action of $\mathbb{R}$ on $C^\infty(M)$.

Theorem 5.2.

1. *$\frac{df_t}{dt} = \{H, f_t\}$ for $f_0 = f$. Now even though the space is infinite-dimensional, the equations of motion are linear.*

2. *$F_t^*(\{f, g\}) = \{F_t^*(f), F_t^*(g)\}$ (it is clear that $F_t^*(fg) = F_t^*(f)F_t^*(g)$, so F_t^* is a Poisson algebra homomorphism).*

[3.] An infinitesimal version: $\partial_H f \overset{def}{=} \{H, f\}$ is a derivation of the Poisson algebra $(C^\infty(M), \{,\})$.

Proof. (1)

$$\begin{aligned}
\frac{df_t}{dt} &= \frac{d}{dt} f(x(t)) \\
&= \big(\dot{x}(t), df(x(t))\big) \\
&= \big(v_H(x(t)), df(x(t))\big) \\
&= \big(\omega^{-1}(dH), df(x(t))\big) \\
&= \omega^{-1}(dH \wedge df_t) = \{H, f_t\}
\end{aligned}$$

(3), then we can exponentiate it to get (2).

$$\begin{aligned}
\partial_H(\{f, g\}) &= \{\partial_H f, g\} + \{f, \partial_H g\} \\
\{H, \{f, g\}\} &= \{\{H, f\}, g\} + \{f, \{H, g\}\}
\end{aligned}$$

which is just the Jacobi identity for $\{\,,\}$. □

Proposition 5.3. *v_H is tangent to the level surfaces of H.*

In other words, the dynamics is parallel to the level surfaces of H; if a trajectory originates on a particular level surface, it will remain there. This is good news because instead of solving differential equations on a $2n$-dimensional manifold, we can solve them on a $(2n-1)$-dimensional manifold. The level surfaces of H are physically surfaces of constant energy. Notice that this proposition is true only for conservative systems, when the Lagrangian (and thus the Hamiltonian) do not depend on time. This is also known as conservation of energy.

This property of the Hamiltonian function inspires the following definition.

Definition 5.4. A function $G \in C^\infty(M^{2n})$ is an *integral* of the Hamiltonian dynamics generated by H if

– $F_t^*(G) = G$

– equivalently (because of the Theorem 5.2), $\{H, G\} = 0$. That is, G is in involution with H. ◇

When we have such a function, it is clear that the level surfaces $G^{(c)} = \{x \in M | G(x) = c\}$ are also preserved by evolution (i.e. v_H is tangent to $G^{(c)}$).

If the system is conservative, we are guarenteed at least one integral (namely H). If another integral G exists, then we know that if a trajectory starts at the intersection $H^{(E)} \cap G^{(c)}$, it will remain there. So we will have reduced our dynamics to a $(2n-2)$-dimensional space. In general, if we have lots of integrals, we can significantly reduce the dimension of our space.

Elements of symplectic geometry 3

What is the maximal number of integrals we can have? What are the extra properties of these integrals we should require?

First let's prepare the ingredients. Let's return to Lagrangian submanifolds. Remember that if we have $L \subseteq M$ such that $\omega|_L = 0$ (i.e. that $T_xL \subseteq T_xM$ is an isotropic subspace for the symplectic form ω_x for all $x \in L$), we call L *isotropic*, and a maximal-dimension isotropic submanifold is called *Lagrangian*, and this maximal dimension is n when M is $2n$-dimensional. Recall that $L \subseteq M$ is called *coisotropic* if $T_xL \subseteq T_xM$ is a coisotropic subspace for the symplectic form ω_x, which means that $(T_xL)^\perp \subseteq T_xL$. Thus, Lagrangian submanifolds are both isotropic and coisotropic. Let's translate this notion to the algebra of functions. What does it mean for the Poisson algebra that a submanifold is Lagrangian?

$C^\infty(M)$ has (i) pointwise commutative multiplication and (ii) a Lie bracket $\{,\}$. From the point of view of the first structure, a submanifold $L \subseteq M$ corresponds to the ideal $I_L \subseteq C^\infty(M)$ of functions which vanish on L. What happens with this ideal if we take into account the second structure. Suppose $f, g \in I_L$ (so f and g vanish on L), then what can we say about $\{f, g\}$? This I_L could be nothing special, a Lie subalgebra, or a Lie ideal. We have

$$\{f, g\}|_L = \omega^{-1}(df \wedge dg)|_L = 0$$

[[★★★ HW; prove this; it's true for L coisotropic]] This means that for an isotropic submanifold L, the vanishing ideal I_L is a Lie-subalgebra. It is *not* a Lie ideal, so $\{I_L, C^\infty(M)\} \not\subseteq I_L$.

Definition 5.5. A *Lagrangian fibration* $M_{2n} \xrightarrow{\pi} B_n$ is a fibration over a base of dimension n, where the generic fibers are Lagrangian submanifolds. ⋄

When we have such a map, we get $\pi^*\colon C^\infty(B_n) \hookrightarrow C^\infty(M_{2n})$, so we have a subalgebra. What special property does this subalgebra have if the fibers are Lagrangian?

Claim. *If $\pi\colon M_{2n} \to B_n$ is a Lagrangian fibration, then $C^\infty(B_n) \hookrightarrow C^\infty(M_{2n})$ is a maximal commutative Lie subalgebra.*

So far I introduced Hamiltonian dynamics on a symplectic manifold, and said that you can think of it as flow lines or as linear dynamics on the space of functions. An isotropic submanifold is an ideal which is a Lie subalgebra.

More generally, if $M_{2n} \to B_k$ such that generic fibers are coisotropic, then $\pi^* : C^\infty(B_k) \hookrightarrow C^\infty(M_{2n})$ is a commutative Lie subalgebra.

A completely integrable system is: if you have a Lagrangian fibration so that the flow lines are parallel to the fibers. Hamiltonian dynamics on symplectic manifold with Lagrangian fibration such that blah is a completely integrable system.

6 NR 09-10

I want to try to finish Hamiltonian dynamics this lecture. Recall that last time, we introduced Hamiltonian dynamics on a symplectic manifold (M, ω), and we defined Lagrangian fibrations. Geometrically, we fix a function $H \in C^\infty(M)$, then we get $v_H = \omega^{-1}(dH)$, and trajectories of our system are flow lines of this vector field: $\dot{x}(t) = v_H(x(t))$. Algebraically, $f_t(x) = f(x(t))$, $\frac{df_t}{dt} = \{H, f\}$.

An integral of motion is a function whose values along these flow lines are constant () $F(x(t))$ =const, () $\{F, H\} = 0$, or () flow lines are paralel to level surfaces of F.

Isotropic subspaces. We have $\omega \in \wedge^2 T^*M$. $L \subseteq M$ is *isotropic* if $\omega(\xi \wedge \eta) = 0$ for all $\xi, \eta \in TL \subseteq TM$. That is, $TL \subseteq (TL)^\perp$ (where the $\perp$ is with respect to the pairing ω). We say L is *coisotropic* if $(TL)^\perp \subseteq TL$. We say L is *Lagrangian* if $(TL)^\perp = TL$ (i.e. if L is both isotropic and coisotropic).

Proposition 6.1. *If L is isotropic $I_L \subseteq C^\infty(M)$ is a Lie subalgebra of the poisson algebra of functions on M.*

Last time I said this for isotropic L, but I was corrected that this should be true for L coisotropic. Let's do an experiment.

Example 6.2. $M = \mathbb{R}^{2n}$, and take $L \subseteq \mathbb{R}^{2n}$ to be the submanifold given by $\{q_1 = 0\}$ (this is $(2n-1)$-dimensional). This is clearly coisotropic. We have that $I_L = \{q_1 f(p, q)\}$, and $\{q_1 f, q_1 g\}$ will be proportional to q_1, so the proposition is true for coisotropic; we have $\{I_L, I_L\} = 0$. ⋄

Proof. [[★★★ HW. At least half of the class should do this.]] □

At the end of this week, we'll start talking about term paper stuff.

Suppose F_1 and F_2 are two integrals of motion. If a point belongs to some level surface $F_1^{(c)}$, it will remain there. The same is true of F_2. If $x \in F_1^{(c_1)} \cap F_2^{(c_2)}$, then it will stay there, but what is the dimension of the submanifold spanned by these trajectories? There are two possibilities. If F_1 and F_2 are integrals, then $\{F_1, F_2\}$ is also an integral. So you can keep taking brackets to get more and more integrals. We want to know how this impacts the dynamics. I don't want to get into this. It is called Hamiltonian dynamics with constraints, and it is important.[[★★★ In the lecture notes, NR will write what this is and give some references]] There is one particular case which is imporatant to us, which is when $\{F_1, F_2\} = 0$. This is the beginning of the notion of integrable systems.

Completely integrable systems

These are the systems where you have the maximal number of Poisson commuting integrals. $F_1, \dots, F_n \in C^\infty(M)$ integrals, with $\{F_i, F_j\} = 0$, such that $dF_1 \wedge \cdots \wedge dF_n \neq 0$ (the F_i are *independent.*

Definition 6.3. The Hamiltonian system generated by $H \in C^\infty(M)$ is *(completely) integrable* if there exists such integrals $F_1, \dots, F_n$. ⋄

Geometrically, this means that we have a Lagrangian fibration $F_1 \times \cdots \times F_n \colon M_{2n} \to \mathbb{R}^n$ (i.e. the generic fiber is a Lagrangian submanifold in M). Why is this definition important?

Theorem 6.4. *For an integrable sustem, we have the following.*

1. *Level surfaces of this fibration map (i.e. fibers) are invariant with respect to flow.*
2. *Each generic fiber has an affine structure (i.e. each generic fiber we can cover by an atlas with transition functions which are rotations and translations).*
3. *Compact fibers are n-dimensional tori.*

Let $(\phi_1, \dots, \phi_n)$ be local coordinates in this affine coordinate system. These are called *angle coordinates*. Coordinates on $\mathbb{R}^n$ given by the values of these functions, $c_1, \dots, c_n$ are called *action coordinates*. The flow lines of any H which Poisson commutes with these F_i are straight lines in these coordinates:

$$\phi_i(t) = \omega_i(H, c_1, \dots, c_n)t + \phi_i$$

These ω_i are called *frequencies* and in the case of a compact fiber, these really are the frequencies of the trajectories around the torus; if we fix a coordinate, the flow is given by the action of $\mathbb{R}$ on the torus. If these frequencies are rational, then you have periodic trajectories around the torus. If only some of them are rational, in which case we get a dense path in the torus. If all of them are irrational, you also get a dense covering. You can also imaging the situation that for every generic fiber, these frequencies are rational. Then the flow lines span one dimensional manifolds (or maybe 2-dimensional).

This is called *superintegrability (or degenerate integrability)*. In this case, the invariant submanifolds have dimension smaller than n. In a normal integrable system, the dimension of invariant submanifolds is at most n. If you relax the condition of commutativity of the integrals, then you can get such a system. The F_i generate a subalgebra of $C^\infty(M)$. Normally, this is a commutative subalgebra, and if we want it to be commutative, it is not possible to get more than n such functions. Instead, we can require (i) the commutative (multiplication) subalgebra F in $C^\infty(M)$ generated by $F_1, \dots, F_k$ (where $k \geq n$) is a Poisson subalgebra (i.e. $\{F_i, F_j\} = G_{ij}(F_i \dots F_k)$, (ii) $\{F_i, H\} = 0$, (iii) the center $Z(F)$ has rank $2n-k$. Note that if $k = n$, condition (iii) says that the whole algebra is commutative. In such a system, the Liouville theorem holds, but level surfaces of the fibration should be replaced by level surfaces of the center $Z(F)$. Then the dimension of invariant tori will be $2n - k \leq n$.

Integrable systems are the ones where one can say something really explicit about the dynamics. When working with non-integrable systems, you try to find an integrable system which is close by, and then use some perturbation theory. I'll put up references for Hamiltonian dynamics and integrable systems.

PT: can you say something about this affine structure. NR: roughly, $\phi_1, \dots, \phi_n$ are given by flow lines of $F_1, \dots, F_n$ (you can think of each of the integrals as a Hamiltonian). $(F_1, \dots, F_n)\colon M_{2n} \to \mathbb{R}^n$. PT: And the coordinate changes are affine linear because of the Poisson bracket? NR: I think that's probably right. If I try to reconstruct the proof now, I'll waste the rest of the time, so I'll talk about these coordinates next time. It follows from the involutivity of the integrals. These are the same reasons you use the center $Z(F)$ for the more general Liouville theorem. PT: and you get disjoint unions of tori? NR: yes. I'll return to this question next time.

7 NR 09-12

From last lecture there was a question about where this affine structure came from on level sets of integrals. $F\colon M_{2n} \to \mathbb{R}^n$. When we have $\{F_i, F_j\} = 0$ with $dF_1 \wedge \cdots \wedge dF_n \neq=$, the corresponding vector fields v_{F_i} commute. On corresponding components of $M^{(c)} = \{x \in M | F_i(x) = c_i\}$, we have an action of $\mathbb{R}^n$ which is locally free and transitive, giving you an affine structure. If there is a subgroup $\Gamma \subseteq \mathbb{R}^n$ that acts trivially (Γ is the stabilizer), then this connected component is isomorphic to $\mathbb{R}^n/\Gamma$. Any such Γ must be isomorphic to $\mathbb{Z}^k$, so any connected component is isomorphic to $\mathbb{R}^{n-k} \times \mathbb{T}^k$.

[[★★★ Project 1: "noncommutative integrability", which we called superintegrablity last time.]] There is a tool which allows you to construct basically all known integrable systems. These are Poisson-Lie groups. [[★★★ This can either be in Project 1, or it can be Project 2]] You probably know about Lie groups. I'll talk about the Poisson part when we talke about Hamiltonian reduction.

States in classical mechanics

Definition 7.1. A *state* on manifold M (doesn't have to be symplectic) is a probability measure on M (i.e. a measure μ so that $\mu(M) = 1$). ◇

Last time, we assigned to a function $f \in C^\infty(M)$ a measure μ_f on $\mathbb{R}$. If $E \subseteq \mathbb{R}$, then $\mu_f(E)$ is supposed to be the probability that f has value in E. In the view of our new definition, $\mu_f(E) = \mu\big(f^{-1}(E)\big)$, or $\mu_f(E) = \int_{M_f(E)} \mu$, where $M_f(E) = f^{-1}(E) = \{x \in M | f(x) \in E\}$.

Remark 7.2. If μ_1 and μ_2 are such measures, then any convex combination $\alpha\mu_1 + (1-\alpha)\mu_2$ is also a probability measure. ◇

Definition 7.3. *Pure states* are states μ that are supported at points. That is, they are measures $\mu_x(U) = \begin{cases} 1 & x \in U \\ 0 & x \notin U \end{cases}$. ◇

$C^\infty(M)$ is the algebra of classical observables. M can be identified with the set of pure states. We get an expectation value $E_\mu(f) = \int_M f\mu =:$

$\langle f\rangle_\mu$ for an observable f. Another important value is the disperssion around $E_\mu(f)$

$$\sigma_\mu(f)^2 = \big\langle (f - \langle f\rangle_\mu)\big\rangle_\mu = \langle f^2\rangle_\mu - \langle f\rangle_\mu^2 \geq 0.$$

Say we have a (Hamiltonian) dynamics[1] on M is $f \mapsto f_t$, with $\frac{df_t}{dt} = \{H, f_t\}$. The dual dynamics on states is $\mu \mapsto \mu_t$, where $E_{\mu(t)}(f) := E_\mu(f_t)$. This evolution is called *Liouville evolution* of states. If M is symplectic with some volume measure ω^n and if our state is continuous with respect to this measure (i.e. if $\mu = \rho\omega^n$ for some continuous function ρ), then $\frac{d\rho_t}{dt} = -\{H, \rho_t\}$.

This is the picture of states in classical mechanics. Q: what does a state mean? do you have a set, and the measure is the number of particles in that set? NR: it depends what measure you take. If you have a pure state, then $E_{\mu_x}(f) = f(x)$. Another example is the Gibbs state. If you have an energy functino (usually the Hamiltonian) $E(x) > 0$, then the Gibbs state is $\mu(x) = \exp\big(-E(x)/T\big)\omega^n$, where T is a parameter analogous to temperature. If you let T go to zero (if you cool the system), then the measure will become concentrated at the points which have minimal energy.

Q: how do we know there is a μ_t like that? NR: This is something like the Rees-Nikodim theorem. It says that if you have a functional on the space of functions, then there is measure giving it. PT: aren't you just pushing forward the measure along the flow? NR: yes, that's a better way to say it: $\mu_t(E) = \mu\big(g_t(E)\big)$, where g_t is the evolution map.

Hamilton-Jacobi equation

Now we return to the situation where our symplectic manifold is T^*N for some smooth manifold N, Hamiltonian $H \in C^\infty(T^*N)$ with corresponding Lagrangian $\mathcal{L}$ (assume strongest possible non-degeneracy). Suppose $\gamma(q, Q)$ is a solution to the Euler-Lagrange equations with fixed end points so that $\gamma_0 = q$ and $\gamma_t = Q$. Let's assume that this solution is unique for every pair (q, Q). Then we have

$$\mathcal{A}[\gamma(q,Q)]_{t_1,t_2} = \int_{t_1}^{t_2} \mathcal{L}\big(\dot\gamma(\tau), \gamma(\tau)\big)\, d\tau =: \mathcal{A}_{t_1,t_2}(q, Q)$$

[1] By "dynamics" I mean a local action of $\mathbb{R}$ telling you how points evolve in time.

PT: actually, in geometry, you just say $\dot\gamma$ because it has a well-defined basepoint. NR: Let's stick to this mixed version.

So we have a family of functions $\mathcal{A}_{t_1,t_2}$ on $N \times N$. These functions are quite remarkable. It defines a Lagrangian submanifold in $T^*N \times \overline{T^*N}$, and this Lagrangian submanifold is $L_t = \{(P,Q),(p,q) | P = dQ\mathcal{A}_{t_1,t_2}, p = -dq\mathcal{A}_{t_1,t_2}\}$. The second wonderful property is that if we want to compute it we don't have to use the Euler-Lagrange equations.

Theorem 7.4 (Hamilton-Jacobi[2])**.** —

$$\frac{\partial \mathcal{A}_{t_1,t_2}}{\partial t_2} + H(d_Q\mathcal{A}_{t_1,t_2}, Q) = 0 \qquad \frac{\partial \mathcal{A}_{t_1,t_2}}{\partial t_1} - H(-d_q(\mathcal{A}_{t_1,t_2}), q) = 0$$

Proof. Use the Legendre transformation.

$$\mathcal{A}_{t_1,t_2} = \int_{t_1}^{t_2} (p\dot q - \underbrace{H(\gamma_*(\tau))})\, d\tau$$

the second term is the image of $(\dot\gamma, \gamma)$ under the Legendre transform, so

$$\mathcal{A}_{t_1,t_2} = \int_{\gamma^*} \alpha - \int_{t_1}^{t_2} H(p(\tau), q(\tau))\, d\tau$$

[[★★★ NR: it is better to use t_1 and t_2 so that we can apply this stuff to non-conservative systems.]] So we have

$$\frac{\partial A_{t_1,t_2}}{\partial t_2} = -H(p(t_2), q(t_2)) = -H(P, Q).$$

Q: the path γ depends on the t_1 and t_2? NR: yes, but the integral $\int_{\gamma^*}\alpha$ doesn't depend. Q:but we only fixed the q's, not the p's. NR: maybe [[★★★ HW1]] □

We'll see how these equations will appear as a kind of justification of the path integral in quantum mechanics.

Another way to think about it: these equations give you the generating function for the Lagrangian submanifolds L_t. There are many ways to get the same information about classical evolution.

[2] Maybe this was proven by somebody else, but these are the names everybody attaches.

Hamiltonian reduction

So far, we've been using symplectic manifolds of the form T^*N with the symplectic form $\omega = d\alpha$, but there is a general source of examples not of this form

Definition 7.5. A *Poisson manifold* is a pair $(M, p \in \wedge^2 TM)$ with M a smooth manifold and p a bivector field such that the bracket

$$\{f, g\} = \langle p, df \wedge dg\rangle$$

induces a Poisson algebra structure on $C^\infty(M)$. ◇

Since this bracket is a bidifferential operator of first order, it acts by derivations on $C^\infty(M)$ (with pointwise multiplication). The Jacobi identity gives a bilinear differential identity for p. In local coordinates it can be written as

[[★★★]]

The coordinate-free approach using the Schouten bracket can be found in [[★★★]].

[[★★★ Project 3: Poisson geometry and the Schouton bracket.]]

Example 7.6. Suppose p is non-degenerate. Then $p\colon T^*M \to TM$ has an inverse, which can be regarded as a 2-form $p^{-1} \in \wedge^2 T^*M$. [[★★★ HW2: show that the corresponding 2-form p^{-1} is a symplectic structure on M. You need to prove that it is closed.]] ◇

Example 7.7 (Lie-Kirillov-Kostant)**.** Let $\mathfrak{g}$ be the Lie algebra of a (finite-dimensional) Lie group G. Let $\mathfrak{g}^*$ be the dual space. Then on $C^\infty(\mathfrak{g}^*)$, we can define the operation

$$\{f, g\}(x) = \big\langle x, [df(x), dg(x)]\big\rangle$$

(note that $x \in \mathfrak{g}^*$ and $df(x) \in \mathfrak{g}$, $dg(x) \in \mathfrak{g}$). This defines a bivector field p on $\mathfrak{g}^*$. If $\{e_i\}$ is a basis for $\mathfrak{g}$, $\{x^i\}$ are coordinate functions on $\mathfrak{g}$, and $\{\frac{\partial}{\partial x^i}\}$ is a basis for the tangent space, then

$$p = \sum_{i,j,k} c^i_{jk} x_i \frac{\partial}{\partial x_j} \wedge \frac{\partial}{\partial x_k}$$

where $[e_i, e_j] = \sum_k c^k_{ij} e_k$ ◇

Equivalently, the Poisson bracket between coordinate functions has the form

$$\{x_i, x_j\} = \sum_k c^k_{ij} x_k.$$

[[★★★ inconclusive … maybe this section should be moved to the next lecture.]]

8 NR 09-14

Definition 8.1. Let $x \in M$. The *symplectic leaf* of M through x is the subset $S_x = \{\text{points connected to } x \text{ by piecewise Hamiltonian paths}\}$, where a Hamiltonian path is a flow line of a Hamiltonian vector field $v_H = p(dH)$ for a smooth function $H \in C^\infty(M)$. ⋄

Theorem 8.2. *$S_x \subseteq M$ is a submanifold. Furthermore, S_x is a symplectic manifold with symplectic structure given by restricting p. Two symplectic leaves either coincide or do not intersect.*

One of the main questions in Poisson geometry is: given a Poisson manifold (M, p), find the symplectic leaves. This is the geometric analogue of classifying isomorphism classes of irreducible representations of a given associative algebra. Deformation quantization deforms an associative algebra in such a way that the first order deformation is given by the Poisson bracket. The symplectic leaves then correspond to ideals in the algebra so that quotienting by them gives irreps. [[★★★ part of Project 3: symplectic leaves in Poisson-Lie groups. These examples can be very involved and complecated, but are very interesting.]]

Example 8.3. If M is symplectic, there is only one symplectic leaf, namely M. ⋄

Example 8.4. $\mathfrak{g}^*$ is a Poisson manifold.

Theorem 8.5 (Lie-Kostant-Kirillov+others)**.** *Symplectic leaves in $\mathfrak{g}^*$ are coadjoint G-orbits.*

G naturally acts on $\mathfrak{g}$ by the adjoint action. The dual action of G on $\mathfrak{g}^*$ is the *coadjoint action.* ⋄

Summary: we extended the notion of symplectic manifolds to Poisson manifolds. Poisson manifolds in many ways behave like associative algebras in the sense that symplectic leaves correspond to irreducible representations.

The moment map and Hamiltonian reduction

Recall that a vector field v on a symplectic manifold (M, ω) is *Hamiltonian* if there exists a function $H \in C^\infty M$ such that $v = \omega^{-1}(dH)$. Since $[v_{H_1}, v_{H_2}] = v_{\{H_1,H_2\}}$, Hamiltonian vector fields form a Lie subalgebra $HVect(M)$ of the Lie algebra $Vect(M)$ of all vector fields on M. Assume a Lie group G is acting on M. This induces a Lie algebra homomorphism $\phi\colon \mathfrak{g} \to Vect(M)$.

Definition 8.6. The action of G on M is *Hamiltonian* if the image of $\mathfrak{g}$ in $Vect(M)$ lies in $HVect(M)$. ⋄

In other words, if $x \in \mathfrak{g}$, then $e^{tx} \in G_e \subseteq G$ (neighborhood of the indentity), and we have

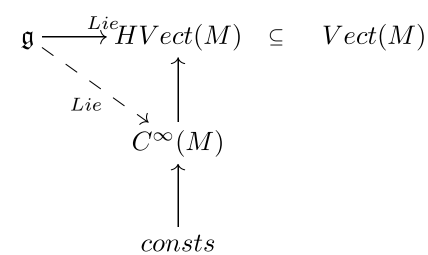
$$\frac{d}{dt} f(e^{tx}m)\Big|_{t=0} = \langle \phi(x), df\rangle(m) = x \cdot f(m)$$

So Hamiltonian action means that for each $x \in \mathfrak{g}$, there exists a function $h_x \in C^\infty(M)$ such that

$$x \cdot f(m) = \{h_x, f\}(m).$$

From the definition, we can see that h_x is linear in x (modulo a constant, so let's require that $h_0 = 0$), so $h_x(m) = \langle \mu(m), x\rangle$ where $\mu\colon M \to \mathfrak{g}^*$. [[★★★ for NR: what is the standard notation here?]]

PT: do you assume $h_{[x,y]} = \{h_x, h_y\}$? NR: it follows. PT: then you need a stronger assumption.



Theorem 8.7. *(1) $\mu\colon M \to \mathfrak{g}^*$ is a Poisson map (i.e. $h_{[x,y]} = \{h_x, h_y\}$) and (2) μ is G-equivariant.*

[[★★★ for NR: clarify the following disscussion.]]

Natural question: what can we say about M/G? This is already a quite complicated question. There are various ways to make it into a manifold if there are geometric problems. Let's assume there are no such geometric problems.

Theorem 8.8. *M/G is a Poisson manifold.*

If we want to study this manifold, one way to do it (corresponding to the notion of a categorical quotient) is to consider $C^\infty(M)^G$ (G-invariant functions). For this, we should be in the algebraic category. The subalgebra $C^\infty(M)^G \subseteq C^\infty(M)$ is a Poisson subalgebra, which means that $M \to M/G$ is Poisson (assuming M/G makes sense as a manifold).

We produced a Poisson manifold out of a symplectic manifold. What are the symplectic leaves of M/G? [[★★★ Project 3 is not for one person. You can include the precise statement of the following and a precise discussion of what kind of quotients we can have when we have a Lie group action on a manifold. One notion that was used effectively is the theory of GIT quotients.]]

Theorem 8.9. *Symplectic leaves of M/G are $\mu^{-1}(\mathcal{O})/G$ (where $\mathcal{O}$ is a coadjoint orbit).*

Remember that symplectic leaves in $\mathfrak{g}^*$ are coadjoint orbits $\mathcal{O} \subseteq \mathfrak{g}^*$.

In particular, we always have the distinguished orbit zero. So in particular, $\mu^{-1}(0)/G$ will be a symplectic leaf. This symplectic leaf is called the *Marsdon-Weinstein reduction* of M by G. The origin of this theory is in physics and angular momentum, when $G = SO(3)$

[[★★★ Refferences]]

Classical field theory

[[★★★ for NR: is there a page of corrections to this section somewhere?]]

If you're a (classical) physicist and all you care about are coordinate functions $q_i(t)$ $1 \le i \le n$. If you have infinitely many particles in a box, then you have infinitely many degrees of freedom. In this case, does it make sense to ask how many particles are in a given region? NO. You should ask what is the density of particles in the region. So we pass from finitely many degrees of freedom to densities of particles. In field theory, you have no individual particles, just fields.

<u>The idea</u>: replace the time interval $[t_1, t_2]$ by some (possibly complicated) manifold M with possibly non-empty boundary ∂M. Classically, $M = I$, and $\partial M = pt \sqcup \overline{pt}$ (the initial point should come with a minus sign). It makes sense to assume M is oriented, so ∂M is oriented.

These densities evolve in time. There is some distinction between the time direction and the other directions. We can either choose to take this into account or to ignore it. We can also choose to think of M as a Riemannian manifold.

Recall that the action can be written in terms of the Hamiltonian

$$\int_{\gamma^*} \alpha - \int_{t_1}^{t_2} H(\gamma_*(t))\, dt$$

The first term is independent of the parameterization. If we take $H = 0$, then a phisicist would say this is an empty system, but the evolution is non-trivial. This is called *topological mechanics*.

At Berekeley, Michael Green (not Brian Greene) gave some lectures. The title of the colloquim talk was "string theory = theory of nothing". His last slide was that in some cases the action is non-zero even though the hamiltonian is zero. Thus, TQFT is "the theory of nothing", which we'll study a lot.

9 NR 09-17

Two things we'll have to come back to: (1) constrained mechanical systems, Lagrangian and Hamiltonian, and (2) systems with degenerate Lagrangian (gauge systems)[[★★★ Project]].

Back to classical field theory. Last time I explained that the main idea is to replace the interval $[t_1, t_2]$ with an oriented manifold M. We will assume that the boundary of M is naturally polarized; that is, it has two connected components $\Sigma_1 \sqcup \overline{\Sigma}_2$. We will replace paths γ by section of some fiber bundle. This is the first approximation of what we want. Let's do the second approximation (still not the final version)

Space-time categories

A *d-dimensional spacetime category* is a category in which objects are oriented closed $(d-1)$-dimensional manifolds Σ (possibly with extra structure: e.g. Riemannian metric or symplectic form) and morphisms $\Sigma_1 \to \Sigma_2$ are d-dimensional oriented compact manifolds M (usually with extra structure) together with an isomorphism $\partial M \cong \overline{\Sigma}_1 \sqcup \Sigma_2$ (respecting any extra structure on M, Σ_1, and Σ_2). The composition is given by gluing.

Example 9.1 (Riemannian category)**.** Objects are $(d-1)$-dimensional Riemannian manifolds. The morphisms are (equivalence classes) of d-dimensional Riemannian manifolds (in the true version, there will be some extra information, "collars" on objects which tell you how to glue an object to a morphism) M with $\partial M \cong \overline{\Sigma}_1 \sqcup \Sigma_2$. Gluing is more involved. ◇

Example 9.2 (Minkowski category)**.** Here there will be similar subtleties. The objects are $(d-1)$-dimensional manifolds (most likely with collars). Morphisms are d-dimensional Minkowski manifolds M (the metric is not unique, just the signature should be $(d-1, 1)$, so one minus sign). ◇

Example 9.3 (Topological category)**.** Objects are compact oriented $(d-1)$-dimensional smooth manifolds Σ and morphisms $M \colon \Sigma_1 \to \Sigma_2$ are homotopy classes of d-dimensional manifolds M with smooth $(d-1)$-dimensional boundary together with isomorphisms $\partial M \cong \overline{\Sigma}_1 \sqcup \Sigma_2$. This is the category of d-cobordisms. ◇

Example 9.4 (Cell decomposition)**.** Objects are $(d-1)$-dimensional manifolds with a cell decomposition and morphisms are d-dimensional manifolds with a cell decomposition $M \colon \Sigma_1 \to \Sigma_2$ together with $\partial M \cong \overline{\Sigma}_1 \sqcup \Sigma_2$ so that the boundary is a subcomplex. The cell decomposition is part of the extra structure. ◇

Example 9.5 (Metrized cell decomposition)**.** A metrized cell decomposition is a cell decomposition where you assign volumes to all cells. Take this as the extra structure. This is an intermediate case between the topological and Riemannian categories. By taking finer and finer approximation, we can obtain the Riemannian category. One of the ideas of dealing with infinite-dimensional field theory is to approximate it by combinatorial approximations on a cell complex. ◇

Example 9.6. One can weaken the previous example by assigning volumes only to d-dimensional cells. So objects are (non-metrized) cell decompositions of $(d-1)$-dimensional manifolds, but morphisms are d-dimensional cell complexes of manifolds together with volumes of the d-cells. ◇

Example 9.7 (Classical mechanics)**.** Objects are (oriented) points, and morphisms are (oriented) intervales connecting points. ◇

Objects in spacetime categories are analogs of the endpoints of time intervals, and morphisms are the analogs of of time intervals.

Now we need the notion of fields to talk about classical field theory. The space of fields is the space of smooth sections $C^\infty(F \to M)$ of a fiber bundle $F \to M$ with fiber X. In mechanics, we take $M = [t_1, t_2]$, and F is the trivial bundle with fiber $X = N$.

In classical mechanics a Lagrangian function $\mathcal{L}$ is a function on TN. We will only take first order Lagrangians (this is a fundamental principal that to fix a trajectory you only need to fix a position and velocity. In general, the Lagrangian could be some function on Jet space). In mechanics, we had $L(\xi, q)$. In field theory, we will have $\mathcal{L}\big(\phi(x), d\phi(x)\big) \in \bigwedge^n T_x^* M$. A classical field ϕ is a section of $\pi \colon F \to M$, so over each point $x \in M$ of space time we have a fiber $F_x \cong_{i_x} X$, which contains the point $\phi(x)$. Then $d\phi(x) \in T_x^*M \otimes_{\mathbb{R}} T_{\phi(x)}F_x \cong_{\mathbb{R}} T_x^*M \otimes T_{i_x(\phi(x))}X$. Q: are you assuming a connection? How did you get into the tangent space to the fiber? AJ: are we secretly using the Jet space. NR: yes, we are secretly using the Jet space. Maybe I should just pick a connection right now. What I mean is that

$$\langle d\phi(x), \xi\rangle = \frac{d}{dt}\phi(\gamma_t)$$

where γ_t has $\dot\gamma_t(0) = \xi$ (in particular $\gamma_t(0) = x$).

The Lagrangian defines the classical action functional as

$$\mathcal{A}[\phi] = \int_M \mathcal{L}(x).$$

It is a functional on the space of fields.

Example 9.8 (Scalar $\mathbb{R}$-valued field in the Riemannian category)**.** $X = \mathbb{R}$. $\mathcal{L}\big(\phi(x), d\phi(x)\big) = \frac{1}{2}\langle d\phi(x), d\phi(x)\rangle - \frac{m^2}{2}\phi(x)^2 - V(\phi(x))$ where V is a finite polynomial $\frac{1}{3!}\lambda_3\phi(x)^3 + \cdots \frac{\lambda_n}{n!}\phi(x)^n$ (this is n-th order). These are the "kinetic term", "massive term", "self-interaction term". ◇

10 NR 09-19 Pure Yang-Mills theory

Fix a principal G-bundle $E \to M$. Assume M is Riemannian and $\mathfrak{g} = Lie(G)$ has a non-degenerate bilinear form $\langle,\rangle$. The space of fields in pure Yang-Mills theory is the space of connections on E (see Appendix 1 [[★★★ does not yet exist]]). The Lagrangian is

$$\mathcal{L}(A) = \frac{1}{2}(F(A), F(A))$$

where $F(A) = dA + A\wedge A \in \Omega^2(M, E^{ad})$ is the curvature of the connection A. If $\{x^i\}$ are local coordinates on M and $\{e_a\}$ is a basis for $\mathfrak{g}$, then

$$F(A) = \sum_{i,j,a} F^a_{ij} dx^i \wedge dx^j e_a \quad \text{where} \quad F^a_{ij}(A) = \partial_i A^a_j - \partial_j A^a_i + c^a_{bc} A^b_i A^c_j$$

There are several names for this: Yang-Mills, gauge theory, chromodynamics. There are observables which are guage invariant when you cannot express in terms of F, like $W(A) = tr(P\exp(\int_\gamma A))$ for $\gamma \subseteq M$. [[★★★ for NR: edit this an write Appendix 1]]

If we assume $G = GL_N$, then $A^g = gAg^{-1}$ and $\omega^R_g = dg\cdot g^{-1}$. This is an infinite dimensional group. It acts on connections: $F(g(A)) = gF(A)g^{-1}$. You can see that the Lagrangian $\mathcal{L}$ is invariant with respect to this action. So we have a Lie group acting on the space of fields and the Lagrangian is invariant. This is very bad news because it means that the Lagrangian is invariant. The Legendre transform assumes that $\det(\frac{\partial^2\mathcal{L}}{\partial\xi_i\partial\xi_j}) \neq 0$, but we have that $\mathcal{L}(\xi^g, q^g) = \mathcal{L}(\xi, q)$ which implies that this determinant is identically zero. I think this was the second project, and it is still open.

Why chromodynamics? According to the accepted theory of strong and weak interactions, these fields are supposed to describe the particles which glue the nucleus together. If you consider the case where $G = U(1)$, then

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}(F(A), F(A))_{\text{metric on } M} \\ &= \frac{1}{2}F_{ij}(A)^2\end{aligned}$$

This is called Pure electrodynamics. [[★★★]] Something about $d = 4$. As an exercise, you can derive the Maxwell equations as the Euler-Lagrange equations for this action. In this case, A is called the vector potential. The electric field induced by this vector postential $A = (A_0, \vec{A})$ (in coordinates x_0 (time), x_1, x_2, x_3) is

$$\vec{E} = \partial_0 \vec{A} \qquad \vec{B} = \vec{\nabla} \times \vec{A}$$

From these you can derive the Maxwell equations. Q: are you assuming Minkowski metric? NR: strictly speaking, I'm assuming we're in dimension $d = 3 + 1$.

The moral is that there are many systems with degenerate Lagrangians. We should be able to somehow reduce the manifold since the Lagrangian is invariant with respect to the guage group, and formulate the mechanics on orbits of the action. Then we can hope the Lagrangian will be non-degenerate so that we can do the Hamiltonian formulation.

Example (3); Chern-Simons (CS) theory. M is a 3-dimensional smooth manifold, $F = \Omega^1(M, E^{ad})$, $E \to M$ is a principal G-bundle, and $\mathfrak{g}$ has $\langle a, b\rangle$ non-degenerate. Let's assume $\mathfrak{g}$ is simple, so this is the killing form $\operatorname{tr}(ab)$.

Remark 10.1. The Lagrangian should always be a top form so that we can integrate it, so you should have a volume form whenever you describe it as a function. ⋄

Take the Lagrangian $\mathcal{L} = \operatorname{tr}(A \wedge dA) + \frac{1}{3}\operatorname{tr}(\wedge^3 A)$ (this is a 3-form on M. PT: isn't it a 3-form on the total space E?), then

$$\mathcal{A} = \int_{M_3} \mathcal{L}(x)$$

Is this guage invariant? What happens if $A \mapsto g(A)$.

PT: isn't A a 1-form on E with values in the Lie algebra?

11 NR 09-21

Functionally, classical Hamiltonian field theory can be regarded as a functor from a spacetime category to the category of symplectic manifolds.

Definition 11.1 (First approximation)**.** A *Hamiltonian classical field theory* in a d-dimensional spacetime category is an assignment of a symplectic manifold $S(N_{d-1})$ to each object N_{d-1} in the spacetime category and an assignment of a lagrangian submanifold $L(M_d) \subseteq S(\partial M_d)$ for each morphism M_d. Axioms:

1. $S(\varnothing) = pt$.
2. $S(\overline{N}_{d-1}) = \overline{S(N_{d-1})}$.
3. $S(N_1 \sqcup N_2) = S(N_1) \times S(N_2)$.

   $L_{M_1 \sqcup M_2} = L_{M_1} \times L_{M_2} \subseteq S(\partial M_1) \times S(\partial M_2)$.
4. If $\partial M_d = N \sqcup \overline{N} \sqcup N'$ and if $M' = M/\langle N \sim \overline{N}\rangle$ (so $\partial M' = N'$), then $L_{M'} = \{\ell \in S(N') |$ there is $m \in S(N)$ with $(m, m, \ell) \in L_M \subseteq S(N) \times \overline{S(N)} \times S(N')\}$. [[★★★ for NR: there needs to be some transversality assumption to get a Lagrangian submanifold]] ⋄

If you ignore the transversality problem, this gives a functor from a spacetime category to the category of symplectic manifolds (where morphisms in $\operatorname{Hom}(M_1, M_2)$ are lagrangian submanifolds of $M_1 \times \overline{M}_2$).

Let $\mathbb{R} \to E \to M_d$ with M Riemannian, and let $\mathcal{L}$ be a first order Lagrangian (written $\mathcal{L}(\phi, d\phi)$). Then we get an action functional. Let's compute it's variation. We can write $\mathcal{L} = \mathcal{L}_0(\phi) + \mathcal{L}_1(\phi, d\phi) + \mathcal{L}_2(\phi, d\phi) + \cdots$

$$\begin{aligned} \delta\mathcal{A}[\phi] &= \int_M \Big(\frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial d\phi} \wedge \delta d\phi\Big) \\ &= \int_M \Big(\frac{\partial \mathcal{L}}{\partial \phi} - d(\frac{\partial \mathcal{L}}{\partial d\phi})\Big) d\phi + \int_{\partial M} (\frac{\partial \mathcal{L}}{\partial d\phi})\delta\phi = \alpha(\delta\phi). \end{aligned}$$

$\delta\phi$ is a vector field on the space of fields and α is a 1-form on the space of fields. But at the same time this is a $(d-1)$-form on ∂M_d.

The Euler-Lagrange equations are the condition that the bulk term of the variation vanishes.

$$\frac{\partial \mathcal{L}}{\partial \phi} - d\Big(\frac{\partial \mathcal{L}}{\partial d\phi}\Big) = 0.$$

In this case $S(\partial M) = \Omega^0(\partial M) \oplus \Omega^{d-1}(\partial M)$. The symplectic structure is

$$\omega = \int_{\partial M} D\varphi \wedge D\pi$$

where $\varphi \in \Omega^0(\partial M)$ and $\pi \in \Omega^{d-1}(\partial M)$. [[★★★ what are these Ds? NR: the symplectic form is a pairing on the cotangent space of the space $S(\partial M)$]] It's the same formula as $\sum dp_i \wedge dq^i$ in local coordinates.

$$\omega((\delta\phi_1, \delta\pi_1), (\delta\phi_2, \delta\pi_2)) = \int_{\partial M} (\delta\phi_1 \wedge \delta\pi_2 - \delta\phi_2 \wedge \delta\pi_1)$$

Now I have to describe $L_M \subseteq S(\partial M)$.

$$L_M = \Big\{(\varphi, \pi) \;\Big|\; \varphi = \phi|_{\partial M}, \pi = \tfrac{\partial \mathcal{L}_{(\phi, d\phi)}}{\partial d\phi}\Big|_{\partial M}, \text{ where } \phi \text{ solves E-L eqns}\Big\}$$

[[★★★ HW: prove this is isotropic. to prove it is maximal isotropic, you might need some extra assumption.]]

In the case of classical mechanics, $M = [t_1, t_2], \partial M = p \sqcup \overline{p}$, $S(\partial M) = S \times \overline{S}$, where $S = T^*N$. Solution to the E-L equations

$$L^{(\gamma)}_{t_1,t_2} = \{(P,Q),(p,q) | P = d_Q\mathcal{A}[\gamma], p = -d_q\mathcal{A}[\gamma]\}$$

In principal you can have several solutions to the Euler Lagrange equations, so you might have many Lagrangian submanifolds.

The primal idea is the variational principle. You can then impose some constraints and the result will be a more complicated dynamics. Hamiltonian dynamics gives a very natural framework for such constraints. For field theories, it is a similar story. You start with a variational field theory, but then you can pass to the Hamiltonian description. If $\mathcal{L}$ is invariant under some group action, one should reduce the symplectic manifold to get a Hamiltonian description.

12 NR 09-24

Last time I gave an outline of classical field theory. Our main goal is Chern-Simons theory, where the fields are connections on a principal G-bundle on a [[★★★ Riemannian?]] manifold. The other example is Yang-Mills theory. So let's talk about connections on principal bundles today.

Connections

Definition 12.1. Let $\pi\colon P \to M$ be a fiber bundle over M. A *connection* on P is a distribution on P which projects isomorphically to TM. ◇

A distribution is a subbundle of the tangent bundle, so for each $p \in P$, we select a linear subspace $A(T_pP) \subseteq T_pP$. We have $d\pi\colon TP \to TM$, which has a kernel. So we get the sequence

$$0 \to \ker(d\pi) \xrightarrow{i} T_pP \to T_{\pi(p)}M \to 0$$

A connection is a choice of a splitting $T_pP = \ker(d\pi)) \oplus A(T_pP)$ with $A(T_pP) \cong T_{\pi(p)}M$ under $d\pi$. Equivalently, a connection is a section $A\colon T_pP \to \ker(d\pi))$ (so $i \circ A = \mathrm{id}_{\ker(d\pi)}$.

An important notion is *parallel transport*. If we have two points $x, y \in M$, and a path γ connecting them, then we can lift γ to an isomorphisms $h(\gamma)\colon P_x \to P_y$. Because $T_pP \supseteq A(T_pP) \cong T_{\pi(p)}M$, we can lift a tangent vector from the base to P. Integrating these lifted tangent vectors gives a lifted path [[★★★ how does this give the isomorphism?]].

Alternatively, you could define $\Gamma_1^{sm}(M)$ to be the category whose objects are points of M with $[-\varepsilon, \varepsilon) \subseteq M$ so that 0 maps to the point (call such a thing c_x) [[★★★ germs of paths]] and $\mathrm{Hom}(x,y)$ consists of smooth paths from x to y with $c_x, \bar{c}_y \subseteq \gamma$ (i.e. with the given germ). Gluing gives the composition. [[★★★ will this be a category . . . any trouble with identity morphisms?]] [[PT: you could take the objects to be $[0, \varepsilon)$ collars and the morphisms look like [[★★★ picture with "outside collar"]] so morphisms are also collars, so the identity is just the object itself. Unfortunately, when you go to infinite-dimensional something, it turns out that the natural direction is different from what you think. This gives you a nice category; it works well. It probably isn't the only choice.]] [[★★★ you have to vary parameterizations to get associativity of composition]]

If P is a fiber bundle over M, a representation of $\Gamma_1^{sm} \to P$ is an assignment $c_x \mapsto P_x$ and $(c_x, c_y \subseteq \gamma) \mapsto (h(\gamma)\colon P_x \xrightarrow{\sim} P_y)$. In otherwords, it is a functor from $\Gamma_1^{sm}(M)$ to the category where objects are fibers of P and morphisms are linear morphisms between the fibers [[★★★ what if we don't require isomorphisms?]]. This gives an equivalent description of a connection. PT: you have to put some subtle conditions to give you smoothness of the distribution. NR: I always ignore these smoothness conditions and we'll see why in a minute. You're right that there should be some conditions on smoothness.

Consider M_T a cell decomposition of M. We can try to modify this definition to define a connection on a fiber bundle over a cell decomposition.

Definition 12.2. A fiber bundle over M_T is an assignment to each vertex (0-cell) $x \in V(M_T)$ a fiber P_x such that $P_x \cong P_y$ (non-canonically) for all $x, y \in V(M_T)$. ⋄

Clearly a fiber bundle over M induces a fiber bundle over M_T by restriction. It is not true that any fiber bundle over the decomposition extends to a fiber bundle over M. If you can construct a fiber bundle for any decomposition M_T, then you can expect that it will give you a fiber bundle on M. Q: isn't any fiber bundle over M_T trivial?

We define $\Gamma_1(M_T)$ to have objects vertices of M_T and morphisms are edge paths (in the 1-skeleton) from x to y. A connection is a choice of isomorphisms $P_{e_+} \xrightarrow{\sim} P_{e_-}$ for every path from e_+ to e_- (in a compatible way). A connection is thus a representation of $\Gamma_1(M_T)$.

Connections on principal G-bundles

Let G be a Lie group and $P \to M$ a principal G-bundle (i.e. G acts simply transitively on the fibers of the bundle).

Definition 12.3. A *connection* on a principal G-bundle $P \to M$ is a G-invariant distribution on P that projects isomorphically to TM. ⋄

Again we have

$$0 \longrightarrow \ker(d\pi) \underset{A}{\overset{i}{\rightleftarrows}} TP \xrightarrow{d\pi} TM \longrightarrow 0$$

so that $i \circ A = \mathrm{id}$ and G-invariance.

I claim that a connection can be viewed as an element $A \in \Omega^1(P, \mathfrak{g})^G$, where G acts on $\mathfrak{g}$ by the adjoint action. For this we need the following lemma.

Lemma 12.4. $\ker(d\pi) \cong P \times \mathfrak{g}$ *G-equivariantly (where there is the diagonal action on the product $P \times \mathfrak{g}$).*

Proof. [[★★★ HW]] □

Now it is completely clear; this is exactly what 1-forms do. A element of $\Omega^1(P, \mathfrak{g})^G$ is a G-equivariant morphism $TP \to \mathfrak{g}$. Fiberwise, this means we have a canonical isomorphism $T_pP \cong \mathfrak{g} \times P_p$. [[★★★]]

Suppose A_1 and A_2 are two connections. Will a linear combination be a connection? No, because of the condition $i \circ A = \mathrm{id}$. However, the difference satisfies $i \circ (A_1 - A_2) = 0$, so $A_1 - A_2 \in \mathrm{Hom}_{\mathsf{Vect}}(T_pP/\ker(d\pi), \mathfrak{g}) =: V$. In other words, the space of connections is not a vector space, but it is an affine space over V.

Definition 12.5. An affine space L over a vector space V is a triple $(L, V, \theta\colon L \times L \to V)$ (we usually denote $\theta(a, b)$ by "$a - b$") so that for any $b \in L$, $a \mapsto a - b$ is a bijection $L \cong V$ and $(a - b) + (b - c) = a - c$. In otherwords, it is a principal V-set. ⋄

Claim. *Connections on a principal G-bundle P form an affine space over $V = \mathrm{Hom}_{\mathsf{Vect}}(T_pP/\ker(d\pi), \mathfrak{g})$.*

Claim. $V \cong \Omega^1(M, \mathfrak{g}^{ad})$.

This was my initial definition of the space of connections. This is wrong because the space of connections is an *affine space* over this space. The easiest way to see this is through the transformation properties with respect to the action of the guage group.

If you have a connection A, you get an action of $g \in G$ given by $A^g = gAg^{-1} + dgg^{-1}$ (this is the action on connections), so $(A_1 - A_2)^g =$

$g(A_1 - A_2)g^{-1}$ (which is how 1-forms transform). Next time we'll go into details about what this means.

Let P be a principal G-bundle over M_T (a cell decomposition of M), so over each $x \in V(M_T)$, we have $P_x \cong G$ (non-canonically).

Example 12.6. Given $P \to M$, it induces $P \to M_T$. ◇

If A is a connection on P and γ is a path connecting x and y, we have a G-equivariant isomorphism $h(\gamma)\colon P_x \xrightarrow{\sim} P_y$. If we fix a trivialization $P_x \cong G$, then we have the action of G on P by right multipication by the inverse ($g\colon h \mapsto hg^{-1}$). The map $h(\gamma)\colon G \to G$ is multiplication by some element $g(\gamma)$.

Moral: a connection on $P \to M_T$ with fixed trivialization is an assignment to each edge e a group element $g(e) \in G$, so a connection is a mapping $E(M_T) \to G$. If you change the trivialization, this mapping changes.

13 NR 09-26

Unfortunately, this class is going slower than I expected. Something should be done about it or we won't get to the point of the class (Chern-Simons theory). I was going to spend two more classes on connections, but I'll just make a handout which will be part of the notes. So we'll skip the stuff about connections for the time being. I'll return to this part when I'll be discussing guage theories. So connections will be on paper, and later we'll spend a bit more time on connections when we talk about Yang-Mills theory and Chern-Simons theory.

If M is a space of fields with an action of a Lie group G, you get guage theory. Another important part of this is classical field theory with degenerate Lagrangians. Recall that when we went from Lagrangian mechanics to Hamiltonian dynamics, we assumed the Lagrangian is non-degenerate. In general, you can expect that the Hessian will be of constant rank less than the dimension of the manifold. The setup we care about is infinite-dimensional. The nature of the problem can be understood in finite dimensions.

First, let's ignore the most important examples and focus again on the very basic structures.

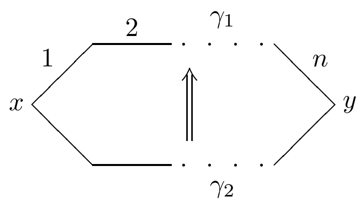
First let's say something about discrete versions of connections. Let M_T be a cell decomposition of M and let $\Gamma = (M_T)^1$ be the 1-skeleton (a graph). A principal G-bundle $P \to M_T$ on M_T is $P \to V(M_T) \cong G \times V(M_T)$ (non-canonically). Two different trivializations are related by

$$\begin{array}{ccc} P \xrightarrow{\sim} G \times V(M_T) & & (h,v) \\ \quad\searrow^{\sim} \quad \downarrow\wr & & \downarrow \\ G \times V(M_T) & & (g(v)h, v) \end{array}$$

$G_{M_T} = \{g\colon V(M_T) \to G\}$ is the discrete version of the guage group.

$\mathcal{A}^G_{M_T} = \{\text{connections on } P\}/G_{M_T}$. A connection on P, $\alpha = \{\alpha(e)\}_{e \in E(\Gamma)}$, with $\alpha(e)\colon P_{e_+} \xrightarrow{\sim} P_{e_-}$ a G-equivariant (with respect to the right G action) isomorhism given by $p \mapsto \alpha(e)p$. Then we see that $\mathcal{A}^G_{M_T} \cong G^{E(\Gamma)}/G^{V(\Gamma)}$; this isomorphism is the trivialization. Herer $G_{M_T} = \{\beta(v)\colon P_v \xrightarrow{\sim} P_v \ G\text{-equivariant}\}$.

Flatness. Suppose we have two graph paths which are related by a homotopy in the cell complex.



Given α, we can define parallel transport along γ as $h_\gamma(\alpha) = \alpha(e_n)\cdots\alpha(e_2)\alpha(e_1)$. We say that α is *flat* if $h_{\gamma_1}(\alpha) = h_{\gamma_2}(\alpha)$. In particular, if you had a connection on M which was flat in the sense of differential geometry, the induced connection on the cell deomposition is flat. This gives a subspace $\mathcal{M}^G_{M_T}$, the moduli space of flat connections, inside of $\mathcal{A}^G_{M_T}$, the moduli space of graph connections.

[[★★★ HW: $\mathcal{M}^G_{M_T} \cong (\pi_1(M) \to G)/G$ (the representation variety of the manifold) when G is a finite group.]]

These moduli spaces will appear again and again. Moduli spaces that will be common will be [[★★★ something]] of 2-dimensional manifolds M.

back to classical field theory

Let's look at the examples of classical field theories which are finite-dimensional but still have all the important features.

Classical Bose field on surface graphs. A surface graph $\Gamma = (M_T)^1$ is the 1-skeleton of a cell decomposition of a compact oriented manifold, possibly with boundary. When there is a boundary, we will assume that the edges of Γ do not include the edges of $(M_T)^1$ which are on the boundary. The vertices of Γ will be all vertices of $(M_T)^1$.

Recall Hamiltonian field theory. We assign to ∂M (possibly with some special structure, such as marked boundary points) a symplectic manifold $S(\partial M)$ and to M we assign a Lagrangian subspace of $S(\partial M)$ so that some axioms are satisfied. We want to construct this from a Lagrangian field theory. That is, we fix fields on M and a Lagrangian and the variational problem will suggest this structure and the Lagrangian subspace will be the space of solutions to the Euler-Lagrange equations.

In our case, the special structure will be the choice of cell decomposition M_T and the special structure on the boundary is the induced cell decomposition of ∂M.

The space of fields F will be maps from $V(\Gamma)$ to $\mathbb{R}$, $v \mapsto \phi(v)$. The Lagrangian will be

$$\mathcal{A} = \sum_{e\in E(\Gamma)} \frac{\big(\phi(e_+) - \phi(e_-)\big)^2}{\ell(e)^2} v(e)$$

where $\ell, v\colon E(\Gamma) \to \mathbb{R}_{>)}$ are a part of the special structure on M ("part" of the Riemannian metric on M; the length of the edges. $v(e)$ is the volume of the dual to e). For now these are just some functions.

[[★★★ picture]]

If ϕ is a smooth function on M, then we have that $\phi(e_+) - \phi(e_-) \approx \ell(e)\cdot\partial_e\phi$. So the action is (in the limit where the graph fills up the surface)

$$\sum_{e\in E(\Gamma)} (\partial_e\phi(x))^2 \longrightarrow \int_M (d\phi, d\phi)\, d^2x$$

E-L equations of this form are solutions to the equation $\Delta\phi = 0$. It is easy to see that

$$\begin{aligned}\delta\int_M (d\phi, d\phi) &= 2\int_M (d\delta\phi, d\phi)\\ &= -\int_M (\Delta\phi)\,\delta\phi\end{aligned}$$

This action is invariant with respect to transformations $x \mapsto \lambda x$ (and in fact invariant with respect to all conformal (angle-preserving) transformations).

What are the Euler-Lagrange equations for this Lagrangian? It should be some different version of the Laplacian.

$$\begin{aligned}(d\mathcal{A}[\phi],\delta\phi) &= \sum_{e\in E(\Gamma)} \frac{\phi(e_+)-\phi(e_-)}{\ell(e)^2}(\delta\phi(e_+)-\delta\phi(e_-))v(e)\\ &= \sum_{v\in V(\Gamma)} \delta\phi(v)\sum_{e\in S_v} v(e)\frac{(-1)^{(e,v)}}{\ell(e)^2}\big(\phi(e_+)-\phi(e_-)\big)\\ &= \sum_{v\in V^{int}}\delta\phi(v)[\cdots] + \sum_{v\in V^{bdry}} \underbrace{\delta\phi(v)\frac{\phi(e_+)-\phi(e_-)}{\ell(e)^2}}_{*} v(e)(-1)^{(e,v)}.\end{aligned}$$

Where S_v is the collection of edges adjacent to v. $\delta\phi(v)$ is a "vector field", an element of $\bigoplus_{s\in E(\Gamma)}\mathbb{R} = \mathbb{R}^{V(\Gamma)}$, with $\phi(v)\in\mathbb{R}$. $(-1)^{(e,v)}=1$ if e starts at v and -1 if e ends at v.

So the Euler-Lagrange equations are

$$\begin{aligned}0 &= \sum_{e\in S_u} v(e)\frac{(-1)^{(e,v)}}{\ell(e)^2}(\phi(e_+)-\phi(e_-))\\ &= \sum_{w-u} v(w,u)\frac{(-1)^{(w,u)}}{\ell(w,u)^2}(\phi(u)-\phi(w))\end{aligned}$$

If the adjacency matrix is $a_{u,w}=0$ if disconnected, -1 if connected (or some other weight), and p if $u=w$. This is the determinant of some weighted adjacency matrix. Under some assumptions, this discrete Laplace-Beltrami operator converges to the smooth Laplace-Beltrami operator.

On solutions to the Euler-Lagrange equations,

$$(d\mathcal{A},\delta\phi) = \text{boundary terms}$$

$$= (\alpha,\delta\phi)$$

$\partial M \mapsto S(\partial M) = (\bigoplus_{v\in\partial\Gamma}\mathbb{R})\oplus(\bigoplus_{bdry edges \Pi(e)}\mathbb{R})$, with $\omega = \sum_{v\in\partial\Gamma} d\phi(v)\wedge d\pi(e)$. The Lagrangian L_M is $\{(\phi(v),\pi(v))|\phi(v)$ is the boundary of a discrete harmonic function such that $\pi(e_v) = *\}$.

This is a finite-dimensional approximation to a very important example in conformal field theory, the Bose field something.

(1) $\omega = d\tilde\alpha$ $\tilde\alpha = \sum_v \pi(e_v)d\phi(v)$, and (2) $\tilde\alpha|_L = d\mathcal{A}$.

14 NR 09-28

Last time I gave an example of the classical Bose field on a surface graph. Today we'll generalize a bit and slightly modify. Last time I arranged the theory in such a way that the fields are defined on vertices and the edges are not on the boundary. Another version of the same theory is where $\Gamma = (M_T)^1$ is the 1-skeleton of a cell decomposition on a 2-dimensional compact oriented manifold. Last time I made some mistakes in the signs. What I described last time was Gaussian field theory (or Linear field theory, because the Euler-Lagrange equations were linear; or Free bose field on Γ, because free propagation is described by linear equations).

Now let's consider a more general theory. The fields are the same (maps $V(\Gamma) \to \mathbb{R}$), and the Lagrangian is a collection of smooth functions $\mathcal{L}_{v,w}\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ ((v,w) is an edge in Γ), which is assumed symmetric. The action is given by

$$\mathcal{A}[\phi] = \sum_{neighboring\ vertices\ v,w} \mathcal{L}(\phi(v),\phi(w))$$

Last time, $\mathcal{L}_{v,w}(x,y) = (x-y)^2\frac{a(v,w)}{\ell(v,w)^2}$. It could be $\mathcal{L}_{v,w}(x,y) = (x-y)^2\frac{a(v,w)}{\ell(v,w)^2} + V(x) + V(y)$, in which case $V(x)$ is called the self-interacting potential, and this describes a wave interacting with itself.

Last time I talked about the continuum limit. In this case, $\ell(e) \to 0$ and $a(e) \to 0$. We assume $\phi(v)$ is the restriction to v of a smooth function ϕ on $\Sigma = M$. Then $\sum_{(v,w)} \mathcal{L}(\phi(v),\phi(w)) \to \int_\Sigma \big(\frac{1}{2}(d\phi(x))^2 + V(\phi(x))\big)$. In this case, we can compute the E-L equations.

$$\delta\mathcal{A}[\phi] = \sum_v \Big(\sum_{(v,w)} \frac{\partial\mathcal{L}}{\partial x}(\phi(v),\phi(w))\Big)\delta\phi(v)$$

The E-L equations are then

$$\sum_{w:w-v} \frac{\partial\mathcal{L}}{\partial x}(\phi(v),\phi(w)) = 0$$

for $v \in V(\Gamma^{int})$. A solution to the E-L equations satisfies

$$\delta\mathcal{A}[\phi] = \sum_{v\in V(\partial\Gamma)} \Big(\sum_{(v,w)} \frac{\partial\mathcal{L}}{\partial x}(\phi(v),\phi(w))\Big)\delta\phi(v)$$

The Hamiltonian interpretation. I'm changing things so that this will work for all dimensions. The symplectic manifold is

$$S(\partial M) = \bigoplus_{v\in V(\partial M)} (\mathbb{R}[[\pi(v)]] \oplus \mathbb{R}[[\phi(v)]])$$

with $\omega = \sum_{v\in V(\partial\Gamma)} d\pi(v)\wedge d\phi(v)$ (this notation is a little different from last time). It is clear that $\omega = d\alpha$ with $\alpha = \sum_v \pi(v)d\phi(v)$. The Lagrangian submanifold is

$$L_M = \{(\pi(v),\phi(v)) \in S(\partial M) | \phi(v) \text{ is the boundary value of a soln to E-L eqns}, \pi = *\}$$

assuming the E-L equation has a unique solution φ for give boundary values of ϕ

$$\pi(v) = \sum_{(v,w)} \frac{\partial\mathcal{L}}{\partial x}(\varphi(v),\varphi(w)) \qquad (*)$$

The 1-form α and $\mathcal{A}$:

$$\alpha|_{L_M} = d\mathcal{A}[\varphi].$$

Remark 14.1. M_T can be of any dimension. ◇

I wanted to do discrete Yang-Mills theory, but this would just be another example of a classical field theory demonstrating that the Lagrangian can be invariant with respect to the action of a big group. So if you want the lagrangian submanifold L_M to make sense, you have to reduce the symplectic manifold. If it reduces to a point, you have a topological field theory, which is what happens in discrete Yang-Mills.

Quantization

I forget who said this, but some great person said, "you cannot really understand quantum mechanics; you can only get used to it". Let's accept this point of view. It's an experimental fact. What is the main concept? You have classical observables (smooth functions) on the phase space M (a symplectic manifold). Somehow, this $C^\infty M$ should be replaced by some non-commutative associative algebra. Why it should be this way is a long story, and there are still people who disagree with it. It was formulated by Dirac that there should be a correspondence. Given a classical observable f, there should be an operator $\hat{f}$ which would represent f. This is the quantum-classical correspondence. Dirac (I think) said that these operators should satisfy

$$[\hat{f}, \hat{g}] = i\hbar\widehat{\{f, g\}}$$

This is a version of the quantum-classical correspondence, but it is too naïve to be right. Mathematically, quantization is a deformation of $C^\infty M$ (commutative algebra with $\{\cdot, \cdot\}$) to a family of associative algebras.

Let me make a detour into some obvious facts about (formal) deformations of algebras. Suppose A is a commutative algebra over $\mathbb{C}$. B over $\mathbb{C}[[h]]$. A_h for $h \in [-\varepsilon, \varepsilon]$ is *a family of deformations* of A if

- A_h is an associative algebra for each h,
- $A_0 = A$, and
- there is an isomorphism of vector spaces $A_h \simeq A$.

We fix such an isomorphism $A_h \simeq A$ as part of the data. Then we have

- A a vector space over $\mathbb{C}$, and
- $a *_h b$ a family of associative multiplications such that $a *_0 b = ab$ (multiplication in A).

Q: is there some naturality condition? NR: no, right now it is very generic. Which deformations should we consider equivalent? We say $*_h$ and $\tilde{*}_h$ are equivalent if there exists $\phi_h \colon A \to A$ a linear isomorphism such that $a *_h b = \phi_h^{-1}(\phi_h(a) \tilde{*}_h \phi_h(b))$.

Natural question: given a commutative associative algebra A, describe equivalence classes of deformations.

All of these space are infinite-dimensional, and the question is too general. There should be the condition that the multiplication is continuous and something else is continuous and smooth. One case where it can be answered is if A is finitely generated and has some nice properties. The other thing we can do is do formal deformations (i.e. use formal power series). $*_h$ as function of h is too difficult, so replace it by a formal power series and study the resulting moduli space. Formal deformations are not studied because they are interesting (they are quite boring), but because you can say quite a lot about them. The problem of formal deformations was resolved in the last 15 years (first by Kontsevich, then several people filled in the picture).

Formal deformations of commutative algebras. $B = A[[h]]$ over $\mathbb{C}[[h]]$ is a *formal deformation* of a commutative algebra A if $(B, *)$ has an associative multiplication (called a *$*$-product*)

$$a * b = ab + \sum_{n=1}^{\infty} m_n(a, b)h^n.$$

where $m_n \colon A \otimes_{\mathbb{C}} A \to A$ extended h-linearly to $B \otimes_{\mathbb{C}[[h]]} B \to B$. $*$ is equivalent to $\tilde{*}$ if there exists $\phi \colon B \to B$ such that $\phi(a) = a + \sum_{n=1}^{\infty} h^n \phi_n(a)$ such that

$$a * b = \phi^{-1}(\phi(a) \tilde{*} \phi(b)).$$

Claim. *If $*$ is as above, then $\{a, b\} := \frac{1}{2}m_1(a, b) - \frac{1}{2}m_1(b, a)$ is a Poisson structure on A.*

We can say that $(A, \{,\})$ is classical mechanics (or at least one of the ingredients). The formal quantization deformation problem is: given $(A, \{,\})$, classify equivalence classes of $*$-products such that this Poisson bracket is induced by the $*$-product.

Let's check that an equivalence doesn't change the induced Poisson bracket. Say $a * b = ab + hm_1(a, b) + O(h^2)$, then

$$\begin{aligned} a \tilde{*} b &= \phi^{-1}(\phi(a) * \phi(b)) \\ &= ab + hm_1(a, b) + \underbrace{\big(-\phi_1(ab) + \phi_1(a)b + a\phi_1(b)\big)h}_{\text{symmetric}} + O(h^2) \end{aligned}$$

since the extra linear stuff is symmetric, it doesn't affect the Poisson bracket.

Say (M, p) is a Poisson manifold. Say $A = C^\infty M$, with $\{f, g\} = (p, df \wedge dg)$. Let's assume we want to study *symmetric* $*$-products, meaning

$$m_n(f, g) = (-1)^n m_n(g, f)$$

Theorem 14.2 (Kontseveich, $\mathbb{R}^d$)**.** *The space of such $*$-products modulo equivalence is in bijection with formal deformations of p modulo formal diffeomorphisms.*

A formal deformation of p is where you try to construct

$$\{f,g\}_h = \{f,g\} + \sum_{n=1}^{\infty} h^n p_n(f,g)$$

such that $\{f,g\}_h$ is still a Poisson bracket on $C^\infty M$. A formal diffeomorphism is: given $\alpha\colon M \to M$, you get $\alpha^*(f)(x) = f(\alpha(x))$, and we forget that this comes from a map. A formal diffeomorphism is $\alpha\colon C^\infty(M)[[h]] \to C^\infty(M)[[h]]$ so that $\alpha(f) = f + \sum_{n=1}^{\infty} h^n \alpha_n(f)$.

Next time I'll continue a bit about formal deformation quantization. Then we'll see that there are actual examples of family deformation quantization. Then we'll return to the quantization procedure and construct quantum obervables.

15 NR 10-01

I'll continue with deformation quantization today. Recall that if you have a Poisson algebra $(A, \{,\})$, then a deformation quantization of A is a family of associative algebras A_h so that

- $A_h \cong A$ (as a vector space)[1]
- Assuming the identification $A_h \cong A$, $\{a,b\} = \lim_{h\to 0} \frac{a*b-b*a}{h}$.

This is very hard because these spaces are typically infinite-dimensional and it is hard to construct a family, so there is an easier version, called *formal deformation quantization.*

Let M be a Poisson manifold (assume $M = \mathbb{R}^d$ with some poisson vector field $p \in \bigwedge^2 TM$), with $\{f,g\} = \langle p, df \wedge dg\rangle$. Consider $A = C^\infty \mathbb{R}^d$. Then a *bidifferential $*$-product* on A is a collection $\{m_n\colon A \otimes_{\mathbb{R}} A \to A\}$ where the m_n are bidifferential operators ($m_n(f,g) = \sum m_n^{\alpha\beta} \partial^\alpha f(x) \partial^\beta g(x)$ for multi-indices α and β of degree $\le n$), such that after extending m_n to $A[[h]] \otimes_{\mathbb{C}[[h]]} A[[h]] \to A[[h]]$ by linearity,

$$f * g = fg + \sum_{n=1}^{\infty} h^n m_n(f,g)$$

is associative. We also require that the $*$-product is *symmetric*, meaning $m_n(f,g) = (-1)^n m_n(g,f)$. Finally, we require $m_1(f,g) = \frac{1}{2}\{f,g\}$.

Let $\phi\colon A[[h]] \to A[[h]]$, with $\phi(f) = f + \sum_{n=1}^{\infty} h^n \phi_n(f)$, where ϕ_n is a differential operator of degree at most n. We say $* \simeq \tilde{*}$ if $f * g = \phi^{-1}\big(\phi(f) \tilde{*} \phi(g)\big)$ for some such ϕ.

Theorem 15.1 (Kontsevich)**.** *Bidifferential $*$-products up to equivalence are in bijection with formal deformations of the Poisson bracket up to equivalence.*

A deformation of the Poisson bracket is a Poisson bracket of the form $\{f,g\}^\sim = \{f,g\} + \sum_{n=1}^{\infty} h^{2n} p_n(f,g)$, where the p_n are bidifferential operators of order $(1,1)$. We say that $\{,\}^\sim \simeq \{,\}^\approx$ if $\{\phi f, \phi g\}^\sim = \phi(\{f,g\}^\approx)$ for some ϕ.

[1] This is sometimes called a *torsion free* deformation quantization.

What was surprising about this theorem is that the bijection was constructed completely explicitly with things that look like Feynman diagrams. It turns out that there is a topological quantum field theory interpretation of this result.

The value $\hbar$ is supposed to be an actual number, not a formal parameter ($\hbar$ is Planck's constant after all, it can't "go to zero"). Can something measured in meters, kilograms, or whatever go to zero? No, what we mean when we say that is that the value goes to zero relative to some unit measure. Poincaré in some sense worked out special relativity before Einstein, but his units were $1 = c = 2\pi$ and he didn't have the physical interpretation. Anyway, the point is that there are relative scales, and $\hbar$ changes value based on which scale you're using.

Family deformations.

Example 15.2. Take $M = \mathbb{R}^2$, $A = Pol_{\mathbb{C}}(\mathbb{R}^2) = \mathbb{C}[p, q]$ with the standard symplectic form $dp\wedge dq$ giving the bracket $\{p, q\} = 1$ (this determines the bracket). We have a natural monomial basis $p^n q^m$ on A. Define

$$A_h = \langle p, q | pq - qp = h\rangle$$

It is clear that this is a family of algebras. To say that this is a deformation of A, observe that $p^n q^m$ is a basis A_h, and identifying the bases gives an isomorphism $\theta\colon A_h \cong A$. Note that we could have chosen a different basis and we would get a different $*$-product, but it would be equivalent. This is why it doesn't make sense to talk about individual $*$-products (rather than equivalence classes). We also have to check that this multiplication is compatible with the bracket. Let's verify that $\{a, b\} = \lim_{h\to 0} \frac{a*b-b*a}{h}$ where $a * b = \theta(\theta^{-1}(a) \cdot_{A_h} \theta^{-1}(b))$. It is enough to check it on generators.

$$\lim_{h\to 0} \frac{p * q - q * p}{h} = \lim_{h\to 0} \theta\Big(\frac{pq - qp}{h}\Big) = 1$$

◇

Example 15.3. Take $M = T^*\mathbb{R}^d = \mathbb{R}^d \times \mathbb{R}^d$ (with coordinates p_i and q^i, respectively), $\omega = \sum_i dp_i \wedge dq^i = d\alpha$, and $A = Pol_{\mathbb{C}}(\mathbb{R}^d_p) \otimes_{\mathbb{C}} C^\infty\mathbb{R}^d_q$. Then $\{p_i, p_j\} = 0 = \{f, g\}$ (for functions f, g) and $\{p_i, f\} = \frac{\partial f}{\partial q^i}$.

We get the deformation quantization

$$A_h = \langle p_1, \dots, p_n, f \in C^\infty\mathbb{R}^d | [p_i, p_j] = 0, p_i f - f p_i = h\frac{\partial f}{\partial q^i}, fg - gf = 0\rangle$$

Take $\theta\colon A_h \xrightarrow{\sim} A$ given by the common basis $p_1^{a_1} \cdots p_n^{a_n} f(q)$.

[[★★★ HW: check that this is a deformation quantization. That is, check that $\{a, b\}$ is the usual limit]]

It is easy to see that A_h can be identified with the algebra of differential operators $Diff_h(\mathbb{R}^d)$ (you have to scale derivatives by h, which is a non-canonical operation).

[[★★★ HW (which could become a projec): if M is d-dimensional and smooth, then $C^\infty_{pol}(T^*M)$ (polynomial in the cotangent direction) has a natural deformation quantization which is the sheaf of differential operators on M]] ◇

Q: what is the multiplication on A_h? NR: I'm defining A_h as a quotient of the free algebra. I can consider the free associative algebra $T(x_1, \dots, x_n)$ and quotient it by some ideal, giving me an associative algebra. We can deform the ideal and get a family $T(x_1, \dots, x_n)/I_h$. Then you have to show that the different algebras you get are isomorphic; this is why we were chosing bases. Q: by $C^\infty\mathbb{R}^d$ is infinite-dimensional. NR: you define it as an algebra over $C^\infty\mathbb{R}^d$. I actually assumed A_h as a space is $Pol(p_1, \dots, p_d) \otimes_{\mathbb{C}} C^\infty(\mathbb{R}^d)$. [[★★★ what is the problem? why can't we just say there are an infinite number of generators?]]

Example 15.4. Let $\mathfrak{g}$ be a Lie algebra, and consider $Pol(\mathfrak{g}^*) = \mathbb{C}[\mathfrak{g}]$. If $\{e_i\}$ is a basis for $\mathfrak{g}$, then we can think of the e_i as coordinate functions x_i on $\mathfrak{g}^*$. A theorem of Kostant, $\{f, g\}(x) = \langle x, [df(x), dg(x)]\rangle$, $\{x_i, x_j\} = \sum_k c^k_{ij} x_k$. $Pol(\mathfrak{g}^*)$ then gets a Poisson bracket.

We can get a deformation quantization

$$A_h = \langle x_1, \dots, x_n | x_i x_j - x_j x_i = h \sum_k c^k_{ij} x_k\rangle$$

Note that $A_h \cong U\mathfrak{g}$ for any $h \neq 0$ (you just have to rescale the x's by h). On the other hand, Chosing the monomial basis $x_1^{a_1} \cdots x_n^{a_n}$ in $\mathbb{C}[x_1, \dots, x_n]$ and the PBW basis in A_h. Identifying them, we get $A_h \cong \mathbb{C}[x_1, \dots, x_n]$, which is how the PBW theorem is usually formulated.

$$U\mathfrak{g} \cong Pol(\mathfrak{g}^*) \cong \mathrm{Sym}(\mathfrak{g})$$

We get a linear isomorphism $\theta\colon A_h \cong A$. It is easy to check that this is a deformation quantization. ◇

There are plenty of examples related to the universal enveloping algebras. [[★★★ Project: report about various aspects of deformation quantization.]]

Quantization of classical mechanics

I didn't talk about all aspects of quantization, just about deformation quantization. You may have heard about geometric quantization and other things. I'll return to them. I want to indoctrinate you that deformation quantization is some how primal (well, not really, there is a way to go back and forth). Geometric quantization gives you representations of deformation quantization. There is a wonderful theorem (the GNS construction) which does something.

Remember that in classical mechanics, we have a symplectic manifold (M, ω) and observables $C^\infty(M)$ (or some algebraic analogue). The deformations we were doing were over $\mathbb{C}$. So if we want to deform, we should complexify: $C^\infty(M)_\mathbb{C} = C^\infty M \otimes_\mathbb{R} \mathbb{C}$. Then we can recover the classical obervables as fixed points of complex conjugation σ. If you open a textbook, it will say that obervables are hermitian operators. In this case, they are elements of this algebra. You run into the problem that the product of two hermitian operators is not hermitian.

16 NR 10-03

Last time I gave examples of deformation quantization. Now I want to discuss the states in quantum mechanics. Before that let's discuss one more example.

By $C(M)$ we mean the algebra of obervables (if M is T^*N, this will be polynomial in the cotangent direction and smooth on N; if M is algebraic, this will be algebraic functions; of it might be $C^\infty M$). This is an algebra over $\mathbb{R}$, but all of our deformations were over $\mathbb{C}$. The first step in quantization is to complexify the algebra of obervables, then $C(M)_\mathbb{R} \subseteq C(M)_\mathbb{C}$ is the fixed point set of complex conjugation σ. Then we form the deformation A_h (with the first jet given by the Poisson bracket). We need one more ingredient in the quantum case, which is the $*$-involution. There is a bit of confusion here; this is different from the $*$-product, so I'll use σ when there could be confusion. $\sigma\colon A_h \to A_h$ is an anti-$\mathbb{C}$-linear anti-involution, so $\sigma(fg) = \sigma(g)\sigma(f)$, $\sigma(\lambda f) = \bar\lambda\sigma(f)$, and $\sigma^2 = \mathrm{id}$. Recall that we imposed the assumtion that $A_h \cong C(M)_\mathbb{C}$. This is actually a very strong assumption. In general you get a sequence of matrix algebras and the best you can hope for is that as h goes to zero, you get some kind of isomorphism. Let's ignore this for the moment and assume this torsion free hypothesis. Finally, we need to require that $A_h^\sigma \cong C(M)$ as a real vector space (this should be the restriction of the isomorphism $A_h \cong C(M)_\mathbb{C}$), so we get a deformation of the whole structure. In this case, A_h^σ is called the *quantum space of obervables*. PT: it's not an algebra any more. NR: that's right. If A, B are hermitian operators on a Hilbert space H, then AB is not hermitian, but $AB + BA$ is and $i(AB - BA)$ is. These are the two structures on A. This is the structure of a *Lie-Jordan algebra* on A_h^σ (I think just the first one gives a Jordan algebra). The traditional abuse of language is to say that A_h is the quantum algebra of observables, but not all of it's elements are observables. PT: why do we classically want obervables to be an algebra? Bruce/NR: for example, energey $E = \frac{p^2}{2m}$.

Now we have a family of associative algebras A_h. The first question you should ask is, "what are isomorphism classes of irreducible representations, and what is the structure of its representations?" We have more than an algebra structure, we also have σ. In this setting, there is a natural notion of Hermitian (or unitary, or $*$-) representation. A representation is a homomorphism $\pi_h\colon A_h \to End(V)$. To have a notion of a hermitian conjugate in V, we have to choose a hermitian bilinear form (a bilinear form on V so that $\langle x,y\rangle = \overline{\langle y,x\rangle}$ and $\langle x,x\rangle > 0$ for $x \neq 0$). This gives us a norm $\|x\| = \langle x,x\rangle$. We can complete V with respect to this norm to get a Hilbert space $H = \overline{V}$.

Definition 16.1. A σ-representation of A_h in $(H, \langle,\rangle)$ is a representation $\pi_h\colon A_h \to End(H)$ such that $\pi_h(\sigma(a)) = \pi_h(a)^*$, where A^* of A is defined by $\langle A^*x, y\rangle = \langle x, Ay\rangle$. ⋄

If f is a classical observable, we represent it as $\pi_h(\hat{f})$, a hermitian operator on H, where $\hat{f}$ is the image of f under the isomorphism $C(M) \cong A_h$.

Assume that $\pi_h(A_h) \subseteq B(H)$ (bounded operators on H). This is a common assumption (but rather brave, and not usually true). $B(H)$ is an algebra with $*$-involution (hermitian conjugation). This is the motivating example for the notion of a C^*-algebra. Andy: are you assuming you have a π_h for each h? NR: yes, and we're assuming $h \in \mathbb{R}$. In geometric quantization, $h = 1/m$ for $m \in \mathbb{N}$.

States

Now let's move on to states. I want to deliver the intuitive notion of a state in quantum mechanics. This can be extended to the notion of a state on a C^*-algebra.

Definition 16.2. A is a *trace-class operator* in H, if $\sum_{n=1}^{\infty} |(Ae_n, e_n)| < \infty$ for any orthogonal basis $\{e_n\}$ of H. In this case, define $\operatorname{tr} A := \sum_{n=1}^{\infty}(Ae_n, e_n)$. ⋄

Theorem 16.3. —

1. *$B_1(H)$, the space of trace-class operators on H, is a Banach space with $\|A\| = \operatorname{tr}(\sqrt{A^*A}$.*
2. *$B_1(H) \subseteq B(H)$ is a two sided ideal.*
3. *$\operatorname{tr}(AB) = \operatorname{tr}(BA)$ for $A \in B_1(H)$ and $B \in B(H)$.*

Any $A \in B_1(H)$ defines a linear functional on $B(H)$, given by $\ell_A(B) = \operatorname{tr}(AB)$. [[★★★ check that AB is a trace-class operator]]

Definition 16.4. For $A \in B(H)$, the *spectrum* $\sigma(A) = \{z \in \mathbb{C} | A - zI$ is not invertible$\}$. ⋄

If $A = A^*$ (i.e. if A is *hermitian*), then $\sigma(A) \subseteq \mathbb{R}$.

Definition 16.5. A hermitian operator A is *positive* if $\sigma(A) \subseteq \mathbb{R}_{\geq 0}$. ⋄

Let $\rho \in B_1(H)$ be a positive trace-class operator which is *normalized* (i.e. $\operatorname{tr}\rho = 1$).

Definition 16.6. The linear functional $\ell_\rho(A) = \operatorname{tr}(\rho A)$ is a *state* on $B(H)$ and ρ is called the *density matrix* of this state. ⋄

You can extend this definition to any operator for which this linear functional is finite.

Definition 16.7. ℓ_ρ is a *pure state* if $\rho = P_\psi$ is the orthoginal projection to $\psi \in H$ (i.e. $P_\psi(\phi) = \frac{\langle \phi,\psi\rangle}{\|\psi\|}\psi$). ⋄

So every vector $\psi \in H$ defines a pure state. There is a natural action of S^1 on H, given by $x \mapsto zx$ where $|z| = 1$. The orthogonal projection is invariant with respect to this action, so when you pass to pure states, you get this extra structure.

Let $S_{un}(H)$ be the space of states (i.e. the subspace of $B(H)$ of positive trace class operators).

Proposition 16.8. *$S_{un}(H) \subseteq B(H)$ is a positive cone. That is, (1) if $\rho_1, \rho_2 \in S_{un}(H)$ (the ρs are the density matrices), then $\rho_1 + \rho - 2 \in S_{un}(H)$ (for $0 < \alpha < 1$, (2) if $\rho \in S_{un}(H)$, then $t\rho \in S_{un}(H)$ for $t \in \mathbb{R}_{>0}$, and (3) $S_{un}(H) \cap (-S_{un}(H)) = \{0\}$*

$S(H) \subseteq S_{un}(H)$, the space of normalized positive trace-class operators (states) is a convex subset in $S_{un}(H)$. That is, $\rho_1, \rho_2 \in S(H)$ implies $\alpha\rho_1 + (1-\alpha)\rho_2 \in S(H)$ for $0 \leq \alpha \leq 1$. Pure states are the extremal points of $S(H)$.

Example 16.9. Assume $\psi_1, \ldots, \psi_N$ is an orthonormal system. Assume that $\rho = \sum_{n=1}^N \rho_n P_{\psi_n}$. ρ is a state if and only if $\rho_n > 0$ and $\sum \rho_n = 1$. In this case, we can interpret ρ_n as the probability that the mixed state ρ is in the pure state P_{ψ_n} during the obervation.

Definition: the *expectation value* of an observable A in a state with density ρ is $\langle A\rangle_\rho = \mathrm{tr}(\rho A)$.

$A = A^*$ is an observable. Say that $A = \sum_{n=1}^M a_n P_{\phi_n}$ (A has finitely many eigenvalues for simplicity). Then $\langle A\rangle_\rho = \sum_{n=1}^N \rho_n (A\psi_n, \psi_n) = \sum_{n=1}^N \sum_{k=1}^M \rho_n a_k |(\phi_k, \psi_n)|^2$. This says that ρ_n is the probability that the system will be found in the pure state P_{ψ_n}. $|(\phi, \psi)|^2$ (≤ 1 since ϕ and ψ are normalized) is the probability that a system in the pure state ψ can be found in the pure state ϕ.

Then $\sum_{n=1}^N \rho_n |(\phi, \psi_n)|^2$ is the probability in a mixed state ρ can be found in the pure state ϕ. ⋄

Under certain reasonable assumptions, we will have that if $\rho = e^{-\beta H_h}$ (H_h hermitian converging to something), then $\mathrm{tr}(\rho \cdot \pi_h(\hat{f}))$ converges as h goes to zero to $\frac{1}{(2\pi h)^n} \int_M e^{-\beta H(x)} f(x)\omega^n$.

17 NR 10-05

Last time I introduced the notion of a state with density matrix ρ. The definition should be that a state with density ρ is the linear functional $\ell_\rho(A) = \mathrm{tr}(\rho A)$, with ρ positve .... More generally, we have $\mathcal{A}_h^\sigma \subseteq \mathcal{A}_h$. Under extra assumptions, we can define a postive subspace $\mathcal{A}_h^+ \subseteq \mathcal{A}_h^\sigma$ (e.g. in a C^*-algebra, this could be the set of A with $\sigma(A) > 0$). In this case, a state is a positive linear functional on $\mathcal{A}_h^\sigma$, which means that (1) $\ell(A) \geq 0$ for every $A \in \mathcal{A}_h^+$, and (2) $\|\ell\| = 1$, which implies that $\ell(1) = 1$. The truth is that we will never use this more general context. In the context of C^*-algebras, the GNS construction constructs a representation out of linear functionals, so these are more or less the same [[★★★]].

More on the probablistic interpretation

Remark 17.1. Suppose P_ϕ and P_ψ are two pure states (orthogonal projections to $\phi, \psi \in H$, respectively), then we can compute $\langle P_\phi\rangle_\psi = \mathrm{tr}(P_\phi P_\psi) = |(\phi, \psi)|^2 = \langle P_\psi\rangle_\phi$, the probability that a system in pure state P_ϕ is found in the state P_ψ. This is the nature of quantum mechanics, that even when you're in a pure state, there is still a probability that you're in some other pure state. ⋄

If $A = A^*$ on $\mathbb{C}^n$, it has n eigenvalues, which we can order increasingly. Then we have the orthogonal projections P_{ϕ_i} to the corresponding eigenvectors: $P_{\phi_i} P_{\phi_j} = P_{\phi_i} \delta_{ij}$. The theorem says that $A = \sum_{i=1}^n a_i P_{\phi_i}$. How do we generalize this to some Hilbert space instead of $\mathbb{C}^n$.

When the spectrum is discrete, we should replace the finite sum by an infinite sum. When the spectrum is continuous, we should get a direct integral.

(1) Projection valued measures on $\mathbb{R}$ are maps $P\colon$ (Borel subsets)$\to B(H)$ such that

- $P(E)^* = P(E)$, $P(E)^2 = P(E)$ for $E \subseteq \mathbb{R}$,
- $P(\varnothing) = 0$, $P(\mathbb{R}) = I$, and
- if $E = \bigcup_{n=1}^\infty E_n$, then $P(E) = \lim_{n\to\infty} P(E_n)$.

P has the meaning: it is the sum of those projections for which the eigenvalue fall into the set E.

(2) Projection valued distributions. Given such a P, $P((-\infty,\lambda)) = P(\lambda)$ has the properties

- $P(\lambda)P(\mu) = P(\min\{\lambda,\mu\})$,
- $\lim_{\lambda\to-\infty} P(\lambda) = 0$, $\lim_{\lambda\to+\infty} P(\lambda) = I$,
- $\lim_{\mu<\lambda,\mu\to\lambda} P(\mu) = P(\lambda)$.

Theorem 17.2 (von Neumann)**.** *For any self-adjoint $A\colon H \to H$, there exists a unique P such that*

1. *$\mathcal{D}(A) = \{\phi \in H | \int_{\mathbb{R}} \lambda^2 d(P_A(\lambda)\phi,\phi) < \infty\}$ (the domain of A, the subspace of H where A is defined), and*
2. *$A\phi = \int_{\mathbb{R}} \lambda dP_A(\lambda)\phi$ for $\phi \in \mathcal{D}(A)$.*

The moral: if we are careful, we can treat operators on Hilbert space as if they are self adjoint something [[★★★]].

Given a state $\ell_\rho(A) = \mathrm{tr}(\rho A)$, we have the distribution of values of an observable A in the state ℓ_ρ defined as follows:

- $\mu_{A,\rho}$ is a measure on $\mathbb{R}$,
- $\mu_{A,\rho}(E)$ is the probability that "A takes values in E".
- (definition) $\mu_{A,\rho}(E) := \mathrm{tr}(\rho\cdot P_A(E))$.

Classical case: we have M_{2n} with symplectic form ω, so we have a measure ω^n. A state is a probabilistic measure on M_{2n}. A state with density ρ (ρ a distribution on M_{2n}) is $\ell_\rho(f) = \int_{M_{2n}} f(x)\rho(x)\omega^n$. Then I defined $\mu_\rho(E) = \int_{f^{-1}(E)} \rho(x)\omega^n = \int_M \chi_{f^{-1}(E)}(x)\rho(x)\omega^n$.

For a pure state, ρ is just a delta distribution at a point. In statistical mechanics, a typical ρ is $\rho(p,q) = e^{-E(p,q)/T}$.

That's not the end of the probabistic interpretation. There is something called the *Uncertainty principle*. We already defined the expectation value of A in state ρ, $\langle A\rangle_\rho = \mathrm{tr}(\rho\cdot A) = \ell_\rho(A)$. We can define the dispersion $\sigma^2 = \langle (A - \langle A\rangle_\rho)^2\rangle_\rho$.

Theorem 17.3. *If ρ is a pure state, $\rho = P_\psi$ for $\psi \in H$, then $\sigma^2_\rho(A)\sigma^2_\rho(B) \geq \frac{1}{4}\langle i[A,B]\rangle^2_\rho$.*

Proof. [[★★★ very nice HW exercise.]] □

If we have two quantum observables, then you cannot measure them at the same time. Once we measure, then we know the value of the observable. This theorem tells us that we cannot narrow the disperssions. Two noncommuting observables cannot be in the same pure state.

Assume $A\phi = a\phi$ for some $A = A^*$, so $a \in \mathbb{R}$. Then $\langle A\rangle_{P_\phi} = \mathrm{tr}(P_\phi A) = a$, and $\sigma^2_{P_\phi}(A) = \langle (A-a)^2\rangle_{P_\phi} = \mathrm{tr}(P_\phi(A-a)^2) = 0$. This means that if a pure state is an eigenvector of the observable, then A has the precise value a in this state—there is no dispersion. If some other B doesn't commute with A, then since $\sigma^2_{P_\phi}(B)$ is finite, we get that $\langle i[A,B]\rangle_{P_\phi} = 0$.

If you have p_i, q_i coordinates on $T^*\mathbb{R}^n$, we have $[p_k, q_\ell] = -i\hbar\delta_{k\ell}$.

Summary: We have the notion of quantization of classical observables. $C^\infty M$ (or $C(M)$) can be quantized to $\mathcal{A}_h$, a family of associative algebras over $\mathbb{C}$, with a $\mathbb{C}$-anti-linear anti-involution σ. We have the real subspaces $\mathcal{A}^\sigma_h \subseteq \mathcal{A}_h$. We have states, linear functionals on the algebra of observables which are positive on the positive cone. States are defined by density matrices for a given representation of $\mathcal{A}_h$ in H.

Quantization is not a functor; the functor goes the other way. You can take classical limits. The states that survive when you take the classical limit are the classical states.

Qunatization of Hamiltonian dymanics

So far I completely ignored the dymanics. On the classical level, with $C^\infty M$, we are given $H \in C^\infty M$. This gives us $v_H = \omega^{-1}(dH)$, and evolution $\frac{df_t}{dt} = \{H, f_t\}$, with $f_0 = \mathrm{id}$ and $f_t(x) = f(x_t)$. This gives us $\mathcal{A}_h$, with $H_h \in \mathcal{A}^\sigma_h$, and evolution $ih\frac{df_t}{dt} = [H, f_t]$, with $f_0 = f$. This is $R\times\mathcal{A}_h \to \mathcal{A}_h$, $f \mapsto f_t$, is an automorphism of $\mathcal{A}_t$. If $\exp(\frac{it}{h}H_h) = U(t)$ makes sense, then $f(t) = U(t)fU(t)^{-1}$. This is the Heisenberg picture.

If we have a representation $\pi_h\colon \mathcal{A}_h \to End(H)$, then the Hamiltonian evolution induces the Schrödinger picture: evolution of vector in H: $\psi \mapsto \psi(t) = e^{\frac{it}{h}\pi(H)}\psi$. More precisely, it is the solution to the infinite dimensional ODE $ih\frac{d\psi_t}{dt} = \pi_h(H)\psi_t$ with $\psi_0 = \psi$.

What is the moduli space of these deformations? We can answer this for formal deformations.

We can require that as h goes to 0, H_h goes to H. However, therer is still no canonical quantization of a given system. If you have an integral system, then you do get some kind of functoriality in quantization.

This finishes up quantum mechanics in general. Next time we'll talk about quantum mechanics on $\mathbb{R}^{2n}$.

18 NR 10-08

Remember that if you have a pure state ψ, then $\sigma_\psi^2(A)\sigma_\psi^2(B) \geq \frac{1}{4}\langle i[A,B]\rangle_\psi^2$. There was a question last time. Imagine that $M = \mathbb{R}^2$, so the corresponding algebra is $\mathcal{A}_h = \langle p = -ih\frac{\partial}{\partial q}, C^\infty(\mathbb{R})\rangle$, so $\hat{p} = -ih\frac{\partial}{\partial q}$, $\hat{q} = q$, and H is $L^2(\mathbb{R})$. We can choose pure states to be $\psi_\varepsilon(q) = c_\varepsilon \exp\left(\frac{-(q-q_0)^2}{\varepsilon}\right)$. This is a sequence of functions which concentrate to q_0 (converge to the delta distribution at q_0, which is not in the Hilbert space), with $\|\psi_\varepsilon(q)\|^2 = 1$. We have that $\langle q\rangle_{\psi_\varepsilon} = q_0$, and $\langle\sigma^2(q)\rangle_{\psi_\varepsilon} \to 0$ as $\varepsilon \to 0$. So by the uncertainty principle, $\sigma_{\psi_\varepsilon}(p)\sigma_{\psi_\varepsilon}(q) \geq \frac{h}{2}$. As $\varepsilon \to 0$, $\sigma_{\psi_\varepsilon}(p) \to \infty$ and $\sigma_{\psi_\varepsilon}(q) \to 0$.

If you open a physics text book, $e^{ipx} = |p\rangle$ is presented as a state with momentum p, but this doesn't make sense because it is not normalized. What you should really do is make a sequence of L^2 states that have more and more localized momenta. If you have an operator on the real line, say $\frac{d^2}{dx^2}$, then you have eigenfunctions, like e^{ipx}, but [[★★★ something about turning the operator into the operator p^2]]. You can have such states if you are working on $\ell^2(\mathbb{Z})$ instead of $L^2(\mathbb{R})$.

There are two types of evolution: Heisenberg and Schrödinger evolution. The Heisenberg evolution is a dynamics on $\mathcal{A}_h$ given by $ih\frac{\partial f}{\partial t} = [H_h, f_t]$, where H_h is the quantum hamiltonian, and $H_h \to H$ as $h \to 0$ via the identification $\mathcal{A}_h^\sigma \cong C(M)$. Schrödinger evolution is a dynamics on a representation H of $\mathcal{A}_h$, given by $ih\frac{\partial \psi_t}{\partial t} = \pi(H_h)\psi_t$, where $\pi\colon \mathcal{A}_h \to End(H)$ is a $*$-representation on a hilbert space H.

If you take the Heisenberg algebra and impose the natural $*$-involution ($p^* = p$, $q^* = q$), then there is some theorem which gives you an equivalence between these two. In general, Heisenberg evolution induces a Schrödinger evolution, but they are equivalent on $T^*\mathbb{R}^n$.

Standard problems in quantum mechanics. In general, we don't know what Hamiltonian to choose. If H is the classical hamiltonian, we could choose anything like $H + o(h)$. Using some context, there is usually a natural choice.

- Given H_h, find the spectrum of H_h. This is the quantum analogue of describing values of H. This is a stationary problem; there is not time dependence.

– Scattering. This is the quantum analogue of a classical scattering problem. Imagine $H = \frac{\vec{p}_1^2}{2m_1} + \frac{\vec{p}_2^2}{2m_2} + \cdots + V(\vec{q}_1, \vec{q}_2) + \cdots$, so there are some interaction terms. You can imagine some particles (or stars, for example) coming in, doing something, and then some particles (or stars) are trapped in the interaction area, and some particles fly out. This is a non-stationary problem. In the quantum case, we should assume that as $t \to -\infty$, $\psi_t \to \psi_{ih}(t)$ should be some states with non-interacting particles (a typical example of such a state is $\exp(i\vec{p}\vec{x} + iEt)$; since we want to satisfy the Schrödinger equation, $E = \frac{p^2}{2m}$). Now describe the outgoing asymptotics (as $t \to +\infty$).

Quantization of $T^*\mathbb{R}^n$

In this case, $M = R^*\mathbb{R}^n \cong \mathbb{R}^{2n}$, with coordinates p and q as usual, with the standard symplectic form $\omega = \sum dp_i \wedge dq^i$. The algebras of functions are $Pol(\mathbb{R}^{2n}) \subseteq C^\infty_{pol}(\mathbb{R}^{2n}) \subseteq C^\infty(\mathbb{R}^{2n})$ (the middle one is polynomial functions in the cotangent direction). In this situation, we can actually construct families of $*$-products. Remember that a $*$-product is not unique; you can apply any automorphism which becomes the identity at $h = 0$. Let me describe two of these star products.

(1) Weyl quantization (Weyl $*$-product).

Theorem 18.1. *The operation* $(f_1 * f_2)(p,q) = \frac{1}{(\pi h)^2} \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} f_1(p_1,q_1) f_2(p_2,q_2) \exp\Big(\frac{4i}{\ell}\int_\Delta \omega\Big) dp_1 dq_1$ *where* Δ *is the Euclidean triangle with vertices* (p,q), (p_1,q_1), *and* (p_2,q_2), *is a family of associative products on* $C^\infty(\mathbb{R}^{2n})$.

Proof. [[★★★ HW: prove it (i.e. prove associativity)]] □

[[★★★ HW; prove that $f_1 * f_2 = f_1 f_2 - \frac{ih}{2}\{f_1, f_2\} + o(h^2)$.]]

Similar thing can be used to quantize Kähler manifolds, but for compact Kähler manifolds, you don't get a family. There is something called deformation quantization with torsion. You have a sequence of $*$-products for $h = \frac{1}{n}$. This is Berezin-Toeplitz quantization. There is $C^\infty(M)$, with a map ϕ_n to $End(H_n)$ (family of representations), and maps $End(H_n) \xrightarrow{\psi_n} C^\infty(M)$. The statement is that $\lim_{n\to\infty}(\psi_n \circ \phi_n) = \mathrm{id}$ and $\lim_{m\to\infty} i[\psi_n, \phi_n(f), \psi_n \circ \phi_n(g)]m = \{f, g\}$. So the case of $\mathbb{R}^{2n}$ is very lucky because we have a torision-free deformation.

Theorem 18.2. *There is an isomorphism of algebras* $\big(Pol(\mathbb{R}^{2n}), *\big) \xrightarrow[\sim]{\Phi} Pol(\hat{p}, \hat{q})$, *where* $\hat{p}_j = -ih\frac{\partial}{\partial q_j}$ *and* $\hat{q}_j = q_j$ *(or think of it as generated by the* $\hat{p}$ *and* $\hat{q}$ *with* $[\hat{p}_j, \hat{q}^k] = -ih\delta_j^k$*), where (for multi-indices* α *and* β*)* $\Phi(p^\alpha q^\beta) = \mathrm{Sym}(\hat{p}^\alpha \hat{q}^\beta)$, *where the symmetrization is the sum of all the things you get by letting the symmetric group act on* α *and* β *(or* $(\sum_i u^i \hat{p}_i + \sum_j v_j \hat{q}^j)^k = \sum_{|\alpha|+|\beta|=k} \frac{k!}{\alpha!\beta!} u^\alpha v^\beta$ *non-commutative binomial formula).*

How to construct a deformation of $Pol(\mathbb{R}^{2n}$ with $\{p_i, q^j\} = \delta_i^j$, ps and qs commute? $\langle \hat{p}_i, \hat{q}^j | [\hat{p}_k, \hat{q}^j] = -ih\delta_k^j$, others commute$\rangle$, $\sigma(\hat{p}_i) = \hat{p}_i$ and $\sigma(\hat{q}_i) = \hat{q}_i$. We need to choose an isomorphism. Choose $P(p,q) \xrightarrow{\Phi} P(\hat{p}, \hat{q})$, then $(P * Q)(p,q) = \exp(\frac{\#}{h}\sum_i(\frac{\partial}{\partial p_{(1)}} \wedge \frac{\partial}{\partial q_{(2)}} - \frac{d}{dq_{(1)}} \wedge \frac{\partial}{\partial p_{(2)}}) P(p_{(1)}, q(q)) Q(p_{(2)}, q_{(2)})$.

A remarkable property of this $*$-product: $\mathrm{tr}(f) = \frac{1}{(2\pi h)^n}\int_{\mathbb{R}^{2n}} f(p,q)\omega^n$ is cyclic for this $*$-product (i.e. $\mathrm{tr}(f * g) = \mathrm{tr}(g * f)$).

Next two lectures there will be a guest lecturer who will talk about quantum Calabi-Yau manifolds, then he'll continue the talk 5-6 in the RTGC seminar. Friday, either I'll continue or he'll continue if there is something left.

19 NR 10-10

Today's speaker is Yan Soibelman.

Today's seminar talk will be related to holomorphic Chern-Simons theory. This talk will be an elementry background talk about this. References: `math/0606241` (A_∞-algebras and categories), draft of book with Kontsevich "Deformation theory" Vol. 1 can be downloaded from www.math.ksu/~soibel.

Two main players in today's lecture: A_∞-algebras and L_∞-algebras over a fixed field k of characteristic 0. I will present very similar points of view on these two structures; the point of view of noncommutative geometry.

Definition 19.1 (Preliminary)**.** An *L_∞-algebra* is a formal pointed manifold in the category of $\mathbb{Z}$-graded vector spaces over k together with a vector field Q of degree $+1$ such that $[Q,Q]=0$ and $Q(pt)=0$ (vanishes at the marked point).

An *A_∞-algebra* is a noncommutative formal pointed manifold in the category of $\mathbb{Z}$-graded vector spaces over k together with a vector field Q of degree $+1$ such that $[Q,Q]=0$ and $Q(pt)=0$ (vanishes at the marked point). ◇

In this preliminary definition we see that there is some kind of space. The point of view going back to Grothendieck is that a "space" is a functor $F\colon \mathcal{C}\to \mathsf{Set}$ [[★★★ not $\mathcal{C}^\circ$]] with some properties. Most of what I'll be talking about can be said for any k-linear symmetric monoidal category $\mathcal{C}$ which admits infinite sums and products. If you don't want to think in such abstract terms, the two main examples will be $\mathcal{C}=\mathsf{Vect}_k$ and $\mathcal{C}=\mathsf{Vect}_k^{\mathbb{Z}}$ ($\mathbb{Z}$-graded vector spaces with grade-preserving morphisms, with ordinary tensor product with induced grading, with commuter $V\otimes W\to W\otimes V$ given by $v_n\otimes w_m\mapsto (-1)^{nm}w_m\otimes v_n$). NR: but these are not super vector spaces ... this is some kind of hybrid. YS: yes.

You can talk about algebras, coalgebras, and all sorts of other things in $\mathcal{C}$ (c.f. Peter Teichner's lectures). An algebra is an object A with morphisms $m\colon A\otimes A\to A$ and $1\colon \mathbb{1}\to A$ (we are assuming $End(\mathbb{1})=k$) so that the usual diagrams commute. We have $\mathsf{Alg}_{\mathcal{C}f}\subseteq \mathsf{Alg}_{\mathcal{C}}$ (finite-dimensional or finite length algebras) and $\mathsf{Coalg}_{\mathcal{C}f}\subseteq \mathsf{Coalg}_{\mathcal{C}}$.

Theorem 19.2. *Let $F\colon \mathsf{Alg}_{\mathcal{C}f}\to \mathsf{Set}$ be a functor that commutes with finite projective limits. Then F is represented by a counital coalgebra. that is, there is a $B\in \mathsf{Coalg}_{\mathcal{C}}$ such that $F(R)\cong \mathrm{Hom}_{\mathsf{Coalg}_{\mathcal{C}}}(R^*,B)$.*

Similar to the category of vector spaces, you can talk about cocommutative coalgebras (this is just throwing in another diagram, which says that the opposite coproduct Δ' is equal to the usual coproduct Δ). If instead of $\mathsf{Alg}_{\mathcal{C}f}$, we take $\mathsf{Alg}_{\mathcal{C}f}^{com}$, then the theorem is still true, with B cocommutative.

If we have a coalgebra, we can take its dual to get an algebra (the opposite isn't true). For this reason, I prefer to work with coalgebras. You can dualize, but then you have to speak about topological coalgebras. If we have a commutative algebra, then we have an affine scheme; if it is not commutative, then we can imagine that there is some noncommutative scheme corresponding to our algebra. That is, we have some "generalized space" according to Grothendieck's point of view.

Definition 19.3. The category of *noncommutative thin schemes in $\mathcal{C}$* is the category equivalent to $\mathsf{Coalg}_{\mathcal{C}}$. The category of *thin schemes in $\mathcal{C}$* is the category equivalent to $\mathsf{Coalg}_{\mathcal{C}}^{cocom}$ ◇

For today's lecture, if we have a coalgebra B, we will denote the corresponding "geometric object" by X_B, and given a thin scheme X, we denote the corresponding coalgebra B_X, so $B_X^*=\mathcal{O}(X)$.

Example 19.4. Fix $V\in\mathcal{C}$, and consider $T(V)=\bigoplus_{n\geq 0}V^{\otimes n}$. We can make it into a coalgebra (the *cofree coalgebra*) by $\delta(v_1\otimes\cdots\otimes v_n)=\sum_{i=1}^{n-1}(v_1\otimes\cdots\otimes v_i)\bigotimes(v_{i+1}\otimes\cdots\otimes v_n)$. The noncommutative thin scheme corresponding to $T(V)$ is a *noncommutative formal graded manifold.*

Similarly, we have the cocommutative version $C(V)=\bigoplus_{n\geq 0}\mathrm{Sym}^n(V)$. [[★★★ Exercise: figure out the Δ (you'll need characteristic zero). It should be cocommutative.]] The corresponding thin scheme is called a *formal graded manifold.* ◇

From now on, let's fix $\mathcal{C}=\mathsf{Vect}_k^{\mathbb{Z}}$. We can almost make sense of the preliminary definition. To implement marked point, we should either fix a morphism from k to the coalgebra, or we can take our direct sums starting at $n=1$ instead of $n=0$. We will denote these things by $T_+(V)$ and $C_+(V)$, and the geometric objects will be (noncommutative) formal pointed graded manifolds (or NCfpg manifold). So a NCfpg manifold corresponds to $B \cong T_+(V)$ for some V. PT: you aren't going to allow things like this to be glued together? YS: no, these are really formal manifolds, with just one closed point. A formal pointed graded manifold corresponds to $B \cong C_+(V)$.

You can't do too much differential geometry on formal manifolds, but you can do something. For example, you can speak about vector fields, which correspond to derivations of the algebra or coalgebra. If X is a (noncommutative) formal pointed graded manifold, then $Vect(X)$ corresponds to $Der(T_+(V))$ as a coalgebra (or without the + if not pointed). A derivation is an element of $\mathrm{Aut}(M \otimes k[\varepsilon]/\varepsilon^2)$ [[★★★]]. Since we are working with a graded coalgebra $T_+(V)$, we can look for derivations of different degrees, and define $[\mathcal{D}_1, \mathcal{D}_2] = \mathcal{D}_1\mathcal{D}_2 - (-1)^{|\mathcal{D}_1||\mathcal{D}_2|}\mathcal{D}_2\mathcal{D}_1$, making $Vect(X)$ into a graded lie algebra. Note that a vector field need not commute with itself because for an odd vector field Q, $[Q, Q] = 2Q^2$.

Definition 19.5. A *noncommutative formal pointed differential graded manifold* is a pair $((X, pt), Q)$, where (X, pt) is a NCfpg manifold and Q is a vector field on X vanishing at the marked point such that $\deg Q = +1$ and $[Q, Q] = 0$. A vector field Q of degree 1 with $[Q, Q] = 0$ will be called a *homological vector field*. If you drop noncommutativity, then you get the notion of a formal pointed differential graded manifold. ⋄

Let $A \in \mathsf{Vect}_k^{\mathbb{Z}}$, then we denote the shifting of the grading by 1 by $A[1]$, so $A[1]^n = A^{n+1}$. Consider $T_+(A[1])$, a noncommutative formal pointed graded manifold. Given a coalgebra B, let's denote the corresponding space $\mathrm{Spc}\, B$.

Definition 19.6. An A_∞ *structure on* A is given by a structure of a noncommutative formal pointed differential graded manifold on $\mathrm{Spc}\, T_+(A[1])$. If you drop noncommutative and change T_+ to C_+, you get the definition of an L_∞-structure on A. ⋄

Algebraically, we have $T_+(A[1]) = \bigoplus_{n\geq 1} A[1]^{\otimes n}$ with $Q^2 = 0$, so the derivation Q respects the coproduct. In order to define a derivation on a free algebra, it is enough to define it on generators. So to define such a Q, it it equivalent to have a collection of maps $Q_n \colon A^{\otimes n} \to A[2-n]$ (for $n \geq 1$). Geometrically, the Q_n are Taylor coefficients of the vector field Q. Abusing language a bit, we write $Q = Q_1 + Q_2 + \cdots$ (we start with Q_1, which corresponds to the fact that $Q(pt) = 0$). The condition $Q^2 = 0$ imposes infinitely many quadratic relations on the Q_n.

We'll finish on Friday.

20 NR 10-12

Today Yan Soibelman is speaking again.

Recall that last time we did noncommutative (resp. commutative) (1) $((X, pt), Q)$ formal pointed differential graded manifolds ($\deg Q = 1$, $Q(pt) = 0$ and $[Q, Q] = 0$), which corresponds to a cofree coalgebra $B \cong \bigoplus_{n\geq 0} V^{\otimes n} = T(V)$ (resp. $\mathrm{Sym}^*(V)$) with a fixed coalgebra morphism $k \to B$, with $\tilde{Q}\colon B \to B[1]$ derivation with $\deg \tilde{Q} = 1$, $\tilde{Q}^2$, and $\tilde{Q}$ vanishes on the image of $k \to B$.

$V = A[1]$ for some graded vector space A. Then $T(V)$ is by definition an A_∞-algebra structure on A. Geometrically, we get the Taylor expansion $Q = Q_1 + Q_2 + \cdots$. Algebraically, this corresponds to a collection of maps $m_n\colon A^{\otimes n} \to A[2-n]$, called *higher multiplications*. We get conditions on the m_i from the condition $Q^2 = 0$.

$\left(\sum m_i\right)^2 = 0$ implies $m_1^2 = 0$, $m_1\colon A \to A[1]$ a derivation. We also get $m_2^2 + m_1 m_3 + m_3 m_1 = 0$, so $m_2^2 = \{m_1, m_3\}$ (anti-commutator), so if $m_{\geq 3} = 0$, then we get $m_2^2 = 0$, which is equivalent to $m_2\colon A \otimes A \to A$ being an associative product. In general, $H^\bullet(A, m_1)$ is an associative algebra with respect to the product m_2.

If we take $V = \mathfrak{g}[1]$, then $C(\mathfrak{g}[1])$. We then get "higher Lie brackets" $b_n\colon \wedge^n \mathfrak{g} \to \mathfrak{g}[2-n]$, with $b_1^2 = 0$, and b_2 defining a Lie bracket. Sometimes, we denote $b_n(\alpha_1 \wedge \cdots \wedge \alpha_n) =: [\alpha_1, \dots, \alpha_n]$.

Recall that we stated last time that all these things are defined as functors $\mathsf{Artin}_k^{(NC)} \to \mathsf{Set}$, and stated a theorem that if such a functor commutes with finite projective limits, then it is represented by a coalgebra.

<u>L_∞-algebras and deformation theory in characteristic 0</u>. Suppose we want to define the formal scheme of zeros of Q. As a functor, given a commutative finite dimensional Artin algebra, we get $\mathrm{Zeros}(Q)(R) = \{R^* \to C(V) | \tilde{Q}$ vanishes on the image of $\mathfrak{m}^*\}$ ($\mathfrak{m}$ the maximal ideal of R).

In the case $V = \mathfrak{g}[1]$, check [[★★★ HW]] that the last condition is equivalent to the following equation on $\gamma \in \mathrm{Hom}(\mathfrak{m}^*, \mathfrak{g}[1]) = \mathfrak{g}^1 \otimes \mathfrak{m}^*$ (where $\mathfrak{g}^1$ is the first graded component of $\mathfrak{g}$):

$$d\gamma + \frac{1}{2!}[\gamma, \gamma] + \frac{1}{3!}[\gamma, \gamma, \gamma] + \cdots = 0 \qquad \text{(Mourer-Cartan)}$$

This is called the (generalized) Mourer-Cartan equation (if $b_{\geq 3} = 0$, this is the Mourer-Cartan equation, in which case this gives a differential graded Lie algebras as a special case of L_∞-algebras).

Geometrically, this is also understandable. Saying that the vector field Q vanishes at a point x means that $Q(f)(x) = 0$ for all f. If $f = f_1 + f_2 + \cdots$ is the Taylor expansion of f, then we see that $f_n = \frac{1}{n!} f_1 \wedge \cdots \wedge f_1$. (this more or less solves the exercise, but you have to prove this formula).

There is a certain equivalence relation on the set of solutions to Mourer-Cartan. Consider the case when $\mathfrak{g} = \bigoplus_{n\geq 0} \mathfrak{g}^n$ is a differential graded Lie algebra, with d, $[,]$. Then $[\mathfrak{g}^0, \mathfrak{g}^0] \subseteq \mathfrak{g}^0$ and $[\mathfrak{g}^0, \mathfrak{g}^1] \subseteq \mathfrak{g}^1$. Since $\mathfrak{g}^0 \otimes \mathfrak{m}$ is a nilpotent algebra, we get a corresponding group $\exp(\mathfrak{g}^0 \otimes \mathfrak{m})$ acting on $\mathfrak{g}^1 \otimes \mathfrak{m}$. The action is given by $\gamma \mapsto g\gamma g^{-1} - dg\, g^{-1}$. [[★★★ Exercise: this gauge action preserve solutions to the Mourer-Cartan equation]]

PT: there must be some analog to flat connections somewhere. YS: if you like, the $\gamma \in \mathfrak{g}^1 \otimes \mathfrak{m}$ is a connection. PT: but on what bundle? YS: [[★★★ something]]

There is a generalization of this picture to an arbitrary L_∞-algebra.

Definition 20.1. The *deformation functor* associated to an L_∞-algebra $\mathfrak{g}$ is a functor on commutative Artin algebras (to Set) $Def_{\mathfrak{g}}(R) = \{$equivalence classes of solutions to MC$\}$. ◇

The corresponding space should be called the moduli space of this deformation theory. General philosophy which goes back to Deligne, Drinfeld, Kontsevich, says that any deformation theory in characteristic zero is described by a deformation functor for some L_∞-algebra $\mathfrak{g}$.

Informally, you can think about it like this. You have some mathematical structure, like a flat connection or a complex structure or a multiplication on a vector space making it into an associative algebra, and it is part of some space of other structures so that you can speak of structures parameterized by some base (Spec of a local artin algebra). Then we can as for parameterizations such that over the closed point of $\mathrm{Spec}\, R$, we have a given structure. These are flat deformations. These families typically form a category, so you can say when they are isomorphic. Consider the naïve deformation functor Def^{X_0} which associates to R isomorphism classes of families over $\mathrm{Spec}\, R$ with fixed fiber over the closed point of $\mathrm{Spec}\, R$. The philosophy is that for any structure X, $Def^X \cong Def_{\mathfrak{g}}$ for some $\mathfrak{g}$.

Note that isomorphic L_∞-algebras will produce isomorphic deformation functors. There is a weaker notion of quasi-isomorphism between L_∞-algebras (a morphism which induces isomorphisms on homology groups).

Theorem 20.2. *If $\mathfrak{g}_1$ is quasi-isomorphic to $\mathfrak{g}_2$, then they give rise to isomorphic deformation functors.*

An L_∞-algebra is *formal* if $b_1 = 0$, it is called *linearly contractible* if $H^1(-, b_1) = 0$ and $b_{\geq 2} = 0$, and it is called *abelian* if it is formal and $[,] = 0$. On can prove that any L_∞-algebra is isomorphic to a product of a formal and a linear contractible (this is the minimal model theorem).

For a differential graded Lie algebra, in the abelian case, there is no MC formula, which tells you that the point in the moduli space is smooth.

Suppose we have $E \to X$ a G-bundle (later X will be a Kähler manifold), and suppose it is flat. Then we have a corresponding vector bundle $ad(E)$, with flat connection ∇_0. I'm interested in the deformation theory of this flat connection, so I'd like to add a 1-form $\gamma \in \Omega^1(X, ad(E))$ so that $\nabla_0 + \gamma$ is flat. We have a differential $d = [\nabla_0, -]$, giving us $\mathfrak{g} = \Omega^\bullet(X, ad(E))$. Then the flatness condition can be written in the form of MC: $d(\gamma) + \frac{1}{2}[\gamma, \gamma] = 0$. So we get a moduli space $\mathcal{M}_{\nabla_0}$ of flat deformations of ∇_0 (this is a local theory, not a global one ... we work with Artin algebras, not arbitrary schemes).

Theorem 20.3 (Goldman-Millson, 1988)**.** *If X is compact Kähler, G compact, then $\mathfrak{g}$ is formal.*

Suppose X is Calabi-Yau (complex Kähler manifold with trivial canonical class), so it admits a nowhere vanishing holomorphic form dz. Suppose we are interested in deformations of a given complex structure. This deformation theory is controlled by $\mathfrak{g} = \big(\Omega^{0,\bullet}(X, T^{1,0}), \overline{\partial}\big)$.

Theorem 20.4 (Tian-Tolozov?)**.** *$\mathfrak{g}$ is quasi-isomorphic to an abelian differential graded Lie algebra.*

21 NR 10-15

Recall that last time we talked about Weyl quantization. This was the story about $T^*\mathbb{R}^n \cong \mathbb{R}^{2n}$ with coordinates p_i and q^i and symplectic form $\omega = \sum dp_i \wedge dq^i$. We gave a $*$-product on $C^\infty\mathbb{R}^{2n}$:

$$f_1 * f_2(x) = \frac{1}{(2\pi h)^{2n}} \iint_{\mathbb{R}^{2n}\times\mathbb{R}^{2n}} \exp\Big(\frac{4i}{h}\int_{\Delta_{x,x_1,x_2}} \omega\Big) f_1(x_1) f_2(x_2) \omega_1^n \omega_2^n$$

Then we have $(Pol(\mathbb{R}^{2n}), *) \cong \langle \hat{p}_i, \hat{q}^i | [\hat{p}, \hat{p}] = 0 = [\hat{q}, \hat{q}], [\hat{p}_i, \hat{q}^j] = \sqrt{-1}\delta_i^j$, and $P(p,q) \mapsto P^{sym}(\hat{p}, \hat{q})$ is an isomorphism of algebras. We also have the trace

$$\operatorname{tr} f = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^{2n}} f(x)\omega^n$$

with the property $\operatorname{tr}(f * g) = \operatorname{tr}(g * f)$.

We have the p-q quantization

$$(f_1 \tilde{*} f_2)(p,q) = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^{2n}} \exp\Big(\frac{i}{h}((p-p_1),(q-q_1)\big) f_1(p, q_1) f_2(p_1, q) d^n p_1 d^n q_1$$

Then we also have $(Pol(\mathbb{R}^{2n}, \tilde{*}) \cong \langle \hat{p}, \hat{q} | \cdots \rangle$, but the isomorphism is different, it is given by $P(p,q) \mapsto P(\hat{p}, \hat{q})|_{p,q \text{ ordered}}$.

This also gives an "explicit" deformation quantization of $C^\infty_{pol}(T^*\mathbb{R}^n)$, which is $\mathit{Diff}_h(\mathbb{R}^n)$ (generated by $h\frac{\partial}{\partial q_j}$); a general element is $\sum_\alpha h^{|\alpha|} f^\alpha(q) \big(\frac{\partial}{\partial q_1}\big)^{\alpha_1} \cdots \big(\frac{\partial}{\partial q_n}\big)^{\alpha_n}$.

Another important property is that there exists a trace for this product $\tilde{*}$

$$\widetilde{\operatorname{tr}} f = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^{2n}} f(x)\omega^n(1 + O(h))$$

such that it is cyclic (given by pushing the other trace across the isomorphisms).

<u>p-q quantization of a Hamiltonian system.</u> Our obersevables are $C^\infty_{pol}(T^*\mathbb{R}^n)$, and the quantization is this algebra $\mathit{Diff}_h(\mathbb{R}^n)$. There are *natural hamiltonian systems* $H = \frac{p^2}{2m} + V(q)$.

Example 21.1. N interacting particles of mass m in $\mathbb{R}^3$. In this case, a typical hamiltonian is $H = \sum_i \frac{\vec{p}^2}{2m} + \sum_{i \neq j} V(\vec{q}_i - \vec{q}_j)$. ⋄

Example 21.2. A particle in $\mathbb{R}^3$ in a potential field V, then $H = \frac{\vec{p}^2}{2m} + V(\vec{q})$. ◇

The idea of Schrödinger was that this Hamiltonian should be replaced by some differential operator $\widehat{H} = \frac{\hat{p}^2}{2m} + V(\hat{q})$. Consider the representation in functions on $\mathbb{R}^n$, where $\hat{q}^i$ is given by multiplication by q^i and $\hat{p}_i = -ih\frac{\partial}{\partial q^i}$. We have an anti-involution (or $*$-structure) σ, given by $\sigma(\hat{p}) = \hat{p}$ and $\sigma(\hat{q}) = \hat{q}$. We want our representation to be a $*$-representation. The representation space is the Hilbert space $H = L_2(\mathbb{R}^n)$. In H, the quantum hamiltonian acts as a differential operator of the form $\widehat{H} = -\frac{h^2}{2m}\Delta + V(q)$. The Schrödinger dynamics (in H) is given by the differential equation (called the Schrödinger equation)

$$ih\frac{\partial\psi(q,t)}{\partial t} = -\frac{h^2}{2m}\Delta\psi(q,t) + V(q)\psi(q,t).$$

The natural product on this hilbert space is $\int_{\mathbb{R}^n} \overline{f}(q)g(q)d^nq$.

All the problems about the spectrum of the hamiltonian and scattering become questions about this differential operator. This is the Schrödinger point of view of quantum mechanics. If we didn't have that i, this would be hyperbolic differential equations.

Semi-classical limit

What can we say about these differential equations? We should be able to recover classical mechanics by letting h go to zero. We should be able to recover the classical evolution from the quantum evolution. Consider the evolution of vectors of the form $\psi(q) = e^{if(q)/h}\phi(q)$.

Why are these vectors significant? This ψ is a pure state. $\langle\hat{p}_k\rangle_\psi = \mathrm{tr}(P_\psi\hat{p}_k) = (\psi, \hat{p}_k\psi) = \int_{\mathbb{R}^n}\overline{\psi}(q)\hat{p}_k\psi(q)d^nq = \int_{\mathbb{R}^n}\big(\overline{\psi}(q)\frac{\partial f}{\partial q^k}\psi(q) + \overline{\phi}(q)(-ih\frac{\partial\phi}{\partial q^k})\big)d^nq$. As $h \to 0$, the second term goes away, so we get $\int_{\mathbb{R}^n}|\phi(q)|^2\frac{\partial f}{\partial q^k}d^nq$. This means that this is a state in which the momentum has a semiclassical limit. We can also compute $\langle\hat{q}^i\rangle_\psi = \int|\phi(q)|^2q^id^nq$. So P_ψ, as $h \to 0$, becomes some classical state. Recall that a classical state is given by a measure on the phase space, so we get $\int_{\mathbb{R}^{2n}}\rho(p,q)f(p,q)\,dp\,dq$. If the classical state were supported on the whole space, we'd have an integral like this, but we only integrate over $\mathbb{R}^n$ (not $\mathbb{R}^{2n}$), so the classical state is supported on a Lagrangian, given by $\rho(p,q) = \delta(p - \frac{\partial f}{\partial q})|\phi(q)|^2$. If you have any differential operator d, then $\langle d\rangle_\psi \xrightarrow{h\to 0} \int_{\mathbb{R}^{2n}}\rho(p,q)d(p,q)\,dp\,dq$.

We are moving towards the path integral from the direction of partial differential equations. I want to motivate the formula for the path integral from the Schrödinger equation.

Let $\psi(q,t)$ be a solution to the Schrödinger equation with the initial condition $\psi(q,0) = \psi(q)$. Lets draw a picture of the evolution of the supporting Lagrangian L_0 ($\{p = \frac{\partial f}{\partial q}\}$)

[[★★★ picture]]

As the Lagrangian evolves according to the Hamiltonian flow, there may be many trajectories which end at a particular value of q. Call them γ_1, γ_2, γ_3.

Theorem 21.3. *As $h \to 0$, the solution $\psi(q,t) = \sum_j\phi(q_j(q,t))\big|\frac{\partial Q_t}{\partial q}(q_j)\big|^{-1/2}\exp\big(\frac{i}{h}S[\gamma_j] - i\frac{\pi}{2}\mu_j\big)(1 + O(h))$, where $S[\gamma_j]$ is the classical Hamilton-Jacobi action for the trajectory γ_j.*

Proof. [[★★★ HW]] You'll have to find that μ_j is the massless index of the trajectory γ_j. The idea is to look for solutions $\psi(q,t) = e^{\frac{i}{h}S}\psi_0(q,t)$ (where $S(q,t)$ is [[★★★]] and ψ_0 is a power series in h.). Substitute this into the Schrödinger equation. The zero order term is $\frac{\partial S}{\partial t}\psi + o(h) = \frac{1}{2m}\big(\frac{\partial S}{\partial q}\big)^2 + V(q) + o(h)$, so we get $\frac{\partial S}{\partial t} = \frac{1}{2m}\big(\frac{\partial S}{\partial q}\big)^2 + V(q) = H(\frac{\partial S}{\partial q}, q)$. This is the Hamilton-Jacobi equation, with $S = S[\gamma]$ (you have to be more careful when there are more γ's). Considering the first order terms, you get

$$\frac{\partial\psi_0}{\partial t} = \frac{\partial S}{\partial q}\frac{\partial\psi_0}{\partial q} \Longrightarrow \psi_0 = \Big|\frac{\partial Q_t}{\partial q}\Big|^{-1/2}.$$

□

So we got something quite familiar. We get something times a rapidly oscilating exponent. Recall where you've seen these before. Consider $Z_h = \int_{\mathbb{R}^n}\exp\big(\frac{i}{h}S(x)\big)d^nx$; assume (1) it converges for all $h \neq 0$, and (2) $S(x)$ has finitely many simple critical points. We want to look at the asymptotics of this integral as $h \to 0$. We should compute the integral using the stationary phase approximation.

(i) find critical points x_α, where $dS(x_\alpha) = 0$. (ii) $S(x) = S(x_\alpha) + \frac{1}{2}\big(x - x_\alpha, S''(x_0)(x - x_\alpha)\big) + O(x - x_\alpha)$. As always, you split the integration region into places that are close to the critical points and far away from the critical points

$$\begin{aligned}\int_{\mathbb{R}^n} \exp\big(\frac{i}{h}S(x)\big)d^nx \\ &= \sum_\alpha \int_{U^\varepsilon_{x_\alpha}} e^{i\frac{S(x_0)}{h}+\frac{i}{2h}(x-x_\alpha,S''(x_0)(x-x_\alpha))+\cdots} d^nx \\ &= \langle y = (x-x_\alpha)\frac{1}{\sqrt{h}}\rangle \\ &= \sum_\alpha \int_{\frac{1}{\sqrt{h}}U^\varepsilon_{x_\alpha}} \exp\big(i\frac{S(x_\alpha)}{h}+\frac{i}{2}(y,S''(x_\alpha)y)+O(\sqrt{h})\big)d^nx \\ &= \int_{\mathbb{R}^n} e^{\frac{i}{2}(y,Ay)}d^nx \\ &= \# \cdot \frac{1}{(\det A)^{1/2}}\end{aligned}$$

Something $|\det(S''(x_\alpha))|^{-1/2}e^{i\frac{\pi}{2}(n_+-n_-)}$. Something about writing something as an integral over all paths.

22 NR 10-17

Anton missed this class. The following are notes were taken by Chris Schommer-Pries.

Recall what happened last time: We considered quantum Mechanics in $\mathbb{R}^n$ (The quantization of classical mechanics on $T^*\mathbb{R}^n$). We had the quantum algebra of observables:

$$Diff_\hbar(\mathbb{R}^n) =< \sum_\alpha \hbar^{|\alpha|}f_\alpha(q) > \text{times derivatives}$$

We represent it on $\mathcal{H} = L^2(\mathbb{R}^n)$. If the Hamiltonian $H = p^2/2m + V(q)$, it's quantization is $\hat{H} = -\frac{\hbar}{2m}\Delta + V(q)$, and the Schrödinger equation is,

$$i\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi$$

With initial condition $\psi(q,o) = \psi(q)$. The evaolution of purre states P_ψ, in the semiclassical limit is

- $\psi(q) = \phi(q)e^{\frac{if(q)}{\hbar}}$
- $\psi(q,t) = \sum_j \phi(q_j(q,t))|\frac{\partial Q_t}{\partial q}(q_j)|^{-\frac{1}{2}}e^{\frac{i}{\hbar}S[\gamma_j]-i\frac{\pi}{2}\mu_j}(1+O(\hbar))$

Where j is a trajectory from the initial Lagrangian $L_0 = \{(p,q) \mid p = df|_q\}$, to the Lagrangian L_t.

The rapidly oscillating integral $Z_\hbar = \int_{\mathbb{R}^n} e^{\frac{i}{\hbar}S(x)}\ dx$ is approximately, in the semicalssical limit:

$$\sum_\alpha e^{i\frac{S(x_\alpha)}{\hbar}}|\det(S''(x_\alpha)|^{-\frac{1}{2}}e^{i\frac{\pi}{2}(n_+-n_-)}(1+O(\hbar))$$

The Evolution Operator: If you just think of this in terms of linear algebra, it's just a linear differential equation and the solutions should be,

$$\psi(q,t) = (e^{\frac{it}{\hbar}\hat{H}})\psi(q) = \int_{\mathbb{R}^n} U(q,q')(t)\psi(q');dq'$$

It is very tempting to say something like this:

1. $\int_{\mathbb{R}^n} U(q_1,q_2)(t_1-t_2)U(q_2,q_3)(t_2-t_3)\ dq_2 = U(q_1,q_3)(t_1-t_3)$

2. $U(q_2, q_1)(-t) = \overline{U}(q_1, q_1)(t) = [U(q_1, q_1)(t)]^{-1}$

We guess/assume

1. $U(q_1, q_1)(t) = \int_{\text{paths}\gamma} e^{\frac{i}{\hbar}S[\gamma]}$, where the paths satisfy $\gamma(0) = q_1$ and $\gamma(t) = q_2$. The left-hand-side is defined using the functional calculus for self-adjoint unbounded operators, and involves some technicalities, but is well defined. The right-hand-side is problematic.

2. $\psi(q, t) = \int_{\mathbb{R}^n} \int_{\{\gamma\}} \phi(q') e^{i\frac{f(q')}{\hbar} + \frac{i}{\hbar}S[\gamma]} \; \mathcal{D}\gamma \; dq'$ The critical points of $f(q) + S[\gamma]$ are the paths starting at L_0 and ending at q.

We don't really know how to do an integral over all paths in $\mathbb{R}^n$. But we have a semiclassical expansion and we have a semigroup law structure. Let γ_C be a classical solution, i.e. a solution to the Euler-Lagrange equations. Let's expand the action around this path γ_C:

$$S[\gamma] = \int_0^t (\frac{m}{2}\dot{\gamma}^2(t) + V(\gamma(t)) \; dt$$

So,

$$S[\gamma_C + x] = S[\gamma_C] + \int_0^t (\frac{m}{2}\dot{x}^2 + < V^{(2)}(\gamma_C)x, x >) \; dt + \sum_{n\geq 3} \frac{1}{n!} \int_0^t V^{(n)}_{\gamma_C}(x(t)) \; dt$$

i.e. we do a Taylor expansion. So when we integrate over all γ (near γ_C) we get,

$$e^{i\frac{S[\gamma_C]}{\hbar}} \int_x e^{\frac{i}{2}<,ky> + i\sum_{n\geq 3} \frac{\hbar^{\frac{n}{2}-1}}{n!} \int_0^t V^{(n)}(x)} \; dx$$

Which becomes,

$$e^{i\frac{S(\gamma_C)}{\hbar}} |\det(K)|^{-\frac{1}{2}} e^{i\frac{\pi}{2}Ind(K)}$$

Where,

$$K = id\frac{m}{2}\frac{d^2}{dt^2} + V^{(2)}(\gamma_C(t))$$

acts on functions $[0, 1] \to \mathbb{R}^n$. This is very similar to the expansion of the operator U.

We have a question: do we have $|det'(K)| = |\frac{\partial Q}{\partial q}|$? Yes. There is a theorem.

Now can we do better? Can we identify all the terms in the asymptotic expansions? On the one hand (for U) we get a sum made out of Feynmann diagrams, and on the other hand the asymptotic expansion associated to the PDE. Are these the same? This is probably an open question.

Further questions:

1. Is it true $U(q_1, q_2)(t)$ [defined by Feynmann diagrams] satisfies the 'semigroup' identities?

If so, then we can assign a vector space to the endpoints on an interval, and to the interval itself we can assign the power series $U(q_1, q_2)(t)$. We don't know if it converges or anything like that.

23 NR 10-19

$U(q_1, q_2|t) = e^{itH/h}(q_1, q_2) = \left(\frac{1}{2\pi i h}\right)^{n/2}\left|\frac{\partial^2 S}{\partial q_1 \partial q_2}\right|^{1/2} \exp(\frac{i}{h}S(q_1, q_2, t))(1 + O(h))$

Assuming a single trajectory connecting two points.

$$\psi(q,t) = \int U(q,q'|t)\phi(q')e^{i\frac{f(q')}{h}}dq'$$
$$= \phi(q_0)|\frac{\partial\gamma(t)}{\partial q_0}|^{-1/2}\exp(iS(q,q_0,t)/h)(1+O(h))$$

Where γ is a path with $\gamma(0) = q_0$ and $\gamma(t) = q$. The Legendre transform of $\dot\gamma(t)$ is $df(q)$.

Proof of formula. Assume $T^*\mathbb{R}$. We have

$$ih\frac{\partial\psi}{\partial t} = -\frac{h}{2m}\frac{\partial^2\psi}{\partial q^2} + V(q)\psi$$

Try solutions of the form $\psi(q,t) = e^{iS(q,t)/h}\psi_0(q,t)(1 + O(h))$ where ψ_0 is a power series in h. Then the equation becomes

$$-S_t\psi_0+ih\frac{\partial\psi_0}{\partial t} = -\frac{h^2}{2m}\frac{\partial^2\psi_0}{\partial q^2}-i\frac{h}{m}\frac{\partial\psi_0}{\partial q}\frac{\partial S}{\partial q}+\frac{1}{2m}(\frac{\partial S}{\partial q})^2\psi_0-i\frac{h}{2m}\frac{\partial^2 S}{\partial q^2}\psi_0+V(q)\psi_0.$$

Looking at the terms

h^0: $-\frac{\partial S}{\partial t} = \frac{1}{2m}(\frac{\partial S}{\partial q})^2 + V(q)$. This is the Hamilton-Jacobi equation $S = \mathcal{A}_f[\gamma_{cl}]$, where $\mathcal{A}_f[\gamma_{cl}] = \int_0^t(\frac{1}{2m}\dot\gamma_{cl}^2 + V(\gamma))d\tau + f(q_0)$.

h^1: $i\frac{\partial\psi_0}{\partial t} = -\frac{i}{m}\frac{\partial\psi_0}{\partial q}\frac{\partial S}{\partial q} - \frac{i}{2m}\frac{\partial^2 S}{\partial q^2}\psi_0$, whic we can write as

$$(\frac{\partial}{\partial t} + \frac{1}{m}\frac{\partial S}{\partial q}\frac{\partial}{\partial q})\log\psi_0 = -\frac{1}{2m}\frac{\partial^2 S}{\partial q^2}$$

From Hamilton-Jacobi, $\frac{\partial\mathcal{A}[\gamma_{cl}]}{\partial q} = p = m\frac{d\gamma(t)}{dt}$ and $\frac{\partial\mathcal{A}[\gamma_{cl}]}{\partial q_0} = p_0$. This is just because $\mathcal{L}(\xi,q) = \frac{\xi^2}{2m} - V(q)$ and $p = m\xi$. This means that this formula is

$$(\frac{\partial}{\partial t} + \frac{1}{m}\frac{\partial\gamma(t)}{\partial t}\frac{\partial}{\partial q})\log\gamma_0(q,t) = -\frac{1}{2}\frac{d}{dt}\log\frac{\partial q}{\partial q_0}$$

because the right hand side is

$$\frac{\partial^2 S}{\partial q^2} = \frac{\partial}{\partial q}\frac{\partial\mathcal{A}[\gamma_{cl}]}{\partial q} = m\frac{\partial}{\partial q}\frac{d\gamma(t)}{dt}$$
$$= m\frac{\partial q_0}{\partial q}\frac{\partial}{\partial q_0}(\frac{d\gamma(t)}{dt})$$
$$= m\frac{\partial q_0}{\partial q}\frac{d}{dt}(\frac{\partial\gamma(t,q_0)}{\partial q_0})$$
$$= m(\frac{\partial q}{\partial q_0})^{-1}\frac{d}{dt}(\frac{\partial q}{\partial q_0}) = \frac{d}{dt}\log\frac{\partial q}{\partial q_0}$$

$$(\frac{\partial}{\partial t} + \frac{1}{m}\frac{\partial\gamma(t)}{\partial t}\frac{\partial}{\partial q})\log\gamma_0(q,t)\Big|_{q=\gamma(t)} = \frac{d}{dt}(\log\psi_0(\gamma(t),t))\underbrace{\frac{\partial q}{\partial q_0}\Big|_{t=0}}_{=1}$$

so

$$\frac{d}{dt}\log\psi_0(\gamma(t),t) = -\frac{1}{2}\log\frac{\partial q}{\partial q_0}$$
$$\psi_0(\gamma(t),t) = \phi(q_0)\Big(\frac{\partial q}{\partial q_0}\Big)^{-1/2}$$

□

From the Schrödinger picture, we have $U(q_1, q_2|t) = C \cdot e^{i\frac{S(q_1,q_2,t)}{h}}\left|\frac{\partial^2 S}{\partial q^2}\right|^{1/2}(1 + O(h))$. From the path integral, we have

$$U(q_1,q_2|t) = C\cdot e^{i\frac{S(q_1,q_2,t)}{h}}|\det K|^{-1/2}(1+O(h))$$

Where $K = -\frac{1}{m}(\frac{d}{ds})^2 + V^{(2)}(\gamma(s))$ for $0 \le s \le t$ on $L^2[0,t]$ with $x(0) = x(t) = 0$.

Computing these coefficients is painful (and grows more painful with higher order terms). Using the path integral, we have

$$\begin{aligned}
U(q_1,q_2|t) &= \int_{\gamma(0)=q_1,\gamma(t)=q_2} \exp(\frac{i}{h}\mathcal{A}[\gamma])\,\mathcal{D}\gamma \\
&= \int_{x(0)=x(t)=0} \exp\Big(\frac{i}{h}\mathcal{A}[\gamma_c+x]\Big)\,\mathcal{D}x \\
&= e^{i\mathcal{A}[\gamma_c]/h}\int_{y(0)=y(t)=0} \exp(i(y,Ky)+\sum_{n\geq 0}\frac{h^{n/2-1}}{n!}(V^{(n)},y^n))\,\mathcal{D}y \\
&= Ce^{i\mathcal{A}[\gamma_c]/h}|\det K|^{-1/2}e^{i\pi\nu(K)/2}\cdot\sum_{n\geq 0}h^n c_n[\gamma_x]
\end{aligned}$$

Say $\gamma = \gamma_c + x$, then use $\mathcal{A}[\gamma_c + x] = \mathcal{A}[\gamma_c] + \int_0^t(\frac{1}{2m}(\dot{x}(t))^2 + V''(\gamma_c(\tau))x^2(\tau))\,d\tau + \sum_{n\geq 3}\frac{1}{n!}\underbrace{\int_0^t V^{(n)}(\gamma_c(\tau))x^n(\tau)\,d\tau}_{=:(V^{(n)},y^n)}$, and $x = \sqrt{h}y$.

$\nu(K)$ is the index of the operator (difference of positive and negative eigenvalues)

Look at

$$\int e^{i(x,Bx)+\sum_{n\geq 3}\frac{1}{n!}(V^{(n)},x)h^{n/2-1}}\,dx = \sum_{n_3\geq 0,n_4\geq 0\cdots}\frac{h^{(3n_3+4n_4+\cdots)/2-n_3-n_4-\cdots}}{n_3!(3!)^{n_3}n_4!(4!)^{n_4}\cdots}\int_{\mathbb{R}^n} e^{i(x,Bx)}(V^{(3)},x^3)^{n_3}(V^{(4)},x^4)^{n_4}\cdots$$

Assume B has some positive imaginary part.

$$\begin{aligned}
\int e^{i(x,Bx)}x_{i_1}\cdots x_{i_n}\,d^n x &= \frac{1}{n!}\frac{\partial}{\partial y_{i_1}}\cdots\frac{\partial}{\partial y_{i_n}}\int e^{i(x,Bx)+(y,x)}d^n x\Big|_{y=0} \\
&= C\cdot\frac{1}{n!}\frac{\partial}{\partial y_{i_1}}\cdots\frac{\partial}{\partial y_{i_n}}(e^{-i/4\cdot(y,B^{-1}y)})\Big|_{y=0}
\end{aligned}$$

You can write this as the sum over all perfect matchings on $(i_1,\dots,i_n)$ of the product of the $(-\frac{\sqrt{-1}}{4}B^{-1})_{ij} =: G_{ij}$ (where i and j are paired).

Example 23.1. $n = 4$, then this integral will be equal to $G_{12}G_{34} + G_{13}G_{24} + G_{14}G_{23}$. ◇

The power series can be written as

$$= \sum_{n_i\geq 0,i\geq 3}\frac{h^{\#}}{\cdots}\sum_{\text{perfect matchings}}\text{perfect matchings with } V^{(3)},\ V^{(4)}, \text{etc attached to vertices}$$

This is the sum over all graphs Γ with valence at least 3 at each vertex of $\frac{1}{|\operatorname{Aut}(\Gamma)|}F(\Gamma)$, where $F(\Gamma)$ assigns a $V_{\cdots}^{(k)}$ to each k-valent vertex and G_{ij} to each edge between i and j.

As far as I know, nobody has bothered to prove that $\int U(q_1,q_2|t)U(q_2,q_3|s)\,dq_2 = U(q_1,q_2|s+t)$.

24 NR 10-22

Last time: I explained that the amplitude in quantum mechanics can be considered as a sum of Feynman diagrams. As far as I know, it is open to verify that the composition law is satisfied. Recall that

$$\int_{\mathbb{R}^n} \exp\bigl(-\frac{1}{2}(x,Kx)+V(x)\bigr)d^n x = \sum_{\Gamma} \frac{F(\Gamma)}{|\operatorname{Aut}\Gamma|}$$

Where $F(\Gamma)$ is computed by assigning elements of your potential, $V_{i_1,i_2,\dots}$, and to each vertex and to each edge assigning $(K^{-1})^{i_1 j_1}$, and then multiplying everything together. Then

$$U(q_!,q_2|t) = \int_{\gamma(0)=q_1,\gamma(t)=q_2} \exp\bigl(\frac{i}{h}S[\gamma]\bigr)$$

which we expand near a classical path γ_c, with $\gamma = \gamma_c + x$. We get $C\exp\bigl(i\frac{\mathcal{A}[\gamma_c]}{h}\bigr)|\det'(K_{\gamma_c})|^{-1/2}\sum_\Gamma \frac{F(\Gamma)}{|\operatorname{Aut}(\Gamma)|}$ with

$$\mathcal{A}[\gamma_c+x] = \int_0^t \bigl(\frac{1}{2m}\dot\gamma(\tau)^2+V(\gamma(\tau))\bigr)d\tau = \mathcal{A}[\gamma_c]+\underbrace{\int_0^t \bigl(\frac{1}{2m}\dots x^2+V''(\gamma_c)x^2\bigr)}_{-\frac{1}{2}(x,Kx)}$$

From now on I'll assume $n=1$. We have that the regularized determinant is $\det'(K)=\prod_{n=1}^\infty \frac{\lambda_n}{\lambda_n^0}$.

What is $F(\Gamma)$? If we have n vertices, and $0\le s\le t$, then K^{-1} will have kernel $K^{-1}(s,t)$, so it will satisfy the differential equation

$$\bigl(-\frac{1}{m}\frac{d^2}{ds^2}+V''(\gamma_c(s))\bigr)K^{-1}(s,u)=\delta(s-u)$$

This $K^{-1}(s_i,s_j)$ is the weight we assign to an edge between vertices i and j in the diagram, and we assign $V'''(\gamma(s_i))$ to the vertices. As far as I know, nobody has bothered to prove that $\int_{\mathbb{R}^n} U(q_1,q_2|t)U(q_2,q_3|s)dq_2 = U(q_1,q_3|s+t)$. Phisicits didn't do this because it is obvious. Mathematicians didn't bother to do this because it doesn't solve any fundamental problem. So this problem, which is probably not very hard, is open.

This is more or less the end of the path integral in classical mechanics. Now we ask the following questions.

1. How is this related to the deformation quantization?
2. What to do if $\frac{\partial^2\mathcal{L}}{\partial\xi_i\partial\xi_j}$ is degenerate? If you have a classical system with a Lie group action so that the Lagrangian is invariant, then you'll run into this problem. In this case, you should reduce the dynamics to the orbits. This is known as Hamiltonian reduction. This is where all these gadgets like Fadeev-Popov ghosts, BRST quantization, and BV quantization appear.

Since I already gave an example of a classical field theory, let me give and example of a quantum field theory which quantizes this scalare Bose field.

Quantum field theory of a scalar Bose field (perturbative)

Classically

- M is Riemannian
- Fields are $\mathbb{R}$-valued functions on M, so the space of fields is $C^\infty(M)$.
- $\mathcal{A}[\phi]=\int_M\Bigl(\frac{1}{2}\bigl(d\phi(x),d\phi(x)\bigr)+V(\phi(x))\Bigr)d^n x$
- Critical points of $\mathcal{A}[\phi]$ with fixed $\phi|_{\partial M}=\varphi$ have $\delta\mathcal{A}[\phi_c]=0$. $\delta\phi$ is a vector field on the space of fields $C^\infty(M)$. Given a functional F on fields, $\delta F[\phi] := \frac{d}{ds}F[\phi+s\delta\phi]|_{s=0}$. So

$$\delta\mathcal{A}[\phi]=\int_M\bigl(-\Delta\phi(x)+V'(\phi(x)\bigr)\delta\phi(x)\,d^n x+\int_{\partial M}\delta\phi(x)\bigl(d\phi(x),d^n x\bigr)$$

This second term will be zero anyway because we assume the fields are φ at the boundary. This is where you run into the problem of renormalization. So the Euler-Lagrange equations are

$$-\Delta\phi(x)+V'(\phi(x))=0$$

with the boundary condition $\phi|_{\partial M}=\varphi$. For good potentials V, this problem has unique solutions. You can kind of see this from $\frac{1}{2}(d\phi,d\phi)+V(\phi)$; if V is good, there is a unique minimum.

We can try to define the amplitude U_M, a functional on the space of possible values of boundary values φ. In a quantum field theory, this would be exactly $\mathcal{H}(\partial M)$. Since this space is very bad, I'll try to define what I can. So we try

$$U_M(\varphi) = \int_{\phi|_{\partial M}=\varphi} \exp\big(\frac{i}{h}\mathcal{A}[\phi]\big)\mathcal{D}\phi$$

How can we define this in a meaningful way? I know how to make sense of formal oscillating integrals like I did before: $\int e^{\frac{i}{h}S(x_c+y)}dy$ using Feynman diagrams. To deal with that, we wrote $S(x_c + y) = S_(x_c) + \frac{1}{2}(y, Ky)) + \sum_n V_n(y)$ ...then everything turned out to $e^{i\frac{S[x_c]}{h}} \det(d^2S(x_c)^{-1/2} \sum_\Gamma \frac{F(\Gamma)}{|\operatorname{Aut}\Gamma|}$. In this case, we will just define this U_M as a sum of diagrams. In the previous case, I could actually "do the diagram integrals", but now we're over an infinite-dimensional space.

In quantum mechanics, we can define $U_M(\varphi)$ as

1. the kernel $(e^{i\frac{t}{h}H})(x, y)$ (this is the honest definition). We can derive this as the $h \to 0$ limit of the Schrödinger equation.

2. A wild project:

$$\begin{aligned} U(q_1, q_2|t) &= \lim_{h\to 0} \int_{x(0)=x(t)=0} \exp\big(i\frac{S[x_c + x]}{h}\big)\mathcal{D}x \\ &\overset{\text{def}}{=} \exp(i\frac{S[x_c]}{h} \det{}'(K_{x_c})^{-1/2} \sum_\Gamma \frac{F(\Gamma)}{|\operatorname{Aut}\Gamma|} \end{aligned}$$

Problem 1: Prove that 2 gives the same notion as 1 (as far as I know, it's open).
Problem 2: $U_s^{p,int} * U_t^{p,int} = U_{s+t}^{p,int}$ [[★★★ I don't know what those superscripts are]].

The idea is to approximate the infinite-dimensional integral as a finite-dimensional integral and hope that there is a limit of these expressions. But this is very hard. I want to go through known facts as much as possible, but this area is like a mine field; you often step on something unproven. We'll use the perturbative approach as much as we can. We'll see how it works in this very simple example. Then we'll do this in Chern-Simons theory. The perturbative approach will give you knot invariants and other good things, but the limitations of this perturbative approach will become clear, so we'll defined things as these power series.

In quantum mechanics we're in very good shape; we have an honest definition. Anywhere else, we only have these guesses. The goal is to organize these guesses as much as we can.

So I define

$$U_M(\varphi) \overset{\text{def}}{=} \exp\Big(i\frac{\mathcal{A}[\phi]}{h}\Big)\det(K_{\phi_c})^{-1/2}\sum_\Gamma \frac{F(\Gamma)}{|\operatorname{Aut}(\Gamma)|}$$

$$\int \exp\Big(i\frac{\mathcal{A}[\phi_c + x]}{h}\Big) = \mathcal{A}[\gamma_c] + \frac{1}{2}(x, K_{\phi_c}x) + \sum_n V^{(n)}(\phi_c)x^n.$$

You can already see a problem with this definition. This $K(x, y)$ is singular at $x = y$ (ultraviolet divergences). This is what Richards course is about. I will make a few comments about this next time. Then we will completely ignore all divergence problems in this course.

25 NR 10-24

Today I wanted to continue talking about the Scalar Bose field. I want to focus on a problem which doesn't exist in any TQFT, but does exist in any realistic QFT: divergences due to ultraviolet divergences and renormalization. RB should be doing this in his class. In this case, fields ϕ are elements of $C^\infty(M)$. The action functional is $\mathcal{A}[\phi] = \int \big(\frac{1}{2}(d\phi, d\phi)+V(\phi)\big)dx$, where $V(\phi)$ is the self-interaction term. We assume M is Riemannian with boundary ∂M. Let ϕ_{cl} be the solution to the Euler-Lagrange equations, assuming we fix $\phi|_{\partial M} = \varphi$, so $\delta\phi|_{\partial M} = 0$. Then we have

$$\delta\mathcal{A}[\phi] = \int_M \big(-\Delta\phi(x) + V'(\phi(x))\big)\delta\phi(x)\,dx$$

For this to vanish, we must have $-\Delta\phi_c + V'(\phi_c) = 0$. We have $U_M(\varphi)$, the analogue of $U(q_1, q_2|t)$ (where $q_1 = \gamma(0)$, $q_2 = \gamma(t)$). We define

$$U_M(\varphi) = \int_{\phi|_{\partial M}=\varphi} \exp\big(\frac{i}{h}\mathcal{A}[\phi]\big)\mathcal{D}\phi$$

We want to have some analogue of the composition law $\int_M U(q_1, q_2|t)U(q_2, q_3|s)dq_2 = U(q_1, q_3|s + t)$, so we require U_M to satisfy the following axiom. If M_1 and M_2 are manifolds, with part of their boundaries identified, say $\partial_1 M_1 \xrightarrow[\sim]{f} \partial_1 M_2$, then

$$\int_{\varphi|_{\partial_1 M_1 = \partial_1 M_2}} U_{M_1}(\varphi)U_{M_2}(\varphi)\mathcal{D}\varphi = U_{M_1\#_f M_2}(\varphi) \qquad (*)$$

We can only make sense of $U_M(\varphi)$ perturbatively as

$$c\sum_{\phi_c} \exp\big(\frac{i}{h}\mathcal{A}[\phi_c]\big)(\det{}' K_{\phi_c})^{-1/2} \sum_\Gamma \frac{F(\Gamma)}{|\operatorname{Aut}(\Gamma)|}$$

The problem: Does $U_M(\varphi)$ satisfy $(*)$?

Remark 25.1. Though the integral itself doesn't make sense, we can try to make sense of it perturbatively, so $(*)$ should be regarded as an identity involving series of Feynman diagrams. $\diamond$

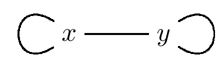
Let me say what are the rules of these Feynman diagrams. The intuitive idea is that we should consider $\mathcal{A}$ near ϕ_c, so

$$\mathcal{A}[\phi_c+\psi] = \mathcal{A}[\phi_c]+\frac{1}{2}\int_M \psi\big(-\Delta+V''(\phi_c(x))\big)\psi\,dx+\sum_{n\geq 2}\frac{1}{n!}V^{(n)}(\phi_c(x))\psi(x)^n\,dx$$

When you have a vertex with valence n, you assign $x \in M$ and assign $V^{(n)}(\phi_c(x))$ to the vertex. To an edge between x and y, you assign $(K_{\phi_c})^{-1}(x, y) =: G(x, y)$.

Let's look at all diagrams of order 1.

Example 25.2. To



we assign

$$\iint_{M\times M} G(x, x)V^{(3)}(\phi_c(x))G(x, y)V^{(3)}(\phi_c(y))G(y, y)\,dxdy$$

Also, we have

[[★★★ add formulas]] $\diamond$

$K_{\phi_c} = -\Delta + \overbrace{V''(\phi_c(x))}^{U}$. The problem with K is that eigenvalues λ_i blow up as $n \to \infty$, so $\det K_\phi$ has no chance to exist. But we also have the operator $-\Delta$, whose eigenvalues λ_i^0 also diverge if you order them like $\lambda_1^0 \leq \cdots$. But λ_n/λ_n^0 converges to 1, so we have some hope of making sense of $\det(\frac{-\Delta+U}{-\Delta}) = \det(I - U\Delta^{-1})$. It turns out that this determinant exists.

Let's see how these expressions behave. For this, we have to understand how $K_{\phi_c}^{-1}(x, y) = G(x, y)$ behave as x goes to y. "x goes to y" means that in the Fourier transform, we should take p to ∞. If x and y are close, then the distance to the boundary is much larger than the distance between them. Let's say M is m-dimensional. I claim that as x goes to y, the asymptotics of this Greens function is $G(x, y) \to G_{\mathbb{R}^m}(x - y) = \int_{\mathbb{R}^m} \frac{\exp(ip(x,y))}{p^2} d^m p = |x - y|^{-m+2} \int_{\mathbb{R}^m} \frac{\exp(iq(\frac{x-y}{|x-y|}))}{q^2}$. In the short distance asymptotics, we're picking up large eigenvalues of the Laplacian Δ, so we can pretend like $U = 0$. We can ignore the integral $\int_{\mathbb{R}^m} \frac{\exp(iq(\frac{x-y}{|x-y|})}{q^2}$ [[★★★ for some reason]], so $G_M^U(x,y) \to \frac{c}{|x-y|^{m-2}} + \cdots$.

If $m = 1$ (quantum mechanics), $G(x,y) \sim |x-y|$, so no divergences.

If $m = 2$, it turns out that $G(x,y) \sim \log|x-y| + \cdots$, so there divergences, but they aren't that bad.

If $m \geq 3$, then $G(x,y) \sim \frac{1}{|x-y|^{m-2}}$, with $m - 2 > 0$, so you have to do something to make the formulas corresponding to Feynman diagrams make sense; all the integrals have divergences at short distances. These are ultraviolet divergences.

So we've failed to define the integral $U_M(\varphi)$ using Feynman diagrams. In any QFT in dimension at least 2, there will be singularities. If we believe that something like this should exist, we can try to regularize.

How should we fix the problem? You could integrate over some neighborhood outside the diagonal in $M \times M$. Another idea is as follows. Let's do something with the action to make these Greens functions non-singular at the diagonal. In $\mathbb{R}^m$, instead of considering $-\Delta$, take $-\Delta + \sum_{i \geq 4} \varepsilon_i \Delta^i$, where we've added some higher order differential operators. If you do the Fourier transform, this becomes $p^2 + \sum_{i=1}^k \varepsilon_i (p^2)$, so as $p \to \infty$, this behaves like $\varepsilon_k p^{2k}$. Then $G_M^{U,\varepsilon}(x,y) \sim \frac{c(\varepsilon)}{|x-y|^{m-2k}}$. So we can kill off the singularities, but at a very heavy price; we'll have to let ε_i go to zero eventually or something. The idea of changing the Laplacian like this is known as Pauli-Williams regularization. Physicists used this for more than 50 years. The idea is: regularize $\mathcal{A}[\phi]$, and then $U_{M,\varepsilon}^{pert}(\varphi)$ is defined. The big question is then, "what happens when $\varepsilon \to 0$?" We know the answer: each of the Feynman diagrams (which now depend on ε) diverges as $\varepsilon \to 0$. The hope is that as $\varepsilon \to 0$, we can choose a modification $V_\varepsilon(\phi)$ of our potential $V(\phi)$ so that the coefficients in $F(\Gamma)$ will be finite. Richard is doing this more carefully.

Q: how does this address the problem of $G(x,x)$? NR: just as in the case of mechanics, we had a power in the denominator of $m - 2k$, so if k is large, this will be negative, so you get a positive power of $|x - y|$. Q: if I regularize the action, I'm really changing the entire problem; the determinant changes as well. Will we have to reregularize that part too? NR: with the determinant we should be more careful. $\det' K_\phi^\varepsilon = \det(K_\phi^\varepsilon / K_0^\varepsilon)$.

So now we have to adjust V_ε to compensate the divergences. It is not at all clear that this is ever possible, but it turns out it is.

Theorem 25.3 (BZZ)**.** *If $m = 2$, then $V(\phi)$ can be any polynomial and the renormalization procedure exists. This is a rather involved statement already. If $m = 3$, then $V(\phi)$ should be a polynomial of degree 6, or it won't be renormalizable. If $m = 4$, then V must be a polynomial of degree 4. A renormalizable theory doesn't for $m > 4$.*

There is another complicated question: there are many many renormalization procedures; Theo proposed the momentum cutoff regularization. And we can produce many more. Will the answer depend on the regularization procedure or not? The statement at the moment is that it depends up to a certain finite renormalization. There is the (infinite-dimensional) group of renormalization schemes. There is a series of papers by Connes and Krimer. They invented some kind of cocommutative Hopf algebra of diagrams, a candidate for the universal enveloping algebra of some Lie algebra. It simplified lots of computations in the proof of this theorem. Observables should be invariant with respect to the action of this group. I don't understand if this has really been resolved, and I haven't gotten a straight answer to it. It is probably not an issue for phisicits.

These lectures will sometimes be quite pessimistic: I'm telling you that people know very little about quantum field theory. Though if you look in textbooks, you'll see more optimistic statements.

Today we did an example of a QFT with non-degenerate Lagrangian (that is, $(d\phi, d\phi)$ was non-degenerate). Next time we'll start talking about systems with degenerate Lagrangian. One question is to make the Hamiltonian formulation, and the other question is how to quantize using the path integral.

26 NR 10-26

Last time, we took M Riemannian and $\mathcal{L} = \int_M \big(\frac{1}{2}(d\phi)^2 + V(\phi)\big)dx$. I talked about a regularization scheme where you first regularize the propagators $G(x,y) = (-\Delta + V''(\phi_c))^{-1}(x,y)$ which you assign to edges in the Feynman diagrams, where ϕ_c is a solution to the Euler-Lagrange equation with given boundary conditions. For $m \geq 2$, you get singularities, so the integrals which we assign to Feynman diagrams don't make sense. To take care of the singularities at $x = y$, you replace $-\Delta$ by $-\Delta + \sum_{i=2}^n \varepsilon_i \Delta^i$ and eventually send ε_i to 0. The first problem is that $F(\Gamma)$ are singular as $\varepsilon_i \to 0$.

The second part of the procedure is renormalization. $V_\varepsilon(\phi) = \sum_{i\geq 3} g_i(\varepsilon)\phi^i$, and adjust $g_i(\varepsilon)$ such that $F_\varepsilon(\Gamma) \to F_0(\Gamma)$. There are many questions, like "what if we regularize the propagators some other way? Will you get different results?" This is what Richard is doing and will be doing. The answer is that there is a group of renormalizations (not "the renormalization group" you find in physics literature). Different regularization schemes are related by the transitive action of the group of renormalizations.

You will always have problems with ultraviolet divergences when you do perturbation theory. In this case, fortunately, they arise in a controllable way.

The work by Kevin Costello; BV quantization, as I understand it (or don't understand it, as the case may be), the goal is to have "$d^2 = 0$" description of working with ultraviolet divergences. The other name for this is BRST, and secretly, it is the same as the Fadeev-Popov trick. All of these involve super analysis, so I'll do a complementary introduction to super geometry.

Grassman algebra

Recall some facts about $\mathbb{R}^{n|k}$ and $\mathbb{C}^{n|k}$. The Grassman algebra is the algebra $\langle c_i, \dots, c_n | c_i c_j + c_j c_i = 0\rangle$. We can consider odd derivations $\frac{\partial}{\partial c_i} c_{i_1}\cdots c_{i_n} = \begin{cases} 0 & i \notin \{i_1,\dots,i_n\} \\ (-1)^k c_{i_1}\cdots \hat{c}_{i_k}\cdots c_{i_n} & i = i_k \end{cases}$. This is the "left derivative" and you get the right derivative using the sign $(-1)^{n-k}$ instead.

Ingegral over $G_n = \bigwedge^\bullet \mathbb{C}^n$: Choose an orientation of $\mathbb{C}^n$, a basis in $\bigwedge^n \mathbb{C}^n$. Choose $c_1\wedge\cdots\wedge c_n$ in $\bigwedge^n \mathbb{C}^n$. If you have $P \in G_n$, you can write it as $p^{top} c_1 \wedge \cdots \wedge c_n +$ lower terms. Then we define

$$\int_{C^{0|n}} P\, dc := p^{top}.$$

Now let's see if this is a useful definition. So far, we've only been using integration very crudely, we only care about integrating gaussians.

Example 26.1. $P = \exp\big(\frac{1}{2}(c, Bc)\big)$, where $(c, Bc) = \sum_{ij} c_i B_{ij} c_j$. Since the c_i anti-commute, we need $B_{ij} = -B_{ji}$, and we should assume n is even (otherwise, we'll never get something in the top degree by exponentiating an even degree function). If n is even, then

$$\int \exp\Big(\frac{1}{2}(c, Bc)\Big)\, dc = \frac{(1/2)^{n/2}}{(n/2)!}.$$

To see this, not that

$$(c, Bc)^{n/2} = \sum_{i_k, j_k, 1\leq k \leq n/2} B_{i_1 j_1}\cdots B_{i_{n/2} j_{n/2}} c_{i_1}\cdots c_{i_{n/2}} c_{j_1}\cdots c_{j_{n/2}}$$

and we get that

$$(c_{i_1}\cdots c_{i_{n/2}} c_{j_1}\cdots c_{j_{n/2}})^{top} = (-1)^{\sigma(i|j)} c_1 \wedge\cdots\wedge c_n$$

(incidently, $\sigma(i|j)$ is the number of perfect matchings on n elements)

$$= \frac{(1/2)^{n/2}}{(n/2)!}\sum_{\sigma(i|j)} B_{i_1 j_1}\cdots B_{i_{n/2} j_{n/2}} (-1)^{\sigma(i|j)}$$

Note that the sign doesn't change when you switch i_a with j_a because the signs come in pairs. Also, the sign doesn't change when you apply a given permutation to both $\{i\}$ and $\{j\}$, so

$$= \sum_{\substack{\sigma(i|j) \\ i_a < j_a \\ i_{a_1} < \cdots < i_{a_n}}} (-1)^{\sigma(i|j)} B_{i_1 j_1}\cdots B_{i_{n/2} j_{n/2}} = Pf(B)$$

which is the Pfaffian of B. This is the formula that every physicists know for $\int_{\mathbb{C}^{0|n}} \exp((c,Bc)/2)\,dc$. This is the combinatorial definition of the Pfaffian. More conceptually, you can define the Pfaffian as

$$\Big(\sum_{i<j} x_i \wedge x_j B_{ij}\Big)^{\wedge \frac{n}{2}} = Pf(B) x_1 \wedge \cdots \wedge x_n$$

over $\bigwedge^\bullet \mathbb{C}^n$. It depends on the basis, but only on the orientation of the basis (i.e. it depends on a choice of orientation of $\bigwedge^n V$). [[★★★ HW: Prove that $Pf(B)^2 = \det B$. All you have to do is take $\bigwedge^{2n}(\mathbb{C}^n \oplus \mathbb{C}^n)$, and compute something like this in two way, one of which is the determinant and one of which is the Pfaffian.]] Given a basis $c_1 \wedge \cdots \wedge c_n \in \bigwedge^n \mathbb{C}^N$, $x_i = \sum_{j=1}^n A_{ij} c_j$, $x_1 \wedge \cdots \wedge x_n = \det A c_1 \wedge \cdots c_n$. ◇

What if we only know determinants, but not Pfaffians. Let $c_1, \ldots, c_n, \bar{c}_1, \ldots, \bar{c}_n$ be a basis for $\mathbb{C}^{0|2n}$ (don't think complex conjugation, these are independent), then

$$\begin{aligned}
\int_{\mathbb{C}^{0|2n}} \exp((\bar{c}, Ac))\,d\bar{c}\,dc &= \frac{1}{n!}\int (\bar{c}, Ac)^n\,d\bar{c}\,dc \\
&= \frac{1}{n!}\int \sum_{\{i,\},\{j\}} A_{i_1 j_1} \cdots A_{i_n j_n} \bar{c}_{i_1} c_{j_1} \cdots \bar{c}_{i_n} c_{j_n}\,d\bar{c}\,dc \\
&= \cdots \\
&= (-1)^{n(n-1)/2+n^2} \frac{1}{n!} \sum_{\sigma,\tau \in S_n} (-1)^{\sigma+\tau} \prod_i A_{\sigma(i)\tau(i)} \\
&= \cdots = \pm \det A
\end{aligned}$$

On the other hand, we can write this as

$$\int_{\mathbb{C}^{0|2n}} \exp\Big(\frac{1}{2}(x, \left(\begin{smallmatrix} 0 & A \\ -A^t & 0 \end{smallmatrix}\right) x)\Big) dx + \pm Pf\left(\begin{smallmatrix} 0 & A \\ -A^t & 0 \end{smallmatrix}\right)$$

So we get $\det B = Pf(B)^2$ and $Pf\left(\begin{smallmatrix} 0 & A \\ -A^t & 0 \end{smallmatrix}\right) = \det A$. This is what you need to know.

So let's derive the formula for the Berezinian using this voodoo. So far, I was integrating over $\mathbb{C}^{0|n}$, but what if I want to integrate over $\mathbb{C}^{n|m}$? Choose a hermitian bilinear form $(,)$ with even coordinates x (with complex conjugates $\overline{x}$) and odd coordinates c and $\bar{c}$ (not related by complex conjugation). Let $(\overline{x}, Bc) = \sum_{i,a} \overline{x}_i B_{ia} c^a$. Consider the following real integral over $\mathbb{R}^{2n}$:

$$\int_{\mathbb{C}^{n|2m}} \exp\big(-(\overline{x}, Ax) + (\overline{x}, Bc) + (B^*\bar{c}, x) + (\bar{c}, Dc)\big)\,d\bar{c}\,dc\,d\overline{x}\,dx$$

$$\begin{aligned}
&+(\overbrace{\overline{x} - A^{-1}B^*\bar{c}}^{\overline{y}}, A(\overbrace{x - A^{-1}Bc}^{x})) + (B^*\bar{c}, A^{-1}Bc) + (\bar{c}, Dc) \\
&\qquad = \int \exp\big(-(\overline{y}, Ay) + (\bar{c}, (D + B^*A^{-1}B)c\big) \\
&\qquad = \det A^{-1} \det(D + B^*A^{-1}B)
\end{aligned}$$

$$\int \exp\Big((\overline{x}, \bar{c})\left(\begin{smallmatrix} A & B \\ B^* & D \end{smallmatrix}\right)\left(\begin{smallmatrix} x \\ c \end{smallmatrix}\right)\Big)\,d\overline{x}\,dx\,d\bar{c}\,dc = Ber\left(\begin{smallmatrix} A & B \\ B^* & D \end{smallmatrix}\right)^{-1}$$

on $GL(n|m)$.

27 NR 10-29

Last time: $\int_{\mathbb{R}^{0|n}} P(c)\,dc := (P(c))^{top}$ is the definition of the integral over the Grassman algebra $G_n = \bigwedge^\bullet \mathbb{R}^n$. Note that we need to choose a basis for the top degree part (we choose basis $c_1 \wedge c_n$ in G_n, where c_i are the usual basis for $\mathbb{R}^n$). We used this to compute that

$$\int_{\mathbb{R}^{0|n}} \exp\Big(-\frac{1}{2}(c, Bc)\Big)\,dc = Pf(B).$$

I want to complexify, so I look at $G_{2n} = \langle a_s, b_s | 1 \le s \le n, \cdots\rangle = \bigwedge^\bullet(\mathbb{R}^n \oplus \mathbb{R}^n)$. Then we consider $\mathcal{G}_{2n,c} = G_{2n} \otimes_{\mathbb{R}} \mathbb{C} = \langle c_s, \bar{c}_s | c_s = a_s + ib_s, \bar{c}_s = a_s - ib_s\rangle = \bigwedge^\bullet \mathbb{C}^{2n}$. In this algebra, we have the following identity for a complex $n\times n$ matrix:

$$\int \exp((\bar{c}, Ac))\,d\bar{c}\,dc = \pm \det A$$

Last time I showed how the Berezinian comes up, but I started in the complexified case. Last time I showed that if $A^* = A > 0$, and if B and C are odd elements,

$$\int_{\mathbb{C}^{n|2m}} \exp\big(-(\overline{x}, Ax) - (\overline{x}, Bc) - (\bar{c}, Cx) - (\bar{c}, Dc)\big)\,d\overline{x}\,dx\,d\bar{c}\,dc$$
$$= \det(A)^{-1}\det(D - CA^{-1}B)$$

This is an identity in $\bigwedge^\bullet\big(M_{n\times m}(\mathbb{C}) \oplus M_{m\times n}(\mathbb{C})\big)$. In general, this is an identity in the corresponding exterior algebra. If $P(c)$ is a polynomial in c with coefficients in $\bigwedge^\bullet V$, then this identity is in $\bigwedge^\bullet V$. In this case, this is the algebra $\mathcal{A}$ generated by odd elements C_{ia} and B_{dj}. The integrand is in $\mathcal{A} \otimes \langle c_s, \bar{c}_s | \cdots\rangle$.

So we get $\mathrm{Ber}\left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right) = \det A \cdot \det(D - CA^{-1}B)^{-1}$. Notice that the $\exp(\cdots)$ in the integrand can be written as $\exp\Big((\overline{x}, \bar{c})\left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right)\left(\begin{smallmatrix} x \\ c\end{smallmatrix}\right)\Big)$. It is clear that for unitary super matrix U, $U\left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right)U^{-1}$ is change of basis by $\left(\begin{smallmatrix} x \\ c\end{smallmatrix}\right) \mapsto U\left(\begin{smallmatrix} x \\ c\end{smallmatrix}\right)$. The "measure" is invariant under such transformations. This explains why the Berezinian is invariant under such transformations. PT: what do you mean by a unitary super matrix? Is it some super Lie group? NR: $U(n|m) \subseteq GL(n|m)$ is the compact real form. It should be the super matrices that preserve the hermitian inner product. It acts on $\mathbb{C}^{n|m}$, with hermitian product something like $\langle x, y\rangle = \sum_i \overline{x}_i y_i + \sum_a \bar{c}_a c_a$. Consider polynomial functions $Pol(\mathbb{C}^{n|m}) = \mathrm{Sym}(\mathbb{C}^n) \otimes \bigwedge(\mathbb{C}^m)$. The scalar product $\langle\cdot,\cdot\rangle$ is an element in $Pol(\mathbb{C}^{n|m} \oplus \mathbb{C}^{n|m}) = Pol(\mathbb{C}^{n|m})^{\otimes 2}$. The usual $\mathbb{C}$-bilinear scalar product is $(\cdot,\cdot) = \sum_i x_i \otimes x_i + \sum_a c_a \otimes c_a$. The hermitian product is $\langle\cdot,\cdot\rangle = \sum_i \overline{x}_i \otimes x_i + \sum_a \bar{c}_a \otimes c_a$. $a \mapsto \overline{a}$ is an anti-linear anti-involution of the algebra $Pol(\mathbb{C}^{n|m})$. PT: this is different from the bar you used before ... I take it back, maybe it's the same bar as before. I think we're doing a functor of points description of $U(n|m)$. NR: you're right. There are no super groups, there are Hopf algebras which are the functions on super groups. This thing that looks like an action is a coaction of the Hopf algebra.

You have the braided monoidal category SVect. In this category, you have an algebra object $Pol(\mathbb{C}^{n|m})$. In the same category, you have $Pol(GL_{n|m})$, a Hopf algebra object. There is a coaction $Pol(\mathbb{C}^{n|m}) \to Pol(\mathbb{C}^{n|m}) \otimes Pol(GL_{n|m})$, making $Pol(\mathbb{C}^{n|m})$ into a comodule. To make this clearer, we'll talk about this last time. PT: in some sense, we talked about this in my class. If you have a super group acting on a super manifold, then this is really a coaction on the level of algebras. You're doing the universal case, where you're taking your base algebra to be generated by all the things that appear in the formula. NR: yes.

You should all know that SVect_k is an abelian monoidal category with braiding given by $v \otimes w \mapsto (-1)^{|v|\cdot|w|} w \otimes v$. You should also know from Peter's class what an algebra object in such a category is. You should also know what a Hopf algebra object is. Recall what is the meaning of a Hopf algebra. We want a Hopf algebra H to coact on an algebra A.

Example 27.1. If Γ is a finite group, then $H = Maps(\Gamma, k)$ is a Hopf algebra object in Vect_k, with $(f\cdot g)(x) = f(x)g(x)$, $(\Delta(f))(x,y) = f(xy)$, $S(f)(x) = f(x^{-1})$, and $\varepsilon(f) = f(e)$. ◇

From a completely algebraic point of view, (functions on) a *super group* is a commutative Hopf algebra object in SVect_k.

Example 27.2. $H = C(M_{n|m}) = \langle a_{ij}, b_{i\beta}, c_{\alpha j}, d_{\alpha\beta} | 1 \le i,j \le n, 1 \le \alpha,\beta \le m$, supercommutative with b, c odd$\rangle$. This has a bialgebra structure. The co-algebra structure is given by $\Delta a_{ij} = \sum_k a_{ik} \otimes a_{kj} + \sum_\alpha b_{i\alpha} \otimes$

$c_{\alpha j}$, and the counit is $\varepsilon(a_{ij} = \delta_{ij}$, $\varepsilon(b) = \varepsilon(c) = 0$, $\varepsilon(d_{\alpha\beta} = \delta_{\alpha\beta}$. You can think of this as

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \boxtimes \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} \Delta a & \Delta b \\ \Delta c & \Delta d \end{smallmatrix}\right)$$

Then you get $\tilde{H}$, the Hopf algebra of $GL_{n|m}$, as $H \otimes \langle A, D\rangle/(A \det a - 1, D \det d - 1)$. In this algebra, there exists an antipode, which you can think of as

$$S\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^{-1}. \qquad \diamond$$

Next time I'll say a few more words about actions and coactions. The eventual goal for this week is to get the Feynman diagrams for fermions. The goal for next week is to get

$$\int_{\mathbb{R}^n} \exp(\frac{i}{h} S) dx$$

where S is invariant under some group action. One way to deal with this is with the Fadeev-Popov trick. The mathematical meaning of this trick is revealed by the BRST quantization or BV quantization.

28 NR 10-31

Let me start by reminding you where we got stuck last time. I want to make clear what $GL_{n|m}$ is and what is the action on $\mathbb{C}^{n|m}$. We have the category SVect_k, in which we have the Hopf algebra $H =$"$Pol(GL_{n|m})$"$=$ $\big(\langle 1, a_{ij}, b_{i\alpha}, c_{\alpha j}, d_{\alpha\beta}\rangle \otimes \mathbb{C}[\Delta_a^{\pm 1}, \Delta_d^{\pm 1}]\big)/\langle \Delta_a = \det a, \Delta_d = \det d\rangle$, where a, d are even and b, c are odd. The coalgebra structure is given by

$$\Delta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \dot{\otimes} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \varepsilon \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

which should be read as saying, for example, that $\Delta a_{ij} = \sum_k a_{ik} \otimes a_{kj} + \sum_\alpha b_{i\alpha} \otimes c_{\alpha j}$.

In a usual group $S(f)(x) = f(x^{-1})$. The axiom for S is [[★★★]]. For $GL_{n|m}$, this axiom says that

$$S \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot S \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

which you should read as saying that $\sum_k S(a_{ij}) a_{kj} + \sum_\alpha S(b_{i\alpha}) c_{\alpha j} = \delta_{ij}$. I claim that this determines S uniquely. Define $(a^{-1})_{ij} := \frac{M_{ij}^{n-1}(a)}{\Delta_a}$. If b and c are even, then we have

$$S \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{M(a)}{\Delta a}.$$

In general, we will have that

$$S \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{(SM)_{ij} \begin{pmatrix} a & b \\ c & d \end{pmatrix}}{\operatorname{Ber} \begin{pmatrix} a & b \\ c & d \end{pmatrix}}.$$

We also have $Pol(\mathbb{C}^{n|m}) \in \mathsf{SVect}_{\mathbb{C}}$. As an algebra, this is isomorphic to $Pol(\mathbb{C}^n) \otimes \bigwedge^\bullet \mathbb{C}^m$. $\mathbb{C}^n$ is a GL_n-module, so $Pol(\mathbb{C}^n)$ is a commutative algebra and a $Pol(GL_n)$-comodule. Similarly, $Pol(\mathbb{C}^{n|m})$ is an algebra and an H-comodule. A comodule structure is an even map $Pol(\mathbb{C}^{n|m}) =$

$A \xrightarrow{\delta} H \otimes A$ satisfying the commutative diagram

$$\begin{CD} A @>{\delta}>> H \otimes A \\ @V{\delta}VV @VV{\mathrm{id}\otimes\delta}V \\ H \otimes A @>{\Delta\otimes\delta}>> H \otimes H \otimes A \end{CD}$$

We have $Pol(\mathbb{C}^{n|m}) = \langle x_i, s_\alpha \rangle$, and the coaction is given by $\delta(x_i) = \sum_j a_{ij} \otimes x_j + \sum_\alpha b_{i\alpha} \otimes s_\alpha$ and $\delta(s_\alpha) = \sum_j c_{\alpha j} \otimes x_j + \sum_\beta d_{\alpha\beta} \otimes s_\beta$.

If we have $H = Pol(G)$, the Hopf algebra of an affine algebraic group G, since it is an infinite-dimensional vector space, we can choose a dual in a couple of different ways. You can choose $H^\vee$ and a pairing $\langle \cdot, \cdot \rangle \colon H^\vee \otimes H \to \mathbb{C}$. Such a triple $(H, H^\vee, \langle , \rangle)$ is called a dual pairing. One of the important dual pairings for $Pol(G)$ is given by taking $H^\vee$ to be distributions supported at the identity. [[★★★ HW: open a textbook on Lie groups and Lie algebras or go to Anton's Lie theory notes and look at the discussion about how this space of distributions can be identified with $U\mathfrak{g}$.]] You can think of $Dist_1(G)$ as left or right invariant differential operators on G.

When $\mathfrak{g}$ is a Lie super algebra, we still have the notion of the universal enveloping algebra and we still have the notion of left and right invariant differential operators. There is *a* dual Hopf algebra to $Pol(GL_{n|m})$ which is $U\mathfrak{gl}_{n|m}$. If you are not familiar with universal enveloping algebras, I strongly encourage you to learn about them. When you have a comodule over one of the guys in the dual, it is always a module over the other one.

PT: I'm a little nervous about evaluating at 1 in the super case. I agree that the universal enveloping algebra still acts on these comodules, but is the pairing still valid? NR: let's do an example. We have $U\mathfrak{gl}_{n|m}$ on one side and $Pol(GL_{n|m})$. I should define $U\mathfrak{gl}_{n|m}$. The standard way to do this is to define $\mathfrak{gl}_{n|m}$. In the usual setting, the notion of the Lie algebra is very natural, it is the subalgebra of left invariant vector fields. In the category of super manifolds, we can still do this and we'll end up with the notion of a Lie super algebra. You can think of $\mathfrak{gl}_{n|m}$ as having a linear basis $e_{ij}, e_{i\alpha}, e_{\alpha j}$, and $e_{\alpha\beta}$, with $[e_{ij}, e_{k\ell}] = \delta_{jk} e_{i\ell} - \delta_{i\ell} e_{kj}$, $[e_{i\alpha}, e_{\beta k}] = \delta_{\alpha\beta} e_{ik} + \delta_{ik} e_{\beta\alpha}$, and so on. You sometimes get a sign. This defines $\mathfrak{gl}_{n|m}$. Now $U\mathfrak{gl}_{n|m}$ is a unital associative algebra generated by the same elements, with the relations $[a, b] = ab - (-1)^{|a|\cdot|b|} ba$. A representation of a Lie super algebra is the same thing as a representation of its universal enveloping algebra. Note, by the way, that $U\mathfrak{gl}_{n|m}$ is just an associative algebra with a $\mathbb{Z}/2$-grading; it doesn't know anything about its super origins.

What is the pairing between $U\mathfrak{gl}_{n|m}$ and $Pol(GL_{n|m})$. First, let's describe the $(n|m)$-dimensional representation of $\mathfrak{gl}_{n|m}$. We have to construct a homomorphism of $\mathbb{Z}/2$-graded algebras $\pi \colon U\mathfrak{gl}_{n|m} \to End(\mathbb{C}^{n|m})$. It is given by taking e_{ij} to $\left(\begin{smallmatrix} E_{ij} & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $e_{i\alpha}$ to $\left(\begin{smallmatrix} 0 & E_{i\alpha} \\ 0 & 0 \end{smallmatrix}\right)$. and so on. It is easy to check that this π extends to a homomorphism of algebras. Now for an element a, I can associate functions $\pi_{i/\alpha, j/\beta}$.

PT: why do you need the representation? NR: the coordinates on $Pol(GL_{n|m})$ were a_{ij}. I want to make the pairing $\langle x, a_{i_1 j_1} \cdots a_{i_n j_n} \rangle = \langle \Delta^{(n)} x, a_{i_1 j_1} \otimes \cdots \otimes a_{i_n j_n} \rangle$, where $\langle x, a_{ij} \rangle = \pi_{ij}(x)$.

PT: maybe I should not have been nervous about evaluating at the identity. Part of the homework from my class was that $\mathfrak{gl}_{n|m}$, left invariant vector fields, really is isomorphic to the tangent space at the identity, so you can let a left-invariant vector field act on a function and then evaluate at the identity.

NR: Recall from PT's class that you came across a Lie super algebra $\langle X \text{ odd}, H \text{ even} | [X, H] = 0, [X, X] = H \rangle$, which is somehow related to the de Rham differential. Let me tell you about something related to the Lie algebra $\mathfrak{gl}_{1|1}$. Consider $[X, Y] = H$ central, $[G, X] = X, [G, Y] = -Y$, and $X^2 = Y^2 = 0$, where $X = e_{12}$, $Y = e_{21}$, $H = e_{11} - e_{22}$, and $G = e_{11} + e_{22}$.

V is a representation of $\mathfrak{gl}_{1|1}$, given by $V = \bigoplus_{n \in \mathbb{Z}} V[n]$, $X \colon V[n] \to V[n+1]$, $Y \colon V[n] \to V[n-1]$, $H \colon V[n] \to V[n]$. $XY + YX = H$.

If you have a Riemannian manifold M , with de Rham operator d, and conjugated by the Hodge star, $*d*$, and you have $\Delta = H = d{*}d{*} + {*}d{*}d$. PT: you could make it a project to explain this $\mathfrak{gl}(1|1)$ action on any Riemannian manifold just like we explained the de Rham differential on a Manifold.

Next time, I will not continue with the representation of the Lie super algebras.

29 NR 11-02

$U\mathfrak{gl}_{n|m}$ is generated as a unital algebra by e_{ij}, $e_{i\alpha}$, $e_{\beta j}$, and $e_{\alpha\beta}$, with the relations you expect (e.g. $e_{\alpha i}e_{j\beta} + e_{j\beta}e_{\alpha i} = \delta_{ij}e_{\alpha\beta} + \delta_{\alpha\beta}e_{ji}$). $\mathbb{C}^{n|m}$ has basis e_i, s_α. If we don't care about the tensor product, this is just a vector space. Once we start dealing with tensor product, we have to decide whether to treat it as a $\mathbb{Z}/2$-graded space or a super vector space. The vector representation on $U\mathfrak{gl}_{n|m}$ in $\mathbb{C}^{n|m}$ is an even linear map $\pi\colon U\mathfrak{gl}_{n|m} \to End(\mathbb{C}^{n|m})$, with $\pi(e_{ij}) = \left(\begin{smallmatrix} E_{ij} & 0 \\ 0 & 0\end{smallmatrix}\right)$, $\pi(e_{i\alpha}) = \left(\begin{smallmatrix} 0 & E_{i\alpha} \\ 0 & 0\end{smallmatrix}\right)$, and so on. It is easy to check that the appropriate relations hold. So for every $x \in U\mathfrak{gl}_{n|m}$, we have linear functions $\pi_{ij}(x)$, $\pi_{i\alpha}(x)$, $\pi_{\beta j}(x)$, and $\pi_{\alpha\beta}(x)$ on $U\mathfrak{gl}_{n|m}$. These are "coordinate functions" on $GL_{n|m}$ in the sense that "$Pol(GL_{n|m})$" is the Hopf (super) algebra forming a dual pair with $U\mathfrak{gl}_{n|m}$. This $Pol(GL_{n|m})$ is generated by $\pi_{i/\alpha,j/\beta}$, with the condition that the matrix is invertible. PT: I don't understand how this works for any other group, because you need a fundamental representation. NR: it doesn't. If you wanted to construct some other group, you'd need to pick a representation. PT: so how would you put the unitary or symplectic conditions into the algebra $Pol(Sp)$ or $Pol(U)$. NR: For $SL_{n|m}$, you would add the relation that the Berezinian is 1. For $U_{n|m}$, you have to say a little more. On $\mathfrak{gl}_n(\mathbb{C})$, we have the involution $\sigma(e_{ij}) = e_{ij}$ and $\sigma(\lambda x) = \overline{\lambda}x$. The fixed points of σ is $\mathfrak{gl}_n(\mathbb{R})$. The general construction is that you pick an involution of the Lie algebra (these are all classified), and then the fixed points give you a real form of the Lie algebra. For example, if you take $\sigma(e_{ij}) = -e_{ji}$, then $\mathfrak{gl}_n(\mathbb{C})^\sigma = \mathfrak{u}_n = \{a^* = -a\}$. The claim is that these involutions carry through the whole story. You can get involutions on the dual Hopf algebra and take the fixed points. In the non-super case, you have that $Pol(SU(n)) = \langle u_{ij}|\overline{u}_{ji} = M_{ij}(u)\rangle$. An algebraic version of Peter-Weyl tells us that this is $\bigoplus_{\lambda \text{ irrep}} \bigoplus_{i,j=1}^{\dim V_\lambda} \mathbb{C}\pi^\lambda_{ij}$. I'll leave it as an exercise to work out how it works in the super case. It is trickier because Lie superalgebras are not simple in general (only $\mathfrak{osp}(n|1)$ is simple).

The main reason we've made this detour about super groups is because I'll want to do some manipulations with them later. Let's return to where we started.

$$\int \exp\big((\overline{x}, AX) + (\bar{c}, BX) + (C\bar{c}, x) + (\bar{c}, Dc)\big)\, d\overline{x}\, dx\, d\bar{c}\, dc = \operatorname{Ber}\begin{pmatrix} A & B \\ C & D\end{pmatrix}^{-1}$$
$$= \det(A)\cdot\det(D - CA^{-1}B).$$

Suppose you have

$$\int_{x\in\mathbb{R}^n, \bar{c}, c\in\mathbb{C}^n} \exp\Big(\frac{1}{2}(x, Ax) + P(x) + (\bar{c}, Bc) + Q^{even}(x, \bar{c}, c)\Big)\, d\bar{c}\, dc\, dx = I$$

Where c and $\bar{c}$ are independent. In general, $\int P(\bar{c}, c)\, d\bar{c}\, dc = P^{\overline{top}, top}(\bar{c}, c)$, where we've picked an orientation. I don't know how to compute this I for general $P(x)$, but I'll take $P(x) = \sum_{n\geq 3} V^{(n)}(x)$, where $V^{(n)}$ is a homogeneous polynomial of degree n, and I assume $Q(x, \bar{c}, c) = \sum_{n\geq 1, m\geq k\geq 1} \underbrace{\frac{1}{n!k!} Q^{(n,k)}(x)_{i_1,\ldots,i_n} \bar{c}_{j_1}\cdots\bar{c}_{j_k} c^{i_1}\cdots c^{i_k}}_{Q^{(n,k)}}$.

We want to write I as a formal power series in V and Q. Say we live 100 years ago and we don't have any computers, but we want to compute these numbers. We know that we get an asymptotic expansion, so we know that the first coefficients give a good approximation in some areas.

$$I = \sum \int \exp\Big(\frac{1}{2}(x, Ax) + (\bar{c}, Bc)\Big) V^{(n_1)}(x)\cdots V^{(n_k)}(x) Q^{(n_1,k_1)}\cdots Q^{(n_\ell,k_\ell)}\, d\bar{c}\, dc\, dx$$

Each term is well defined, though we know that the series diverges. We can use the Feynman diagram technique. We can say that $V^{(n)}(x)_{i_1,\ldots,i_n}$ is a vertex with valence n, with labels $i_\bullet$ on the edges. $Q^{(n,k)}$ is associated to some edges labelled $i_1, \ldots i_n$ and some directed edges $a_1, \ldots a_\ell$, $b_1, \ldots b_\ell$ (a's go in, b's go out). Then we compute (using Wick's theorem)

$$\int \exp((\bar{c}, Bc))\bar{c}_{b_1}\cdots c_{b_\ell} c_{a_1}\cdots c_{a_\ell}\, d\bar{c}\, dc$$
$$= \sum \det(B) \sum_{\substack{\text{bipartite}\\ \text{perfect}\\ \text{matchings } \sigma}} (-1)^\sigma (B^{-1})_{a_1\sigma(a_1)}\cdots(B^{-1})_{a_\ell\sigma(a_\ell)}.$$

You usually write this with pictures. Let's do an example to see how this shows up.

Example 29.1.

$$\int \exp((\bar{c}, Bc))\bar{c}_b c_a \, d\bar{c}\, dc = \Big(\frac{1}{(m-1)!}(\bar{c}, Bc)^{m-1}\bar{c}_b c_a\Big)^{top}$$
$$= [[\bigstar\bigstar\bigstar \text{ HW: is a minor}]] \qquad \diamond$$

Example 29.2.

$$\int \exp((\bar{c}, Bc))\bar{c}_{b_1}\bar{c}_{b_2}c_{a_1}c_{a_2}\, d\bar{c}\, dc = \frac{1}{(m-2)!}\big((\bar{c}, Bc)\bar{c}_{b_1}\bar{c}_{b_2}c_{b_1}c_{b_2}\big)^{top}$$

We can at least check that the degrees are both $m-2$. In general, the total degree is $m-\ell$. ◇

$$\int \exp\Big(\frac{1}{2}(x, Ax)\Big)x^{i_1}\cdots x^{i_n}\, dx = Pf(A)^{-1}\sum_{\text{perf match}} \cdots$$

After this long exercise, we have

$$I = \sum_{\Gamma_{b,f}} \frac{(-1)^F F(\Gamma_{b,f})}{|\operatorname{Aut}\Gamma_{b,f}|}$$

where the weight is computed by [[★★★ picture]] and F is computed by the number of loops formed by the fermionic variables.

Next time we'll see how this formal power series can be used to approximate the asymptotics of oscillating integrals. This is the Fedeev-Popov trick.

30 NR 11-05

Last time I wrote the formula for the Feynman diagram expansion of an oscillatory integral which includes some Grassman variables. Let me repeat part of it.

$$\int_{\mathbb{R}^{0|m}} \exp((c, Bc)/2)\, dc = Pf(B).$$

Now I want to do

$$\int_{\mathbb{R}^{0|m}} \exp((c, Bc)/2 + V(c))\, dc$$

where $V = \sum_{k\geq 4}\sum_{\{a\}} V^{\{a\}}c_{a_1}\cdots c_{a_k}$. If you do the expansion in the integral, you get

$$= \sum_{\ell\geq 0}\frac{1}{\ell!}\exp((c, Bc)/2)\cdot V(c)^\ell\, dc.$$

You can write $V(c)^\ell = \sum V^{\{a\}}V^{\{b\}}c_{a_1}\cdots c_{a_k}c_{b_1}\cdots c_{b_k}$. So the question is, what is

$$\int \exp((c, Bc)/2)c_{a_1}\cdots c_{a_k}\, dc \overset{\text{def}}{=} I_{a_1\cdots a_k} \tag{30.1}$$

This is something like the square root of what we computed last time, so we expect some kind of sum over perfect matchings. As Bruce suggested, we could integrate this by parts. Another thing we could notice is that we could compute the generating function

$$I(\eta) = \int_{\mathbb{R}^{0|m}} \exp\big((c, Bc)/2 + \sum_a \eta^a c_a\big)\, dc \in {\textstyle\bigwedge}^\bullet \mathbb{R}^m = \langle \eta_1, \dots, \eta_m\rangle \tag{30.2}$$

We can do a change of variables $c = c' - B^{-1}\eta$, completing the square $(\frac{1}{2}(c, Bc) + (\eta, c) = \frac{1}{2}(c + B^{-1}\eta, B(c + B^{-1}\eta)) - \frac{1}{2}(\eta, B^{-1}\eta))$ to get

$$\int \exp\big((c', Bc')/2 - (\eta, B^{-1}\eta)/2\big) = Pf(B)\exp\Big(-\frac{1}{2}(\eta, B^{-1}\eta)\Big) = I(\eta)$$

And we have the relationship

$$I(\eta) = \sum_{k\geq 0}\frac{1}{k!}I_{a_1\cdots a_k}\eta^{a_1}\cdots\eta^{a_k}(-1)^{k(k-1)/2} \tag{30.3}$$

So all we have to do is expand the power series $\exp(-(\eta, B^{-1}\eta)/2)$.

$$\frac{I(\eta)}{Pf(B)} = \sum_{k\geq 0} \frac{(-1/2)^k}{k!} \underbrace{(\eta, B^{-1}\eta)\cdots(\eta, B^{-1}\eta)}_{k}$$
$$= \sum_{k\geq 0} \frac{(-1/2)^k}{k!} \sum_{a_1,\dots,a_k} \eta_{a_1}\cdots\eta_{a_k} \sum_{\text{peft match on } \{a\}} (-1)^\mu (B^{-1})_{i_1 j_1}\cdots(B^{-1})_{i_{k/2} j_{k/2}}$$

where $m\colon (a_1,\dots,a_k) \to (i_1,\dots,i_{k/2},j_1,\dots,j_{k/2})$, and μ is the sign of the perfect matchings

$$(\eta, B^{-1}\eta)\cdots(\eta, B^{-1}\eta) = \sum_{i.j.} \eta_{i_1}(B^{-1})_{i_1 j_1}\eta_{j_1}\cdots\eta_{i_{k/2}}(B^{-1})_{i_{k/2} j_{k/2}}\eta_{j_{k/2}}$$
$$= \sum_{i.,j.} \overbrace{\eta_{i_1}\cdots\eta_{i_{k/2}}\eta_{j_1}\cdots\eta_{j_{k/2}}}^{\text{odd}} \overbrace{(B^{-1})_{i_1 j_1}\cdots}^{\text{even}}$$

So

$$\frac{I_{a_1\cdots a_k}}{Pf(B)} = (-1)^{k(k+1)/2}\underbrace{(1/2)^k \sum_{\mu \text{ perf match}} (-1)^\mu \prod_\alpha (B^{-1})_{i_\alpha j_\alpha}}_{Pf_{k\times k}((B^{-1})_{a_\alpha a_\beta})}$$

Maybe I should take a break from these computations and explain why I need them. It will be a homework to derive the formula

$$\int_{\mathbb{R}^{0|m}} \exp\big((c,Bc)/2 + V(c)\big)\,dc = Pf(B)\cdot \sum_\Gamma \frac{(-1)^\ell F(\Gamma)}{|\operatorname{Aut}\Gamma|}$$

where the Γ have only even-valent vertices. The weight of the graph Γ =[[★★★ two vertices connected by 4 edges]] by asigning weights $i_1,\dots,i_4$ and $j_1,\dots,J_4$ to the halves of edges, then the weight is $\sum V^{i_1\cdots i_4}V^{j_1\cdots j_4}\prod_{e=1}^4 (B^{-1})_{i_\ell j_\ell}(-1)^{3+2+1}$ (the $3+2+1$ is the number of loops). Let's leave it as part of the homework to derive this formula for the Grassman integral with the prescription for computing the weights for Feynman graphs.

What is the real reason we want to have all these strange integrals? The physical reason is that there are elementary particles which have fermionic statistics, where each state has at most one particle in it. We model this mathematically by taking generating functions in a Grassman algebra.

Consider the following problem which seems completely irrelevant. Let's try to compute

$$I = \int_{\mathbb{R}^m} \exp\big(iS(x)/h\big)\,d^m x$$

but let's assume there is a Lie group G acting on $\mathbb{R}^m$ so that $S(gx) = S(x)$. The measure $d^m x$ is also G-invariant. If G is compact, we replace this integral by

$$|G|\int_{\mathbb{R}^m/G} \exp\big(iS([x])/h\big)\,d[x]$$

where $d[x] = dx/dg$, where dg is the Haar measure on G. Assume the action has trivial stabilizers. The only way we've been able to study these so far is with specific asymptotic expansions.

If G is not compact, then we don't have this, but we know that the original integral is meaningless, so we should still try to study the second integrals.

How can we make sense of the perturbative expansion of such integrals (we want to get some Feynman diagrams and so on). Say a cross section [[★★★ level hypersurface?]] through the space of orbits is given by $f_a(x) = 0$, $a = 1,\dots d = \dim G$. We have that $\dim(\mathbb{R}^m/G) = m - d$ (since we assume the action is free). I claim that

$$J_f(x)\int \delta\big(f(gx)\big)dg = 1$$

where $\delta(f(x))$ is the distribution on $\mathbb{R}^m$ which is supported at the cross section given by the equations $f_a(x) = 0$, and $J_f(x)$ is the Jacobian $\det\big(\frac{\partial f^a}{\partial \xi^b}\big)$, where the ξ are the vector fields of the G-action. I'll return to this next time but let me just give the answer.

Assume G is compact, then

$$\begin{aligned}\int_{\mathbb{R}^m} \exp(iS/h)\,dx &= \int_{\mathbb{R}^m} \exp(iS/h)\cdot J_f(x) \int_G \delta(f(gx))\,dg\,dx \\ &= \int_G \int_{\mathbb{R}^m} \exp(iS(x)/h)\delta(f(gx))J_f(x)\,dx\,dg \\ &= \int_G dg \int_{\mathbb{R}^m} \exp(iS/h)J_f(x)\delta(f(x))\,dx \qquad (x \mapsto gx)\end{aligned}$$

But we know that the result of the integral doesn't depend on the choice of the cross section. We can say that $\delta(f(x)) = \int_{\mathbb{R}^d} \exp(i\lambda f(x)/h)\,d\lambda$, and we can write $J_f(x)$ as the Grassman integral $\int \exp\left(\bar{c}^a \frac{\partial f_a}{\partial \xi^b} c^a\right) d\bar{c}\,dc$, so the result is

$$\int_{\mathbb{R}^m/G} \exp(iS)\,dx = \int_{\mathbb{R}^m\times\mathbb{R}^d\times\mathbb{C}^{0|d}} \exp\Big(iS/h + \sum_a \bar{c}^a \frac{\partial f^a}{\partial \xi^b} c_b + \sum_a \lambda_a f^a(x)/h\Big)\,dx\,d\bar{c}\,dc\,d\lambda$$

Theorem 1: this is what we should understand as the integration over the quotient space

Theorem 2: it doesn't depend on the choice of cross section. This uses BRST.

This is the only reason we went through the trouble of Grassman variables.

31 NR 11-07

Last time we started to discuss oscillating integrals $\int_{X=\mathbb{R}^m} \exp(iS(x)/h)\,dx$ as $h \to 0$. On X we have a Lie group G acting so that $S(gx) = S(x)$ and dx is G-invariant. If we do the standard variational analysis, we run into trouble because the Hessian is zero. Fadeev and Popov suggested the following trick.

Suppose G is compact of dimension k, and suppose $f_a(x)$ for $a = 1, \ldots, k$ such that $\{f_a(x) = 0\}$ is a cross section of the G action on X (i.e. $X/G \simeq f_a^{-1}(0)$). Then we have vector fields $\frac{\partial}{\partial \xi^a}$ on X representing the action of the basis elements e_a in $\mathfrak{g} = Lie(G)$. Then we have

$$\det\Big(\frac{\partial f_a}{\partial \xi^b}\Big) \int_G \delta(f(gx)) = 1 \qquad (*)$$

where $\delta(f(x))$ is a δ distribution on X supported at $f^{-1}(0)$.

Consider $\mathbb{R}$, and $f\colon \mathbb{R} \to \mathbb{R}$. Then $\int_{\mathbb{R}} g(x)\delta(x)\,dx = g(0)$ by definition. Changing the variables to $t = f(x)$, we get $\int g(x)\delta(f(x))\,dx = \int g(f^{-1}(t))\delta(t)\frac{dx}{dt}\,dt = g(f^{-1}(0))/f'(0)$.

Similarly, we get a δ distribution on $\mathbb{R}^n$, so $\int_{\mathbb{R}^n} g(x)\delta(x)\,dx = g(0)$. Again changing variables (in such a way that the Jacobian is non-zero), we get $\int_{\mathbb{R}^n} g(x)\delta(f(x))\,dx = g(x_0)\det(df(x_0))^{-1}$, where $f(x_0) = 0$.

Now we have to generalize this to the situation where the distribution is supported on a submanifold. Assume $X = \mathbb{R}^n \supseteq S$ a submanifold with some chosen measure μ on S. We say that $\delta_S(x)$ is *the δ distribution supported on S with measure μ* if $\int_X g(x)\delta_S(x)\,dx = \int_S g(x)\,d\mu$. Ok, sorry; this is not relavent.

$$\begin{aligned}\int_G \delta(f(gx))\,dg &= \int_{U_e^\varepsilon} \delta\Big(f(x) + \sum_a t^a \frac{\partial f}{\partial \xi^a}(x)\Big)\,dt \\ &= \int \delta\Big(\sum_a t^a \frac{\partial f}{\partial \xi^a}(x)\Big)\,dt \\ &= \det\Big(\frac{\partial f_a}{\partial \xi^b}\Big)^{-1}\end{aligned}$$

on $f(x) = 0$ where t^a are local coordinates near $e \in G$.

Now

$$\int_X \exp(iS/h)\,dx = \int_X \exp(iS/h)\det\Big(\frac{\partial f_a}{\partial \xi^b}\Big)$$
$$= \int_G \delta\big(f(gx)\big)\,dg\,dx$$
$$= \int_G \left(\int_X \exp(iS(x)/h)\det\Big(\frac{\partial f_a}{\partial \xi^b}(x)\Big)\delta\big(f(gx)\big)\,dx\right)dg$$
$$= \int_G \left(\int_X \exp(iS(x)/h)\det\Big(\frac{\partial f_a}{\partial \xi^b}(g^{-1}x)\Big)\delta\big(f(x)\big)\,dx\right)dg$$
$$= |G|\int_X \exp(iS(x)/h)\det\Big(\frac{\partial f_a}{\partial \xi^b}\Big)\delta(f(x))\,dx$$

The Jacobian is G-invariant if the identity $(*)$ is true for all x, not just x for which $f(x)=0$. X is covered by the set of all $g\cdot f^{-1}(0)$ (because $f^{-1}(0)$ is a cross section of the action). Let's assume the Jacobian is G-invariant so that we don't get stuck here. We'll clear it up next time. PT: it depends on the choice of f. For a general f, it won't be invariant. You're saying that there exists an f so that it is invariant. NR: ok. Let's assume it's true for now. Theo/Bruce: you should should takde $\tilde{x}$ to be the point on $f^{-1}(0)$ which is in the same orbit as x, then $J_f(\tilde{x})\int_G \delta(f(gx))\,dx = 1$ is automatically gauge-invariant.

We can write

$$\det\Big(\frac{\partial f_a}{\partial \xi^b}\Big) = \int \exp\Big(\sum_{a,b}\bar{c}^a\frac{\partial f_a}{\partial \xi^b}c^b\Big)\,d\bar{c}\,dc$$
$$\delta(f(x)) = h^k\int_{\mathbb{R}^k}\exp\Big(i\frac{1}{h}\sum_a \lambda^a f_a(x)\Big)\,d\lambda$$
$$\int_{\mathbb{R}} g(x)\delta(x)\,dx = g(0) = \int_{\mathbb{R}}\hat{g}(\lambda)\,d\lambda$$
$$\delta(x) = \int_{\mathbb{R}} e^{ix\lambda}\,d\lambda$$
$$\hat{g}(\lambda) = \int_{\mathbb{R}} e^{ix\lambda}g(x)\,dx$$

$$\int_X \exp(iS/h)\,dx = \int_{X\times\mathbb{R}^k\times\mathbb{R}^{0|k}\times\mathbb{R}^{0|k}} \exp\underbrace{\Big(i\frac{S(x)}{h} + i\sum_a\lambda^a f_a(x)/h + \sum_{a,b}\tau^a\frac{\partial f_a}{\partial\xi^b}c^b\Big)}_{S_{FP}}\,dx\,d\lambda\,d\bar{c}\,dc$$

Now what to do with this integral? If we consider $\widetilde{S}(x,\lambda) = S(x) + \sum_a \lambda^a f_a(x)$, what are the critical points? $\frac{\partial\widetilde{S}}{\partial\lambda^a} = 0$ implies $f_a(x)=0$, and $\frac{\partial\widetilde{S}}{\partial\lambda^a} = \frac{\partial S}{\partial x^i} + \sum_a \lambda^a\frac{\partial f_a}{\partial x^i} = 0$, so critical points are critical points of S which are on $f^{-1}(0)$. $d^2\widetilde{S}$ is non-degenerate. You can expand near the critical points and evaluate the Gaussian integral as an asymptotic series, so we get

$$S_{FP}(x_0+X;\lambda_0+\lambda,\bar{c},c) = S_{FP}(x_0,\lambda_0) + (d^2S(x_0,\lambda_0)z,z) + \bar{c}K(x_0)c + \sum_k z^k,\bar{c}z^nc$$

where $z=(x,\lambda)$.

$$\int_X e^{iS/h}\,dx = |G|\sum_\Gamma \frac{(-1)^{F(\Gamma)}}{|\operatorname{Aut}\Gamma|}F(\Gamma)$$

Where solid (bosonic) edges get weight $(d^2S_{FP}(x_0))^{-1}$, dashed (fermionic) edges get weight $K(x_0)^{-1}$. There will be vertices coming from the expansion of the action (giving the n-th derivative $\widetilde{S}^{(n)}$) and vertices with dashed edges $K^{(n)}(x_0)$. This $(-1)^F$ will cancel the most severe divergences.

Next time I will probably have to return to some of these questions, but then I want to explain that there is another way to think about all this. This Fadeev-Popov action can we written as $S_{FP} = S(x) + Q\psi$, where $Q^2=0$ and ψ is odd. This is the BRST approach to gauge theory. A more sophisticated version is known as BV quantization. You can see that this is true if $Q\bar{c}^a = \lambda^a$, $Q\lambda^a = 0$, $Qc^a = \frac{1}{2}\sum_{a,b,c}c^a_b d^b c^c$, $Qf(x) = \frac{\partial f(x)}{\partial\xi^a}c^a$. You can try to interpret this Q as the derivation in some cohomology theory.

32 NR 11-09

Last time we had a formula for the asymptotic expansion of $I_h = \int_X \exp(iS(x)/h)\,dx$. As $h \to 0$, this is given by Feynman diagrams. When S is invariant under the action of some Lie group G of dimension k (assume $X = \mathbb{R}^m$ for simplicity). We got that

$$I_h = \int_{X\times\mathbb{R}^k\times\mathbb{R}^{0|2k}} \exp\Big(\frac{iS}{h} + \sum_a \frac{\lambda^a f_a(x)}{h} + \sum_{a,b} \bar{c}^a \frac{\partial f_a(x)}{\partial \xi^b} c^b\Big)\, d\bar{c}\, dc\, d\lambda\, dx$$

$$\overset{h\to 0}{=} \sum_\Gamma \frac{(-1)^F}{|\operatorname{Aut}\Gamma|} F(\Gamma)$$

where $f(x) = 0$ is a cross section of the action of G. In quantum field theory, such expressions cannot be derived, so they are taken as definitions. A main goal (which is still largely open) is to construct a QFT which is not perturbative. The progress has been largely disappointing. You can go beyond perturbation theory in some cases which are not physical but still quite interesting to mathematicians.

PT: where are you hiding the h's in the notation? NR: in the $F(\Gamma)$. Say z is an even coordinate on $X \times \mathbb{R}^k$. Then we expand the action in powers of z. Every vertex with n even (solid) lines will have weight $h^{n/2-1}$. There will also be fermionic edges.

$$S(z)/h = S(z_c)/h + A(z-z_c)^2/h + \frac{1}{h}\sum_n \Big(\frac{z-z_c}{\sqrt{h}}\Big)^n V_n \cdot h^{n/2}.$$

Theorem 32.1. *Let Q be an operation on $\langle \bar{c}^a, c^a, C^\infty(X\times\mathbb{R}^k)\rangle \cong \bigwedge^\bullet(\mathfrak{g}^* + \mathfrak{g}^*)\otimes C^\infty(X\times\mathbb{R}^k)$, given by $Q\lambda^a = 0$, $Qc^a = \frac{1}{2}\sum_{bc} c^a_{bc}c^bc^c$ [[★★★ unfortunate notation, the c^a_{bc} are the structure constants of $\mathfrak{g}$.]], $Q\bar{c}^a = \lambda^a$, and $(Qf)(x) = -ih\sum_a c^a\partial_a f(x)$, where $\partial_a f(x) = \frac{d}{dt} f(e^{te_a}x)|_{t=0}$ for a basis $\{e_a\}$ for $\mathfrak{g}$ (note that $[\partial_a, \partial_b] = \sum_c c^c_{ab}\partial_c$). Then*

$$S_{FP} = S - Q\psi \qquad \psi = \sum_a c^a f_a(x) \qquad Q^2 = 0.$$

You can think of this Q as follows. We have the super manifold $X_{FP} = X\times\mathbb{R}^k\times\mathbb{R}^{0|2k}$, and Q is an odd vector field on X_{FP} such that $Q^2 = 0$.

You can write Q as a vector field:

$$Q = \sum_a \lambda_a \bar{c}^a + (-ih)\sum_a c^a\partial_a + \frac{1}{2}\sum_{a,b,c} c^a_{bc}c^bc^c \frac{\partial}{\partial c^a}$$

where $\frac{\partial}{\partial c^a}$ is the "left derivation" in $\langle c^a\rangle$. This is known as the BRST operator.

Let's go over Lie algebra cohomology a bit. Consider the *standard complex $C^\bullet(\mathfrak{g}, M)$ for $\mathfrak{g}$ with coefficients in M* (M a representation of $\mathfrak{g}$, defined as $\sum_{\ell=0}^{k=\dim\mathfrak{g}} \operatorname{Hom}_\mathfrak{g}(\bigwedge^\ell \mathfrak{g}, M)$. We define $C^\ell(\mathfrak{g}, M) = \operatorname{Hom}_\mathfrak{g}(\bigwedge^\ell \mathfrak{g}, M) = \bigwedge^\ell \mathfrak{g}^*\otimes M$. Then we define $d\colon C^\ell \to C^{\ell+1}$ by

$$dc_\ell(x_1,\dots,x_{\ell+1}) = \sum_{i<j}^{\ell} (-1)^{i+j-1} c_\ell([x_i,x_j], x_1,\dots\hat{x}_i,\dots,\hat{x}_j,\dots,x_{\ell+1}) + \sum_{i=1}^{\ell}(-1)^{i-1}x_i\cdot c(x_1,\dots\hat{x}_i\dots x_{\ell+1})$$

[[★★★ HW: check that $d^2 = 0$.]] Now we can define $H^\bullet(\mathfrak{g}, M)$ as the homology of the complex.

Example 32.2. Let $M = \mathbb{C}$ be the trivial representation of $\mathfrak{g}$, so $x\cdot c = 0$. Then let $c_2 \in C^2(\mathfrak{g},\mathbb{C})$ and require that

$$dc_2(x,y,z) = c_2([x,y],z) - c_2([x,z],y) + c_2([y,z],x) = 0.$$

And we have that for $c_1 \in C^1(\mathfrak{g},\mathbb{C})$,

$$dc_1(x,y) = c_1([x,y]).$$

So $H^2(\mathfrak{g},\mathbb{C}) = \{c_2 \in C^2 | dc_2 = 0\}/\{dc_1\}$.

Claim. *Given $c_2 \in H^2(\mathfrak{g},\mathbb{C})$, we can define a central extension $\hat{\mathfrak{g}} = \mathfrak{g}\oplus\mathbb{C}k$ by $[x,y]^\sim = [x,y] + kc_2(x,y)$ and $[k,x] = 0$. The condition $dc_2 = 0$ implies the Jacobi identity.*

We can define a trivial central extension $c_2(x,y) = c_1([x,y])$. [[★★★ HW: find a basis for $\hat{g}$ in this case such that $\hat{g} \cong \mathfrak{g}\oplus\mathbb{C}k$ as a Lie algebra]].

The conclusion is that $H^2(\mathfrak{g},\mathbb{C})$ classifies central extensions of $\mathfrak{g}$ by $\mathbb{C}$. PT: you can replace $\mathbb{C}$ by any representation M to get extensions of $\mathfrak{g}$ by M (with the abelian Lie algebra structure). NR: yes. ⋄

Example 32.3. Let $\mathfrak{g} = Vect(S^1)$. There is a unique non-trivial central extension of this Lie algebra given by

$$c_2(f(t)\frac{d}{dt}, g(t)\frac{d}{dt}) = \frac{1}{2\pi i}\int_{S^1}(fg''' - f'''g)\,\frac{dt}{t}.$$

If you choose the basis $L_n = t^{-n-1}\frac{d}{dt} = ie^{in\theta}\frac{d}{d\theta}$ (where $t = e^{i\theta}$, then $[L_n, L_m] = (n-m)L_{n+m} + \frac{k}{12}(n^3 - n)\delta_{n,-m}$. ◇

Example 32.4. Let $M = \mathfrak{g}$ with the adjoint action. Then $H^0(\mathfrak{g},\mathfrak{g}) = \{c \in C^0(\mathfrak{g},\mathfrak{g}) \cong \mathfrak{g} | dc = 0\}$. Since $dc(x) = x\cdot c = [x,c]$, $H^0(\mathfrak{g},\mathfrak{g}) = \mathfrak{g}^{inv}$ is the invariant part of the adjoint representation, the center of $\mathfrak{g}$. This is true for any module; $H^0(\mathfrak{g}, M) = M^{\mathfrak{g}}$, the invariant part of M (the part killed by the action of $\mathfrak{g}$).

Now let's consider $H^1(\mathfrak{g}, M) = \{c_1 \in \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g},\mathfrak{g}) | dc_1(x,y) = c_1([x,y]) - x\cdot c_1(y) + y\cdot c_1(x) = 0\}/\{dc_0\}$. This says that $c_1([x,y]) = [c_1(x), y] + [x, c_1(y)]$, so c_1 is a derivation. Elements of the form dc_0 are inner derivations, so $H^1(\mathfrak{g},\mathfrak{g})$ is the space of outer derivations of $\mathfrak{g}$.

One can consider $H^2(\mathfrak{g},\mathfrak{g})$. Remember that I was talking about deformation quantization of Poisson manifolds. I formulated Kontsevich's theorem. There is a commutative product which we don't deform and a Poisson bracket that we do deform. It turns out that $H^2(\mathfrak{g},\mathfrak{g})$ is the space of equivalence classes of infinitesimal deformations of $\mathfrak{g}$.

[[★★★ HW: check that $H^3(\mathfrak{g},\mathfrak{g})$ is the space of possible obstructions to extending an infinitesimal deformation one step further]] PT: what is a deformation? NR: it is a bracket $[x,y]_t = [x,y] + \sum_{n\geq 1} t^n c^{(n)}(x,y)$. We require that the Jacobi identity hold for this bracket and we consider such deformations up to equivalence $\phi\colon \mathfrak{g}[[t]] \to \mathfrak{g}[[t]]$, $\phi = \mathrm{id} + \sum_{n\geq 1} t^n\phi^{(n)}$.

First statement: it turns out that $c^{(2)} \in Z^2(\mathfrak{g},\mathfrak{g})$. Second statement: ϕ equivalences act on the possible c_2 as elements of B_1, $\phi\colon c_2(x,y) \mapsto c_2(x,y) + [\phi_1(x), y]$, so $[c_2] \in H^2(\mathfrak{g},\mathfrak{g})$.

We want that $[[x,y]_t, z_t] + alt = 0$. The coefficient in t^n is $\big\{[c^{(n)}(x,y),z] + alt + c^{(n)}([x,y],z) + alt\big\} + \overbrace{c^{(1)}(c^{(n-1)}(x,y),z) + alt + \cdots}^{\text{other}} = 0$. This is related to the Gerstenhabber algebra and the Schouten bracket, which appear naturally. Only the terms in $\{\}$ have $c^{(n)}$.

This can be written as $dc^{(n)} + \sum_{K+\ell=k}^{n-1}[c^{(k)}, c^{(\ell)}] = 0$ (the bracket is called the Gerstenhabber bracket of Schouten bracket). This "other stuff" (other than $dc^{(n)}$) is a class in H^3, which is an obstruction to the existence of $c^{(n)}$. ◇

33 NR 11-14

I updated my website, so now there are references. There is also a link to projects. If you want a project for this class, you should talk to me, preferably this week. Today 2-3, Friday 9-11 and 2-3. You can also find homework problems there.

Last time I was talking about the standard complex $C^\bullet(\mathfrak{g},\mathfrak{g}) = \bigoplus_{n\geq 0}\mathrm{Hom}_\mathfrak{g}(\bigwedge^n \mathfrak{g},\mathfrak{g})$. There is another approach to this complex which is helpful for thinking about BRST quantization. Let X be a finite dimensional vector space and consider $\bigwedge^\bullet X$. Choose a basis $\{a_\alpha\}$ of X. Let $Q\colon \bigwedge^\bullet X \to \bigwedge^\bullet X$ be a derivation of this algebra (i.e. $Q(ab) = Q(a)b + (-1)^{|a|\cdot|Q|}aQ(b)$). Let $Der(X)$ be the linear space of derivations. It has a natural Lie super algebra structure: $[Q_1,Q_2] = Q_1\circ Q_2 + (-1)^{|Q_1|\cdot|Q_2|}Q_2\circ Q_1$.

Proposition 33.1. *$Der(X)\cong \mathrm{Hom}(X,\bigwedge^\bullet X)$ as a vector space.*

The proof is obvious: it is because Q is completely determined by how it acts on X, and any choice of action on X can be extended uniquely to an endomorphism of $\bigwedge^\bullet X$.

If $X=\mathfrak{g}^*$, then $Der(\mathfrak{g}^*)\cong C^\bullet(\mathfrak{g},\mathfrak{g})$ is a vector space isomorphism. Let $[\cdot,\cdot]\colon \mathfrak{g}\wedge\mathfrak{g}\to\mathfrak{g}$. It defines $c\colon \mathfrak{g}^*\to\mathfrak{g}^*\wedge\mathfrak{g}^*$. What does it mean for the bracket to satisfy the Jacobi identity? It defines an odd derivation of $\bigwedge^\bullet \mathfrak{g}^*$.

Proposition 33.2. *The Jacobi identity for $[\,,]$ holds if and only if $[c,c]=2c^2=0$.*

Proof. [[★★★ HW: direct computation]] □

Corollary 33.3. *A Lie algebra structure on $\mathfrak{g}$ defines a differential on $Der(\mathfrak{g}^*)=\mathrm{Hom}(\mathfrak{g}^*,\bigwedge^\bullet\mathfrak{g}^*)=C^\bullet(\mathfrak{g},\mathfrak{g})=\mathrm{Hom}(\bigwedge^\bullet\mathfrak{g},\mathfrak{g})$, given by $d_c(a)=[c,a]$ for $c,a\in C^\bullet(\mathfrak{g},\mathfrak{g})\cong Der(\mathfrak{g}^*)$.*

You can think of $Der(X)$ as $Vect(X[1])$, vector fields on X (thought of as an odd vector space). You can generalize this to the case where X is a super vector space.

Last time I used $C^\bullet(\mathfrak{g},\mathfrak{g})$ to talk about deformation theory, using mysterious words like "Gerstenhabber". The bracket on $Der(\mathfrak{g}^*)$ is actually the Gerstenhabber bracket. The Schouten bracket is another example of such a construction.

So $c\in C^2(\mathfrak{g},\mathfrak{g})$ is a Lie algebra structure on $\mathfrak{g}$. We want to deform it. A formal deformation is $c^h = c+\sum_{n\geq 1}h^n c^{(n)}$ where $c^{(n)}\in C^2(\mathfrak{g},\mathfrak{g}) = \mathrm{Hom}(\bigwedge^2\mathfrak{g},\mathfrak{g})$. We want $[c^h,c^h]=0$ again (i.e. we still want a Lie algebra structure). What does this mean for the coefficients $c^{(n)}$. The coefficient of h^n is

$$\overbrace{[c,c^{(n)}]+[c^{(n)},c]}^{2[c,c^{(n)}]}+\sum_{k=1}^{n-1}[c^{(k)},c^{(n-k)}]=0 \qquad (*)$$

We write $d_c(c^{(n)})=[c,c^{(n)}]$.

Lemma 33.4 ([[★★★ HW]]). *$d_c\left(\sum_{k=1}^{n-1}[c^{(k)},c^{(n-k)}]\right)=0$ assuming $c^{(1)},\dots,c^{(n-1)}$ satisfy $(*)$.*

So $(*)$ is $d_c(c^{(n)})+z^{(n)}=0$ for some $z^{(n)}\in Z^3(\mathfrak{g},\mathfrak{g})$. Assume inductively that $[z^{(n)}]=0$ in $H^3(\mathfrak{g},\mathfrak{g})$. The inductive step is to construct a $c^{(n)}$ so that $dc^{(n)}=-z^{(n)}$ such that $[z^{(n+1)}]=0$. If $H^3(\mathfrak{g},\mathfrak{g})=0$, then there is no problem; you can choose $c^{(n)}$ without any trouble. In general it is a very non-trivial problem. The whole deformation theory of associative algebras can be formulated this way.

When Kontsevich classified $*$-products, the result itself wan't the striking thing. He constructed a very explicit $*$-product, which has to do with some field theory.

Now I want to return to quantization of guage systems. Reminder about L_∞-algebras. Say X is a super vector space. Consider $P(X)=X\oplus \mathrm{Sym}^2(X)\oplus\cdots$ (like functions on X^* vanishing at 0). This is a super commutative (non-unital) algebra. Then a super derivation $Q\colon P(X)\to P(X)$ is something satisfying $Q(ab)=Q(a)b+(-1)^{aQ}aQ(b)$.

1. $Der(P(X))\cong \mathrm{Hom}(X,P(X))$ as before (because the action on a monomial is determined by the action on degree 1 elements).
2. $Der(P(X))$ forms a Lie super algebra, with the usual bracket.

Definition 33.5. A *formal pointed differential graded manifold* (or an *L_∞ algebra*) is a pair $\big(P(X),Q\big)$ where Q is an odd differential with $Q^1=\frac{1}{2}[Q,Q]=0$. ⋄

Example 33.6. If $\{c_\alpha\}$ is a basis for X, then if $Q(c_\alpha) = \sum_{\beta,\gamma} c_\alpha^{\beta\gamma} c_\beta c_\gamma$, this is where X^* is a Lie algebra. ◇

Example 33.7. If $Q(c_\alpha) = \sum_\beta d_\alpha^\beta c_\beta + \sum_{\beta,\gamma} c_\alpha^{\beta\gamma} c_\beta c_\gamma$, where $d\colon X^*[1] \to X^*[1]$, $c\colon X^*[1] \wedge X^*[1] \to X^*[1]$ Lie bracket. Then $Q^2 = 0$ implies $d^2 = 0$, so d is a derivation of c.

This is the notion of X^* being a *differential graded Lie algebra*. $x \in X$, $Q(x) = d(x) + c(x)$. If X is a super vector space then this gives you a super differential graded Lie algebra. ◇

Suppose $E = E_0 \oplus E_1$ is a super vector space. Then it is natural to assume we have $d\colon E \to E$ with $d^2 = 0$, with $d(E^i) \subseteq E^{1-i}$. Then we can construct the cohomology space $H = \ker d/\operatorname{im} d$. There is a general fact from linear algebra. After some work, it is also known as the Lipschitz theorem.

Assume $A\colon E \to E$ so that $[A, d] = 0$. Then A also defines a map $[A]\colon H \to H$. Fact: $\operatorname{str}_E(A) = \operatorname{str}_H([A])$. We are given the right hand side by some path integral, and we're trying to construct an A with some properties on some larger space with all these fermions. All these things (Fadeev-Popov, BRST, etc.) are different constructions of such an A.

34 NR 11-16

Last time I explained a construction from linear algebra. If $[A, d] = 0$, for A and d operators on a super vector space Σ, with d odd and $d^2 = 0$, then A defines an operator $[A]$ on $H := \ker d/\operatorname{im} d$ and $\operatorname{str}_E A = \operatorname{str}_H [A]$.

Remember that the reason I brought this up was to make sense of integrals of the form $\int \exp(iS(x)/h)\,\mathcal{D}x$. For this we need to make a connection between path integrals and traces. But I haven't yet explained what a quantum field theory is.

Classical field theory

We'll talk about Hamiltonian d-dimensional field theory. It is the following assignment

- to a $(d-1)$-dimensional (compact oriented) manifold N, we assign a symplectic manifold $S(N)$,
- to a d-dimensional M, we assign a Lagrangian $L_M \subseteq S(\partial M)$.

The axioms are

- $S(\varnothing) = *$,
- $S(N_1 \sqcup N_2) = S(N_1) \times S(N_2)$,
- $S(\overline{N}) = \overline{S(N)}$ (opposite orientation goes to opposite symplectic structure),
- $L_{M_1 \sqcup M_2} = L_{M_1} \times L_{M_2} \subseteq S(\partial M_1) \times S(\partial M_2)$.
- $L_\varnothing = *$.
- (gluing axiom) $\partial M = \partial_1 M \sqcup \partial_2 M \sqcup \partial_3 M$, with an orientation reversing isomorphism $\partial_1 M \cong_f \partial_2 M$, then let M_f be the gluing, then $S(\partial M) \supseteq L_M \to L_{M_f} \subseteq S(\partial_3 M)$.

This can be formulated as a functor.

Now let's consider Lagrangian classical field theory of first order (i.e. the Lagrangian only depends on the first jet), which is an example of this more general Hamiltonian picture. It consists of

- $F(M)$, the space of fields on M,
- $F(\partial M) = F(M)|_{\partial M}$, and $S(\partial M) = T^*F(\partial M)$,
- $L_M \subseteq S(\partial M)$ is the set of solutions to the Euler-Lagrange equations.

One can then argue that all of the axioms hold assuming the Lagrangian is non-degenerage. This is a very strong assumption; in all guage theories, the Lagrangian will be degenerate.

Ok, this is the classical picture. How do we quantize it? Let's start with classical mechanics. Waht did we discuss? If S is a symplectic manifold, then $C(S)$ is a Poisson algebra, which we can deform (at least formally) to an associative algebra $C_h(S)$, with the first jet of the deformation equal to the Poisson structure. If we have a Lagrangian submanifold L, then it gives us an ideal $I_L \subseteq C(S)$ (it is an ideal in the commutative algebra, and a Lie subalgebra, but not a Lie ideal). This quantizes to I_L^h, a one-sided (say right) ideal in the algebra $C_h(S)$.

So we have the representation (left module) $H_L = C_h(S)/I_L^h$ of $C_h(S)$. Observe also that if the quantization is flat (i.e. $C_h(S) \cong C(S)$ and $I_L^h \cong I_L$ as vector spaces), then $H_L \cong C(L)$ as a vector space (at least in the smooth case). PT: is L a Lagrangian or a polarization? NR: a Lagrangian; this is not geometric quantization. Geometric quantization has the huge advantage that from the geometry and the polarization, you produce a representation. PT: this is quite similar. NR: I think they are related as [[★★★ I didn't catch how]], so they are morally the same thing because in a small neighborhood you can always choose a polarization where these Lagrangians are the fibers.

Classically, we can have a Lagrangian $L \subseteq S$ in a symplectic manifold. Upon quantization, it becomes a left ideal I_L^h in an associative algebra $A(S)$ (which in some sense looks like the algebra of endomorphisms of some vector space, so you expect a trivial center for example, so you don't have families of irreps).

Heisenberg picture of Quantum mechanics

Strictly speaking, in quantization, we have to complexify at some point, but let me not pay attention to this at the moment. We have $C_h(S) = A$ and a family of algebra automorphisms $u_t\colon A \to A$ such that $u_0 = \mathrm{id}_A$, $u_t u_s = u_{s+t}$, and (when it makes sense) $u_t(a) = e^{iHt/h} a e^{-iHt/h}$.

Now let's try to translate this picture as a 1-dimensional quantum field theory. Classical 1-dimensional field theory is an assignment:

- To a point, assign a symplectic manifold S.
- To an interval $[t_1, t_2]$, assign $L_{t_2,t_1} \subseteq S \times \overline{S}$ defined by the Hamilton-Jacobi action (assuming the solutions are unique).

$u_{t_2-t_1}$ is the evolution operator that we assign to the segment $[t_1, t_2]$. It is a mapping from A to A, so we can think of it as an element of $A \otimes A^*$ (where A^* is a dual vector space to A; we don't try to make any completions in any topology). This can be regarded as an assignment:

- To a point (which is a compact orientated 0-manifold), we assign $A = C_h(S)$. To a point with the opposite orientation, we assign A^{op}, the algebra A with the opposite multiplication.
- To an interval $[t_1, t_2]$, assign the ideal $I^h_{L_{t_2,t_1}} \subseteq C_h(S) \otimes C_h(S)^{op}$ (assuming $C(S \times \overline{S}) = C(S) \otimes C(S)^{op}$, which is a very strong assumption; in general, I must take the Frechét completion of the tensor product).

How is this ideal realted to the automorphism u_t? The relation is the following. Given $u\colon A \to A$ an algebra automorphism, I can define $I_u \subseteq A \otimes A^{op}$ by the following formula: I_u is the right ideal generated by elements $u(a) \otimes 1 - 1 \otimes a$. The idea is that if $A = C(X)$, then if $u = \phi^*$ for $\phi\colon X \to X$, then I_u is the ideal of functions generated by $\{(x, y) \mapsto f(\phi(x)) - f(y)\}$ on $X \times X$ which vanish at $(x, \phi(x))$ [[★★★]]. You can write this ideal as $\langle u(f) \otimes 1 - 1 \otimes f \rangle \subseteq A \otimes A$.

Claim. $I_{u_{t_2-t_1}} = I^h_{L_{t_2,t_1}} \subseteq A \otimes A^{op}$.

Where L_{t_2,t_1} is the Lagrangian submanifold generated by the Hamilton-Jacobi function for the classical limit $[H]$ of H (if the deformation is flat, we can consider any element of $C_h(S)$ as an element of $C(S)$, which is what we call the classical limit). PT: we use Kontsevich's result to quantize the associative algebra, does he also show that we can quantize these ideals? Is there some analogous statement? NR: The main strength of his result is that you can quantize any Poisson manifold. For symplectic manifolds, it is easier to quantize, and [[★★★ student of Weinstein]] showed how to quantize a Lagrangian submanifold to a left ideal. PT: what is the statement? NR: for any $L \subseteq S$ Lagrangian, and for a given star product $(C_h(S), *)$ with chosen isomorphism $C_h(S) \cong C(S)$, there exists an ideal $I_L^h \cong I_L[[h]]$ as a vector space. I think there is a stronger statement that by changing the star product [[★★★ to an equivalent one?]], you can leave the ideal constant. PT: do you have something other than formal deformation in mind? NR: you can do it for family deformation in algebraic cases.

Now let's talk about the gluing. It should be the analogue of the property $u_t(a) = e^{iHt/h} a e^{-iHt/h}$. Suppose we have $[t_1, t_2]$ and $[t_2, t_3]$, with ideals $I_{t_2,t_1} \subseteq A \otimes A^*$ and $I_{t_3,t_2} \subseteq A \otimes A^*$. At some point, I did something strange. I said the automorphism u defines an element of $A \otimes A^*$, which is just linear algebra. The property $u_t u_s = u_{s+t}$ is equivalent to the following. In $A \otimes A^* \otimes A \otimes A^*$, we have $u_{t_2 - t_1} \otimes u_{t_3 - t_2}$; pairing the middle two, this element should map to $u_{t_3 - t_1}$. So we should have some composition of ideals $I_{t_2 t_1} * I_{t_3, t_2}$ (which should be $I_{t_3 t_1}$. What is this composition? The only natural thing is trace.

We have a trace in $C^\infty(S)$, given by $\operatorname{tr} f = \int_S f \cdot \omega^n$. Assume it quantizes to a trace on $C_h(S) = A$.

Next time I'll prove that $\mathrm{id} \otimes \mathrm{tr} \otimes \mathrm{id}$ is really the composition map. It will just be the translation of the old gluing property into the new setting. I'm pretty sure this is true, but I haven't seen it written down.

35 NR 11-19

There will be no lecture on Wednesday.

Last time I made an anouncement but I didn't really explain it. Another point of view of Heisenberg evolution is ideals in the algebra of observables. Let's first have an algebraic lemma.

Let A be a unital algebra, and let $I \subseteq A \otimes A^{op}$ be a left ideal. I is also a left-right (LR) ideal in $A \otimes A$ (i.e. a sub-A-bimodule). Let I_1 and I_2 be two such LR ideals in $A \otimes A$. Define $I_1 \circ I_2 = (I_2 \otimes 1)(1 \otimes I_1) \cap A \otimes 1 \otimes A \cong A \otimes A$. [[★★★ This should be the tensor product of bimodules over A followed by intersection , $I_1 \otimes_A I_2 \cap A \otimes 1 \otimes A$]]

Lemma 35.1. *$I_1 \circ I_2$ is a LR ideal in $A \otimes A$.*

Proof. Suppose I_L is a left ideal and I_R is a right ideal in A. Then $I = I_L I_R \subseteq A$ is a two-sided ideal. So $I_{12} = (I_2 \otimes 1)(1 \otimes I_1) = \{\sum x^i \otimes y^i x_i \otimes y_i | x^i \otimes x_i \in I_1, y^i \otimes y_i \in I_2\}$ is a LDR (left, double-sided,right) ideal in $A \otimes A \otimes A$. So $I_{12} \cap A \otimes 1 \otimes A$ is naturally a LR ideal in $A \otimes A$. □

The conclusion is that we have "composition" of LR ideals in $A \otimes A$. [[★★★ nevermind checking associativity. If we define the composition as tensor product over A, it should clear]]

Heisenberg evolution in Quantum mechanics. We have $A = C_h(M)$, $u_t(a) = e^{iHt/h} a e^{-iHt/h}$ gives an algebra automorphism $u_t \colon A \to A$. We define the left ideal $I_t = \langle u_t(a) \otimes 1 - 1 \otimes a \rangle_{\text{left}} \subseteq A \otimes A^{op}$. It quantizes the vanishing ideal I_{L_t}, for $L_t = \{(x, y) \in M \times M | x = \phi_t(y)\}$. L_t is a Lagrangian subspace in $(M, \omega) \times (M, -\omega)$, so this ideal $I_{L_t} \subseteq C(M) \otimes C(M)^{op}$ (opposite Poisson bracket) is an ideal (in the commutative assiciative algebra) and a Lie subalgebra (in the Poisson structure). I_{L_t} quantizes to the left ideal I_t in $C_h(M) = A$. The Heisenberg evolution is a family of LR ideals $I_t \subseteq A \otimes A$ such that $I_t \circ I_s = I_{t+s}$. We can check that this is true:

$$\begin{aligned} I_t \circ I_s &= (I_t \otimes_A I_s) \cap A \otimes 1 \otimes A \\ &= [[\bigstar\bigstar\bigstar \text{ HW}]] \end{aligned}$$

Q: composition of Lagrangians doesn't work unless the intersection is clean; do you have the same sort of issue here? NR: yes. We'll discuss this more later. The intersections here should be clean.

Schrödinger picture. We have (1) our usual algebra of obervables $A = C_h(M)$. We choose a representation $\pi\colon A \to End(\mathcal{H})$ (H should be a Hilbert space if you want the probablistic interpretation). $U_t = \exp(it\pi(H)/h)\colon \mathcal{H} \to \mathcal{H}$, where H is the quantum Hamiltonian. (2) Let's return to the classical picture. Classically, we have the symplectic manifold (M, ω). Let $L_1, L_2 \subseteq M$ be two Lagrangian subspaces. We can choose a modified action functional $\mathcal{A}_{L_1,L_2} = \mathcal{A} + F_{L_1} - F_{L_2}$. It has the extremum on a solution to the Euler-Lagrange equations such that $\gamma(0) \in L_1$ and $\gamma(t) \subseteq L_2$. This is the flow from L_1 to L_2. We also have $L_t \subseteq (M, \omega) \times (M, -\omega)$. If the evolution and boundary conditions are good, $L_t \cap (L_1 \times L_2)$ is a single point. In the case $M = T^*N$, we can choose $L_1 = L_2$ to be the zero section $N \subseteq T^*N$.

Quantization. $L_1 \subseteq (M, \omega)$ gives the left ideal $I^h_{L_1} \subseteq A = C_h(M)$, which gives the representation $\mathcal{H}(L_1) = C_h(M)/I^h_{L_1}$. Similarly, we get $\mathcal{H}(L_2)$, another left A-module. So once we choose boundary conditions (L_1, L_2), there is a unique trajectory connecting them (if things are good). When we quantize, we get two representation of our algebra of observables. $L_t \subseteq (M, \omega) \times (M, -\omega)$ quantizes to $I_t \subseteq A \otimes A$. This produces a vanishing subspace $V_t \subseteq \mathcal{H}(L_1) \otimes \mathcal{H}(L_2)$, the subspace where the ideal I_t acts trivially: $I_t V_t = 0$. The dimension of V_t will be the number of points in $L_t \cap (L_1 \times L_2)$. So in good cases, $V_t = \mathbb{C} v_t$ is one-dimensional, with $v_t \in \mathcal{H}(L_1) \otimes \mathcal{H}(L_2)$. The is related to the Schrödinger picture like this. $\mathcal{H}(L_i)$ are Hilbert spaces, so v_t gives a linear map $\widetilde{U}_t\colon \mathcal{H}(L_1) \to \mathcal{H}(L_2)$. It is an unproven theorem that this is a linear isomorphism coinciding with U_t (in good cases, provided $\mathcal{H}(L_1) = \mathcal{H}(L_2) = \mathcal{H}$).

In the case $L_1 = L_2 = N \subseteq T^*N$, then $\mathcal{H} = C^\infty(M)$ and the evolution operator can be regarded as an integral operator with kernel $U_t(q_1, q_2)$. PT: are you thinking of N as linear or curved? NR: curved, but I'm assuming unique trajectories. PT: I think this may behave poorly in the curved case. NR: when the geodesic is not unique, it should be more complicated. Instead of a function, you get a sheaf, where for generic points you get a function, but I haven't thought about this carefully.

Summary:

1-dimensional QFT (Heisenberg): to a point, we assign A. to the opposite orientation of the point, we assign A^{op}. To the interval $[t_1, t_2]$, we assign an LR ideal $I_{t_2 - t_1} \subseteq A \otimes A$. We then get gluing $I_t \circ I_s = I_{s+t}$.

(Schrödinger) to a point we assign $\mathcal{H}(L)$ for $L \subseteq M$ Lagrangian. to the point with opposite orientation, we assign $\mathcal{H}(L)^\vee$. To the interval $[t_1, t_2]$, we get $V_{t_2 - t_1} \in \mathcal{H}(L_1) \otimes \mathcal{H}(L_2)$ a 1-dimensional subspace.

36 NR 11-26

[[★★★ Kolya was ill, so there was no class]]

37 NR 11-28

Last time we had a discussion of Heisenberg evolution in quantum mechanics and the Schrödinger picture of evolution. In the Heisenberg picture,

- To a point, we assign an associative $*$-algebra $A(pt) = C_h(M)$.
- To an interval $[t_1, t_2]$, we assign the LR ideal $I_{t_1,t_2} \subseteq A \otimes A$. This is the quantization of the vanishing ideal for the Lagrangian $L_{t_1,t_2} \subseteq M \times M^{op}$ (which appears in Hamilton-Jacobi). This is if there is a unique solution to the Euler-Lagrange equations. If there are serveral solutions, $L_{t_1,t_2} = \bigcup_\gamma L^\gamma_{t_1,t_2}$.

In the Schrödinger picture,

- To a point, we assign a vector space H (thought of as $C_h(M)/I_L$, where $L \subseteq M$ is the Lagrangian submanifold of boundary conditions. There are two kinds of points: in points and out points. To a point with opposite orientation, we assign the dual representation $H^\vee$.
- To the interval $[t_1, t_2]$, we assign the 1-dimensional subspace $\mathbb{C}v_{t_1,t_2} \subseteq H_1 \otimes H_2$ (the zero subspace for I_{t_1,t_2}), where H_i corresponds to $(I_{L_i}, L_i \subseteq M)$. So we are looking at Hamiltonian flows connecting the two Lagrangians L_1 and L_2. The H is a representation of $A = C_h(M)$. If A were not a $*$-algebra, we would not have the notion of a dual representation. Since A is a $*$-algebra, we have the dual representation $(\pi^\vee, H^\vee)$, where $H^\vee$ is the dual vector space to H, and $\pi^\vee(a) = \pi(a^*)^t$.

Semiclassically, this means the following. As $h \to 0$, $H \to C(L)$. First, the classical picture. We have a Hamiltonian $\mathcal{H}$. The Heisenberg picture: to a point, we assign $C(M)$, and to an interval $[t_1, t_2]$, we assign $I_{t_1,t_2} \subseteq C(M \times M^{op})$. The Schrödinger picture: to a point, we assign a Lagrangian $L \subseteq M$, and to an interval, we assign the intersection $L_{t_1,t_2} \cap (L_1 \cup L_2)$, where the L_1 and L_2 are boundary condition Lagrangians. PT: so this is an extra choice? NR: yes. PT: for geometric quantization, you choose a polarization. NR: geometric quantization gives you a hilbert space given a polarization. This is a different procedure. The problem is this. If we already know $C_h(M)$, then we can construct this H, but this is quite rare. Geometric quantization constructs the H without giving you the algebra C_h which it represents. PT: so you don't quantize the observables, you only get some of them. Are there examples where you don't get a polarization, but this $C_h(M)$ thing works. NR: yes. The simplest example is the quantum torus $\mathbb{T}^2$.

How can we connect the quantum picture with the classical picture? This is the semiclassical picture. We have a nondegenerate Lagrangian $\mathcal{L}$, and $\mathcal{H}$ is the Legendre transform of $\mathcal{L}$. First the Schrödinger picture (and only for the case where $M = T^*N$, "Lagrangian quantum mechanics"). To a point, we assign the space $H = C(L)$ for the classical $L \subseteq M$. This L is extra boundary condition data ... think of it as the zero section $N \subseteq T^*N$. We'll assume L is parallel to the zero section (i.e. $L = \{(p,q)|p = dF(q)\}$ for $F \in C^\infty N$). To the interval, we get $v_{t_1,t_2} \in C(L_1 \times L_2)$. That is, for each L_1, L_2, t_1, t_2, we get $U_{t_1,t_2}^{L_1,L_2}(q_1,q_2)$ for $q_i \in L_i$. The formula for this function is a sum of Feynman diagrams.

$$U_{t_1,t_2}^{L_1,L_2}(q_1,q_2) = \sum_\gamma \exp\bigl(iS_\gamma^{L_1L_2}(q_1,q_2)/h\bigr)(Hes?_\gamma)^{-1/2}\sum_\Gamma \frac{F_\Gamma}{|\operatorname{Aut}\Gamma|}$$

where γ varies over solutions to the Euler-Lagrange equations (we assume there are a finite number of solutions) and where $S_\gamma^{L_1L_2}(q_1,q_2) = S_\gamma(q_1,q_2) + F_1(q_1) - F_2(q_2)$. The composition (gluing) rule is that

$$\int_{L_2} U_{t_1,t_2}^{L_1,L_2}(q_1,q_2)U_{t_2,t_3}^{L_2,L_3}(q_2,q_3)\,dq_2 = U_{t_1,t_3}^{L_1,L_3}(q_1,q_3).$$

In the example where all $L_i = N$, then the F's are zero, and this is the usual composition. PT: is this a theorem or a conjecture? NR: it is a conjecture; I couldn't find it in the litarature. PT: I don't think this is true; you should have to use the path integral to define the U's. It looks like you're just taking classical solutions. NR: The question is how to define the path integral. Here is another conjecture. You can try to approximate the paths by piecewise geodesics and take a limit. What is known about this? Bruce: I'm not sure about the i in the exp. NR: [[★★★ I didn't catch all this stuff]] In the cases where you have another definition of these U's, the conjecture is that it will agree with this one.

PT: could you remind me how you compute the F_Γ? What do you put on the edges and vertices? NR: If there are n edges coming from a vertex (labelled t), you assign $\frac{\partial^n V(q)}{\partial q_{i_1}\cdots\partial q_{i_n}}\Big|_{q=\gamma(t)}$. To an edge between vertex i (with t_1) and j (with t_2), we assign K^{-1}, where $K_{ij}(t) = \bigl(-\frac{d^2}{dt^2}\delta_{ij} + \frac{\partial^2 V}{\partial q_i \partial q_j}\gamma(t)\bigr)$, $t_1 \le t \le t_2$ action on $L_2[t_1,t_2]$. $U_{t_1t_2} = \exp(iH(t_2 - t_2)/h)$ acts on $L_2(N)$.

You can try to get rid of the condition $M = T^*N$. It is natural to assume that (with some corrections), this kind of procedure should work. You should expect that to ta point, we assign $H = C(L)$ for a fixed $L \subseteq M$, and to the interval, we assign $U_{t_1t_2}^{L_1L_2}(q_1,q_2)$. There should be some composition law, but it will be more complicated for a general symplectic manifold.

Now we have some idea of what is quantum mechanics. What is quantum field theory? It can be considered as a functor from the cobordism category of Riemannian manifolds to some other category (like Vect or SVect). Let me describe the ingredients. The Heisenberg picture is the following. To a $(d-1)$-dimensional manifold N_{d-1}, we assign an associative $*$-algebra $A(N_{d-1})$. To the same manifold with opposite orientation, we assign the opposite algebra (using the $*$ structure). To a d-dimensional manifold M_d, we assign an ideal $I(M_d) \subseteq \bigotimes_i A((\partial M)_i)$ (connected components of boundary of M). We also have a gluing axiom.

The Schrödinger picture. To N_{d-1}, we assign a vector space $H(N_{d-1})$. The opposite orientation gives the dual vector space. To M_d, we assign a 1-dimensional subspace $\mathbb{C}v(M_d) \subseteq \bigotimes_i H((\partial M)_i)$. We also have a gluing axiom.

Semiclassical (Schrödinger). Given a classical Lagrangian field theory, which means that to M_d, we assign $\Phi(M_d)$, and $\mathcal{L}$ a function on the jet space of fields. Remember I discussed the classical Bose field and I discussed how the Lagrangian picture gives the Hamiltonian picture. Roughly, the symplectic manifold we assign to N_{d-1} (think of N as part of ∂M) is $T^*\Phi(N_{d-1})$. Classically, we have to choose a Lagrangian subspace $L \subseteq S(N_{d-1})$ (e.g. the zero section). Then the picture is very similar to classical mechanics. To N_{d-1}, we assign H = functionals on L (on $\Phi(N_{d-1})$) on boundary values of fields. To M_d, we assign the 1-dimensional space $\mathbb{C}v(M_d) \subseteq \bigotimes H((\partial M)_i)$, with $v(M_d)[b] = \int \exp(i\mathcal{A}(\phi)/h)\mathcal{D}\phi$ where the integral is over fields ϕ in $\Phi(M_d)$ with given boundary values b. When $\mathcal{A}$ is non-degenerate, we can try to define this integral as a formal sum over solutions to the Euler-Lagrange equations and Feynman diagrams.

Next time I'll keep discussing this subject.

38 NR11-30

Lagrangian QFT. I want to keep track of the quantization procedure, so this is the semiclassical view of QFT. To a compact oriented Riemannian $(d-1)$-manifold N_{d-1} we assign a vector space $H(N_{d-1})$, and to a d-manifold M_d we assign $\mathbb{C}v(M_d) \subseteq \bigotimes_i H((\partial M_d)_i)$. To the manifold $\overline{N}_{d-1}$, we assign the dual vector space $H(N_{d-1})^\vee$.

Recall classical field theory. The idea is this. In the Hamiltonian picture, to N_{d-1} we assign a symplecic manifold $S(N_{d-1})$ and to the opposite orientation, we assign the same manifold with opposite symplectic structure. To M_d, we assign a Lagrangian $L(M_d) \subseteq S(\partial M)$. We require that $S(N_1 \sqcup N_2) = S(N_1) \times S(N_2)$, and the other axioms.

An example of such a construction comes from Lagrangian mechanics. We have

– Φ, the space of fields on M_d, and

– the Lagrangian function $\mathcal{L}$, a local function on Φ (we assume it depends only on the first jet $\mathcal{L}(\phi, d\phi)$).

Then we have the action functional $\mathcal{A}[\phi] = \int_{M_d} \mathcal{L}(\phi, d\phi)$ (if the volume form on M_d is not given, we have to take the output of $\mathcal{L}$ to be a top form). Classical trajectories are solutions to the Euler-Lagrange equations $\delta\mathcal{A} = 0$. So where is the classical field theory in this? The idea is this. We can restrict Φ to $\Phi^b(\partial M)$, fields on the boundary of M (fields are usually sections of a bundle). Now let $S(N_{d-1})$ is $T^*\Phi^b(N_{d-1})$. PT: this is where collars are really useful; you don't have any trouble restricting anything to a collar because it is an open subset. In my class, we take $S(N_{d-1})$ to be the space of solutions to the Euler-Lagrange equations on the collar. In the case where fields on the boundary make sense, the two definitions agree. NR: yes, collars do the job quite nicely. Collars are absolutely natural, even in classical mechanics, but because of the lack of time, we won't do it. For me, it is easier to think about fields on the boundary because of one example: field theory on graphs. We let $L(M_d) = \{\text{solutions to the Euler-Lagrange equations}\} \subseteq S(\partial M_d)$. PT: in my language, the restriction to the collar is the inclusion of this Lagrangian into the symplectic manifold. NR: ok.

So this is the classical Hamiltonian picture coming from Lagrangian mechanics (with a non-degenerate Lagrangian). Fix the boundary conditions. For each N_{d-1}, we fix a boundary Lagrangian $L(N_{d-1}) \subseteq S(N_{d-1})$, requiring that $L(N_1 \sqcup N_2) = L(N_1) \times L(N_2)$. Then $L(M_d) \cap L(\partial M_d)$ is a collection of points. We can modify the action to

$$\mathcal{A}_L[\phi] = \int_{M_d} \mathcal{L}(\phi, d\phi) + F_L[\phi]$$

where F is the generating function for L. Solutions of $\delta\mathcal{A}_L = 0$ are then points of $L(M_d) \cap L(\partial M_d)$.

If we are on a cotangent bundle, taking $L = \{p = dF(q)\}$, then L_0 gives a representation of $[\hat{p}, \hat{q}] = ih$ on $C^\infty(\mathbb{R})$ given by $\hat{p} = ih\frac{\partial}{\partial q}$ and $\hat{q} = q$. L_F gives the representation $\hat{p} = \frac{\partial F}{\partial q} + ih\frac{\partial}{\partial q}$, $\hat{q} = q$.

The "naïve" quantization. Let $L = L_0$. We can choose $H(N_{d-1}) = C[\overline{L(N_{d-1})}]$, so $v(M_d) \in C[L(\partial M_d)]$. Let's denote points of $L(\partial M_d)$ by b.

$$v(M_d)(b) = \int_{\phi|_{\partial M} = b} e^{i\mathcal{A}/h}\,\mathcal{D}\phi.$$

This is the "naïve" quantization because we don't know what this integral is. In the finite-dimensional case, we can make sense of it. If the spaces are infinite-dimensional, we can sometimes employ some functional analysis (at least in the non-oscillating case, where we don't have the i and the exponenential is rapidly decreasing). This v should be thought of as a quantization of the points in $L(M_d) \cap L(\partial M_d)$.

Let's consider the formal asymptotics of this integral as $h \to 0$. Remember that we just want to get something which satisfies all the axioms of quantum field theory. We can try to construct formal power series that do this.

$$\int_{\phi|_{\partial M} = b} e^{i\mathcal{A}/h}\,\mathcal{D}\phi = \sum_\gamma e^{i\mathcal{A}_\gamma(b)/h} \det{}'(K_\gamma(b))^{-1/2} \sum_\Gamma \frac{F(\Gamma)(b)}{|\operatorname{Aut}\Gamma|}$$

where γ is a solution to the Euler-Lagrange equations with boundary condition b. We expand $\mathcal{A}[\gamma + \psi]/h = \mathcal{A}[\gamma]/h + (K_\gamma\psi, \psi)/h +$ (higher order terms)$/h$. We scale ψ by $\sqrt{h}$, so the higher order terms have $h^{\frac{n}{2}-1}$ with $n \geq 3$, so they are small.

If we want to be consistantly naïve, we have to tensor our H with $\mathbb{C}[[h]]$. We now want to check if this satisfies gluing axiom. Consider Lagrangian mechanics on N with non-degenerate Lagrangian $\mathcal{L}(\xi, q)$. Exercising this philosophy, we get

$$U_t(q_1, q_2) = (e^{i\hat{H}t/h})(q_1, q_2)$$

where $\hat{H}$ is the quantization of the Legendre transform of $\mathcal{L}$. I want to emphasize that we don't need to choose a Riemannian structure on N. Since $\mathcal{L}$ is non-degenerate, it gives a metric on the tangent space, $d^2_\xi\mathcal{L}(\xi, q)$. I want to use this to define a natual measure on Lagrangian submanifolds. PT: in geometric quantization, you use the metaplectic structure on S and use half-forms. NR:

If we have M, and we want to glue together two parts of the boundary (b and b'; call the rest of the boundary b''). We want critical points $\delta_{b'}\mathcal{A}^M_\gamma(b', b'') = 0$.

$$\int v(M_d)(b', b'')\,\mathcal{D}b' = \sum_\gamma \cdots \sum_{\Gamma_{int}} \sum_{\Gamma_b} \quad (b'')$$

you get Feynman diagrams from the interior and Feynman diagrams from the boundary. We then would like to check the gluing rule.

Next time I'll say what to do if the Lagrangian is degenerate. We'll end up with super spaces with differentials which annihilate $v(M_d)$.

39 NR 12-03

Last time we discussed quantum field theory in terms of path integrals. Today I want to return to the degenerate systems and give a more conceptual look at Fadeev-Popov quantization.

Cohomological Field Theories

1. In the Schrödinger picture, we assign to N_{d-1} a vector space $H(N_{d-1})$ and to M_d a vector (really a 1-dimensional subspace) $\mathbb{C}v(M_d) \subseteq H(\partial M_d)$. *Cohomological field theory* simply means that we have one more operation. In addition, we have a differential $d\colon H(N_{d-1}) \to H(N_{d-1})$ (which depends on N_{d-1}) with $d^2 = 0$. $H(N)$ is a super vector space and d is odd. Furthermore, we require that $d(v(M)) = 0$.

2. We can construct QFTs on cohomologies. We can assign to N_{d-1} the cohomology $H^*(N_{d-1})$ (unfortunately, this is $H^*(H(N))$), and to M_d we assign $[v(M)] \in H^*(\partial M)$.

3. Two cohomological QFTs are *quasi-isomorphic* if the resulting cohomology QFTs are isomorphic.

If we start with a guage theory, it is impossible to use the Feynman diagram technique because the theory is degenerate. The idea is to replace the guage theory, which doesn't have any fermionic part, thought of as a cohomology QFT. Suppose G acts on a variety X (let's assume X is linear), then you can construct $C^\bullet(\mathfrak{g}, X)$, and then $C(X/G) \cong H^0(\mathfrak{g}; X)$ in many interesting cases. We know that $C^\bullet(\mathfrak{g}, X)$ is functions on $X \oplus \mathfrak{g}[1]$ (the [1] means to shift the grading, so think of the $\mathfrak{g}$ as odd). The differential is the odd vector field $Q = \sum_\alpha c^\alpha x_\alpha + \sum_{\alpha,\beta,\gamma} c^\gamma_{\alpha\beta} c^\alpha c^\beta \frac{\partial}{\partial c^\gamma}$, where $\{e_\alpha\}$ is a basis for $\mathfrak{g}$ and $\{x^i\}$ are coordinates on X (and $x_\alpha = x^i_\alpha(x)\frac{\partial}{\partial x^i}$). We can replace the guage theory on X/G by a quasi-isomorphic guage theory on $X \oplus \mathfrak{g}[1]$, which has some odd degrees. This is why there are fermionic variables in Fadeev-Popov, BRST, BV quantizations. The idea is to replace these complicated path integrals over complicated spaces by path integrals over linear spaces. PT: X is the space of fields? Do we assume it is linear? NR: yes. The fields are usually sections of some vector bundle, so it will be linear. It need not be linear in general.

The second step is to this replacement in such a way that we get a non-degenerate field theory (the action should be non-degenerate). I will return to this (BV quantization) on Wednesday, at which point I will hopefully have figured out a good way to explain it.

Example of a QFT

Today, I'll give a finite dimensional model to illustrate how these constructions can be achieved with finite dimensional approximations instead of Feynman diagrams.

We'll talk about the "discrete" Bose field. By discrete, I mean that the spacetime is a (finite) graph Γ. The spacetime category: spaces are collections of points and spacetimes are graphs Γ. To a collection of points n points N, we assign $H(N) = L^2(\mathbb{R}^n)$, and to Γ, we assign $v(\Gamma)(b) = \int_{\mathbb{R}^{v_{in}}} \exp(-A(\phi, b)/h)\, d\phi$ where v_{in} is the number of interior vertices of Γ. We assume that $A(\phi)$ is a positive polynomial in ϕ. Locallity of the action on the graph: $A(\psi) = \sum_{v\in\Gamma} A_v(\psi_v) + \sum_{e\in\Gamma} A_e(\psi_{e_+} - \psi_{e_-}) + \cdots$ where A_e is an even degree polynomial (this is the analogue of saying that the action is the integral of a Lagrangian which depends only on the jet). The natural first order action is given by

$$A(\mathfrak{p}[si) = \sum_{u,v \text{ adjacent}} A_{u,v}(\psi_v, \psi_u).$$

We want to check the gluing axiom. Suppose se have Γ, and we want to glue together parts of the boundary $\partial_+\Gamma$ and $\partial_-\Gamma$.

$$\begin{aligned} v(\Gamma)(b)|_{\text{diag}} &= \int d\psi_{\text{from }\partial_+\Gamma} \int_{\mathbb{R}^{v_{in}(\Gamma)}} d\psi_{u_0} \exp\Big(- \sum_{u,v\in\partial_\pm\Gamma} + A(\psi_u, \psi_{u_0})\Big) \\ &= \int_{\mathbb{R}^{v_{in}(\Gamma)}} \exp\Big(-\sum_{u,v \text{ adj}} A(\psi_u, \psi_v)\Big)\, d\psi. \end{aligned}$$

This graph QFT is like a universal example, thinking of the graphs as skeletons of spacetimes (and letting them grow infinite). If you want to do $d > 2$, you have to allow different things for boundaries of the graphs.

1. How to find the limits $|\Gamma| \to \infty$. 2. Compute various quantities (in the limit where $|\Gamma| \to \infty$). 3. Find functions $A(\psi, \psi)$ such that the resulting theory is a TQFT for cell complexes. This would mean that on a surface, this $v(\Gamma)$ is invariant under the standard moves of the cell

decomposition (thinking of Γ as embedded in M). It is not clear that this theory exists.

So the idea is to either try to make the theory topological or to make it approximate something else as $|\Gamma| \to \infty$.

2-dimensional Yang-Mills theory

Suppose Σ is a compact oriented surface (possibly with boundary). Let Γ be a spine of a cell decomposition of Σ. Consider a trivialized G-bundle B over Γ (i.e. a copy of G over each vertex). A connection in B is a map $E(\Gamma) \to G$, which you think of as parallel transport along the edges. You can think of Γ as a subgroupoid of the fundamental groupoid $\Gamma_1(\Sigma)$. Let $g = \{B(e)\}_{e\in E(\Gamma)}$ [[★★★ let $g(e) = B(e)$]], then

$$A(g) = \sum_{\substack{f=\text{2-cell}\\ \text{of }\Sigma}} w(f)\,\mathrm{tr}(h(f))$$

where $h(f)$ is the holonomy around f (read the edges around the 2-cell f using the orientation and take the product of the $g(e_i)^{\pm 1}$) and the trace is taken in the adjoint representation.

Spacetime is $\Sigma \supset \Gamma$, space is the closed 1-dimensional cell decomposition. $H(C) = L^2(G^{E(C)})$, and $v(\Sigma) = \int_{G^{E_{in}(\Gamma)}} e^{-A(g)}\,dg$, where G is simple compact.

This action is clearly invariant under $G^{V_{in}(\Gamma)}$, and the transformation is $h\colon g(e) \to h(e_+)g(e)h(e_-)^{-1}$, corresponding to changing the trivialization.

40 NR 12-05

Last time I was explaining 2-dimensional discrete Yang-Mills theory. We started with the classical case. The objects of the spacetime category are 1-dimensional oriented closed manifolds (disjoint unione of copies of S^1) with marked points. The morphisms are cell complex decompositions Γ of 2-dimensional compact oriented manifolds. You should think of a morphism as a surface bordism between collections of circles.

Classically, the space of fields is the space of sections of the trivial principal G-bundle B over Γ (the total space is $G^{V(\Gamma)}$). A connection on this bundle B is a choice of parallel transport isomorphism for each edge. So a connection is a map $E(\Gamma) \to G$, so the space of connections is $G^{E(\Gamma)}$. Let $g(e)$ be "parallel transport along the edge e". A connection is *flat* if the ordered product $\prod^{\to}_{e\in\partial f} g(e)^{\sigma(e,f)} = h_{x_0}(f) = 1$ for each face f ($\sigma(e,f) = \pm 1$ depending on the relative orientations of f and e).

The classical Yang-Mills theory on $\Gamma \subseteq \Sigma$ has the following action.

$$A(g) = \sum_{f\subseteq(\Gamma\subseteq\Sigma)} w(f)\,\mathrm{tr}(h(f))$$

where $w(f) > 0$ is [[★★★]]. This action is gauge invariant. The gauge group can be identified with $G(\Gamma) = G^{V(\Gamma)}$. It acts on connections by $h\colon g(e) \mapsto h(e_+)g(e)h(e_-)^{-1}$, where e_- is the source vertex and e_+ is the target vertex. I don't have time to describe the Hamiltonian structure, but it is a good exercise to describe it.

What is the Quantum version? To 1-dimensional manifolds N_1, objects of the spacetime category, we should identify some space $H(N_1) = L_2(G^{E(N_1)})$ (remember that N_1 is a collection of circles with marked points, so edges are pieces of circle between marked points). To $(\Gamma \subset \Sigma)$, we assign a vector $v(\Gamma \subset \Sigma) \in L_2(G^{E(\partial(\Gamma))})$.

Our path integral philosophy tells us how to choose this vector $v(\Gamma \subset \Sigma)$.

$$v(\Gamma \subset \Sigma)(b) = \int_{G^{E_{in}(\Gamma)},\, g|_{\partial\Sigma}=b} \exp(-A(g)/h)\,\mathcal{D}_h g$$

If we have the i in the exp, it is a more realisitc theory (but still not realistic because Σ should have more structure). Removing the i is more like statistical mechanics [[★★★ I missed the explanation]].

Let's recall what we want from this quantum field theory. (1) We want locallity, which is equivalent to the gluing principle. That is, if we have two parts of the boundary of Σ and an isomorphism between the two parts (call fields b, and let b' denote fields on the rest of the boundary). We want

$$\int_{G^{E_{in}(\Gamma)}} \exp(-A(g,b,b'))\,\mathcal{D}_h g\,\mathcal{D}_h b = \int_{G^{E_{in}(\tilde{\Gamma})}} \exp(-A(g,b')\,\mathcal{D}_h g$$

(2) We require Guage invariance of $v(\Gamma \subset \Sigma)(b)$. (3) We can try to find $\mathcal{D}_h g$ such that the result is a TQFT. So $v(\Gamma \subset \Sigma)(b)$ will only depend on $h(C_i)$ (C_i are the connected components of the bounary) and the genus of Σ. Since we require gauge invariance, v should only depend on the conjugacy classes $[h(C_i)]$, the genus, and the number of components of $\partial\sigma$. (4) The classical limit should recover the classical field theory.

There is no reason to expect that we should be able to find such a thing. A stronger gluing axiom: given two components of the boundary of Σ and given an isomorphism f between *part* of the components. For example, you could take two disks (each with 1 2-cell and a cell decomposition of the boundary) and try to glue together some of the 1-cells on the boundary. How about (3), can we choose the measure so that the result is topological? It turns out the answer is yes. Assume G is simple compact. Take the measure $e^{-A(g)}\mathcal{D}_h g = \prod_f w([h(f)]|A) \prod_e dg_e$, where $w([g]|A) = \sum_\lambda e^{-c_2(\lambda)A_f} \dim(V_\lambda)\chi_\lambda(g)$, where the sum is over all irreducible representations and c_2 is the second Casimir.

Then the formula for $v(\Gamma \subset \Sigma)$ is

$$v(\Gamma \subset \Sigma) = \int_{G^{E_{in}(\Gamma)}} \prod_f w([h(f)]|A_f) \prod_{e \in E_{in}(\Gamma)} dg(e).$$

Now let me explain where this measure came from. Start with disks with one 2-cell. Start with one disk, with one real number A. Then $v(\Gamma \subset D) = w([h(\partial D)]|A)$. To glue, we have to choose a measure on the boundary. We choose $\mathcal{D}_h b = \prod_{e \in \text{gluing edges}} db(e)$.

Then you can check the following identity. We want to glue two oriented disks (with real numbers A and B and opposite orientations on the gluing edges) along an edge. Say we have b assigned to the edge, the holonomy around the rest of the A-disk is g, and the holonomy around the rest of the B-disk is h. HW: check that

$$\int_G w(bg|A) w(b^{-1}h|B)\, db = w(gh|c_{A,B})$$

when $c_{A,B} = A + B$. So the natural interpretation for the real numbers A and B is area.

$$\int_G \chi_i(bg)\chi_j(b^{-1}h)\, db = \#(i,j)\chi_\#(gh).$$

So it is almost topological, except that we have to add up A and B when we do the gluing. There is a formula

$$v(\Gamma \subset \Sigma)(b) = \sum_\lambda \prod_{i \in \text{bd. comp. of } \Sigma} \chi_\lambda(h(C_i)) e^{-c_2(\lambda)\sum_f A_f} (\dim V_\lambda)^{\chi(\Sigma)}.$$

This is kind of an amazing identity showing that v doesn't depend on Γ at all. This is called 2-dimensional QCD (reference is Witten, 2D Yang-Mills theory revisited).

This demonstrates that using the ideology of the path integral, one can construct 2-dimensional TQFTs. By allowing Γ to get finer and finer, Witten argues that as the size of the approximation grows, this goes to the Yang-Mills path integral and the weight goes to $e^{A_{YM}}$, where A_{YM} is the smooth Yang-Mills action. Smooth Yang-Mills theory is almost topological. You don't need a metric, just a volume form (just like we needed the area).

It turns out that 3-dimensional Yang-Mills theory is not topological, but there is a 3-dimensional TQFT, which is Chern-Simons theory. Unfortunately, it is not known how to replace the infinite-dimensional path integral by a finite-dimensional one for Chern-Simons theory. We can try to develop some formal power series which resembles what we have. The power series should (1) satisfy gluing, (2) be gauge invariant, (3) topological, and (4) classical limit reproduces the classical field theory. In Chern-Simons, it is possible to find such perturbative expansions.

Nobody has produces a gluing procedure for these invariants yet. All of these results were developed for closed 3-manifolds, but not for 3-manifolds with boundary. Even for closed manifolds, the results that exist are for "acyclic flat connections".

Based on surgery, with Turaev we developed [[★★★ didn't catch all of this]].

Next time I'll return to BV quantization.

41 NR 12-07

Last time I talked about discrete 2-dimensional Yang-Mills theory.

In 4-dimensional Yang-Mills theory, the fields are connections on a principal G-bundle on a 4-dimensional manifold M. For physical applications, M is Minkowski (it has a non-degenerate 2-form of signature $(1,3)$). The action functional is

$$\mathcal{A}(A) = \int_M \operatorname{tr}(F(A)^2)\,dx$$
$$\overbrace{+\frac{1}{2}\int_M \langle \nabla\phi, \nabla\phi\rangle\,dx + \int_M V(\phi)\,dx}^{\text{matter}}$$
$$\underbrace{+\int_M (\psi, \mathcal{D}\psi)\,d^2x + m\int_M (\psi,\psi)[[\text{mass}]] + \int_M F((\psi,\psi))\,dx}_{\text{fermions}}$$

($\mathcal{D}$ the dirac operator) We can add matter fields (add sections of a G-bundle where fibers are some representations of G). We can add fermions (sections of a G-bundle of super vector spaces).

Can we quantize this? The machinery for (perturbative) quantizing is Feynman diagrams. The major obstruction is that the Feynman diagrams diverge. The question (for physics) is how to fix this to get something that can be tested. Mathematically, the whole problem requires redefinition. There are two kinds of divergence. Some of the divergences can be absorbed in the fact that you have infinitely many degrees of freedom. The first thing you have to do is deal with the gauge symmetry. This is what FP, BRST, and BV do. Then you have to deal with renormalization.

The standard game played in high energy physics is to start with a large symmetry group ($G = SU(N)$, $N = 4$) and try to break them.

How many orders of perturbation theory should we compute? We know the series diverges, so you should compute more than, say, 10 orders.

There are also theories that are interesting from the mathematical perspective, like Chern-Simons theory. Probably the most interesting is 3-dimensional Chern-Simons. The fields are connections on the trivial G-bundle on M_3.. The action functional is

$$\mathcal{A}(A) = \int_M \operatorname{tr}\big(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\big)$$

This is not invariant, but it is invariant under infinitesimal transformations $A \mapsto [\gamma, A] + d\gamma$, where $\gamma\colon M \to \mathfrak{g}$. $e^{2\pi i \mathcal{A}(A)}$ is invariant under global transformations because the difference is an integer (if you pick the coefficients right). One can, for any integer n, try to make sense of

$$\int_{\text{connections}} e^{2\pi i n \mathcal{A}(A)}\, \mathcal{D}A$$

If n is finite, it is hard. For $n \to \infty$, first you should find critical points. (1) it is easy to see that the critical points are flat connections, for which $F(A) = 0$. (2) the integral should be the "sum" over all guage classes of flat connections "$\sum \int \exp(2\pi i n \mathcal{A}(A+\alpha))\, \mathcal{D}\alpha$", where α is a 1-form on M with coefficients in G. α is small in the sense that that

$$n\mathcal{A}(A+\alpha) = n\mathcal{A}(A) + \int_M \operatorname{tr}(\alpha\wedge d\alpha) + n\int_M \operatorname{tr}(\alpha\wedge A\wedge\alpha) + \frac{2}{3}\int_M \operatorname{tr}(\alpha\wedge\alpha\wedge\alpha)$$

Rescale $\alpha \to \alpha/\sqrt{n}$, and consider the formal power series expansion of the integral:

$$\text{“}\sum\text{”}\, e^{2\pi i n \mathcal{A}(A)} \int e^{2\pi i \int(\alpha\wedge\mathcal{D}_A\alpha) + 2\pi i \frac{2}{3\sqrt{n}}\int \operatorname{tr}(\alpha\wedge\alpha\wedge\alpha)}\, \mathcal{D}\alpha = \sum_\Gamma \frac{F(\Gamma)}{|\operatorname{Aut}\Gamma|}$$

if such expressions existed. The propagator in the diagrams would be $(\mathcal{D}_A)^{-1}$. If we had expressions that made sense, it would overcome the problem (by introducing a metric, say) that each diagram diverges (because of the gauge invariance), then we'd have to show that the result is independent of metric or whatever. This is how finite type invariants of 3-manifolds were started.

Let's focus on the problem of how to deal with gauge theory. I promised to talk about BV, so that's what I'll do now. Next time I'll write the answer for Feynman diagrams describing Chern-Simons theory.

So what does BV do? It deals with the problem that you might have integrals

$$\int_X e^{A/h}\, dx$$

where there is a group G acting (locally freely, say) on X. We want to find a formal power series expansion for this integral. Assume that X has a volume form dx. We can try to write the integral as

$$\int_{X/G} e^{[A]/h}\, [dx]$$

The problem is that even if X is a linear space, the quotient can be quite bad. The idea is to replace the integral using the spirit of cohomological field theories by an integral over some huge superspace E:

$$\int_E e^{\tilde{A}/h} dy$$

where $d\colon F(E) \to F(E)$ is a vector field on E and $d(e^{\tilde{A}/h}) = 0$. Cattaneo and Felder define BV-style cohomological field theory, Poisson sigma model, which gives the star product for Poisson manifolds.

What are the main ingredients of BV theory?

- A manifold X with a volume form dx.
- G acting on X, preserving the volume form dx. If $\{e_\alpha\}$ is a basis for $\mathfrak{g} = Lie(G)$, let the action be given by vector fields $X_\alpha = \sum_i X^i_\alpha(x)\frac{\partial}{\partial X^i}$ in local coordinates.
- $\widetilde{X} = X \times \mathfrak{g}[1]$, the super manifold where X is the even part and $\mathfrak{g}$ is the odd part.
- because G acts on X, it induces an odd vector field Q on $\widetilde{X}$. Q acts on functions on $\widetilde{X}$, which are $Fun(X) \otimes \bigwedge^\bullet \mathfrak{g}^*[1]$. As a super vector space, it is $C^\bullet(\mathfrak{g}, Fun(X))$. It has a differential, which is Q. We assume that $H^i(\mathfrak{g}, Fun(X)) = \delta^{i,0} H^0(\mathfrak{g}, X) = Fun(X/G)$.

  In local coordinates $\{x^i\}$ on X and $\{c^\alpha\}$ on $\mathfrak{g}$, with $[e_\alpha, e_\beta] = \sum_\gamma e^\gamma_{\alpha\beta} c_\gamma$. In local coordinates, the BRST operator is

  $$Q = \sum_{\alpha,i} c^\alpha X^i_\alpha(x)\frac{\partial}{\partial x^i} + \frac{1}{2}\sum_{\alpha,\beta,\gamma} c^\gamma_{\alpha\beta} c^\alpha c^\beta \frac{\partial}{\partial c^\gamma}.$$

- The odd cotangent bundle $E = \pi T^*\widetilde{X}$. X has a volume form. [Assume $\mathfrak{g}$ has a $\mathfrak{g}$-invariant scalar product, so that $\widetilde{X}$ has a volume form.] We have the odd $\omega = \sum_i dx^i d\xi_i + \sum_\alpha dc^\alpha d\lambda_\alpha$ (ξ and λ are the cotangent directions to x and c). We have $\{F, G\}$ on $Fun(\pi T^*\widetilde{X})$.

- Q lifts of an odd vector field on E. This lift is Hamiltonian, with $h_Q = \sum_\alpha c^\alpha X^i_\alpha(x)\xi_i + \frac{1}{2}\sum_{\alpha,\beta,\gamma} c^\gamma_{\alpha\beta} c^\alpha c^\beta \lambda_\gamma$. Denote $Q_E = Q$. This is still odd, and $Q^2 = 0$. So $QF = \{h_Q, F\}$.

- If A is a G-invariant function on X, it pulls back to a function on $\widetilde{X}$ and on E. Let's denote the pull back to E also by A. What can we say about it? (i) $\{h_Q, A\} = 0$ because A is G-invariant. (ii) $\widetilde{X}$ is a Lagrangian in E, so $\{A, A\} = 0$. (iii) $\{h_Q, h_Q\} = 0$ because $Q^2 = 0$ (this is non-trivial because Q is odd). So $\{h_Q + A, h_Q + A\} = 0$.

  We have an L_∞-algebra. We have (E, Q), $\tilde{A} = A + h_Q \in Fun(E)$. This has the properties that $Q\tilde{A} = 0$ and $\{\tilde{A}, \tilde{A}\} = 0$ (this is an indication that it came from some Lagrangian submanifold).

42 NR 12-10

I'll (NR) keep working on the lecture notes. After January, they should be edited a fair amount.

Today I'll try to outline the perturbation theory of Chern-Simons.

First let me explain the concept of BV quantization. The problem whne you read the papers is that you don't see that this is equal to that. Instead, you have things that you know don't quite make sense, like $\int_{X/G} e^{A/h}[dx]$, which you have to replace by something else in order to compute it.

You can have two different quasi-isomorphic field theories. The BV concept is to replace these integrals $\int_{X/G} e^{A/h}[dx]$ by

$$\int_{L\subseteq E} e^{\tilde{A}/h}\, d\tilde{x} \qquad (*)$$

where $\widetilde{X} = X \times \mathfrak{g}[1]$ and $E = \pi T^*\widetilde{X}$. The action of G on X (which has a volume form) induces an odd vector field Q on $\widetilde{X}$. This Q lifts to a Hamiltonian vector field on the odd cotangent bundle E. E has the odd symplectic form $\omega = d\alpha$. This lift of Q is Hamiltonian, with $h_Q = \sum_{\alpha,i} c^\alpha X^i_\alpha(x)\xi_i + \frac{1}{2}\sum_{\alpha,\beta,\gamma} c^\gamma_{\alpha\beta} c^\alpha c^\beta \lambda_\gamma$, where x are coordinates on X, c^α are coordinates on $\mathfrak{g}[1]$, ξ are coordinates on the fibers of T^*X, and λ are coordinates on $\mathfrak{g}^\vee$. [[★★★ some discussion among NR, PT, and BD]] Given $Q\colon \widetilde{X} \to T\widetilde{X} \cong T^*\widetilde{X}$, $dQ\colon T\widetilde{X} \to TT\widetilde{X} \cong T(T^*\widetilde{X})$. h_Q should be a canonical construction.

We're doing all this for one reason. We want to compute $\int_X e^{A/h}\, dx$ (if G is compact). We cannot compute the perturbative expansion because the G-symmetry makes stuff degenerate. There are isolated critical points, and we only need formal neighborhoods of these critical points to compute with, and in these neighborhoods, we can use local coordinates.

The action A is G-invariant. We can pull it back to $\widetilde{X}$ and to E. Let A also denote the pullback of A to E. G-invariance says that $A|_{\widetilde{X}}$ is such that $QA = 0$. G-invariance on E means that $\{h_Q, A\} = 0$, where $\{F, G\}$ is the odd Poisson bracket on $Fun(E)$ induced by ω.

$Q^2 = 0$, which means that $\{h_Q, h_Q\} = 0$ (this is non-trivial because the bracket is odd).

$\widetilde{X} \hookrightarrow E = \pi T^*\widetilde{X}$ (the zero section) is a Lagrangian submanifold. This means that $\{A, A\} = 0$. Again, this is non-trivial because the bracket is odd.

When we combine all this, we have that $\{h_Q + A, h_Q + A\} = 0$. Let $\tilde{A} = h_Q + A$. The goal is to choose $\tilde{A}$ so that it is non-degenerate at critical points and "$\int_L e^{\tilde{A}/h}\, d\tilde{x} = \int_X e^{A/h}\, dx$". Assume $\tilde{A}$ has a critical point at zero (let X be a pointed manifold). In a neighborhood of this point, we want to look at $\tilde{A}(m) = \frac{1}{2}\big(m, q(m)\big) + S(m)$, where $S(m)$ is $O(m^3)$ and $\big(m, q(m)\big)$ is the pairing induced by ω and q is some linear operator (m is a tangent vector). Let $\hat{q}(m) = \big(m, q(m)\big)$.

Then we have $\{\tilde{A}, \tilde{A}\} = 0$. This is equivalent to $q^2 = 0$ as a linear operator, and $\{\hat{q}, S\} + \{S, S\} = 0$.

Claim. *Generically (cohomologies of $Fun(E)$ are non-zero only in degree 0), q is non-degenerate on L. That is, $\int_L e^{\tilde{A}/h}\, d\tilde{x}$ defines a formal power series that is a candidate for $(*)$.*

This handles the problem of degeneracy, but creates another problem. Such integrals, as they are written, depend on L. We should also argue why this integral has anything to do with what we started with. There are two lemmas. [[I'm very happy to have learned all this stuff, but it is still settling. I advise you to look at the lecture notes at the end of January.]] We want the integral to only depend on the cohomology class of $\tilde{A}$. We want to argue that the proposed integral is $\int_{L_0} e^{A/h}\, d\tilde{x}$ where L_0 is $X \times \mathfrak{g}[1] = \widetilde{X}$ (the zero section). Then we can argue that this is $\int_X d^{A/h}\, dx$, but not now.

X has a volume form, and G has Haar measure on it, so we get a volume form on $\widetilde{X}$. Locally, assume $\{a_i\}$ are coordinates on $\widetilde{X}$ and $\{\alpha_i\}$ are coordinates in the cotangent direction. Assume a's are even and α's are odd (this is a brave assumption; it is not true in our case). Then we have the operator (called the BV Laplacian) $\Delta = \sum_i \frac{\partial^2}{\partial a^i \partial \alpha_i}$. Locally, $Fun(\pi T^*\widetilde{X}) = \{\sum_{\{i\}} f^{i_1 \cdots i_k}(a)\alpha_{i_1} \cdots \alpha_{i_k}\}$, which are forms on $\widetilde{X}$. $\Delta\colon \Omega^\bullet(\widetilde{X}) \to \Omega^\bullet(\widetilde{X})$ is the Hodge dual of the differential $d*$. PT: how do you get the Hodge operator from just the volume form? I know how to do it with a metric. NR: We could equip X with a metric, but I think it is possible to get the Hodge operator just from the volume form.

Lemma 42.1 (α)**.** *If $\Delta F = 0$, $\int_L F\, \widetilde{vol} = \int_{L_\sigma} F\, \widetilde{vol}$ where L_σ is a Lagrangian homotopic to L. Here F is a function and we integrate against the volume form on L (and some deformaed volume form on L_σ).*

The proof I know is not very satisfactory. Assume $L = L_0$ and let $L_\sigma = \{\xi = d\sigma(\tilde{x})\}$ for some function σ on $\widetilde{X}$.

Lemma 42.2 (β)**.** $\int_L \Delta F \cdot vol = 0$.

Once we have these two lemmas, we can use Lemma α to get $\int_L e^{\tilde{A}/h}\, d\tilde{x} = \int_{L_0} e^{A/h}\, d\tilde{x}$, and Lemma β to replace A by anything in the same cohomology class.

It seems like it isn't very clear how this formalism works for manifolds with boundary.

Let me say how this changes the naïve perturbation for Chern-Simons. We want to make sense of $\int e^{ikCS(A)} \mathcal{D}A$, where the integral is over connections on a G-bundle on M_3. We want to make sense of this integral as $k \to \infty$.

$$CS(A_0 + \alpha) - CS(A_0) = S_2(\alpha) + S_3(\alpha)$$

gives Feynman diagrams with trivalent vertices. Still some trouble, so we use BV. BV is too general, so use FP. The Feynman diagrams you get after applying this machinery will have the same trivalent vertices and the same propagators. The difference is that they will not be bosonic diagrams. There will be an extra factor of $(-1)^{\#}$. This happens because we used the technique of making a perturbative expansion on a theory which is a quasi-isomorphic theory. You then check that the result is independent of all the choices you made. This is what is called perturbative, or finite type, invariants of 3-manifolds. This stuff has only been done for manifolds without boundary, but for QFT we certainly need to do it for manifolds with boundary. So many things are still open.

References

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