Contents

1 ???

This will be a *basic* course in homological algebra, which is used in various areas. We won't focus on topology and we won't get too fancy. There will be homework every week, to be done in groups of two or three (no groups of one). I will give you very precise definitions and some intuition from geometry, but you have to do calculations for yourself. The most interesting stuff will be on the homework. Chris will be your GSI. We will grade only one of the problems, so you only have to write up one. There will be a meeting once a week with Chris where you'll have to present solutions. The purpose of this is to learn to present mathematics. The grade will be based on the written and presented homework. Each of us will have one office hour. Mine will be Tuesday 2-3.

A little bit of history. The origins of homological algebra. There are two main routes.

One due to Poincaré, which might be called "combinatorial topology" today. The goal is to understand qualitative features of spaces. At the time, they didn't have nice definitions for topological spaces. Poincaré thought of them as built out of simplices. Some of the features are

0. number of connected components,
1. number of holes in a plane,
2. number of holes in 3-space.

Today, we'd measure these with Betti numbers; the number of components is b_0, the number of holes in a plane is b_1 (the holes are "caught" by loops), and the number of holes in 3-space ("caught" by spheres) is b_3. The n-th Betti number is $b_n(X) = \operatorname{rk} H_n(C_*X)$.

Today, we'd say that Poincaré started with a topological space and constructed a chain complex, whose homology we can look at: $\mathsf{Top} \to \mathsf{Chain} \xrightarrow{H_i} \mathsf{Ab}$ (the first arrow is sometimes not a functor)

Definition 1.1. A *chain complex* is a sequence of abelian groups $C_* = (\cdots C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} C_{-1} \xrightarrow{\partial_{-1}} \cdots)$ such that $\partial_{i+1} \circ \partial_i = 0$ for all i. The *n-th homology* is $H_n(C_*) = \ker(\partial_n)/\operatorname{im}(\partial_{n+1})$. ◇

The tricky part is to get the chain complex. That's the part I'd call the art form. The main focus of the class, on the other hand, is to start with chain complexes. But sometimes we'll get some intuition from topology for why we do certian things with chain complexes. What Poincaré found is that even though the chain complex of the space is not an invariant, the homology is.

The other approach is due to Hilbert, and it lead to commutative algebra. He studied ideals $I \subseteq k[x_1, \dots, x_n] = R$. Let's assume I is generated by finitely many homogeneous polynomials. That is, we have a surjection of R-modules $R^m \twoheadrightarrow I$. Then Hilbert defined the *syzygy* $Z_0(I) := \ker(R^m \twoheadrightarrow I)$. This is measuring relations between the generators of I. Then Hilbert repeated this construction (you have to prove Z_0 is finitely generated ... it is). Let $F_0 = R^m$.

$$\begin{tikzcd}
0 \arrow[r] & F_n \arrow[dr, two heads, "\sim"'] & & \cdots & F_2 \arrow[r, "\partial_2"] \arrow[dr, two heads] & F_1 \arrow[r, "\partial_1"] \arrow[dr, two heads] & F_0 \arrow[r, two heads] & R/I \\
& & Z_{n-1} & & Z_2 \arrow[u, hook] & Z_1 \arrow[u, hook] & Z_0 = I \arrow[u, hook] &
\end{tikzcd}$$

Hilbert's theorem is that this terminates: Z_{n-1} is free! You all know this for $n = 1$: $k[x]$ is a PID. Maybe I should have talked about R/I instead of I to get the indices shifted so that they are right. The generalization of being a PID is that any module has a free resolution of length n. We say that the ring has *homological dimension* n in this case.

$F_* = (F_n \to \cdots \to F_0)$ is a chain complex (check that the compositions are zero!). The great thing about homological algebra is that it fits in the plane. What are the homology groups of F_*? I claim that

$$H_i(F_*) = \begin{cases} R/I & i = 0 \\ 0 & i \neq 0 \end{cases}$$

This is what it means to be a free resolution of R/I; a complex of free modules with this property. Since this is an iteration, it is enough to check that $H_1 = 0$. Well, $\ker \partial_1 = Z_1 = \operatorname{im} \partial_2$.

Here, we've used R-modules, not just abelian groups. What kind of other categories have chain complexes? An important one is sheaves of abeilan groups. You can make chain complexes whenever you can make sense of the chain condition $\partial_{i-1} \circ \partial_i = 0$. For example, you can do this in additive categories. But to define homology groups, you need to have kernels and images. So the categories that people study chain complexes in are *abelian categories*. We'll generalize this in the third part of this class, when we talk about derived categories (or model categories).

This ends the historic introduction to homological algebra.

[[break]]

There are some pretty good references. The references and homework will be posted on the website. Now let's start with Part 1: Group homology.

Group homology

These days, you can form homology of anything (spaces, algebras, etc.). We'll focus on one thing: groups. We'll define homology functors $H_n\colon \mathsf{Gp} \to \mathsf{Ab}$ (there won't be any negative homology groups).

$H_0(G)$ is always $\mathbb{Z}$. $H_1(G) = G^{ab} = G/[G,G]$. The higher cohomology groups are not that easy. You can get them in two different ways, following Poincaré (there is a way to make a space from a group) or following Hilbert (using the module $\mathbb{Z}$ over $\mathbb{Z}G$). The module structure on $\mathbb{Z}$ is given by the augmentation map $\mathbb{Z}G \to \mathbb{Z}$, given by $\sum a_n g_n \mapsto \sum a_n$. We go to chain complexes by applying the syzygy construction to get a free resolution F_* of $\mathbb{Z}$ (which will not terminate in general). Q: but then there aren't any interesting homology groups. PT: you're right, I forgot to apply a functor. You take coinvariants of the free resolution. If M is a G-module (i.e. there is a group homomorphism $G \to \operatorname{Aut}(M)$. This is the same as a $\mathbb{Z}G$-module). The *invariants* are $M^G = \{m \in M | g \cdot m = m \forall g \in G\}$, the largest submodule on which G acts trivially. The coinvariants are $M_G = M/\langle g \cdot m - m\rangle_{\text{submod}}$, the largest quotient on which G acts trivally. Coinvariants is not an exact functor, so we get some interesting cohomology. Note that $(\mathbb{Z}G)_G \cong \mathbb{Z}$ (this is actually the augmentation map). In this free resolution, all the terms are $\mathbb{Z}G^n$, and coinvariants behaves well with direct sums, so $(\mathbb{Z}G^n)_G = \mathbb{Z}^n$. You can make F_1 finitely generated if and only if G is finitely generated and you can make F_2 finitely generated if and only if G is finitely presented. If all the F_i are finitely generated, then they are $\mathbb{Z}G^n$ for some n.

What is group homology good for? What is the higher group homology measuring? It turns out you can define cohomology as well (modules are like sheaves). $H^2(G; M)$ is in bijective correspondence with extensions $1 \to M \to \widetilde{G} \to G \to 1$ (modulo the obvious equivalence relation). You know that finite simple groups are classified. Does this mean that we understand all finite groups? Well, if a group is not simple, then there is a proper normal subgroup $N \subseteq Q$. Then N and Q/N are of smaller order, so we might understand them by induction, but we also have to understand the extension $1 \to N \to Q \to Q/N \to 1$. These extensions will be parameterized by $H^2(Q/N; Z(N))$. Q/N acts on the center of N by pulling back and conjugating (?). The example I prepared, but don't have time to do, is a classification of groups of order 8 (I would have told you the relevant cohomology groups).

First we'll define group cohomology and understand them up to H^3. Then we'll do an application: free group actions on spheres (we'll prove that a group acting freely on a sphere of dimension n must have periodic cohomology with period $n+1$).

Part 2 will be about computational tools for $H_*(G; M)$:

- Mayer-Vietoris sequences calculate cohomologies of things like $G_1 *_H G_2$.
- Künneth theorem for $G_1 \times G_2$.
- long exact sequences from short exact sequences of coefficient modules (just like in sheaf cohomology).
- Leray-Serre spectral sequence, which computes the homology of an extension in terms of the smaller pieces. This is actually the main tool for computing these homologies.

2 ???

Basics of chain complexes

I'll write down a definition, you tell me what it means. Fix some associative ring R, and we'll look at the category R-mod (left or right). You can play this game for abelian categories, but let's not do that now.

A sequence

$$M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3$$

is *exact* (at M_2) if $\operatorname{im} f = \ker g$. Note that this is more than saying $g \circ f = 0$ (which just says $\operatorname{im} f \subseteq \ker g$). If $M_3 = 0$, then this is exact if f is onto. If M_1 is zero, the sequence is exact if and only if g is injective.

A *long exact sequence* (LES) is a sequence

$$\cdots \to M_1 \to M_2 \to M_3 \to M_4 \to \cdots$$

which is exact at each place. Note that a LES is a chain complex (actually, since the indices go up, we call it a *cochain complex*[1]). In fact, a LES is exactly a (co)chain complex M_* with $H^i(M_*) = 0$ for all i. This is also called an *acyclic* (co)chain complex.

Another thing that comes up a lot is a *short exact sequence* (SES), which is an exact sequence of the form

$$0 \to M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \to 0$$

The sequence is exact at all places, so f is injective, g is surjective, and $\operatorname{im} f = \ker g$.

Given any chain complex $C_* = (\cdots C_2 \to C_1 \xrightarrow{d_1} C_0 \to C_{-1} \to \cdots)$, we have $Z_i = \ker d_i \subseteq C_i$, the *$i$-cycles*[2], and $B_i = \operatorname{im} d_{i+1} \subseteq C_i$, the *$i$-boundaries*. Because it is a complex, $B_i \subseteq Z_i$. The homology is $H_i(C_*) = Z_i/B_i$.

[1] To get from a chain complex to a cochain complex, you could apply any contravariant additive functor, or you could just re-index (define $M_i = C_{N-i}$ for any N). Another way is to define $M_i = C_i^* = \operatorname{Hom}_R(C_i, R_i)$, but there is a problem. If C_i are left R-modules, then C_i^* are right R-modules. Let R-mod be the category of left modules, and let mod-R be the category of right modules. So we have a functor $\operatorname{Hom}_R(-, R)\colon \mathsf{mod}\text{-}R^\circ \to R\text{-}\mathsf{mod}$ (the circle means opposite category). Poincaré duality says that for a closed oriented manifold of dimension N, the two cochain complexes C_i^* and M_i are homotopy equivalent. So $H^i(M) \cong H_{n-i}(M) = H_i(C_{n-*}(M))$.

[2] It's a Z because in another language, "cycle" is "zykel".

Let me draw some pictures so remind you where this terminology comes from.

Example 2.1 (Poincaré)**.** Let K be a simplicial complex, i.e. K is a union of simplices along faces in some $\mathbb{R}^N$ (I don't want to make this precise because I don't think this is a nice notion[3]). We want to define the homology of this simplicial complex. We define $C_n(K)$ to be the free abelian group generated by the n-simplices. Choose some ordering on the vertices, so C_0 has ordered bases. We orient the edges (going from the smaller number to the bigger number). Similarly, we orient the 2-simplices and so on. Now what are the boundary maps?

[[★★★*picture*]]

$$\cdots \to C_2(K) \to C_1(K) \to C_0(K)$$

First, let's decide what the boundary of an edge is. Let's say $d(ij) = j - i = 1\cdot j + (-1)\cdot i \in C_0(K)$ (you have to have the signs on the boundary pieces to get $d^2 = 0$). For 2-simplices, we read off the edges on the boundary with sign determined by whether you go along the orientation of the edge or against it. So $d(ijk) = jk - ik + ij$.

Now you can kind of see why things in B_i are called boundaries (because you take some linear combination of boundary simplices). To see why Z_i are called cycles, consider Z_1. To get the boundary to be zero, every time you have an edge coming into a vertex, you have to have an edge going out of it. ⋄

[[break]]

Quick review of CW-complexes

A *CW-complex* X is a Hausdorff space with a decomposition (as a set) $X = \coprod_n \coprod_{i \in I_n} e_i^n$, where $e_i^n \cong \mathbb{R}^n$ are n-cells (simplices, if you remove the boundary, are n-cells) such that there exist continuous maps $\phi_i^n\colon D^n \to X$ such that $\phi_i^n\colon \mathring{D}^n \cong e_i^n$ such that

(C) (closure finite) $\phi_i^n(S^{n-1})$ is contained in a finite number of e_i^k for $k < n$, and

[3] Later, we'll make this stuff more precise with simplicial sets.

(W) (weak topology) $A \subseteq X$ is closed (or open) if and only if $(\phi_i^n)^{-1}(A) \subseteq D^n$ is closed (or open).

The intuitive thing to remember is that X is a disjoint union of open cells whose boundaries meet only a finite number of lower-dimensional cells.

Lemma 2.2. *If X and Y are CW-complexes, then $(X \times Y)_{cg}$ (compactly generated) is a CW-complex with cells $e_X^p \times e_Y^q$ for $p+q=n$.*

Now we can define the *cellular chain complex* $C_*(X)$. Take $C_n(X)$ to be the free abelian group on the n-cells, with $d(e^n)$ a linear combination (with correct signs) of the $(n-1)$-cells that appear in $\phi_i^n(S^{n-1})$. This is not precise, but if you took algebraic topology, you know how to make this precise. Q: how do you remember the signs (in simplicial sets, we just ordered the vertices and that took care of it)? PT: you orient each cell however you like and you just deal with those orientations.

Assume that a group G acts (on the right) on our CW-complex X, permuting the cells freely (i.e. $g(e_i^n) = e_j^n$, with $i \neq j$ if g not the identity).

Example 2.3. Take $X = \mathbb{R}$ with a vertex at each integer and 1-cells in between (btw, you can't think of it as a single 1-cell because there is no map $D^1 \to \mathbb{R}$; in fact, you can prove that if you have finitely many cells, the CW-complex is compact) and take $G = \mathbb{Z}$, acting by translation. ⋄

Example 2.4. $X = \mathbb{R}^n$ with $G = \mathbb{Z}^n$ by taking the product of the previous example with itself n times. ⋄

Lemma 2.5. *X/G is again a CW-complex with an n-cell for each G-orbit of n-cells in X.*

Theorem 2.6. *If X is contractible with such a G-action, then $H_*(X/G) \cong H_*(G)$ (defined by $\mathbb{Z}G$ projective resolutions of $\mathbb{Z}$).*

The upshot is that you can use topological homology to compute group homology.

3 ???

Starting Thursday, we'll be in 71 Evans. Chris will have office hours (1060 Evans) and discussion meeting starting this week. The office hours will be Wednesday 9-10am and the meeting will be Friday 12-1 (in a room to be announced). You only need to write up problem 1. The other problems will be discussed.

Remember the following theorem.

Theorem 3.1. *If a group G acts on a contractible space X so that the cells (of some CW decomposition) are permuted freely, then $H_*(X/G) \cong H_*(G)$.*

We took the example $X = \mathbb{R}$ and $G = \mathbb{Z}$ with the action given by translation. So we get that $H_*(S^1) \cong H_*(\mathbb{Z})$, and we know $H_*(S^1)$ really well. For now, we haven't defined group homology, so the theorem is not so impressive, but this allows you to compute group homology using topological homology. One thing that is perhaps surprising is that this homology does not depend on X. For example we could take $X = \mathbb{R}^2$ with traslation action of $\mathbb{Z}$, and the homology of the quotient is the same.

For this discussion, you just have to know that $H_*(G)$ is defined purely algebraically. I'll define it very soon.

Why not just use free G-actions on a contractible space X instead of worrying about cell decompositions? Because of the following example. $\mathbb{Z}^2 \subseteq \mathbb{R}$ (pick two relatively irrational numbers, like 1 and π), so you can think of $\mathbb{Z}^2$ acting on $\mathbb{R}$ by translation. This is a free action, but the orbits are dense, so it doesn't act freely on any CW decomposition. The quotient $\mathbb{R}/\mathbb{Z}^2$ has $H_2 = 0$ (the top non-zero homology is roughly the dimension of the space, and $\mathbb{R}/\mathbb{Z}^2$ is roughly 1-dimensional). However, $H_2(\mathbb{Z}^2) = H_2(S^1 \times S^1)$ (since $\mathbb{Z}^2$ acts on the plane freely), which is non-zero.

Remark 3.2. TG permutes the cells freely if and only if (i) X/G is a CW complex, and (ii) $X \to X/G$ is a regular G-covering. ⋄

Remark 3.3. X is contractible, so $\pi_n(X) = 0$ for all n. This implies that $\pi_1(X/G) \cong G$ and $\pi_n(X.G) \cong 0$ for $n \neq 1$ (I assume G is a discrete group). So X/G is a CW complex of type $K(G, 1)$. The cool thing is that the homotopy type of X/G is totally determined by G (you have to know a little obstruction theory in topology). The theorem is slightly weaker than this (it says that homology type is determined by G). ⋄

It's really important that we have CW structure.

Theorem 3.4 (Whitehead)**.** *If $X \to Y$ induces isomorphisms on all homotopy groups (for all choices of base point in X), then f is a homotopy equivalence* provided *that X and Y are CW complexes.*

You need a map to induce the isomorphisms. Having isomorphisms between the homotopy groups is not enough. You prove the theorem by explicitly constructing the homotopy inverse explicitly (π_n roughly controls the n-cells).

Corollary 3.5. *If K is another $K(G, 1)$ (in particular, is a CW complex), then $K \simeq X/G$.*

This is a corollary as soon as we get a map $X/G \to K$ or the other way, inducing an isomorphism. Since there is only one non-trivial homotopy group, it turns out that you can construct the map cell by cell. If there are multiple non-zero homotopy groups, you get things called K invariants which obstruct such a construction. There is a whole story about Postnikov (?) towers. We won't use this obstruction theory stuff for the proof of the theorem; we'll prove the theorem algebraically.

The first thing we need to do is define group homology. We'll do one more application of the theorem in a bit. I'm starting on $H_*(X/G)$ not because I'm a topologist, but because it is the easier one to compute.

Definition 3.6. Given a group G, the *Bar complex* for G is given by taking $C_n^{Bar}(G)$ to be the free abelian group on G^n. $d\colon C_n^B(G) \to C_{n-1}^B(G)$ is given by $\sum_{i=0}^n (-1)^i d_i$, where

$$d_i(g_1|\cdots|g_n) = \begin{cases} g_2|\cdots|g_n & i = 0 \\ g_1|\cdots|g_i g_{i+1}|\cdots|g_n & 0 < i < n \\ g_1|\cdots|g_{n-1} & i = n \end{cases}$$

Now we define $H_n(G) := H_n(C_*^B(G))$. ⋄

This is called the Bar complex either because Cartan and Eilenberg came up with it in a bar, or because of all the bars in the notation. You need to prove $d^2 = 0$. It should be fairly obvious that $C_n^B(G) \cong \mathbb{Z}G^n \cong \mathbb{Z}G^{\otimes n}$. You could define the map in these terms, in which case you'd see that this d makes sense if you change $\mathbb{Z}G$ to any algebra; this is called the *Hodgechild homology* (sp?) of the algebra.

Example 3.7 ($G = \mathbb{Z}/2$)**.** This complex is really hard to compute with. I never tried it; maybe you could do it. However, we can compute the homology using the theorem. The Brower fixed point theorem says that any map on a disk has a fixed point, so there is no fixed-point-free action on the disk. We start with a point, and add cells to make the space contractible and avoid fixed points. Take $X = \bigcup_{n\geq 0} S^n$, with $\mathbb{Z}/2$ acting by the antipodal map and CW structure given by two cells in each dimension (so $X^{(n)} = S^n$). This X is contractible; use Whitehead's theorem with $X \to *$. We have to show that any map $S^k \to X$ is contractible (i.e. that $\pi_k(X) = 0$). Up to homotopy, the image lands in $X^{(k+1)} \cong S^{k-1}$ (by the CW approximation theorem), so you can contract it (homotopy equivalent to a differentiable map, then by Sard's theorem there is a dense set of regular values, but since the dimension is too small, the only regular values are missed points and once you miss a point, you can contract).

Now we need to compute $H_*(X/G)$. X/G has a single n-cell for each n; it is sometimes called $\mathbb{R}P^\infty$. So we get a chain complex $\cdots \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}$. The maps are given by multiplication by some number. It turns out that they alternate between multiplication by 2 and multiplication by 0 (the zeros come out of the odd-dimensional $\mathbb{Z}$'s). This has to do with the degree of the antipodal map (which is ± 1 depending on dimension). Compare the nice complex $\cdots \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$ to the terrible complex $C_*^B(G)$. We get

$$H_n(\mathbb{Z}/2) = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}/2 & n \text{ odd} \\ 0 & n \neq 0 \text{ even} \end{cases}$$

◇

We already get a nice result from the theorem and this example. The group homology keeps going (there is no top homology), so we get the following result.

Corollary 3.8. *$\mathbb{Z}/2$ does not act freely on $\mathbb{R}^n$ or any finite-dimensional contractible CW complex.*

Sketch Proof. If X is CW and a finite group acts freely on it, then there is a CW decomposition on which the group acts freely. □

Remark 3.9. We'll show that this is actually true for any group with torsion. ◇

I invite you to do the same argument for $\mathbb{Z}/n$. Maybe we'll make this a homework. Since I don't have time to start the proof of the theorem, I'll just give you a hint about how to do these calculations.

Lemma 3.10. *If a finite group G acts freely S^d, then the (reduced)*[1] *homology of G is $(d+1)$-periodic. That is, $H_n(G) \cong H_{n+d+1}(G)$.*

Example 3.11. $\mathbb{Z}/k$ acts freely on S^1 by rotation, so it's homology must be 2-periodic. We already know that $\tilde{H}_0(\mathbb{Z}/k) = 0$, so we just need to compute $H_1(\mathbb{Z}/k)$ to know everything (it turns out to be $\mathbb{Z}/k$). We'll do the proof on Thursday. ◇

[1] This just removes the $\mathbb{Z}$ in degree zero. We just define $\tilde{H}_0(G) = 0$ and the other groups are the same as the usual homology.

4 ???

Discussion section will be Friday 12-1 in 939.

Today we'll make some definitions purely algebraically. Later, we'll show the definitions are motivated by geometry.

The homotopy category of chain complexes

Note that Chain_R is a category (of chain complexes of R-modules). A morphism between two chain complexes is what you think it is (morphisms in all degrees so that the squares commute). It should be clear that a chian map induces a morphism on homology. This is because a chain map sends cycles to cycles and boundaries to boundaries. Another way to say this is that $H_n\colon \mathsf{Chain}_R \to R\text{-}\mathsf{mod}$ is a functor.

There is another category, $\mathsf{h}\text{-}\mathsf{Chain}$, whose objects are chains, but the morphisms are different. The morphisms in $\mathsf{h}\text{-}\mathsf{Chain}$ are homotopy classes of chain maps.

Definition 4.1. Two chain maps, f and g, are *chain homotopic* (written $f \sim g$) if there exists a "map of degree -1" (which isn't a chain map), h, such that $f - g = hd + d'h$. ⋄

$$\begin{tikzcd} \cdots \arrow[r, "d"] & C_{n+1} \arrow[r, "d"] \arrow[d, "f-g"] \arrow[dl, "h"'] & C_n \arrow[r, "d"] \arrow[d, "f-g"] \arrow[dl, "h"'] & C_{n-1} \arrow[r, "d"] \arrow[d, "f-g"] \arrow[dl, "h"'] & \cdots \\ \cdots \arrow[r, "d'"'] & C'_{n+1} \arrow[r, "d'"'] & C'_n \arrow[r, "d'"'] & C'_{n-1} \arrow[r, "d'"'] & \cdots \end{tikzcd}$$

Let's check that $hd + d'h$ is a chain map. We have to check that $d'(hd + d'h) = (hd + d'h)d$. Since $d^2 = 0$ and $d'^2 = 0$, so both sides are equal to $d'hd$.

[[$\mathrm{Hom}_R({}_RM, {}_RM')$ is not an R-module (unless R is commutative)]]

I claim that the homology functor $H_n\colon \mathsf{Chain}_R \to R\text{-}\mathsf{mod}$ factors through $\mathsf{h}\text{-}\mathsf{Chain}$. To show this, it is enough to show that two homotopic maps induce the same map on homology. For this, we need to show that $hd + d'h$ induces the zero map on homology. Let z be an n-cycle (so $dz = 0$). Then $(hd + d'h)(z) = 0 + d'(h(z))$, which is a boundary, so cycles get taken to boundaries (which are zero in homology).

Notation: A chain map $f\colon C_* \to C'_*$ is a *homotopy equivalence* if it is an isomorphism in the homotopy category (i.e. there is a chain map the other way so that the compositions are homotopic to the identity maps on C_* and C'_*). This implies that f is a *quasi-isomorphism* (i.e. that it induces isomorphisms on all homologies). We'll see that being a quasi-isomorphism is weaker than homotopy equivalence. If you invert quasi-isomorphisms, you get the derived category. In the homotopy category, the homotopy equivalences are already inverted.

Definition 4.2. A chain complex C_* is *contractible* if it is homotopy equivalent to the zero complex. ⋄

Since the zero complex is initial and terminal, we can explictly say what this means. It means that the identity on C_* is homotopic to the zero map. That is, C_* is contractible if and only if there is a map h of degree -1 so that $dh + hd = \mathrm{id}_{C_*}$.

Lemma 4.3. *If C_* is contractible, then it is* acyclic *(i.e. all homology groups are zero).*

Warning 4.4. A topological space X is contractible if $X \simeq *$, which implies $C_*(X) \simeq C_*(*)$, and $C_*(*)$ is not quite the zero complex (it is $\mathbb{Z}$ in degree zero; I'm using the cellular complex). There are a couple of ways to fix this. On way is to work with pointed spaces (so that the point is initial and terminal in your category). Another way is to work with the augmented chain complex of the topological space.

Whatever chain complex $C_*(X) = (\cdots \to C_1 \to C_0)$ you use, and add an agumentation map $\varepsilon\colon C_0 \to \mathbb{Z}$. Usually, C_0 is some free abelian group. Define ε by taking all free generators to 1. ⌟

Example 4.5 (Simplex with vertex set V)**.** Let V be any set. Define a chain complex $S_*(V)$ by taking $S_n(V)$ to be the free R-module (or abelian group) on V^{n+1} (think of these as ordered vertices of a (possibly degenerate) n-simplex). Define $d\colon S_n(V) \to S_{n-1}(V)$ by $d = \sum_{i=0}^{n}(-1)^i d_i$, where $d_i(v_0, \dots, v_n) = (v_0, \dots, \hat{v}_i, \dots, v_n)$. As usual, $d^2 = 0$ (we'll actually do this calculation when we do simplicial sets).

To show that this is contractible, we need to produce a map $h\colon S_{n-1}(V) \to S_n(V)$ so that $hd + dh = \mathrm{id}$. Define $h(v_0, \dots, v_{n-1}) :=$

$(w, v_0, \ldots, v_{n-1})$, where w is some fixed element of V. Now we check that

$$\begin{aligned}(hd+dh)(v_0,\ldots,v_n) &= h\Big(\sum(-1)^i(\ldots,\hat{v}_i,\ldots)\Big) + d(w,v_0,\ldots,v_n)\\ &= \sum(-1)^i(w,v_0,\ldots,\hat{v}_i,\ldots,v_n) + (v_0,\ldots,v_n)\\ &\qquad - \sum(-1)^i(w,v_0,\ldots,\hat{v}_i,\ldots,v_n)\\ &= (v_0,\ldots,v_n)\end{aligned}$$

So we've proven $S_*(V) \simeq 0$. ◇

[[break]]

If $V = G$ is a group, then $S_*(V) \in {}_{\mathbb{Z}G}\mathsf{Chain}$ (left $\mathbb{Z}G$-modules) by defining $g\cdot(g_0,\ldots,g_n) = (gg_0,\ldots,gg_n)$. This makes $S_*(G)$ into a free $\mathbb{Z}G$-module with basis $\{(1,h_1,\ldots,h_n)\} = \{(1,g_1,g_1g_2,\ldots,g_1\cdots g_n)\}$. We'll define $(g_1|g_2|\cdots|g_n) := (1,g_1,g_1g_2,\ldots,g_1\cdots g_n)$. Then it turns out that we get the Bar complex. So we have $G^n \cong \{(g_1|\cdots|g_n)\} \subseteq G^{n+1}$.

We see that $d_0(g_1|\cdots|g_n) = g_1\cdot(g_2|\cdots g_n)$. This is a little different from the differential for the Bar complex (we didn't have the g_1). Now let's calculate $d_1(g_1|\cdots|g_n) = (g_1g_2|\cdots|g_n)$, and in general, $d_i(g_1|\cdots|g_n) = (g_1|\cdots|g_ig_{i+1}|\cdots|g_n)$ for $0 < i < n$. Finally, $d_n(g_1|\cdots|g_n) = (g_1|\cdots|g_{n-1})$. So d_i for $i \neq 0$ are all exactly like in the Bar complex. I claim that it is a good thing that the d_0 is different; it makes $S_*(G)$ into a complex of $\mathbb{Z}G$-modules.

In fact, we have that $C^B_*(G) \cong \mathbb{Z}\otimes_{\mathbb{Z}G} S_*(G)$ (including the differentials!). This is an isomorphism in Chain.

Definition 4.6. If R is any ring, and M is an R-module, then a *resolution* of M is an exact sequence $\cdot \to F_1 \to F_0 \to M \to 0$. It is called a free (resp. projective) resolution if the F_i are free (resp. projective). ◇

Lemma 4.7. *$\cdots S_2G \to S_1G \to S_0G \to \mathbb{Z}$ is a free $\mathbb{Z}G$ resolution of $\mathbb{Z}$.*

We've already proved the lemma. We already showed that the S_iG are free $\mathbb{Z}G$-modules (with free basis $\{(g_1|\cdots|g_n)\}$), and we showed that $S_*(G)$ is contractible (the $\mathbb{Z}$ is the free abelian group on G^0).

Moreover, $H_n(G) := H_n(C^B_*(G)) = H_n(\mathbb{Z}\otimes_{\mathbb{Z}G} S_*(G))$.

Theorem 4.8. *Given any two free resolutions $F_* \to M$ and $F'_* \to N$ and a module homomorphism $f_{-1}\colon M \to N$, there is a chain map $f\colon F_* \to F'_*$ extending f_{-1}. Moreover, f is unique up to homotopy.*

We'll prove the theorem next week, but we can get some results from it now.

Corollary 4.9. *The following definition makes sense.*

Definition 4.10. $\mathrm{Tor}^R_n(Q,M) := H_n(Q\otimes_R F_*)$ for any right R-module Q and any free resolution of the left R-module M. ◇

In particular, $H_n(G) = \mathrm{Tor}^{\mathbb{Z}G}_n(\mathbb{Z}_{\mathbb{Z}G}, {}_{\mathbb{Z}G}\mathbb{Z})$.

5 ???

Now we're in 3 LeConte.

Today we'll prove the following theorem.

Theorem 5.1. *$H_n(X/G) \cong H_n(G)$ if X is a contractible space with a CW structure which G permutes freely.*

Recall that $H_n(G) := H_n(C_*^B(G))$.

We've shown that $C_*^B(G) \cong S_*(G) \otimes_{\mathbb{Z}G} \mathbb{Z}$, where $S_*(G)$ is the "simplex with vertices G," so $S_n(G) = \mathbb{Z}\langle G^{n+1}\rangle$ with the usual boundary maps $d = \sum(-1)^i d_i$.[1] We showed that $S_*(G)$ is $\mathbb{Z}$-contractible (but the contraction is *not* $\mathbb{Z}G$-equivariant). In particular, the homology is trivial: $H_n(S_*(G)) = 0$ for all n. We also saw that $S_*(G)$ is a free $\mathbb{Z}G$ resolution of the trivial module $\mathbb{Z}$.

Theorem 5.2. *Given free resolutions (over a ring R) $F_* \to M$ and $F'_* \to M'$ and an R-module homomorphism $f_{-1}\colon M \to M'$, there exists a chain map $f\colon F_* \to F'_*$ extending f. Moreover, f is unique up to chian homotopy.*

We'll actually prove this for projective modules (not just free modules).

Lemma 5.3. *If M is a right $\mathbb{Z}G$-module, then $M^G = \{m \in M | m \cdot g = m\} = \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}, M)$ and $M_G = M/\langle m \cdot g - m\rangle = M \otimes_{\mathbb{Z}G} \mathbb{Z}$.*

Proof. Obvious. □

Proof of Theorem 5.1. We have a G-covering $X \to X/G$. Then $C_n^{cell}(X)$ is a free $\mathbb{Z}G$-module with one generator for each n-cell in X/G and $C_n^{cell}(X/G) = C_n^{cell}(X)_G$. By the lemma, this is $C_n^{cell}(X) \otimes_{\mathbb{Z}G} \mathbb{Z}$. Moreover, X is contractible, so $C_*^{cell}(X)$ is a free $\mathbb{Z}G$ resolution of $\mathbb{Z} = H_0(X)$ (since X is connected, the action of G on H_0 is trivial). So $H_n(X/G) = H_n(C_n^{cell}(X) \otimes_{\mathbb{Z}G} \mathbb{Z})$.

But $H_n(G) = H_n(C_*^B(G)) = H_n(S_*(G) \otimes_{\mathbb{Z}G} \mathbb{Z})$. By Theorem 5.2, the identity map $\mathbb{Z} \to \mathbb{Z}$ extends to a chain map $S_*(G) \to C_*^{cell}(X)$. Similarly, we get a chain map the other way. The uniqueness part of the theorem tells us that these two maps are homotopy inverses, so $S_*(G) \simeq C_*^{cell}(X)$ (over $\mathbb{Z}G$). But homotopy equivalences are preserved by additive functors, so

$$\begin{aligned} H_n(G) &= H_n(C_*^B(G)) = H_n(S_*(G) \otimes_{\mathbb{Z}G} \mathbb{Z}) \\ &\cong H_n(C_*^{cell}(X) \otimes_{\mathbb{Z}G} \mathbb{Z}) = H_n(C_*^{cell}(X/G)) = H_n(X). \end{aligned}$$ □

[1] This $S_*(G)$ is actually obtained from a simplicial set by taking free abelian groups on the simplices.

Definition 5.4. If R is any ring, with modules M_R and ${}_RN$. We define $\mathrm{Tor}_n^R(M, N) := H_n(F_* \otimes_R N)$ where $F_* \to M$ is a free resolution of M. ◇

Think of M as a chain complex concentrated in degree zero, and think of $F_* \to M$ as a chain map. Since F_* is a resolution, this map is a quasi-isomorphism. If you like topology, you can think of M as an arbitrary topological space, F_* as a CW-complex, and the map $F_* \to M$ is like a weak homotopy equivalence.

By Theorem 5.2, this definition makes sense. If F'_* is another resolution of M, then it must be homotopy equivalent to F_*, so the resulting homology groups are the same (canonically isomorphic).

Tony: last time you used a resolution of the other guy. PT: yes. Once you prove the lemma in the homework, we'll be able to show that the two definitions are equivalent.

[[break]]

There was a good question during the break. Recall the homotopy $h_n\colon S_n(G) \to S_{n+1}(G)$, given by $(g_1, \dots, g_n) \mapsto (m, g_1, \dots, g_n)$. This does not imply that $S_*(G) \otimes_{\mathbb{Z}G} \mathbb{Z} \simeq 0$ because the homotopy is not $\mathbb{Z}G$-linear, it is only $\mathbb{Z}$-linear, so it doesn't induce a contraction on $S_*(G) \otimes_{\mathbb{Z}G} \mathbb{Z}$.

Now I want to prove Theorem 5.2, replacing free by projective.

Definition 5.5. A module P is *projective* if $\mathrm{Hom}_R(P, -)$ is exact. ◇

That is, if $N \to M$ is a surjection and $P \to M$ is a morphism, then there exists a factorization $P \to N$.

$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists"'] \arrow[d] & \\ N \arrow[r] & M \arrow[r] & 0 \end{tikzcd} \qquad \begin{tikzcd} & P \arrow[dl, dashed, "\exists"'] \arrow[d] \arrow[dr, "0"] & \\ N \arrow[r] & M \arrow[r] & M' \end{tikzcd}$$

Equivalently, if the row on the right is exact, there is a dashed arrow. This is exactly the condition that $\mathrm{Hom}_R(P, -)\colon R\text{-}\mathsf{mod} \to \mathsf{Ab}$ is exact.

The reason this is called projective is because any such P is the image of a projection from a free module to itself. That is, it is a direct summand of a free module. Let's prove that this is equivalent to projectiveness. Clearly, if $P \oplus Q$ is free, then any map $P \to M$ can be extended to $P \oplus Q$. Since this is free, we can solve the mapping problem in the diagram and restrict to P. On the other hand, if P is projective, we can choose a surjection from a free module (take $M = P$ and N free). By the mapping property, this surjection splits.

It is possible to have projective modules which are not free. Take $R = M_n(k)$, and let $M = k^n$. This cannot be free because its dimension is too small, but $R \cong M^{\oplus n}$, so M is projective but not free. If $R = R_1 \times R_2$, then R_1 is projective but not free. If G is a finite group, then any $\mathbb{Q}[G]$-module is projective!

Definition 5.6. J is *injective* if $\operatorname{Hom}_R(-, J)\colon R\text{-}\mathsf{mod}^\circ \to \mathsf{Ab}$ is exact. ◇

Definition 5.7. ${}_RN$ is *flat* if $- \otimes_R N \colon \mathsf{mod}\text{-}R \to \mathsf{Ab}$ is exact. ◇

There is an analouge to Theorem 5.2 for injective modules.

Theorem 5.8. *If $0 \to M \to J_0 \to J_{-1} \to \cdots$ is an injective coresolution of M, $M' \to J'_*$ is an injective coresolution of M', and $f_1\colon M \to M'$, then there is a chain map $f\colon J_* \to J'_*$ extending f_1. Moreover, f is unique up to homotopy.*

Proof of Theorem 5.2. First we construct f_n by induction.

$$\begin{tikzcd}
P_{n+1} \arrow[r] & P_n \arrow[r] \arrow[d, "f_n"'] & P_{n-1} \arrow[d, "f_{n-1}"] & \cdots \\
P'_{n+1} \arrow[r] & P'_n \arrow[r] & P'_{n-1} & \cdots
\end{tikzcd}$$

Since the composition $P_{n+1} \to P_{n-1}$ is zero, there is an extention $P_{n+1} \to P'_{n+1}$ (using projectivity of P_{n+1}.

Now let's assume we have two such chain maps f and g. We construct the homotopy by induction.

$$\begin{tikzcd}
P_{n+1} \arrow[r] \arrow[d, "f_{n+1}-g_{n+1}"'] & P_n \arrow[r] \arrow[d, dashed, "f_n-g_n"'] & P_{n-1} \arrow[d, "f_{n-1}-g_{n-1}"] \arrow[dl, "h_{n-1}"'] & \cdots \\
P'_{n+1} \arrow[r] & P'_n \arrow[r] & P'_{n-1} & \cdots
\end{tikzcd}$$

Assume $d'h + hd = f - g$ so far. Consider the map $f - g - hd\colon P_* \to P'_*$. We see that $d'(f - g - hd) = (f - g)d - d'hd = (hd + d'h)d - d'hd = 0$, so we can use the mapping property to get h_n. ☐

6 ???

The first part of today's class, we'll calculate $H_n(G)$. We spent all this time identifying it with $H_n(X/G)$. You can actually compute H_0 and H_1 from the Bar resolution, but beyond that it gets messy.

We know $H_0(G) = H_0(X/G) = \mathbb{Z}$ since X/G is connected, and $H_1(G) = H_1(X/G) = \pi_1(X/G)^{ab} = G^{ab} = G/[G,G]$. These are the two obvious functors $\mathsf{Gp} \to \mathsf{Ab}$. The others are non-obvious.

So far, we know that $\widetilde{H}_n(\{e\}) = 0$ for all n since X is contractible. We also saw that $\widetilde{H}_n(\mathbb{Z}/2)$ is $\mathbb{Z}/2$ for n odd and 0 for n even.

Example 6.1 ($G = \mathbb{Z}/n$)**.** Consider the n-fold covering $S^1 \xrightarrow{n} S^1/G$, with the decomposition $S^1/G = e^0 \cup e^1$ downstairs and an n-gon upstairs. $\mathbb{Z}/n$ acts freely on S^1 by rotating. S^1 is not contractible, but we can still use it. The (cellular) chain complexes for S^1 and S^1/G are

$$\begin{tikzcd} \mathbb{Z}[G] & = & C_1(S^1) \arrow[r, "(t-1)"] \arrow[d] & C_0(S^1) \arrow[d] & = & \mathbb{Z}[G] \\ \mathbb{Z} & = & C_1(S^1/G) \arrow[r, "0"] & C_0(S^1/G) & = & \mathbb{Z} \end{tikzcd}$$

where t is the generator of $G = \mathbb{Z}/n$. Since we know the homology of S^1, the cokernel of $(t-1)$ is $\mathbb{Z}$ and the kernel is $\mathbb{Z}$. The generater of H^1 is the sum of all the 1-cells; the generator for H^0 is given by a single 0-cell. You could do this algebraically if you like, but we already know this from topology.

This chain complex doesn't compute the homology of G, it computes the homology of S^1. By definition, we have an exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Z}[G] \xrightarrow{(t-1)} \mathbb{Z}[G] \to \mathbb{Z} \to 0$$

This allows us to product an exact sequence

$$\begin{tikzcd} \cdots & \mathbb{Z}[G] \arrow[r, "t-1"] & \mathbb{Z}[G] \arrow[rr, "N"] \arrow[dr] & & \mathbb{Z}[G] \arrow[r, "t-1"] & \mathbb{Z}[G] \arrow[dr, "\varepsilon"] & \\ \mathbb{Z} \arrow[ur] & & & \mathbb{Z} \arrow[ur] & & & \mathbb{Z} \end{tikzcd}$$

The norm map N is given by $\lambda \mapsto N \cdot \lambda$, where $N = \sum_G g$ (for finite groups). It happens that this norm map factors as $\mathbb{Z}[G]_G \to \mathbb{Z}[G]^G$. Repeating, we get a free resolution of $\mathbb{Z}$.

Tensoring over $\mathbb{Z}[G]$ with $\mathbb{Z}$, all the $\mathbb{Z}[G]$'s turn into $\mathbb{Z}$'s, so we get the complex

$$\cdots \xrightarrow{N} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{N} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$$

where $N = |G|$. So we get

$$\widetilde{H}_k(\mathbb{Z}/n) = \begin{cases} \mathbb{Z}/n & k \text{ odd} \\ 0 & k \text{ even} \end{cases}$$

This complex (before tensoring) is the cellular complex of some contractible CW-complex on which G acts freely, by the way. I invite you to try to find it. ⋄

More generally, if a finite group G acts freely on S^{2k-1}, then

(a) $\widetilde{H}_{2k+n}(G) \cong \widetilde{H}_n(G)$ for all n, and

(b) $\widetilde{H}_{2k-1}(G) \cong \mathbb{Z}/|G|$.

We'll prove (b), but first I'll announce a theorem.

Theorem 6.2 (to be proven later)**.** *(b) implies (a)! Moreover, (b) is equivalent to the statement "every abelian subgroup of G is cyclic."*

(b) is equivalent to $|G|$ having periodic homology, and this is how you usually define "G has periodic homology" because otherwise the isomorphism isn't given.

What about the converse? If G has periodic homology, does it follow that it acts freely on a sphere? Let's first do some examples.

Example 6.3. Which groups G can act freely on S^{2k}? $\mathbb{Z}/2$ acts freely by the antipodal map and the trivial group acts trivially, and the claim is that is it. The Euler characteristic of S^{2k} is 2. If G acts freely, then the Euler characteristic of S^{2k}/G must be $2/|G|$, so $|G|$ must be 1 or 2. ⋄

If $f\colon S^d \to S^d$ is a homeomorphism with no fixed points, then it must be orientation preserving if d is odd and orientation reversing if d is even. That is, orientation-wise, f is like the antipodal map.

Proof. Use Lefschetz' fixed point theorem. The number of fixed points can be computed from the traces of the maps on homology. So $0 = L(f) := tr(f_0) + (-1)^d \operatorname{tr}(f_d)$, where f_i is the induced map on $H_i(S^d)$. Well, $H_0 = \mathbb{Z}$ and $tr(f_0) = 1$, and $tr(f_d) = (-1)^d \deg(f)$, so $\deg(f) = (-1)^{d+1}$. □

So if we are on an odd sphere, and there is a free group action, every group element preserves the orientation.

If G (finite) acts freely on S^1, we'll see that G must be a cyclic group. This is because $H_1(G) = G^{ab} = \mathbb{Z}/|G|$, [[★★★ and some size argument that says $G = G^{ab}$]].

What if G acts freely on S^3? By the way, the only spheres which are groups are S^1 and S^3. A Lie group always has a trivial tangent bundle, so only S^1, S^3, and S^7 are possible, but since the octonians are not associative, it turns out that S^7 cannot be a group. So $S^3 = SU(2) \subseteq \mathbb{H}^\times$. The quaternion group $Q_{4m} = \langle e^{\pi i/m}, j\rangle$ is an extension

$$1 \to \mathbb{Z}/2m \xrightarrow{gen \mapsto e^{\pi i/m}} Q_{4m} \xrightarrow{j \mapsto gen} \mathbb{Z}/2 \to 1$$

Note that $j^2 = -1 \in \mathbb{Z}/2m$, so this extension does not split. This is not a semi-direct product. The easiest quaternion group is $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$.

[[break]]

I got a couple of questions over the break. Why does a non-vanishing section of the tangent bundle TM imply that $\chi(M) = 0$? Integrating the non-vanishing vector field to flow for ε time, so you get a fixed-point-free automorphism of M (because the vector field is non-vanishing). Since the flow is homotopic to the identity and L is homotopy invariant, so $0 = L(F_\varepsilon) = L(F) = \chi(M)$.

Ok, back to subgroups of $SU(2)$. There is a double covering $SU(2) \to \mathbb{R}P^3 = SO(3)$, and $SO(3)$ has some nice finite subgroups, the symmetries of your favorite platonic solids, the tetrahedral group, octahedral group, and dodecahedral group. So you get these binary extentions, which turn out to be $SL_2(\mathbb{F}_3)$, [[★★★ something]], and $SL_2(\mathbb{F}_5)$. We have that $|SL_2(\mathbb{F}_p)| = |GL_2(\mathbb{F}_p)|/(p-1) = (p^2-1)(p^2-p)/(p-1) = (p^2-1)p$, so the orders for $p = 3, 5$ are 24 and 120.

It happens that in $SL_2(\mathbb{F}_p)$, every abelian subgroup is cyclic (I invite you to check this), which implies that it has periodic homology (we'll prove this). Moreover, $SL_2(\mathbb{F}_p)$ acts freely on the sphere, but no such *linear* action exists

If $G \subseteq O(2k)$ (in fact, $SO(2k)$ because it must be orientation preserving), then G acts freely on S^{2k-1}. The claim is that there is no embedding $SL_2(\mathbb{F}_p) \to SO(2k)$ for $p \geq 7$.

It is a deep theorem of Wall and Madison that periodic homology implies a free action on a sphere (which may be non-linear).

If you quotient S^3 by one of these binary groups, it is the same as $SO(3)$ modulo the corresponding platonic group.

Ok, let's go back and proof observation (b). We have the covering map $S^{2k-1} \to S^{2k-1}/G$, and $S^{2k-1}/G = M$ is a closed oriented manifold. Write down the chain complexes like before

$$\begin{tikzcd}
C_0(S^{2k-1}) \arrow[rr, "N"] \arrow[dr] & & C_{2k-1}(S^{2k-1}) \arrow[r] & \cdots \arrow[r] & C_0(S^{2k-1}) \arrow[dr] & \\
& \mathbb{Z} \arrow[ur] \arrow[d, "\cdot|G|"'] & C_{2k-1}(M) \arrow[r] & \cdots \arrow[r] & C_0(M) \arrow[dr] & \mathbb{Z} \\
& \mathbb{Z} \arrow[ur] & & & & \mathbb{Z}
\end{tikzcd}$$

As before, $C_n(S^{2k-1})$ are free $\mathbb{Z}[G]$-module, and as before, we splice many copies together to get a free resolution of $\mathbb{Z}$. Tensoring down and computing homology, we get something periodic with period $2k$. Moreover, since tensoring is right exact, the homology $H_{2k}(G) = 0$ (the homology at $C_0(S^{2k-1})$. We can also see that $H_{2k-1}(G) = \mathbb{Z}/|G|$. We know that the top complex is exact at $C_{2k-1}(S^{2k-1})$. Some diagram chase gives you the $H_{2k-1} = \mathbb{Z}/|G|$.

7 ???

The statement $\mathrm{Ext}^1(A,\mathbb{Z}) \cong \mathrm{Hom}(Z,\mathbb{Q}/\mathbb{Z})$ for torsion groups A is useful for thinking about Poincaré duality. If M is a closed oriented manifold of dimension n, you might remember that there is a nice isomorphism $H_{n-p}(M) \xrightarrow{\sim} H^p(M)$. The is one of the main features of manifolds. How do you think of this map geometrically? Use the universal coefficient theorem, which says that the following bottom sequence is exact.

$$\begin{tikzcd}
0 \arrow[r] & torsion\ H_{n-p}(M) \arrow[r] \arrow[d, "\psi"'] & H_{n-p}(M) \arrow[r] \arrow[d, "\wr"'] & H_{n-p}(M)/torsion \arrow[r] \arrow[d, "\phi"'] & 0 \\
0 \arrow[r] & \mathrm{Ext}^1(H_{p-1}(M),\mathbb{Z}) \arrow[r] \arrow[d, "HW"', "\sim"] & H^p(M) \arrow[r] & (\mathrm{Hom}(H_p(M),\mathbb{Z}) \arrow[r] & 0 \\
 & \mathrm{Hom}(tor\ H_{p-1}(M),\mathbb{Q}/\mathbb{Z}) & & &
\end{tikzcd}$$

This is a special case of the universal coefficient spectral sequence. For $\mathbb{Z}$, it collapses right away. The ϕ gives a bilinear pairing $H_{n-p}(M)/tor \times H_p(M)/tor \to \mathbb{Z}$, called *intersection pairing*. I think of homology of degree $n-p$ (for starters) as submanifolds of dimension $n-p$. So if I have submanifolds X^{n-p} and Y^p of dimension $n-p$ and p. Generically, they meet in a finite number of points transversely, which is the number given by the pairing. $\phi(X,Y)$ is exactly the number of intersections (this is an algebraic count, so you have to take signs into account).

ψ gives a bilinear pairing $tor\ H_{n-p}(M) \times tor\ (H_{p-1}(M) \to \mathbb{Q}/\mathbb{Z}$, called the *linking pairing*. How do we think of this geometrically? Take some X^{n-p} and Z^{p-1}. You can't take intersections (because they don't intersect generically). We need to use that these things are torsion. There is a k so that $kZ^{p-1} = \partial Y^p$. We define $\psi(X,Z) = \phi(X,Y)/k \in \mathbb{Q}/\mathbb{Z}$. The quotient by $\mathbb{Z}$ is needed to make the number well defined.

Ok, so that was motivation for the homework.

Definition 7.1. Let $M,N \in R\text{-}\mathsf{mod}$, with $P_* \to M$ a projective resolution, then $\mathrm{Ext}^p_R(M,N) := H^p\big(\mathrm{Hom}(P_*,N)\big)$. ⋄

As for Tor, we can resolve either M (by projectives) or N (by injectives). We'll formulate this precisely later. From the same lemma we used before, the result is independent of projective resolution.

Example 7.2. Let $R = \mathbb{Z}[G]$ and let $M = N = \mathbb{Z}$, then we get the group cohomology $H^p(G) := \mathrm{Ext}^p_{\mathbb{Z}G}(\mathbb{Z},\mathbb{Z}) \cong H^p(C^B_*(G)^*)$. ⋄

Notice that Ext is a functor in M and N. If you had a map $M \to M'$, you'd get an induced map on projective resolutions, then once you apply $\mathrm{Hom}(-,N)$, you get a chain map, which induces a map on homologies.

If $\alpha\colon G \to G'$ is a group homomorphism, then we get an induced map $\alpha_*\colon H_n(G) \to H_n(G')$. Remember that $H_n(G) \cong \mathrm{Tor}_n^{\mathbb{Z}G}(\mathbb{Z},\mathbb{Z})$ (the map is clear if you use the Bar complex). If $P_* \to \mathbb{Z}$ is a $\mathbb{Z}G$-projective resolution and $P'_* \to \mathbb{Z}$ is a $\mathbb{Z}G'$-projective resolution. Now we have resolutions by modules over different rings. α induces a ring homomorphism $\mathbb{Z}G \to \mathbb{Z}G'$, which makes any $\mathbb{Z}G'$-module into a $\mathbb{Z}G$-module, so $P'_* \to \mathbb{Z}$ is a resolution by $\mathbb{Z}G$-modules (it is not projective anymore, but it is still a resolution).

Lemma 7.3 (Projective to acyclic lemma)**.** *If $P_* \twoheadrightarrow M$ is a projective complex and $A_* \twoheadrightarrow N$ is an acyclic complex, then any map $f\colon M \to N$ has a lift $f\colon P_* \to A_*$ which is unique up to homotopy.*

We didn't formulate it this way before, but you'll see that the proof we gave before works. So now we know that $\mathrm{id}\colon \mathbb{Z} \to \mathbb{Z}$ induces a map $P_* \to P'_*$ which is unique up to homotopy. This induces the map $\mathrm{Tor}_n^{\mathbb{Z}G}(\mathbb{Z},\mathbb{Z}) \to \mathrm{Tor}_n^{\mathbb{Z}G'}(\mathbb{Z},\mathbb{Z})$.

Where does "chain homotopy" come from? There are two ways of motivating it, and they both lead somewhere.

(1) Recall that a homotopy in topology is a continuous map $X \times I \to Y$. I want to abstract this to get the concept of chian homotopy. Apply your favorite chain functor (mine is cellular chains). We assume some CW structures on X and Y, and we use the CW structure $I = [0,1] = 2e^0 \cup e^1$. If X and X' are CW complexes, then $X \times_{cg} X'$ is also a chain complex, and we have that $C_n(X \times X') = \bigoplus_{p+q=n} C_p(X) \otimes C_q(X')$. The tensor comes from the fact that the tensor product of free abelian groups is a free abelian group on pairs of basis elements. What is the differential? $d^{X\times X'}(c\otimes c') = d^X c \otimes c' + (-1)^{|c|} c \otimes d^{X'} c'$ (you can remember the signs by

drawing a picture of a rectangle). This leads us to the following definition.

Definition 7.4. If C_* and C'_* are two chain complexes, $C_* \otimes C'_* := \mathrm{Tot}^{\oplus}(C_p \otimes C'_q, d^h = d \otimes \mathrm{id}, d^v = (-1)^p \mathrm{id} \otimes d')$. ◇

Now we go back to our homotopy $X \times I \to Y$. Define $I_* = C_*(I) = (\mathbb{Z} \xrightarrow{(1,-1)} \mathbb{Z}^2)$.

Lemma 7.5. *A chain homotopy between maps $f, g\colon C_* \to D_*$ is the same thing as a chain map $C_* \otimes I_* \to D_*$ restricting to f and g and the two copies of C_*. $C_* \rightrightarrows C_* \otimes I_* \to D_*$.*

I'll leave the proof as an exercise. Note that $(C_* \otimes I_*)_n = (C_n \oplus C_n) \oplus C_{n-1}$. The map h to D_* is exactly $f \oplus g \oplus h$. The condition $dh + hd = f - g$ is exactly the condition that this is a chain map.

[[break]]

Corollary 7.6 (of HW2, prob. 2)**.** *The two ways of computing $\mathrm{Tor}_n^R(M,N)$ are isomorphic. In particular, $\mathrm{Tor}_n^R(M,N) \cong \mathrm{Tor}_n^R(N,M)$ when R is commutative.*

Proof. Let $M \leftarrow P_*$ and $N \leftarrow Q_*$ be projective resolutions. Then we get a double complex

$$\begin{tikzcd}
0 & M \otimes Q_1 \arrow[l] \arrow[d] & P_0 \otimes Q_1 \arrow[l] \arrow[d] & P_1 \otimes Q_1 \arrow[l] \arrow[d] \\
0 & M \otimes Q_0 \arrow[l] \arrow[d] & P_0 \otimes Q_0 \arrow[l] \arrow[d] & P_1 \otimes Q_0 \arrow[l] \arrow[d] \\
0 & M \otimes N \arrow[l] \arrow[d] & P_0 \otimes N \arrow[l] \arrow[d] & P_1 \otimes N \arrow[l] \arrow[d] \\
& 0 & 0 & 0
\end{tikzcd}$$

The claim is that there is a canonical isomorphism $\mathrm{Tor}_n(M,N) \cong H_n(P_* \otimes Q_*) = H_n\big(\mathrm{Tot}^{\oplus}(P_* \otimes Q_*)\big)$ (no matter which definition of Tor you use). If we delete the bottom row of the bicomplex, the rows are exact because the Q_i are projective and therefore flat. So by the homework, the total homology is zero. The homology of the bottom row is exactly the definition of Tor. But we have a short exact sequence of complexes, including the bottom row into the total complex, with quotient the total complex with the bottom row deleted. So we have $0 \to A_* \to B_* \to C_* \to 0$, with C_* acyclic. So we get a long exact sequence in homology, which shows that $A_* \to C_*$ is a quasi-isomorphism. Similarly, if you cut off the leftmost column, you get the other definition of Tor.

To show that $\mathrm{Tor}(M,N) \cong \mathrm{Tor}(N,M)$, just flip everything in the double complex. □

Now we can play the same game for Ext.

Corollary 7.7 (to the same HW prob)**.** *The two ways of computing $\mathrm{Ext}_R^n(M,N)$ are isomorphic.*

Proof. Let $N \to I_{-*}$ be an injective resolution and $P_* \to M$ be a projective resolution. We get a bicomplex

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & \\
\mathrm{Hom}(P_*, N) & \mathrm{Hom}(M, N) \arrow[l] \arrow[d] & 0 \arrow[l] \\
\mathrm{Hom}(P_*, I_*) \arrow[from=u] & \mathrm{Hom}(M, I_*) \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

Cutting off the top row, we get a bicomplex with exact rows (because the I_* are injective, so $\mathrm{Hom}(-, I_*)$ are exact). Make the same sort of short exact sequences. And do the same sort of thing for the rightmost column. □

If C and C' are two complexes, there is a chain complex $\underline{\mathrm{Hom}}(C,C')$, defined to be $\mathrm{Tot}^{\prod}(D_{*,*})$, where $D_{p,q} = \mathrm{Hom}(C_{-p}, C'_q)$, and $d(f)(c) = d'(f(c)) - (-1)^{|f|} f(dc)$, where $|f| = p + q$. So $d^v = d \circ -$, but $d^h = (-1)^? - \circ d$.

A chain homotopy is a 1-chain in $\underline{\mathrm{Hom}}(C,C')$. A chain map is the same as a 0-cycle.

8 ???

Recall the Hom complex. If C_* and C'_* are chain complexes, there is an inner Hom (which is a chain complex) $\underline{\mathrm{Hom}}_*(C_*, C'_*)$, defined as $\mathrm{Tot}^\Pi(D_{*,*})$, where $D_{m,n} = \mathrm{Hom}(C_{-m}, C'_n)$ with the differentials [[★★★ d^v and d^h]]. So $\underline{\mathrm{Hom}}_k(C_*, C'_*) = \prod_{m+n=k} \mathrm{Hom}(C_{-m}, C'_n) = \prod_n \mathrm{Hom}(C'_{n-k}, C'_n)$. That is, an element of $\underline{\mathrm{Hom}}_k$ is a collection of maps (f_n) which raise the degree by k (and need not behave well with the differential). The differential is $d^{\underline{\mathrm{Hom}}}(f) := d' \circ f - (-1)^{|f|} f \circ d$.

If (f_n) is a 0-cycle, then $f_n\colon C_n \to C'_n$ and $d^{\underline{\mathrm{Hom}}}(f) = d' \circ f - f \circ d = 0$. That is, the f_n form a chain map. So 0-cycles are exactly the chain maps from C_* to C'_*. A chian homotopy is just a 1-chain (h_n) in $\underline{\mathrm{Hom}}_1(C_*, C'_*)$. We see that $d^{\underline{\mathrm{Hom}}}(h) = d' \circ h + h \circ d$ which is by definition homotopic to zero.

This $\underline{\mathrm{Hom}}_*$ models the singular chain complex $S_*(C^0(X,Y))$ (where $C^0(X,Y)$ has the compactly generated compact open topology). A 0-cycle in $S_*(C^0(X,Y))$ is a continuous map $X \to Y$. A 1-chain is map from the 1-simplex to $C^0(X,Y)$, which is a homotopy between maps; $h\colon I \to C^0(X,Y)$ is equivalent to $h\colon X \times I \to Y$. The connection here is the adjunction formula

$$\mathrm{Hom}_{\mathsf{Chain}}(A_* \otimes B_*, C_*) \cong \mathrm{Hom}_{\mathsf{Chain}}(A_*, \underline{\mathrm{Hom}}_*(B_*, C_*))$$

This should be familiar for abelian groups. On the level of elements, in degree k we have $\mathrm{Hom}\big(\bigoplus_{m+n=k} A_m \otimes B_n, C_k\big)$. Using the usual adjunction, we see that this is the same as $\prod_{m+n=k} \mathrm{Hom}\big(A_m, \mathrm{Hom}(B_n, C_k)\big)$, which is the degree m part of $\mathrm{Hom}(A_*, \underline{\mathrm{Hom}}(B_*, C_*))$ (or part of it at least; you have to product over all k). [[★★★ clean this up later]]

The analogue in Top, the category of compactly generated topological spaces (a set is open if and only if it intersects each compact subset in an open subset). Top is a closed symmetric monoidal category. We have (I don't like to write Hom in non-linear categories)

$$\mathrm{Mor}_{\mathsf{Top}}(X \times_{cg} Y, Z) \cong \mathrm{Mor}_{\mathsf{Top}}(X, C^0(Y,Z)).$$

"Monoidal" means that we have the product $\times_{cg}$, "symmetric" means that this product is symmetric, and "closed" means that we have an *inner Mor* $C^0(-,-)$.

More generally, if C is a category, a monoidal structure is a functor $\otimes\colon \mathsf{C} \times \mathsf{C} \to \mathsf{C}$ with an associator natural transformation satisfying the pentagon axiom.[1] C is *symmetric* if in addition there is a natural isomorphism

$$\begin{tikzcd} \mathsf{C} \times \mathsf{C} \arrow[r, "\otimes"] \arrow[d, "\text{flip}"'] \arrow[dr, phantom, "\Downarrow c"] & \mathsf{C} \\ \mathsf{C} \times \mathsf{C} \arrow[ur, "\otimes"'] & \end{tikzcd}$$

such that (1) $c^2 = \mathrm{id}$ and (2) S_n acts on $X_1 \otimes \cdots \otimes X_n$.

Example 8.1. If C has a product, the product gives an example of a monoidal structure. ◇

A monoidal category $(\mathsf{C}, \otimes)$ is *closed* if there exist inner Mor objects $\underline{\mathrm{Mor}}(Y,Z)$ such that there is a natural isomorphism

$$\mathrm{Mor}_{\mathsf{C}}(X \otimes Y, Z) \cong \mathrm{Mor}_{\mathsf{C}}(X, \underline{\mathrm{Mor}}(Y,Z)).$$

That is, C is closed if $- \otimes Y$ has a right adjoint $\underline{\mathrm{Mor}}(Y,-)$ for all objects $Y \in \mathsf{C}$.

[[break]]

I haven't been careful about the ring we're working over. The important adjunction is, for a fixed $N \in R\text{-}\mathsf{mod}$

$$\begin{tikzcd} \mathsf{mod}\text{-}R \arrow[r, bend left, "- \otimes_R N"] & \mathsf{Ab} \arrow[l, bend left, "\mathrm{Hom}_{\mathsf{Ab}}(N,-)_R"] \end{tikzcd} \qquad \mathrm{Hom}_{\mathsf{Ab}}(M \otimes_R N, A) \cong \mathrm{Hom}_R(M, \mathrm{Hom}_{\mathsf{Ab}}(N, A)).$$

That is, there is a natural isomorphism as on the right. More generally, $F\colon \mathsf{C} \to \mathsf{D}$ and $G\colon \mathsf{D} \to \mathsf{C}$ are adjoint if there is a natural isomorphism

$$\mathrm{Mor}_{\mathsf{D}}(Fc, d) \cong \mathrm{Mor}_{\mathsf{C}}(c, Gd).$$

Adjoint functors are really nice. For example, right adjoint functors preserve limits and left adjoint functors preserve colimits. This actually

[1] You can think of this as just saying "$\otimes$ is associative"; there is a theorem of Mac Lane which says there is always an equivalent strictly associative monoidal category.

implies that if F is left adjoint (as above), it preserves cokernels, so it is right exact. And if G is right adjoint, it is left exact.

In particular, $\mathrm{Tor}_0^R(M,N) = M \otimes_R N$. To see this, take a resolution $P_* \to M \to 0$, then tensor with N and take homology. But since $- \otimes N$ is right exact, $P_0 \otimes N/\operatorname{im}(P_1 \otimes N) \cong M \otimes N$. Tor_n^R are called the *derived functors* of $\otimes$. The higher Tor measure the failure of $\otimes$ to be exact.

Lemma 8.2. *If $0 \to M \to M' \to M'' \to 0$ is exact, then there is a long exact sequence*

$$\mathrm{Tor}_{n+1}(M,N) \to \mathrm{Tor}_n(M,N) \to \mathrm{Tor}_n(M',N) \to \mathrm{Tor}_n(M'',N) \to \cdots$$
$$\cdots \to \mathrm{Tor}_1^R(M,N) \to M \otimes N \to M' \otimes N \to M'' \otimes N \to 0$$

Proof. Pick projective resolutions $P_* \to M$ and $P''_* \to M''$. Then check that $P_* \oplus P''_*$ gives a projective resolution of M'.

$$\begin{tikzcd}
0 \arrow[r] & P_1 \arrow[r] \arrow[d] & P_1 \oplus P''_1 \arrow[r] \arrow[d, dashed] & P''_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_0 \arrow[r] \arrow[d] & P_0 \oplus P''_0 \arrow[r] \arrow[d, dashed] & P''_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r] & M' \arrow[r] & M'' \arrow[r] & 0
\end{tikzcd}$$

So we get a short exact sequence of chain complexes. When we tensor with N, we still have a short exact sequence of complexes because the direct product sequence splits. So we get a long exact sequence in homology. □

Similarly, we get a long exact sequence for the other variable. We can also get long exact sequences in Ext, but you have to be careful about which way the arrows go.

Next I wanted to show you a nice application of our double complex technique. I want to show that Čech cohomology is isomorphic to de Rham cohomology.

Recall de Rham cohomology. Given a smooth manifold M, we can define $\Omega^k(M)$ as the sections of $\bigwedge^k T^*M$, the global k-forms on M. The 0-forms are sections of the trivial bundle $\mathbb{R} \times M$, so they are just smooth functions. In local coordinates $x_1, \dots, x_n \colon M \to \mathbb{R}$, a basis (over $C^\infty(M)$) of $\Omega^1(M)$ is $dx_1, \dots, dx_n$. A basis for $\Omega^k(M)$ is given by $\{dx_{i_1} \wedge \cdots \wedge dx_{i_k}\}_{i_1<\cdots<i_k}$. So locally, Ω^k is a free module, but not globally. We have the usual differetial $d \colon \Omega^k(M) \to \Omega^{k+1}(M)$. The cool thing about the de Rham d is that if we define it on functions, it extends uniquely. For $f \in \Omega^0(M)$, $df = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i \in \Omega^1(M)$. There is a cool formula for extendining this to a derivation (something so that $d(a \wedge b) = d(a) \wedge b + (-1)^{|a|} a \wedge d(b)$), but it escapes me right now.

Recall Čech cohomology. If X is a topological space with an open covering $\mathcal{U} = \{U_i\}$ and a presheaf of abelian groups $\mathcal{F}$ on X (i.e. a functor $\mathit{Open}(X)^\circ \to \mathsf{Ab}$, with $\mathcal{F}(\varnothing) = 0$ [[★★★ why do we do this?]]). For example, $\mathcal{F}$ could be the constant sheaf; the sections over an open set U are just continuous maps $U \to A$ (we could take A to be a topological abelian group). Or $\mathcal{F}(U)$ could be $\Omega^k(U)$. We define the Čech cohomlogy $\check{H}^k(\mathcal{U}, \mathcal{F})$ as the k-th cohomology of the following cochain complex. $\check{C}^i(\mathcal{U}, \mathcal{F}) = \prod_{(j_1,\dots,j_i)} \mathcal{F}(U_{j_1} \cap \cdots \cap U_{j_i})$, with boundary maps $\sum (-1)^k d_k$, where d_k are the restriction maps.

Now we've defined the two sides. Now the claim is that $\check{H}(X, \mathbb{R}_\delta) \cong \Omega^k(X)$ (the δ means discrete topology). Next times, we'll do this proof.

9 Lecture 9

Today I'll give a quick review of sheaves and a quick review of limits and colimits, but first let me finish something from last time.

Theorem 9.1. *If X is a smooth manifold and $\mathcal{U} = \{U_i\}_{i\in I}$ is a good cover (all $U_{i_0\dots i_k} := U_{i_0}\cap\cdots\cap U_{i_k}$ are contractible), then $H^k_{dR}(X) \cong \check{H}^k(\mathcal{U}, \mathbb{R}_\delta)$.*

Recall that for a presheaf $\mathcal{F}\colon \mathit{Open}(X)^\circ \to \mathsf{Ab}$, we defined Čech cohomology. It is the cohomology of

$$\begin{tikzcd} \check{C}^0(\mathcal{U};\mathcal{F}) \arrow[r] \arrow[d, equal] & \check{C}^1(\mathcal{U};\mathcal{F}) \arrow[r] \arrow[d, equal] & \check{C}^0(\mathcal{U};\mathcal{F}) \cdots \arrow[d, equal] \\ \prod \mathcal{F}(U_i) \arrow[r, "\delta"] & \prod \mathcal{F}(U_{ij}) \arrow[r, "\delta"] & \prod \mathcal{F}(U_i) \end{tikzcd}$$

where δ is the alternating sum of the restriction maps.

Remark 9.2. If $\mathcal{F}$ is a sheaf, then $\check{H}^0(\mathcal{U};\mathcal{F})$ is exactly global sections of $\mathcal{F}$. This is the sheaf axiom:

$$\mathcal{F}(X) \to \prod \mathcal{F}(U_i) \rightrightarrows \prod \mathcal{F}(U_{ij})$$

In fact, it is clear that $\mathcal{F}$ is a sheaf if and only if $\check{H}^0(\mathcal{U};\mathcal{F}) = \mathcal{F}(X)$ for all $\mathcal{U}$. You can use this as the definition of a sheaf. Actually, you need to say that $\check{H}^0(\mathcal{U};\mathcal{F}) \cong \mathcal{F}(U)$ for any open subset U and any open cover $\mathcal{U}$ of U.

Another way to write the sheaf axiom is that the following is an equalizer

$$\mathcal{F}\big(\bigcup U_i\big) \to \prod_i \mathcal{F}(U_i) \rightrightarrows \prod \mathcal{F}(U_{ij})$$

$\diamond$

There always exists a good cover because you can choose a Riemannian metric and take convex neighborhoods. For general spaces, there need not be a good cover; you have to take a limit over all covers to define Čech cohomology.

Example 9.3. You could take $\mathcal{F}(U) = C^0(U;A)$, where A is a topological abelian group. In the theorem, we're taking $A = \mathbb{R}$ with the discrete topology. More generally, if $\pi\colon P \to X$ has abelian groups as fibers, then you could take $\mathcal{F}(U) = \Gamma\big(\pi^{-1}(U) \to U\big)$. It is clear that this $\mathcal{F}$ is always a sheaf.

If you take $P = \bigwedge^q T^*X$, where T^*X is the cotangent bundle, then $\mathcal{F}(U) = \Omega^q(U)$. $\diamond$

Proof. We'll draw a big double complex and argue like before.

$$\begin{tikzcd} & & & \\ \Omega^2(X) \arrow[r] \arrow[u] & \prod_i \Omega^2(U_i) \arrow[r] \arrow[u] & \check{C}^1(\mathcal{U};\Omega^2) \arrow[r] \arrow[u] & \cdots \\ \Omega^1(X) \arrow[r] \arrow[u] & \prod_i \Omega^1(U_i) \arrow[r] \arrow[u] & \check{C}^1(\mathcal{U};\Omega^1) \arrow[r] \arrow[u] & \cdots \\ \Omega^0(X) \arrow[r] \arrow[u] & \prod_i \Omega^0(U_i) \arrow[r, "\delta"] \arrow[u] & \check{C}^1(\mathcal{U};\Omega^0) \arrow[r] \arrow[u] & \cdots \\ & \prod C^0(U_i;\mathbb{R}_\delta) \arrow[r] \arrow[u] \arrow[d, equal] & \prod C^0(U_{ij};\mathbb{R}_\delta) \arrow[r] \arrow[u] \arrow[d, equal] & \cdots \\ & \check{C}^0(\mathcal{U};\mathbb{R}_\delta) & \check{C}^1(\mathcal{U};\mathbb{R}_\delta) & \end{tikzcd}$$

Vertically, we compute de Rham cohomology. Horizontally, we compute Čech cohomology.

Claim: all the rows except the bottom are exact. That is, $\check{H}^k(\mathcal{U};\Omega^q)$ is $\Omega^q(X)$ for $k = 0$ and vanishes for $k > 0$. This follows from the lemma below (and that $\mathcal{U}$ is a good cover) once we prove that Ω^q is soft. To see that Ω^q is soft, use partitions of unity! So the rows are exact by the lemma.

I claim that all the columns except the first are exact. Exactness at the bottom is clear because functions whose derivatives are zero are exactly the locally constant functions. The columns are exact because the de Rham cohomology of a contractible set vanishes.

$\check{H}^k(\mathcal{U}; \mathbb{R}_\delta)$ is the k-th cohomology of the bottom row, and also the k-th cohomology of the total complex, which is also the k-th cohomology of the first column, which is $H^k_{dR}(X)$. □

Definition 9.4. A sheaf $\mathcal{F}$ is *soft* if the restriction map $\mathcal{F}(X) \to \mathcal{F}(A)$ is surjective for all closed subsets A, where $\mathcal{F}(A)$ is defined as the direct limit (the colimit) of all $\mathcal{F}(U)$ for $U \supseteq A$. If you take A to be a point, $\mathcal{F}(A)$ is called the *stalk* of $\mathcal{F}$ at A. ◇

Lemma 9.5. *If $\mathcal{F}$ is a soft sheaf and X has partitions of unity, then $\check{H}^k(X, \mathcal{F}) = 0$ for all $k > 0$.*

[[break]]

There is a forgetful functor $\mathsf{Sh}(X) \to \mathsf{PreSh}(X) = \mathsf{Fun}(Open(X)^\circ, \mathsf{C})$, where C is some category with products. This forgetful functor has a left adjoint, called *sheafification*. There are two approaches. One is to do a limit process which forces the sheaf axiom to work. The other way is a little more concrete, which is what we'll do. Let's assume for now that $\mathsf{C} = \mathsf{Set}$.

If $\mathcal{F}$ is a presheaf on X, we define a topological space $F = \coprod_{x \in X} \mathcal{F}_x$, where $\mathcal{F}_x$ is the stalk at x. We have a projection map $\pi \colon F \to X$. For an open set $U \subseteq X$ and $s \in \mathcal{F}(U)$, we get a section $\tilde{s} \colon U \to \pi^{-1}(U)$, given by taking $\tilde{s}(x)$ to be the stalk of s at x. Define the topology on F to be the finest topology so that all such $\tilde{s}$ are continuous. Now define $\mathcal{F}^+(U)$ to be sections of $\pi|_U$. We saw in the example that this must be a sheaf, and we see that we have a map $\mathcal{F} \to \mathcal{F}^+$.

Lemma 9.6. *π is continuous. In fact, it is a local homeomorphism.*

Lemma 9.7. *$\mathcal{F}$ is a sheaf if and only if the canonical map $\mathcal{F} \to \mathcal{F}^+$ is an isomorphism.*

Lemma 9.8. *$\mathcal{F} \mapsto \mathcal{F}^+$ is left adjoint to the forgetful functor:* $\mathrm{Mor}_{\mathsf{Sh}(X)}(\mathcal{F}^+, \mathcal{G}) \cong \mathrm{Mor}_{\mathsf{PreSh}(X)}(\mathcal{F}, \mathcal{G}_{forget})$.

For this construction to work for $\mathsf{C} \neq \mathsf{Set}$, you need some more work. You at least need a functor $\mathsf{C} \to \mathsf{Set}$ which probably needs to have a left adjoint.

limits

Given a small category I (the collection of objects is a set) and a functor $D \colon I \to \mathsf{C}$ for some category C (called D because this is an "I-diagram in C"), we define the *limit* $L = \lim_I D \in \mathsf{C}$ to be an object in C with morphisms to all the things in the diagram so that all triangles commute, and L is terminal with respect to this property. This limit may not exist. If C is something nice like Set or Ab, then all limits exist.

A nice way to see this to define a new category, the *left cone of I*, $I^\triangleleft$ which has one more object than I (call the extra object ∞) and there is exactly one morphism from ∞ to each object. A limit is a terminal extention of your orginal functor D to $\tilde{D} \colon I^\triangleleft \to \mathsf{C}$.

10 Lecture 10

We were in the middle of doing limits and colimits. We have an index category I (which is usually small) and a category C. We have a diagram in C, given by a functor $D\colon I \to \mathsf{C}$. So for each object $i \in I$, we have an object $D(i) \in \mathsf{C}$ and for each morphism in I, you get a morphism in C so that some things commute (not everything needs to commute).

$\lim_I D$ (if it exists) has maps to each $D(i)$ making all the triangles involving two of these maps commute, and it is final with respect to this property.

A sneaky way to define $\operatorname{colim}_I D$ is $\lim_{I^\circ} D^\circ$. More concretely, $\operatorname{colim}_I D$ (if it exists) has maps from each $D(i)$ making all the triangles involving two of these maps commute, and it is initial with respect to this property.

Example 10.1 (products and coproducts)**.** If $\operatorname{Mor} I$ consists of identities (i.e. I is just a set), then $\lim_I D = \prod_{i\in I} D(i)$ and $\operatorname{colim}_I D = \coprod_{i\in I} D(i)$. ◇

Example 10.2 (inverse and direct limits)**.** Let I be a partially ordered set (with objects in the partially ordered set and $|\operatorname{Mor}(i,j)| = 1$ if $i \le j$ and 0 otherwise). For example, $I = \mathbb{Z}$. Say $D\colon \mathbb{Z} \to \mathsf{Set}$ with $D(i) \hookrightarrow D(j)$ (in general, these need not be injections), then $\lim_I D = \bigcap D(i)$ and $\operatorname{colim}_I D = \bigcup D(i)$. ◇

Example 10.3 (equalizers and coequalizers)**.** If I is the category with two objects 1 and 2, with $\operatorname{Mor}(1,2) = \{f,g\}$ and no other non-identity morphisms. A diagram $D\colon I \to \mathsf{C}$ is just a pair of objects $D(1)$ and $D(2)$ with two morphisms $D(1) \rightrightarrows D(2)$ between them (with no restrictions). The limit of this diagram is exactly the equalizer of the two maps and the colimit is exactly the coequalizer. ◇

Example 10.4 (final and initial objects)**.** If I is the empty category, with $D\colon \varnothing \to \mathsf{C}$, then $\lim_I D$ is the final object in C and $\operatorname{colim}_I D$ is the initial object. ◇

Our original diagram is $D \in \mathsf{Fun}(I, \mathsf{C})$ and we have the restriction functor $\mathsf{Fun}(I^\triangleright, \mathsf{C}) \to \mathsf{Fun}(I, \mathsf{C})$, where $I^\triangleright$ is the right cone of I. The colimit $\operatorname{colim}_I D$ is given by some object $D^\triangleright$ lying over D under the forgetful functor. The colimit is given by $D^\triangleright(\infty)$. $D^\triangleright$ is the initial object in the preimage of D (and its identity morphism) in $\mathsf{Fun}(I^\triangleright, \mathsf{C})$ (this is the subcategory of $\mathsf{Fun}(I^\triangleright, \mathsf{C})$ whose objects map to D and whose morphisms map to id_D under the fogetful functor). You can make a similar analysis for limits.

Example 10.5 (pullbacks and pushouts)**.** If $I = (\cdot \to \cdot \leftarrow \cdot)$, then the limit of the diagram $X \xrightarrow{f} Z \xleftarrow{g} Y$ is the pullback $X \times_Z Y$ and the colimit is just Z. If $I = (\cdot \leftarrow \cdot \to \cdot)$, then the limit of the diagram $X \xleftarrow{f} Z \xrightarrow{g} Y$ is Z and the colimit is the pushout. The usual notation is

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\ulcorner"] & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & Z \end{tikzcd} \qquad \begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d] \\ X \arrow[r] & P \arrow[ul, phantom, very near start, "\lrcorner"] \end{tikzcd}$$

◇

Lemma 10.6. *All limits and colimits exist in* $\mathsf{C} = \mathsf{Set}$. *More generally, if products and equalizers exist in* C*, then all limits exist. Similarly, if all coproducts and coequalizers exist, then so do all colimits.*

Proof. $\lim_I D = Eq\big(\prod_{i\in Ob(I)} D(i) \rightrightarrows \prod_{\phi\in \operatorname{Mor} I} D(t(\phi))\big)$, where the two maps are given by the identity $(D(t(\phi)) \xrightarrow{\mathrm{id}} D(t(\phi)))$ and the other arrow is given by composing with $D(\phi)$. □

[[break]]

Definition 10.7. Given a functor $F\colon \mathsf{A} \to \mathsf{B}$ and an object $B_0 \in B$. The fiber category $F^{-1}(B_0)$ or $(F \downarrow B_0)$ has objects pairs $(A \in \mathsf{A}, \phi\colon F(A) \to B_0)$ and the morphisms $(A, \phi) \to (A', \phi')$ are maps $\alpha\colon A \to A'$ such that $\phi = \phi' \circ F(\alpha)$. ◇

This is like constructing a homotopy fiber of a map, where the morphisms are paths.

If we use the constant functor $\Delta\colon \mathsf{C} \to \mathsf{Fun}(I, \mathsf{C})$ (Δ_X sends all objects to X and all morphisms to id_X), then the fiber of the forgetful functor we talked about before is just $(\Delta \downarrow D)$. The limit of D is the terminal object in $(\Delta \downarrow D)$. Similarly, the colimit of D is the initial object in $(D \downarrow \Delta)$.

There is an adjunction formula which is good to know:

$$\mathrm{Mor}_{\mathsf{C}}(\operatorname{colim}_I D, C) = \mathrm{Mor}_{\mathsf{Fun}(I,\mathsf{C})}(D, \Delta_C).$$

We saw that limits and colimits exist in Set. They also all exist in Ab by the same argument. What is the product in Gp? It is the usual product. What is the coproduct? It is the free product. How about Top? Products are usual products (with the product topology), and coproducts are disjoint unions. Equalizers and coequalizers are the same as in Set, with the topologies you'd expect.

Given a category C, define the category of *presheaves on* C, $\mathsf{PreSh}(\mathsf{C}) = \mathsf{Fun}(\mathsf{C}^\circ, \mathsf{Set})$. There is something called the *Yoneda embeding* $r\colon \mathsf{C} \to \mathsf{PreSh}(\mathsf{C})$ given by $r_Y(X) = \mathsf{C}(X,Y) = \mathrm{Mor}_{\mathsf{C}}(X,Y)$. Functors isomorphic to some r_Y. It turns out that r is fully faithful. That is, $\mathsf{C}(X,Y) \cong \mathrm{Mor}_{\mathsf{PreSh}(C)}(r_X, r_Y)$.

Theorem 10.8. $\mathsf{PreSh}(\mathsf{C})$ *is* cocomplete *(has all colimits) and any presheaf can be canonically written as a colimit of representable functors* $F \cong \operatorname{colim}_{(r\downarrow F)} D$ *where* D *is the diagram* $(r \downarrow F) \to \mathsf{C}$.

11 Extensions (of modules, rings, groups)

Extensions are usually done with explicit cocylces in the Bar resolution. This doesn't make for a good lecture, so we'll do that in the homework.

Theorem 11.1. *Let R be a ring, and let $M, N \in R\text{-}\mathsf{mod}$. Then there is a natural bijection*[1] $\operatorname{Ext}^1_R(M,N) \cong \{$*extensions in* R-mod *of* M *by* $N\}/\sim$. *An extension is a short exact sequence*

$$0 \to N \to E \to M \to 0$$

and two extensions E and E' are equivalent if there is an isomorphism which induces the identity on the subobject N and the quotient M:

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d, "\wr"] & M \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & N \arrow[r] & E' \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

Example 11.2. $\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/p, \mathbb{Z}) \cong \mathbb{Z}/p$. In this case, the exentions are given by the trivial extension $\mathbb{Z} \times \mathbb{Z}/p$ and the extentions $0 \to \mathbb{Z} \xrightarrow{p} \mathbb{Z} \xrightarrow{1 \mapsto k} \mathbb{Z}/p \to 0$, where $k \in (\mathbb{Z}/p)^\times$.

Note that if we have the map $\mathbb{Z}/p \xrightarrow{k} \mathbb{Z}/p$, you can "pull the extension back" along this map. We can get all the extensions from the obvious extension $0 \to \mathbb{Z} \xrightarrow{p} \mathbb{Z} \to \mathbb{Z}/p \to 0$ by pulling back along such maps (including $k = 0$). ⋄

More generally, if we have a map $N \to N'$, we can push out the extension, and if we have a map $M' \to M$, we can pull it back.

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near end] & E \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & N' \arrow[r] & P \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

Since we have the map $E \to M$ and $N' \xrightarrow{0} M$, we get a map $P \to M$. A diagram chase shows that you get a short exact sequence.

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r] \arrow[d, equal] & Q \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & M' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & N \arrow[r] & E \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

Example 11.3. Let G be a group. Then $\operatorname{Ext}^1_{\mathbb{Z}G}(\mathbb{Z},\mathbb{Z}) =: H^1(G;\mathbb{Z})$. By the universal coefficient theorem [[★★★ see lecture 25 or so]], this is $\operatorname{Hom}_{\mathbb{Z}\text{-}\mathsf{mod}}(H_1(G),\mathbb{Z}) \cong \operatorname{Hom}_{\mathsf{Gp}}(G,\mathbb{Z})$. Starting with such a homomorphism ϕ, we can try to try to get an extension of $\mathbb{Z}$ by $\mathbb{Z}$ as a push out along $H_1 G \xrightarrow{\phi} \mathbb{Z}$.

$$\begin{tikzcd} 0 \arrow[r] & H_1 G \arrow[r] \arrow[d, "\phi"'] \arrow[dr, phantom, "\lrcorner", very near end] & ? \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathbb{Z} \arrow[r] & E \arrow[r] & \mathbb{Z} \arrow[r] & 0 \end{tikzcd}$$

It turns out that there is a way to fill in the question mark so that any E is obtained in as a pushout in this way.

$$0 \to I_G/I_G^2 \to \mathbb{Z}G/I_G^2 \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

It turns out that $I_G/I_G^2 \cong H_1(G) \cong G^{ab}$. This does the trick. [[★★★ how to see this?]] ⋄

Theorem 11.4. $\operatorname{Ext}^n_R(M,N)$ *is naturally in bijection with length n extensions of M by N up to equivalence. A length n extension is an exact sequence E_**

$$0 \to N \to E_n \to E_{n-1} \to \cdots \to E_1 \to M \to 0.$$

Two extensions E_* and E'_* are equivalent when there is a chain map (need not be an isomorphism) $E_* \to E'_*$ restricting to equality on M and N. The whole equivalence relation is generated by these.

[1] In fact, it is an isomorphism of abelian groups. The zero element is the product. You can try figuring out how to "add" two extensions to get another extension.

Again, you can push out and pull back extensions, so the naturality makes sense. When you push out along $N \to N'$, you can either push out all the E_i, or just E_n (there is a map between the two, so they are equivalent). Similarly, when pulling back along $M' \to M$, you can either pull back all the E_i, or just E_1.

Let K_i be the kernel of $E_i \to E_{i-1}$ (so $N = K_n$ and $M = K_0$). These break up the long exact sequence into a bunch of short exact sequences $0 \to K_i \to E_i \to K_{i-1} \to 0$ as usual (with $1 \le i \le n$).

Proof of Theorem 11.4. Recall dimension shift for Ext. We can think of E_* as giving a map $\mathrm{Ext}^k_R(A, M) \to \mathrm{Ext}^{k+n}_R(A, N)$. To do this, use the connecting maps $\mathrm{Ext}^{k+i}(A, K_i) \to \mathrm{Ext}^{k+i+1}(A, K_{i+1})$ in the long exact sequences for all the short exact pieces making up E_*.

In particular, we can look at the image of $\mathrm{id}_M \in \mathrm{Ext}^0(M, M)$ in $\mathrm{Ext}^n(M, N)$. This is the map from extensions of length n to Ext^n. Call this map ϕ. Note that ϕ is well defined by the naturality of the connecting homomorphisms in the long exact sequences.

Now we'll write down the inverse of ϕ. Start with a class $[\alpha] \in \mathrm{Ext}^n_R(M, N)$, so $\alpha \in \mathrm{Hom}(P_n, N)$, where $(\cdots \to P_1 \to P_0 \to M)$ is a projective resolution, and α induces the zero map in $\mathrm{Hom}(P_{n+1}, N)$ (under composition with $P_{n+1} \to P_n$). It follows that α factors through K_{n-1}, the cokernel of $P_{n+1} \to P_n$ (or the kernel of $P_n \to P_{n-1}$). But then we have an extension of length n

$$0 \to K_{n-1} \to P_{n-1} \to \cdots \to P_0 \to M \to 0$$

Pushing out along $\overline{\alpha}\colon K_{n-1} \to N$, you get an extension of M by N. You have to check that this extension is independent of cohomology class. To see that this procedure is onto, use something like the projective to acyclic lemma. You can go home and check that these two maps are natural and inverse. □

[[break]]

Remark 11.5. Splicing extensions gives the *Yoneda product*

$$\mathrm{Ext}^n_{\mathsf{A}}(M, N) \times \mathrm{Ext}^m_{\mathsf{A}}(N, K) \to \mathrm{Ext}^{m+n}_{\mathsf{A}}(M, K).$$

Where A is an abelian category. This turns into the composition in the derived category via the isomorphism $\mathrm{Ext}^n_{\mathsf{A}}(M, N) \cong \mathrm{Hom}^n_{\mathsf{D(A)}}(M, N)$. ⋄

(square zero) Extensions of rings

Let $\alpha\colon P \to R$ be a surjective ring homomorphism such that the kernel $J \subseteq P$ is a square zero ideal.

Lemma 11.6. *J is an (R, R)-bimodule.*

Proof. J is a (P, P)-bimodule because it is an ideal in P, and J acts on itself by zero since $J^2 = 0$, so it is a $(P/J, P/J)$-bimodule. □

Definition 11.7. Given a ring R and an (R, R)-bimodule J, we define $\mathrm{Ext}_{\mathsf{Ring}}(R; J)$ to be the square zero extensions of R by J up to equivalence. ⋄

Lemma 11.8. *Given an (R, R)-bimodule I and a derivation*[2] $d\colon R \to I$, *there is an induced map* $\tilde{d}\colon \mathrm{Ext}^1_{(R,R)\text{-}\mathsf{mod}}(I, J) \to \mathrm{Ext}_{\mathsf{Ring}}(R; J)$.

Remark 11.9. (R, R)-bimodules are the same thing as $R \otimes R^{op}$-modules, we Ext makes sense. [[★★★ We can do Ext in any abelian category.]] ⋄

Proof. Take the pullback of an extension $0 \to J \xrightarrow{i} E \xrightarrow{\pi} I \to 0$ along the derivation $d\colon R \to I$ to get some short exact sequence of abelian groups

$$0 \to J \to P \to R \to 0.$$

The claim is that this is a square zero extension. To see this, we have to define the ring structure on P. Remember that $P = \{(e, r) | \pi(e) = d(r)\}$. The product structure is given by $(e, r) \cdot (e', r') = (e \cdot r' + r \cdot e', r \cdot r')$. I invite you to check that this still in P (this uses the derivation property). The unit is $(0, 1_R)$.

J is the set of elements of the form $(i(j), 0)$, so it is clear that $J^2 = 0$ in P. □

Remark 11.10. We had to go to bimodules because that is where derivations are naturally defined. Observe that if R is a k-algebra (where k is a commutative ring) we have an augmentation map of k-algebras $\varepsilon\colon R \to k$, then every left R-module can be turned into a bimodule via $m \cdot r = \varepsilon(r) \cdot m$. ⋄

[2] $d(rs) = d(r)s + rd(s)$.

Theorem 11.11. *Let $\varepsilon\colon R \to k$ be an augmented k-algebra, with $I_r = \ker\varepsilon$ (thought of as an (R,R)-bimodule* via *ε; well, for this theorem, we don't need the bimodule structure). For any R-module J (turned into a bimodule via ε), there is an isomorphism $\mathrm{Ext}^1_R(I_R, J) \cong \mathrm{Ext}_{\mathsf{Ring}}(R; J)$.*

I'll prove this next time. I'll show you the map now. The idea is to pull back along some interesting derivation $R \to I_R$, given by $d(r) = r - \varepsilon(r)$. We'll check that this is a derivation.

Theorem 11.12. $\mathrm{Ext}_{\mathsf{Ring}}(\mathbb{Z}G; J)$ *is in bijection with group extensions of G by J. This is $\mathrm{Ext}^1_{\mathbb{Z}G}(I_G; J) = \mathrm{Ext}^2_{\mathbb{Z}G}(\mathbb{Z}, J) \cong H^2(G; J)$.*

12 More on extensions

You could call the following the "main extension theorem".

Let J be a left G-module. Let J_ε be the corresponding $(\mathbb{Z}G, \mathbb{Z}G)$-bimodule. Group extensions of G by J correspond do square zero extensions of $\mathbb{Z}G$ by J_ε (Theorem 3),[1] which correspond to left $\mathbb{Z}G$-module extensions of I_G by J (Theorem 2) (by Theorem 1, this is $\mathrm{Ext}^1_{\mathbb{Z}G}(I_G, J)$), which correspond to $\mathbb{Z}G$-module extensions of length 2 of $\mathbb{Z}$ by J (splicing with $I_G \to \mathbb{Z}G \to \mathbb{Z} \to 0$) (by Theorem 1, this is $\mathrm{Ext}^2(\mathbb{Z}, J)$). Let $\delta\colon \mathrm{Ext}^1_{\mathbb{Z}G}(I_G, J) \xrightarrow{\sim} \mathrm{Ext}^2(\mathbb{Z}, J)$ be the induced map (this comes from the long exact sequence from $I_G \to \mathbb{Z}G \to \mathbb{Z}$ and uses that $\mathbb{Z}G$ is a free $\mathbb{Z}G$-module). We talked about Theorems 1 and 2 last time [[★★★ figure out which ones they are]]. We didn't prove theorem 2 yet. Finally, we define $H_2(G; J)$ to be $\mathrm{Ext}^2_{\mathbb{Z}G}(\mathbb{Z}, J)$. Putting it all together, group extensions of G by J are in bijection with $H^2(G; J)$.

J is abelian, so if we have an extension of groups

$$1 \to J \to Q \to G \to 1$$

J gets a G-action (because the conjugation action of Q on J factors through G, since J is abelian). When we say "group extensions of G by J," we are fixing J as a G-module, not just as an abelian group.

If I have a ring extension $0 \to J \to P \xrightarrow{\alpha} R \to 0$ with $J^2 = 0$, then J is an (R, R)-bimodule, as we saw last time. The content of Theorem 3 is that extensions of G by J are in bijection with extensions of $\mathbb{Z}G$ by J_ε (the right action is given by the augmentation ε).

Remark 12.1. If M is a super manifold, we have the ring of functions $C^\infty(M)$, which contains the nilpotent ideal N. The quotient is $C^\infty(M_{\text{red}})$. Replacing N by $J = N/N^2$ and $C^\infty(M)$ by $C^\infty(M)/N^2$, we have a square zero extension. We can get all the way to $C^\infty(M)$ by a series of square zero extensions (if N is nilpotent). I challange those of you who've seen supermanifolds to figure out which element of $HH^2(C^\infty(M_{\text{red}}, J)$ this corresponds to. ◇

[1] We defined this as $\mathrm{Ext}_{\mathsf{Ring}}(\mathbb{Z}G, J_\varepsilon)$, which we'll see is isomorphic to $HH^2(\mathbb{Z}G, J_\varepsilon)$ (this will be true for any ring).

Remark 12.2. Q: what if you want to do extensions by non-abelian groups? PT: Let's say you have

$$1 \to N \to Q \to G \to 1$$

Then you don't get a G-action on N in general. You do get a map ρ from G to $Out(N) = Aut(N)/Inn(N)$, and you get an induced action of G on the center of N. Given (G, N, ρ), there is an obstruction in $H^3(G; C(N))$ for existence of an extension Q that induces ρ. This obstruction vanishes exactly when there is such an extension. Moreover, choosing some extension Q gives a bijection between all extensions of G by (N, ρ) and $H^2(G; C(N))$. We'll see this next week. As you can imagine, we'll do it by relating everything to length 3 extensions. The center $C(N)$ appears because there is a canonical exact sequence

$$1 \to C(N) \to Inn(C) \to Aut(N) \to Out(N) \to 0$$

Just as three term extensions have to do with H^2, four term extensions have to do with H^3. This exact sequence will be an element of $H^3(Out(N); C(N))$. The ρ will induce a map $H^3(Out(N); C(N)) \to H^3(G; C(N))$, and the image of this element will be the obstruction. ◇

Theorem 12.3 (Theorem 2)**.** *Let $I_R \to R \xrightarrow{\varepsilon} k$ be an augmented k-algebra (k can be any commutative ring), and let J be some left R-module (made into a bimodule using ε). Then square zero extensions of R by J_ε are in bijection with left R-module extensions of I_R by J.*

Proof. We'll construct the two maps, and not completely say why they are inverse.

($\leftarrow$) We saw that if $d\colon R \to I$ is a derivation, then if we pull back an R-bimodule extention E of I by a bimodule M, you get a square zero ring extension of R by M. The key formula is that the product is given by $(e, r)(e', r') = (er' + re', rr')$.

Using the derivation $d_\varepsilon\colon r \mapsto r - \varepsilon(r)$, we get the map from left R-module extensions of I_R by J to square zero extensions of R by J_ε. (If we have an extension of left modules, we get an extension of bimodules using the right action given by ε)

($\rightarrow$) If we have a square zero extension $J_\varepsilon \to P \to R$, we have an augmentation $P \to R \to k$, so we get an augmentation ideal I_P, which surjects onto I_R. By the snake lemma (or something), the kernel of this surjection is J_ε. You can check that this I_P is an R-module extension of I_R by J_ε.

Now you can check that the two maps are inverse. □

Proof of Theorem 3. Given a square zero extension $J \to P \to R$ where J is an arbitrary R-bimodule. Inside of the rings, we have the groups of units $P^\times \to R^\times$. I claim that this map on units is surjective. If I have a unit $r_1 \in R$, so $r_1 r_2 = 1$ for some r_2. Pick p_1 and p_2 mapping to r_1 and r_2. Then $p_1 p_2 - 1 = j \in J$. Then $p_1 p_2(1-j) = (1+j)(1-j) = 1-j^2 = 1$ because $J^2 = 0$. The kernel of $P^\times \to R^\times$ is the set $1+J$ (we've already seen that everything in $1+J$ is invertible). As a group, $1+J \cong J$ (since $(1+j)(1+j') = 1+(j+j')$).

Now consider $R = \mathbb{Z}G$. We have $G \in \mathbb{Z}G^\times$, and we can pull back along the inclusion to get an extension of G by J. Thus, we've started with a square zero ring extension and obtained a group extension.

If we have a group extension $1 \to J \to Q \xrightarrow{\pi} G \to 1$, we get

$$0 \to \ker/\ker^2 \to \mathbb{Z}Q \xrightarrow{\mathbb{Z}\pi} /\ker \mathbb{Z}G \to 0$$

I claim that $J_\varepsilon \cong \ker/\ker^2$ by $j \mapsto 1-j$. I'll let you check this yourself.[2]

Now prove that the two maps are inverse (you'll use the adjunction between units and group-rings). □

[[break]]

Remark 12.4. Notice that everything going from $\mathrm{Ext}^2_{\mathbb{Z}G}(\mathbb{Z}, J)$ to group extensions was constructive. The only tricky part is "unsplicing". But if you use the Bar resolution, you can see that there is a canonical unsplicing. I'm cheating a little bit because I'm choosing a cocycle, not just a cohomology class to begin with. ◇

[2] If G is the trivial group, you have that $\mathbb{Z}\pi$ is the augmentation map of Q. We saw before that $I_Q/I_Q^2 \cong H_1(Q)$.

Hochschild (co)homology

If R is a ring, we get a semi-simplicial (R,R)-bimodule:

$$\begin{array}{ccccccc} 2 & & 1 & & 0 & & -1 \\ \cdots \Rrightarrow & & R\otimes R\otimes R & \overset{d_0}{\underset{d_1}{\rightrightarrows}} & R\otimes R & \xrightarrow{\mu} & R \end{array}$$

where $d_0(a\otimes a'\otimes a'') = aa'\otimes a''$ and $d_1(a\otimes a'\otimes a'') = a\otimes a'a''$. The dimension n piece has $(n+1)$ tensor product symbols and $(n+1)$ arrows coming out of it.

We can make a complex out of this by taking alternating sums:

$$T_*(R) = \cdots \xrightarrow{d_0-d_1+d_2} R\otimes R\otimes R \xrightarrow{d_0-d_1} R\otimes R \xrightarrow{\mu} R$$

From the theory of semi-simplicial sets, $d^2 = 0$ (this follows from the relation $d_i d_j = d_{j-1} d_i$. It turns out that all chain complexes appear in this way (this is the Dold-Kan theorem). Notice two things. First, this is a chain complex of (R,R)-bimodules. Secondly, this is much easier than the Bar resolution.

Lemma 12.5. *$T_*(R)$ is contractible in R-*mod *and* mod-*R (but not in (R,R)-bimodules).*

Proof. $h\colon R^{\otimes n} \to R^{\otimes(n+1)}$ is given by $a \mapsto 1\otimes a$ (or $a\otimes 1$ if you're working in R-mod instead of mod-R). □

Definition 12.6. Given an (R,R)-bimodule M, we define

$$HH_n(R;M) = H_n(T_*R \otimes_{R,R} M), \text{ and}$$
$$HH^n(R;M) = H^n(\mathrm{Hom}_{R,R}(T_*R, M)).$$

◇

13 Hochschild (co)homology

Recall Hochschild (co)homology. Given a k-algebra R (k could be $\mathbb{Z}$ if R is just a ring), we get an (augmented) semi-simplicial (R,R)-bimodule

$$\begin{array}{ccccc} & 1 & & 0 & \\ \cdots \overset{d_0}{\underset{d_2}{\xrightarrow{d_1}}} R\otimes R\otimes R & \overset{d_0}{\underset{d_1}{\rightrightarrows}} & R\otimes_k R & \xrightarrow{\mu} & R \end{array}$$

where $d_i(a_0\otimes\cdots\otimes a_n) = a_0\cdots\otimes a_ia_{i+1}\otimes\cdots\otimes a_n$.

Recall that for any category C, we have $\mathsf{PreSh}(\mathsf{C}) = \mathsf{Fun}(\mathsf{C}^\circ, \mathsf{Set})$, and we defined semi-simplicial sets as objects of $\mathsf{PreSh}(\mathsf{C}_\Delta)$. A semi-simplicial object in A is an object in $ss\mathsf{A} = \mathsf{Fun}(\mathsf{C}_\Delta^\circ, \mathsf{A})$. The objects of C_Δ are $[n] = \{0,\dots,n\}$ with $n\geq 0$, and the morphisms are strictly order preserving set maps. The morphisms are generated by the face maps $\partial_i\colon [n-1]\to[n]$ (the "skip i" map) for $0\leq i\leq n$. Any morphism $f\colon[m]\to[n]$ has a unique decomposition $f=\partial_{i_1}\circ\cdots\circ\partial_{i_k}$ with $i_1>\cdots>i_k$. The relations are $\partial_j\circ\partial_i = \partial_i\circ\partial_{j-1}$ for $i<j$.

So a semi-simplicial object in A is a sequence of objects X_n (the images of the $[n]$) and morphisms $d_i\colon X_n\to X_{n-1}$ (the images of the ∂_i, which go the other way), satisfying the relations $d_i\circ d_j = d_{j-1}\circ d_i$ when $i<j$.

Lemma 13.1 ("main lemma of homological algebra")**.** *If $X_\bullet$ is a semi-simplicial object in an abelian category A (an object in $ss\mathsf{A}$), then (X_*, d) is a chain complex, where $X_n = X_n$ and $d=\sum_{i=0}^n(-1)^i d_i$.*

[[★★★ Dold-Kahn correspondence is some equivalence of categories to this effect]] In particular, we get a chain complex of (R,R)-bimodules $T_*(R)$ (including the -1 piece). We showed that as a chain complex of left (or right) R-modules, $T_*(R)$ is contractible.

Let M be a left R-module, then $T_*(R)\otimes_R M$ is an acyclic chain complex of left R-modules (it is contractible as a complex of abelian groups). That is, we have a resolution of M by things of the form $R^{\otimes n+2}\otimes_R M\cong R^{\otimes n+1}\otimes_k M$.

Lemma 13.2. *If R and M are flat modules over k, then $T_*(R)\otimes_R M$ is an R-flat resolution of M as a left R-module.*

This is true just because for a right R-module N, $N\otimes_R R\otimes_k B\cong N\otimes_k B$, so if B is flat as a k-module, $R\otimes_k B$ is flat as an R-module. You also have to use that the tensor product of flat modules is flat (because tensor product is associative).

So we may use the resolution $T_*(R)\otimes_R M$ to compute $\mathrm{Tor}_n^R(N,M)$ for a right R-module N. More precisely,

$$\mathrm{Tor}_n^R(N,M)\cong H_n(N\otimes_R T_*(R)\otimes_R M) = HH_n(R; {}_RM\otimes_k N_R)$$

(here we are taking $*\geq 0$, not the augmented guy).

If B is an (R,R)-bimodule, then recall that $HH_n(R;M) = H_n(T_*(R)\otimes_{R,R} B)$. If A and B are bimodules, then $A\otimes_{R,R} B = A\otimes_R B/(ra\otimes b - a\otimes br)$ (this is just an abelian group).

So we've proven the following.

Proposition 13.3. *If R is k-flat and either M or N is k-flat, then $\mathrm{Tor}_n^R(N,M)\cong HH_n(R; {}_RM\otimes_k N_R)$.*

Theorem 13.4. *$HH_n(\mathbb{Z}G; M_\varepsilon)\cong H_n(G;M)$ and $HH^n(\mathbb{Z}G;M_\varepsilon)\cong H^n(G;M)$.*

Proof. $H_n(G;M) := \mathrm{Tor}_n^{\mathbb{Z}G}(\mathbb{Z},M)$. Since $\mathbb{Z}$ and $\mathbb{Z}G$ are $\mathbb{Z}$-flat (torsion-free; in fact, they are free), we can apply the Proposition to tell us that $H_n(G;M)\cong HH_n(\mathbb{Z}G; M\otimes_{\mathbb{Z}}\mathbb{Z})$, but $M_\varepsilon = M\otimes_{\mathbb{Z}}\mathbb{Z}$ by definition.

[[★★★ The other assertion please]] □

[[break]]

Remark 13.5. $HH_n(R;R)$ is given by the homology of $T_*(R)\otimes_{R,R} R$. Note that $R^{\otimes_k n+2}\otimes_{R,R} R\cong R^{\otimes_k n+1}$. This is sometimes just called the "Hochschild homology of R, $HH_n(R)$". You can think of it as $HH(R) := R\overset{L}{\otimes}_{R,R} R$. The reason that the Hochschild homology of R is the value of the circle in many 2-dimensional QFTs is that $\otimes_{R,R}$ "loops together two bimodules". ◇

Homotopy theory in homological algebra

Remember from the homework that we have a pair of adjoint functors $ss\,\mathsf{Set} \underset{\Delta_\bullet}{\overset{|\cdot|}{\rightleftarrows}} \mathsf{Top}$, where $\Delta_n(X) = \{f\colon \Delta^n\to X\}$.

For a semi-simplicial set $X_\bullet$, you should think of elements of the set X_n as the labels of n-simplices. Then the map $d_i\colon X_n \to X_{n-1}$ takes a label x to the label of the i-th face of the simplex labelled by x. The relations come from the fact that there are two ways to get to a codimension 2 face by going through a codimension 1 face first.

(1) The homotopy groups $\pi_n(X)$ are actually the homotopy groups $\pi_n(\Delta_\bullet(X))$ for some notion of π_n of certain semi-simplicial sets. I'll explain this next time.

(2) $H_n^{\text{sing}}(X) = H_n(S_*(X))$, but $S_*(X)$ is the chain complex corresponding to $Free_{\mathsf{Ab}}(\Delta_\bullet(X))$. We'll see that in general, this is $\pi_n(Free\Delta_\bullet(X))$.

So there is some notion of π_n which in one instance gives $\pi_n(X)$ and in some other instance gives $H_n(X)$. The cool thing is that the Hurewicz map is the map on π_n induced by $\Delta_\bullet(X) \to Free(\Delta_\bullet(X))$.

14 ???

Homological algebra $\rightsquigarrow$ Combinatorics of semisimplicial sets $\rightsquigarrow$ homotopy theory.

Example 14.1. Consider the circle S^1 as a vertex and one edge. This is homeomorphic to the realization $|X_\bullet|$ of the semisimplicial set $X_0 = \{v\}$ and $X_1 = \{e\}$, with the two maps $X_1 \rightrightarrows X_0$. ⋄

Example 14.2. Consider S^2 as a vertex and a 2-cell. Is this the realization of a semi-simplicial set? No, because any 2-simplex would have to have three faces, which could be 1-simplices.

What if we have $S^2 = v \cup e \cup f_1 \cup f_2$? There are no choices for the face maps (because there is only one 1-cell and only one 0-cell). The attaching map of the 2-cells is not the identity map, but it has degree 1, so we at least have a homotopy equivalence $|X_\bullet| \simeq S^2$.

How would we check that the degree is 1? We can compute homology.

$$H_n(X_\bullet) := H_n(Free_*(X_\bullet)) \cong H_n^{\text{cell}}(|X_\bullet|)$$

To see the isomorphism, you have to check the boundary maps. In our case, we have $Free_*(X_\bullet) = (\mathbb{Z} \times \mathbb{Z} \xrightarrow{??} \mathbb{Z} \xrightarrow{0} \mathbb{Z})$. Clearly $H_0 = \mathbb{Z}$ and $H_2 = \mathbb{Z}$ (because ?? is either $(1,1)$ or $(3,3)$). Depending on what ?? is, H^1 is either 0 or $\mathbb{Z}/3$. Anyway, the right answer is $(1,1)$.

If you play a little more, you can get something homeomorphic to S^2. ⋄

The representable elements of $ss\,\mathsf{Set} = \mathsf{Fun}(\mathsf{C}_\Delta^\circ, \mathsf{Set})$ are of the form $\Delta_\bullet[n] = \mathsf{C}_\Delta(-,[n])$. There are $\binom{n+1}{k+1}$ k-simplices, the same as the k-dimensional faces of Δ^n. If you think about it a bit, $|\Delta_\bullet[n]| = \Delta^n$ with it's usual cell decomposition. Now we're in good shape to get S^n.

Corollary 14.3. *$S^{n-1} \cong \partial\Delta^n \cong |X_\bullet|$, where $X_\bullet$ is the same as $\Delta_\bullet[n]$, except with the one point set $\Delta_n[n]$ replaced by the empty set (i.e. with the top-dimensional simplex removed).*

Let $K \subseteq \mathbb{R}^N$ be a finite (or at least discrete) ordered simplicial complex (meaning that K is actually the union of simplicies meeting nicely along faces). Define a semi-simplicial set $X_\bullet$ by defining X_n to be the set of all n-simplicies in K, with $d_i \colon X_n \to X_{n-1}$ being the i-th face map (this is where we needed the ordering).

Lemma 14.4 (Definition, actually)**.** $K \cong |X_\bullet|$. *Moreover, this is a cellular isomorphism.*

Now I should say something about the "semi". When people realized that there is this complex underlying everything, they eventually called them semi-simplicial sets. Then something happened (we'll get to it soon) that said that these things are not quite good enough. They are missing some information (degeneracy maps). The "semi" was originally because the $X_\bullet$ was just the combinatorial data, not the whole chain complex, but we use "semi" for a second reason, which is that we're missing the degeneracy maps. We have $H_n^{\text{simp}}(K) := H_n(X_\bullet)$.

$$\begin{tikzcd}
\mathsf{Chain} & ss\mathsf{Ab} \arrow[l, "Alt"'] \arrow[r, shift right, "Forget"'] & ss\,\mathsf{Set} \arrow[l, shift right, "Free"'] \arrow[r, shift left, "|\cdot|"] & \mathsf{Top} \arrow[l, shift left, "\Delta_\bullet"] \\
S_*(X) & S_\bullet(X) \arrow[l, mapsto] & \Delta_\bullet(X) \arrow[l, mapsto] & X \arrow[l, mapsto] \\
C_*^{\text{simp}}(K) & & X_\bullet^K \arrow[ll, mapsto] \arrow[r, mapsto] & K
\end{tikzcd}$$

Left adjoints are on top. $S_*(X)$ is the usual singular chain complex of X. So we've broken up the usual story into many steps. Next we'll play with some of the other arrows in the diagram. What we've done so far is the bottom row.

[[break]]

Another thing we can go is go back and forth in the diagrams. If $X \in \mathsf{Top}$, then we get a natural map $|\Delta_\bullet(X)| \to X$. The following theorem follows from some very general theory.

Theorem 14.5. *This is a map from a CW complex, and it turns out it is a weak homotopy equivalence (it induces isomorphisms an all π_n for all base points). So this is a canonical CW approximation to X.*

Remark 14.6. X is homotopy equivalent to a CW complex if and only if this map is an actual homotopy equivalence (the converse is Whitehead's theorem). ⋄

If R is a ring, then we had the semi-simplicial (R,R)-bimodule $T_\bullet(R)$. Say $R = \mathbb{Z}G$. We have

$$\begin{tikzcd}
\mathsf{Chain} & ss\mathsf{Ab} \arrow[l, "Alt"'] \arrow[r, shift right, "Forget"'] & ss\,\mathsf{Set} \arrow[l, shift right, "Free"'] \arrow[r, shift left, "|\cdot|"] & \mathsf{Top} \arrow[l, shift left, "\Delta_\bullet"] \\
T_*(\mathbb{Z}G) & T_\bullet(\mathbb{Z}G) \arrow[l, mapsto] & T_\bullet(G) \arrow[l, mapsto] & \\
T_*(\mathbb{Z}G)\otimes_{\mathbb{Z}G}\mathbb{Z} & T_\bullet(\mathbb{Z}G)_G \arrow[l, mapsto] & T_\bullet(G)/G \arrow[l, mapsto] \arrow[r, mapsto] & EG \\
\mathbb{Z}\otimes_{\mathbb{Z}G} T_*(\mathbb{Z}G)\otimes_{\mathbb{Z}G}\mathbb{Z} & {}_GT_\bullet(\mathbb{Z}G)_G \arrow[l, mapsto] & G\backslash T_\bullet(G)/G \arrow[l, mapsto] \arrow[r, mapsto] & BG
\end{tikzcd}$$

Note that $T_\bullet(\mathbb{Z}G)_n \cong \mathbb{Z}[G^{n+2}]$, and the maps actually come from maps in a semi-simplicial (G,G)-set $T_\bullet(G)$. We can take the quotient by the right action to get a semi-simplicial G-set $T_\bullet(G)/G$. Then we get $T_*(\mathbb{Z}G)\otimes_{\mathbb{Z}G}\mathbb{Z}$, which was a free $\mathbb{Z}G$-resolution of $\mathbb{Z}$. Then group homology is just given by tensoring this with some module and then taking homology. We can define EG as the geometric realization of $T_\bullet(G)/G$. Since $T_*(\mathbb{Z}G)\otimes_{\mathbb{Z}G}\mathbb{Z}$ is acyclic, EG has no homology (because simplicial homology agrees with usual homology), and it still has a free G-action. What is the quotient $G\backslash EG =: BG$? It is the geometric realization of $G\backslash T_\bullet(G)/G$.

Now we know that the complex $\mathbb{Z}\otimes_{\mathbb{Z}G} T_*(\mathbb{Z}G)\otimes_{\mathbb{Z}G}\mathbb{Z}$ computes $H_n(G)$ (by definition). We've just shown that BG is a canonically associated topological space so that $H_n(BG)\cong H_n(G)$. The punchline is that you should recognize that the homology of G can be computed by BG.

An obvious quesion: is EG contractible? That is, is BG a $K(G,1)$? How did we prove that $T_*(\mathbb{Z}G)\otimes_{\mathbb{Z}G}\mathbb{Z}$ is acyclic? We showed that $T_*(\mathbb{Z}G)$ is contractible as a complex of left (or right) $\mathbb{Z}G$-modules. Wouldn't it be nice if all our maps were compatible with contractions? To do this, we need to introduce the notion of homotpy in $s\,\mathsf{Set}$ and $s\mathsf{Ab}$ (it doesn't work well in $ss\,\mathsf{Set}$ and $ss\mathsf{Ab}$) and there are actually functors

$$\begin{tikzcd}
\mathsf{Chain} \arrow[r, shift right, "\simeq"'] \arrow[d, equal] & s\mathsf{Ab} \arrow[l, shift right, "N_*"'] \arrow[r, shift right, "Forget"'] \arrow[d, shift right] & s\,\mathsf{Set} \arrow[l, shift right, "Free"'] \arrow[r, shift left, "|\cdot|"] \arrow[d, shift right] & \mathsf{Top} \arrow[l, shift left, "\Delta_\bullet"] \arrow[d, equal] \\
\mathsf{Chain} & ss\mathsf{Ab} \arrow[l, "Alt"'] \arrow[r, shift right, "Forget"'] \arrow[u, shift right] & ss\,\mathsf{Set} \arrow[l, shift right, "Free"'] \arrow[r, shift left, "|\cdot|"] \arrow[u, shift right] & \mathsf{Top} \arrow[l, shift left, "\Delta_\bullet"]
\end{tikzcd}$$

$\mathsf{C}_\Delta \subseteq \Delta$, where Δ is the category of objects $[n]$, with increasing (but not necessarily strictly increasing) maps. Then $s\mathsf{C} = \mathsf{Fun}(\Delta^\circ, \mathsf{C})$.

15 ???

Our goal is to understand all the functors in the following diagram. We've already understood all the functors in the bottom row.

$$\begin{tikzcd}
\mathsf{Chain} \arrow[d, equal] \arrow[r, shift right, "R"'] \arrow[r, phantom, "\cong"] & s\mathsf{Ab} \arrow[l, shift right, "N_*"'] \arrow[r, shift right, "U"'] \arrow[d, shift right, "U"'] & s\,\mathsf{Set} \arrow[l, shift right, "Free"'] \arrow[r, shift left, "|\cdot|"] \arrow[d, shift right, "U"'] & \mathsf{Top} \arrow[l, shift left, "\Delta_\bullet"] \arrow[d, equal] \\
\mathsf{Chain} \arrow[r, shift right, bend right, "R"'] & ss\mathsf{Ab} \arrow[l, shift right, "Alt"'] \arrow[u, shift right, "L"'] \arrow[r, shift right, "U"'] & ss\,\mathsf{Set} \arrow[l, shift right, "Free"'] \arrow[u, shift right, "L"'] \arrow[r, shift left, "Re"] & \mathsf{Top} \arrow[l, shift left, "\Delta_\bullet"]
\end{tikzcd}$$

[[★★★ note the distinction between the two realization functors. U means forgetful functor]]

Remember that we're trying to relate chain complexes and topological spaces. It turns out that you can do everything in homological algebra in simplicial sets. It also turns out that simplicial sets will be equivalent to topological spaces as homotopy categories.

Why are semi-simplicial sets not quite good enough? All limits and colimits exist in all these categories. Let's see how these functors behave with respect to limits and colimits. (compactly generated) topological spaces has products. Since $\Delta_\bullet$ is right adjoint, it preserves products. Similarly, $|\cdot|$ respects coproducts (disjoint union). Does $\Delta_\bullet$ respect coproducts? Yes! A map from Δ^n to a disjoint union is the same thing as a map to one of the things in disjoint union. However, it turns out that $Re(X_\bullet \times Y_\bullet) \not\cong Re(X_\bullet) \times Re(Y_\bullet)$.

Consider the case where $X_\bullet = Y_\bullet = \Delta^1_\bullet$, the standard 1-simplex, so $X_0 = \{v_0, v_1\}$, $X_1 = \{e\}$, $Y_0 = \{w_0, w_1\}$, and $Y_1 = \{f\}$. Then $X_\bullet \times Y_\bullet$ is the pointwise product, so $(X \times Y)_0 = \{v_0w_0, v_0w_1, v_1w_0, v_1w_1\}$, $(X \times Y)_1 = \{ef\}$. It turns out that single edge goes between v_0w_0 and v_1w_1, leaving two vertices hanging.

I'm about to talk about homotopy, where you cross with the interval, which goes terribly wrong. So how can we fix it? We need four more edges and two new 2-simplices. So what if we throw in two more edges for X and two more for Y, so $X_1 = \{e, e_0, e_1\}$ and $Y = \{f, f_0, f_1\}$. You also have to throw in some faces.

The upshot is that simplicial sets exactly fill in the picture the way we want it to be. We'll see that the realization functor from simplicial sets to topological spaces will respect products and coproducts.

Definition 15.1. $s\,\mathsf{Set} = \mathsf{Fun}(\Delta^\circ, \mathsf{Set})$, where Δ is the category with objects $\{0, \dots, n\} = [n]$ and morphisms weakly order-preserving maps. ⋄

We have $\mathsf{C}_\Delta \subseteq \Delta$, so we can restrict a functor to C_Δ to get a forgetful functor $U\colon s\,\mathsf{Set} \to ss\,\mathsf{Set}$.

Lemma 15.2. *Any morphism α in Δ can be written uniquely as a composition $i \circ s$, where i is injective and s is surjective.*

If α is injective, it is a morphism of C_Δ, so it is uniquely a composition of $\partial_{i_0} \cdots \partial_{i_s}$ where $0 \le i_s < \cdots < i_0$. Here, ∂_i is the "skip i" map. There are corresponding maps $\sigma_i\colon [n] \to [n-1]$ which is the surjective map so that $\sigma_i^{-1}(i)$ has two elements. Note that there are $n-1$ σ_i and n ∂_i. If α is sujective, then it is uniquely a composition $\sigma_{i_0} \cdots \sigma_{i_r}$ with $0 \le i_0 < \cdots < i_r$

The picture of $X_\bullet$ is the following

$$\begin{tikzcd}
\cdots & X_2 \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & X_1 \arrow[l, bend right, shift right] \arrow[l, bend right, shift right=3] \arrow[r, shift left] \arrow[r, shift right] & X_0 \arrow[l, bend right, shift right=2]
\end{tikzcd}$$

There are some relations among the maps, but I won't write them all out. You can figure them out if you like. There are three sets of relations (∂/∂, σ/σ, and σ/∂).

Example 15.3 (representable functors)**.** Let $\Delta^n_\bullet = r[n]$. Then $(\Delta^n)_m$ is the set of maps in Δ from $[m]$ to $[n]$. There are $\left(\!\binom{m}{n}\!\right)$ of these (this is multi-choose notation).

In particular, we have $\Delta^0_m = \{*\}$ for all m. Similarly, $\Delta^1_m = \Delta([m], [1])$, and such a thing is determined by the preimage of 0. In particular, we get three 1-simplices $\Delta^1_1 = \{e, e_0, e_1\}$ (here e is injective, and e_0 and e_1 are not). ⋄

What will the geometric realization functor be? Recall the homework assignment: If $\mathsf{C} \to \mathsf{Fun}(\mathsf{C}^\circ, \mathsf{Set})$ is the Yoneda embedding and $\gamma\colon \mathsf{C} \to \mathsf{D}$, where D is a cocomplete category, then there is a unique realization functor $Re\colon \mathsf{Fun}(\mathsf{C}^\circ, \mathsf{Set}) \to \mathsf{D}$ making the triangle commute. Moreover, this Re is left adjoint to some other functor $Sing$. Apply the homework to the case $\mathsf{C} = \Delta$, $\mathsf{D} = \mathsf{Top}$ and $\gamma([n]) = \Delta^n$. You have to check that the new (degeneracy) maps give you maps in Top; they are just given by extending the map on vertices linearly. Call the realization $|\cdot|$.

Let's unravel this $|\cdot|$ functor a bit. We have that

$$|X_\bullet| = \Big(\coprod_{n\in\mathbb{N}_0, x\in X_n} \Delta^n\Big)/\sim \quad = \quad \coprod_{n\in\mathbb{N}_0} X_n \times \Delta^n/\sim$$

The equivalence relation is given by the maps, so for every $\alpha\colon [m]\to[n]$, you have $(X(\alpha)(x),t)\sim(x,\gamma(\alpha)(t))$. This is equivalent to saying that $(d_i(x),t)\sim(x,\partial_{i*}(t))$ and $(s_j(x),t)\sim(x,\sigma_{j*}(t))$. If σ_j is the surjection which repeats j, then $\sigma_{j*}\colon\Delta^n\twoheadrightarrow\Delta^{n-1}$. This is a linear collapsing of an n-simplex onto a $(n-1)$-simplex (given by linearly extending a map on vertices, where you repeat the j-th vertex).

The punchline is that simplices of the form $s_j(x)$ don't actually appear as new cells; they get crushed.

Definition 15.4. The simplices $\alpha(x)$ with α a non-identity surjection are called *degenerate.* ◇

Theorem 15.5. *$|X_\bullet|$ is a CW complex with exactly one n-cell for each non-degenerate n-simplex.*

Corollary 15.6. *$|\Delta^n_\bullet\times\Delta^m_\bullet|\cong\Delta^n\times\Delta^m$ (as CW complexes, for some canonical subdivision of $\Delta^n\times\Delta^m$ into simplices) and $|\cdot|$ preserves products.*

[[break]]

Let $\Delta_\bullet[n]$ be the semi-simplicial n-simplex, let $\Delta^n_\bullet$ be the simplicial n-simplex, and let $\Delta^n=|\Delta^n_\bullet|=Re(\Delta_\bullet[n])$ be the standard n-simplex. This is the notation we'll try to use throughout the class.

The functor $\Delta_\bullet\colon\mathsf{Top}\to s\,\mathsf{Set}$, given by $\Delta_n(X)=\{f\colon\Delta^n\to X\}$. We have the forgetful functor $U\colon s\,\mathsf{Set}\to ss\,\mathsf{Set}$. I claim this has a left adjoint L. This L can be constructed by the same homework problem we used earlier, using $\gamma\colon \mathsf{C}_\Delta\subseteq\Delta\xrightarrow{r} s\,\mathsf{Set}$.

We can concretely construct this left adjoint by throwing in a bunch of degenerate simplices. If $X_\bullet\in ss\,\mathsf{Set}$, then $L(X_\bullet)_m=\coprod_{\sigma\colon[m]\twoheadrightarrow[n]}X_n^\sigma$. Given $\alpha\colon[m']\to[m]$, factor $\sigma\colon\alpha$ as a surjection followed by an injection $[m']\xrightarrow{\sigma'}[n']\xrightarrow{\alpha'}[n]$; then we get a map $X_n^\sigma\xrightarrow{X(\alpha')}X_{n'}^{\sigma'}$.

Remark 15.7. This works if we replace Set by any category C which has coproducts. For example, we could take $\mathsf{C}=\mathsf{Ab}$. In particular, we get a pair of adjoint functors between $s\mathsf{Ab}$ and $ss\mathsf{Ab}$. ◇

You get R using the same homework problem again. Note some commutativity relations in the diagram: $Re\circ U\ncong|\cdot|$, but $|\cdot|\circ L=Re$ and $U\circ\Delta_\bullet=\Delta_\bullet$. One of the homework problems (3) is that $Alt_*\circ U=N_*$.

Ok, now we know all the functors. How do the monoidal structures behave. $\Delta_\bullet$ and $|\cdot|$ respect products and coproducts. Between $s\mathsf{Ab}$ and $s\,\mathsf{Set}$, the coproducts are preserved by $Free$, but the coproduct in $s\,\mathsf{Set}$ is taken to $\otimes$ in $s\,\mathsf{Set}$. The direct sum (coproduct) in $s\mathsf{Ab}$ is taken to $\oplus$ in Chain, but the tensor product only goes to $\tilde\otimes$, which is tensor product *up to chain homotopy*, but not exactly. This isomorphism $\otimes\cong\tilde\otimes$ is this Eilenberg-Zilber theorem.

Theorem 15.8 (Dold-Kan correspondence)**.** $\mathsf{Chain}\rightleftarrows s\mathsf{Ab}$ *is an equivalence of abelian categories and of homotopy cateogries.*

We'll talk about this more next time.

Theorem 15.9 (Quillen)**.** $\mathsf{Top}\rightleftarrows s\,\mathsf{Set}$ *is an equivalence of homotopy categories.*

16 ???

I owe you some definitions for the homework.

Definition 16.1. The *nerve* of small[1] category C is a simplicial set $N_\bullet(\mathsf{C})$, with $N_n(\mathsf{C}) = \{X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} X_n\}$, with $d_i \colon N_n(\mathsf{C}) \to N_{n-1}(\mathsf{C})$ is given by

$$d_i(f_1, \dots, f_n) = \begin{cases} (f_2, \dots, f_n) & i = 0 \\ (f_1, \dots, f_i \circ f_{i+1}, \dots, f_n) & 0 < i < n \text{ (compose } at\ X_i) \\ (f_1, \dots, f_n) & i = n \end{cases}$$

and the degeneracy maps are given by

$$s_j(f_1, \dots, f_n) = (f_1, \dots, f_j, \mathrm{id}_{X_j}, f_{j+1}, \dots, f_n). \qquad \diamond$$

Note that $N_0(\mathsf{C}) = Obj(\mathsf{C})$ and $N_1(\mathsf{C}) = Mor(\mathsf{C})$. $d_1, d_0 \colon N_2(\mathsf{C}) \to N_1(\mathsf{C})$ are the source and target maps (respectively). Pictorially, $(f_1, \dots, f_n)$ gives you an n-simplex of morphisms in C (all other edges are uniquely determined as compositions of the f_i, well-defined because composition is associative).

Let $N_n = N_n(\mathsf{C})$. Then we have $N_2 \xrightarrow{d_0 \times d_1} N_1 \times_{N_0} N_1$, with $(f_1, f_2) \mapsto (f_2, f_1)$. This map is an isomorphism, and $d_1 \colon N_1 \times_{N_0} N_1 \cong N_2 \to N_1$ is the composition in the category!

Now we can see that associativity is expressed by simplicial relations. Associativity means that the front face of the following diagram commutes.

$$\begin{tikzcd}
N_3 \arrow[rr, "d_2"] \arrow[dr, "\sim"] \arrow[dd, "d_1"'] & & N_2 \arrow[dr, "\sim"] \arrow[dd, "d_1"] & \\
& N_1 \times_{N_0} N_1 \times_{N_0} N_1 \arrow[rr, "\mathrm{id} \times \mu"] \arrow[dd, "\mu \times \mathrm{id}"] & & N_1 \times_{N_0} N_1 \arrow[dd, "\mu"] \\
N_2 \arrow[rr, "d_1"] \arrow[dr, "\sim"] & & N_1 \arrow[dr, equal] & \\
& N_1 \times_{N_0} N_1 \arrow[rr, "\mu"] & & N_1
\end{tikzcd}$$

Commutativity of the front face is equivalent to commutativity of the back face (you have to check that the "side faces" commute), which is the relation you get from the simplicial-ness of $N_\bullet$. So we have that a simplicial set $X_\bullet$ comes from a small category ($X_\bullet \cong N_\bullet(\mathsf{C})$) if and only if for all $n \geq 1$, $X_n \cong \overbrace{X_1 \times_{X_0} X_1 \cdots \times_{X_0} X_1}^{n \text{ times}}$.

Well, really, we're missing the simplicial-ness, we only have semi-simplicial-ness. This corresponds to the fact that we haven't included identities in our categories (you could define a category without identities, and this somehow corresponds to simplicial sets without degeneracy maps).

We haven't really shown that $N_\bullet(\mathsf{C})$ is a simplicial set. To show this, note that $N_n(\mathsf{C}) = \mathsf{Fun}([\![n]\!], \mathsf{C})$, where $[\![n]\!]$ is the category whose objects are $0, \dots, n$ and $Mor(i, j) = *$ if $i \leq j$ and $\varnothing$ otherwise. Observe that $\Delta(m, n) = \mathsf{Fun}([\![m]\!], [\![n]\!])$. If we wanted strictly order-preserving maps, we would have to define $[\![n]\!]$ to be a category without identities! You can check that the maps between the $N_i(\mathsf{C})$ (the maps coming with the simplicial structure) come from composition with functors between the categories $[\![i]\!]$.

Remark 16.2. The *classifying space* of C is $|N_\bullet(\mathsf{C})|$. The is related to the classifying space of a group G by thinking of a group as a category ΣG with one object $*$, with $\Sigma G(*, *) \cong G$. It will be a homework exercise to show that $BG = B(\Sigma G)$. $\diamond$

[1] A *small* category is a category where the objects form a set. A *large* category is a category where the Hom sets are allowed to be proper classes. In large categories, you have to worry about whether composition of morphisms works like you want. We will run into these large cateogries when we do derived categories.

Example 16.3. $\Delta^m_\bullet = N_\bullet(\llbracket m \rrbracket)$. ◇

[[break]]

Let me show you a good motivation for this cocompletion homework problem we keep using. If you want the simplicization functor $L\colon ss\mathsf{Ab} \to s\mathsf{Ab}$, I said you can do it similar to $L\colon ss\,\mathsf{Set} \to s\,\mathsf{Set}$, but it's a bit trickier. If you have an additive category C (the Hom sets are abelian groups), then instead of the usual Yoneda embedding, you should take $\mathsf{C} \hookrightarrow \mathsf{Fun}_{add}(\mathsf{C}, \mathsf{Ab})$. Let's look at an application.

If R is a ring, then we can make a category ΣR, with one object, with the composition given by multiplication and addition is addition. A functor $M\colon \Sigma R \to \mathsf{Ab}$ is the same thing as a left R-module M (it would be a left R-module if it were contravariant). What is the realization?

$$\begin{tikzcd} \Sigma R \arrow[r, "r"] \arrow[dr, "M"'] & \mathsf{Fun}_{add}(\Sigma R^\circ, \mathsf{Ab}) \cong \mathsf{mod}\text{-}R \arrow[d, "N\mapsto N\otimes_R M"] \\ & \mathsf{Ab} \end{tikzcd}$$

It is exactly tensoring with M! You can check for yourself what the right adjoint is.

Next week, I'll (PT) be gone and Chris will talk about non-abelian extensions, crssed modules, and associators. Then we have spring break. Then we have four weeks and May left

1. Dold-Kan correspondence ($\mathsf{Chain} \cong s\mathsf{Ab}$), spectral sequences and applications

2. spectral sequences and applications

3,4. derived categories, derived functors, and triangulated categories

May. I'm gone again and Chris will do something interesting.

$$S_*(X) \longleftarrow\!\!\shortmid X$$

$$\begin{tikzcd} & \mathsf{Chain} \arrow[r, shift left] & s\mathsf{Ab} \arrow[l, shift left] \arrow[r, shift left] & s\,\mathsf{Set} \arrow[l, shift left] \arrow[r, shift left, "|\cdot|"] \arrow[d, twoheadrightarrow, "?"] & \mathsf{Top} \arrow[l, shift left, "\Delta_\bullet"] \arrow[d, twoheadrightarrow] \\ h\mathsf{Chain}^{proj}\mathsf{A} \arrow[d, equal, "\wr"'] & & & hs\,\mathsf{Set} \arrow[r, "|\cdot|"] \arrow[d, equal, "\wr"'] & h\mathsf{Top} \arrow[d, equal, "\wr"'] \\ d\mathsf{Chain}(\mathsf{A}) & & & h\mathsf{Kan} \arrow[r, "|\cdot|"] & h\mathsf{CW}h\mathsf{Kan} \end{tikzcd}$$

We know what the homotopy categories of Chain and Top are. We'd like to relate the notion of chain homotopy and the notion of homotopy of topological spaces.

Definition 16.4. Two morphisms of simplicial sets[2] $f_0, f_1\colon X_\bullet \to Y_\bullet$ are *homotopic* if there is a homotopy $H\colon X_\bullet \times \Delta^1_\bullet \to Y_\bullet$ such that $H \circ i_j = f_j$, where $i_0, i_1\colon X_\bullet \to X_\bullet \times \Delta^1_\bullet$ are the maps you expect. ◇

Geometric realization preserves products, so a simplicial homotopy gives a topological homotopy, so we get a descended functor.

Warning 16.5. Homotopy is not an equivalence relation in $s\,\mathsf{Set}$. Moreover, $hs\,\mathsf{Set} \xrightarrow{|\cdot|} h\mathsf{Top}$ is not essentially surjective. Everything in the image is a CW complex (and not all topological spaces are CW complexes). Both of these problems can be resolved as follows. ⌟

Remember that for any topological space, we have a canonical map $|\Delta_\bullet X| \to X$. This map is a weak equivalence (it induces isomorphisms on all π_i for all basepoints), but I won't prove it. I'm sure it goes back to Quillen.

Definition 16.6. The *derived category* $d\mathsf{Top}$ is obtained from Top by formally inverting all weak equivalences. ◇

The problem is that this might not be a category! This is the problem of small versus large. $Obj(d\mathsf{Top}) = Obj(\mathsf{Top})$, and morphisms from X to Y are given by chains $X \xleftarrow{weq} X_1 \xrightarrow{f} X_2 \xleftarrow{weq} \cdots X_n \to Y$ modulo some

[2]These are just natural transformations of functors $\Delta^\circ \to \mathsf{Set}$.

relations.[3] Even if we fix X and Y, the X_i can vary over all over the place, so it is not obvious $d\mathsf{Top}$ is a (non-large) category.

Theorem 16.7 (Whitehead's Theorem + CW approximation)**.** *$d\mathsf{Top} \cong h\mathsf{CW}$ as categories.*

Note that homotopy *is* an equivalence relation on a subcategory of simplicial sets $\mathsf{Kan} \subseteq s\,\mathsf{Set}$. The claim is that $\mathsf{Top} \xrightarrow{\Delta_\bullet} s\,\mathsf{Set}$ factors through Kan, just like $s\,\mathsf{Set} \xrightarrow{|\cdot|} \mathsf{Top}$ factors trough CW. Quillen's theorem is that $|\cdot|\colon h\mathsf{Kan} \to h\mathsf{CW}$ is an equivalence of categories.

[3]The universal property is that for any category D and any map $\mathsf{Top} \to \mathsf{D}$ taking weak equivalences to isomorphisms, it factors uniquely through $d\mathsf{Top}$.

17 Group extensions

[1]

Today we get to talk about one of the highlights of the course; we'll prove and understand something we couldn't understand before. We have a classification of finite simple groups, but then what do we know about finite groups? We need to understand how group extentions work.

Given groups Q and N, when can we have an exact sequence of groups

$$1 \to N \to E \to Q \to 1$$

We can make a few observations right away. If N is abelian, we kind of understand this problem. It turns out to be $H^2(Q; N)$. In general, we know that $N \triangleleft E$ and $Q = E/N$. So conjugation by elements of E gives automorphisms of N, so we have a map $E \to \operatorname{Aut} N$. Before, if we had an element $q \in Q$, we took a preimage $e \in E$ and conjugate with it. But now that N is non-abelian, the action of q depends on the choice of preimage e (a different choice will act by something differing by an inner automorphism of N). So we get

$$\begin{tikzcd} 1 \arrow[r] & N \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d, "\rho"] & 1 \\ & N \arrow[r, "\alpha"] & \mathrm{Aut}(N) \arrow[r] & Out(N) \arrow[d] & \\ & & & \mathrm{Aut}(Z(N)) & \end{tikzcd}$$

The goal for the day is to prove the following theorem.

Theorem 17.1. *Given N, Q, and $\rho\colon Q \to Out(N)$,*

(a) there is a functorial obstruction class *$\nu_\rho \in H^3(Q; Z(N))$ such that there exists an extension E inducing ρ if and only if $\nu_\rho = 0$, and*

(b) if $\nu_\rho = 0$, and E_0 is an extension inducing ρ, then $\{extensions\}/\cong$ is in bijection with $H^2(Q; Z(N))$.

[1] This lecture was given by Chris Schommer-Pries.

Consider the case of extensions by abelian groups $N = A$. In this case, $\alpha\colon A \to \mathrm{Aut}(A)$ is the trivial map. PT had an elaborite way of relating extensions to some other stuff, but there is a more direct way to do it, and that is what we'll do today.

If we have an extension

$$\begin{tikzcd} 1 \arrow[r] & A \arrow[r] & E \arrow[r] & Q \arrow[r] \arrow[l, bend right, "s"'] & 1 \end{tikzcd}$$

we can choose a set-theoretic section s, which will not be a group homomorphism in general. We will have

$$s(q)s(q') = f(q, q')s(qq')$$

for some function $f\colon Q \times Q \to A$. Computing $s(q)s(q')s(q'')$ in two different ways, we see that $f \in Z^2(Q; A)$. That is, [[★★★ explicitly write the cocycle condition]]. If we'd chosen a different section, the two sections differ by a map $Q \to A$. Such a map is a 1-cochain. The boundary of this 1-cochain is exactly the difference between the resulting 2-cocycles. So an extension gives us an element of cohomology. We can show that an element of cohomology gives an extension.

What changes in the non-abelian case? Suppose we have

$$\begin{tikzcd} & 1 \arrow[r] & N \arrow[r] \arrow[d, equal] & E \arrow[r, "\pi"'] \arrow[d] & Q \arrow[r] \arrow[d, "\rho"] \arrow[l, bend right, "s"'] \arrow[dl, "\xi"'] & 1 \\ 0 \arrow[r] & Z(N) \arrow[r] & N \arrow[r] & \mathrm{Aut}(N) \arrow[r] & Out(N) & \end{tikzcd}$$

Composing the section with $E \to \mathrm{Aut}(N)$, we get a map ξ (which won't be a homomorphism). In the abelian case, $\mathrm{Aut}(N) \cong Out(N)$, so we had to have $\xi = \rho$. As before, we get a function $f\colon Q \times Q \to N$, so that

$$\xi(q)\xi(q') = \alpha\big(f(q, q')\big)\xi(qq').$$

Then we can verify that f is a *non-abelian cocycle*. That is,

$$f(q, q')f(qq', q'') = {}^{\xi(q)}[f(q', q'')]f(q, q'q'').$$

If you happen to find an f and a ξ that satisfy this condition, then you can build an extension. You just take $E = N \times Q$ (as a set), and define the multiplication using your f and ξ. However, it is not clear that you can find a ξ and an f satisfying these conditions (the cocycle condition and ξ extends ρ and s).

Remark 17.2 (You can use your favorite ξ)**.** If you had two different lifts ξ and ξ' of ρ, they will differ by some inner automorphism, which is induced by some element of N. Then we can change s by that element of n, then we've changed ξ' into ξ. So we can fix ξ and ask, "does such an f exist?" ⋄

Suppose for the time being that we've found such an f somehow. How unique is this f? That is, how do we tell when two different f's give the same extension. Before, we could change our section around and see how f changes. But now we've already chosen our favorite ξ, so we can only try to change s without changing ξ. Changing s is the same as giving a map from Q to N (then s changes by the image of this map under α). Thus, we can change s by any element in the kernel of α without messing with ξ. That is, we can change s by any map from Q to $Z(N)$ (i.e. a 1-cochain). So this gives us one way to get two equivalent f's.

What if somebody gives us (f,ξ) and $(\tilde f,\xi)$, and we want to know if they give the same extension. How do we tell? Consider the difference $f(q,q')^{-1}\tilde f(q,q')$. Because of the conditions f and $\tilde f$ must satisfy, we must have that $f(q,q')^{-1}\tilde f(q,q')$ is in the kernel of α, which is $Z(N)$. The upshot is that any two f's are related by a function $\beta\colon Q\times Q\to Z(N)$. Furthermore, because of associativity in $Z(N)$, β must satisfy a cocyle condition, so $\beta\in Z^2(Q;Z(N))$. By the previous paragraph, we know that 2-coboundaries give the same extension. Since we know how to go from cocycles to extensions in a way that inverts this, we've basically proven part (b) of the theorem.

[[break]]

As you can see, these cocycles can very quickly connect extensions with H^2, but they are "dirty." We'll use the cocycle stuff a little more, but then we'll have a more elegant way to think about it.

Definition 17.3. A *crossed module* is a pair of group homomorphisms $\partial\colon K\to E$ and $\rho\colon\ \to \mathrm{Aut}(K)$ such that

(1) The diagram
$$\begin{tikzcd} & E \arrow[dr, "\rho"] & \\ K \arrow[ur, "\partial"] \arrow[rr, "\alpha"'] & & \mathrm{Aut}(K) \end{tikzcd}$$
commutes, and

(2)
$$\begin{tikzcd} E \arrow[r, "\rho"] \arrow[d, "\alpha"'] & \mathrm{Aut}(K) \arrow[d, "\partial_*"] \\ \mathrm{Aut}(E) \arrow[r, "\partial^*"] & \mathrm{Hom}(K,E) \end{tikzcd}$$
commutes (i.e. $\partial({}^{\rho(e)}k)=e\partial(k)e^{-1}$)

⋄

Some facts. (1) If $A=\ker(\partial)$, then A is abelian and contained in the center of K (by condition (1)). (2) the image of ∂ is normal [[★★★ by condition (2)?]]; call the quotient Q. (3) E preserves A via the action ρ (by condition (2)). Moreover, we see that the action of E on A factors as $E\to Q\to \mathrm{Aut}(A)$.

Example 17.4. If $N\triangleleft E$, then E acts on N. In this case, A is trivial. ⋄

Example 17.5. We have $N\xrightarrow{\alpha}\mathrm{Aut}(N)$, which acts on N. In this case, $Q=Out(N)$ and $A=Z(N)$. ⋄

Example 17.6. If $K\twoheadrightarrow E$ has abelian kernel, it is another example. ⋄

Example 17.7. If $F\to E\to B$ is a fibration, you get a long exact sequence in homotopy groups, and the map $\pi_1(F)\to\pi_1(E)$ is an example of a crossed module. ⋄

Example 17.8. If $Y\subseteq X$ is a CW pair, there is a long exact sequence in relative homotopy groups, and $\pi_2(X,Y)\to\pi_1(Y)$ is a crossed module. ⋄

Given a crossed module $K\xrightarrow{\partial}E$, we have

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d] & K \arrow[r, "\partial"] \arrow[d, equal] & E \arrow[r, "\pi"'] \arrow[d] & Q \arrow[r] \arrow[d] \arrow[l, bend right, "s"'] & 1 \\ 0 \arrow[r] & Z(K) \arrow[r] & K \arrow[r] & \mathrm{Aut}(K) \arrow[r] & Out(K) \arrow[r] & 1 \end{tikzcd}$$

You can choose a section s as before. The resulting $f\colon Q\times Q\to\ker\pi$ gives us an extension

$$1\to\ker\pi\to E\xrightarrow{\pi}Q\to 1$$

but that's not what we want ($\ker \pi$ is properly contained in K if $A \neq 0$). We can try to find a lift $F\colon Q \times Q \to K$ so that $\partial F = f$. In general, F *will not* satisfy the cocycle condition, but it fails in a controlled way. We get

$$c(q, q', q'')F(q, q')F(qq', q'') = {}^{\xi(q)}[F(q', q'')]F(q, q'q'')$$

for some $c\colon Q \times Q \times Q \to A$. We then can check that c satisfies a cocycle condition (because the action of E on A factors through Q?), so $c \in Z^3(Q; A)$. If you chase through it, you see that c is only defined up to a coboundary. So given a crossed module, you can get an element of $H^3(Q; A)$.

Suppose we had two crossed modules with the same A and Q, and suppose you have a map between them

$$\begin{tikzcd}
 & & K \arrow[r] \arrow[dd, "\phi"] & E \arrow[dr] \arrow[dd, "\phi"'] & & \\
0 \arrow[r] & A \arrow[ur] \arrow[dr] & & & Q \arrow[r] \arrow[ul, bend right] & 1 \\
 & & K' \arrow[r] & E' \arrow[ur] & &
\end{tikzcd}$$

Then you chase through it and you see that you get the same cocycle. This generates an equivalence relation on crossed modules. Let $CM(Q; A)$ be such crossed modules up to this equivalence. So we've shown that if we have a map $CM(Q; A) \to H^3(Q; A)$.

The following theorem will imply part (a) if the theorem.

Theorem 17.9. *$CM(Q; A) \to H^3(Q; A)$ is a bijection. In fact, it is an isomorphism of groups.*

The idea is to first show that crossed modules are functorial. If you have a crossed module, you can pull them back along a map $Q' \to Q$:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] & K \arrow[r] \arrow[d, equal] & E' \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & Q' \arrow[r] \arrow[d] & 1 \\
0 \arrow[r] & A \arrow[r] & K \arrow[r] & E \arrow[r] & Q \arrow[r] & 1
\end{tikzcd}$$

Similarly, you can push forward along maps $A \to A'$. Now you can define an addition on $CM(Q; A)$. Given two crossed modules, you take their direct sum, pull back along the diagonal map $Q \to Q \times Q$ and push forward along the sum map $A \to A \oplus A$ (this is a group homomorphism because A is abelian). This makes $CM(Q; A)$ into an abelian group.

With a little more care, you can see that the map $CM(Q; A) \to H^3(Q; A)$ is compatible with the addition in H^3, so this is a morphism of groups.

We want to show that the map is an isomorphism. The interesting part is to show that the kernel is trivial. We want to show that the zero in H^3 is only hit by "the trivial crossed module". The trivial crossed module is

$$0 \to A = A \xrightarrow{const} Q = Q \to 1$$

Suppose we have a crossed module that goes to zero in H^3.

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, equal] & A \arrow[r] & Q \arrow[r, equal] & Q \arrow[r] & 1 \\
0 \arrow[r] & A \times 1 \arrow[r] \arrow[u, equal] \arrow[d, equal] & A \times K \arrow[r] \arrow[u] \arrow[d] & \tilde{E} \arrow[r] \arrow[u] \arrow[d] & Q \arrow[r] \arrow[u, equal] \arrow[d, equal] & 1 \\
0 \arrow[r] & A \arrow[r] & K \arrow[r, "\partial"] & E \arrow[r] & Q \arrow[r] & 1
\end{tikzcd}$$

The $c \in Z^3$ we get from $F\colon Q \times Q \to K$ is trivial (this is the assumption), so we get the middle row (we've thrown in a little extra A in the first two terms)

We've actually done

$$\begin{tikzcd}
0 \arrow[r] & 1 \arrow[r] \arrow[d] & N \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d, dotted] & Q \arrow[d, equal] \arrow[dd, bend left, "\rho"] \\
0 \arrow[r] & Z(N) \arrow[r] \arrow[d, equal] & N \arrow[r] \arrow[d, equal] & \tilde{E} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & Q \arrow[d, "\rho"'] \\
0 \arrow[r] & Z(N) \arrow[r] & N \arrow[r, "\alpha"] & \mathrm{Aut}(N) \arrow[r] & Out(N)
\end{tikzcd}$$

If we had an extension E, then we get a map of crossed modules to the canonical pullback $\tilde{E}$.

Inside $H^3(Out(N); Z(N))$, there is a canonical class α corresponding to the crossed module $\mathrm{Aut}(N)$ (bottom row). We've shown that there exists an extension only if the pullback class in $H^3(Q; Z(N))$ is zero. On the other hand, [[★★★ the other way somehow]]

18 ???

[1]

Today will be really cool, but there is an unfortunate piece of nomenclature. Also, we will ignore all set-theoretic issues (or assume all categories are small).

A group is a set with an associative binary operation with unit and inverses. A monoid is a generalization of a group where we don't require inverses. But there is another description of what a group is. A group is a category with one object, where all morphisms are invertible. This leads to another generalization, where we don't require there to be only one object. So these are two different generalizations, where the "oid" means completely different things. Normally, you don't talk about both of these things at the same time, but today we will. If you generalize in both directions, you get a category (possibly with many objects) where morphisms may not be invertible. This is just a category.

A *strict monoidal category* is a category C with a functor $\otimes\colon \mathsf{C}\times\mathsf{C}\to\mathsf{C}$ and an object 1 such that $\otimes$ is associative and 1 is a unit (i.e. $1\otimes X = X = X\otimes 1$). This is a bad definition for a subtle reason. Often two constructions are isomorphic (even canonically so), but not equal.

Example 18.1. Consider the set $\{(a,x)|a\in A, x\in B\times C\}$ and the set $\{(y,c)|y\in A\times B, c\in C\}$. These are not the same set, but there is an obvious isomorphism between them. ◇

Example 18.2. Consider the category whose objects are vector spaces with ordered bases, and the morphisms are linear maps respecting the ordered bases. Then $V\otimes W$ has a natural ordering on it's basis (lexicographic ordering). In this case, $V\otimes(W\otimes U)$ is quite different from $(V\otimes W)\otimes U$. ◇

Definition 18.3. A *monoidal category* is a category C, a functor $\otimes\colon \mathsf{C}\times\mathsf{C}\to\mathsf{C}$, a unit 1, and natural isomorphisms $a\colon(-\otimes-)\otimes- \cong -\otimes(-\otimes-)$, $l\colon 1\otimes - \cong \mathrm{id}_{\mathsf{C}}$, and $r\colon -\otimes 1\cong \mathrm{id}_{\mathsf{C}}$. So that all diagrams you want to commute, commute. A monoidal functor is more or less what you think it is. ◇

[1]This lecture was given by Chris Schommer-Pries.

This second to last sentence is equivalent to just two diagrams commuting (Mac Lane's coherence theorem)

$$\begin{tikzcd}
& {[-\otimes(-\otimes-)]\otimes-} \arrow[dr] & \\
{[(-\otimes-)\otimes-]\otimes-} \arrow[ur] \arrow[d] & & {-\otimes[(-\otimes-)\otimes-]} \arrow[d] \\
(-\otimes-)\otimes(-\otimes-) \arrow[rr] & & {-\otimes[-\otimes(-\otimes-)]}
\end{tikzcd}$$

$$\begin{tikzcd}
(-\otimes 1)\otimes- \arrow[rr] \arrow[dr] & & -\otimes(1\otimes-) \arrow[dl] \\
& -\otimes- &
\end{tikzcd}$$

For any category C, we can define $\pi_0(\mathsf{C})$ to be the set (class) of isomorphism classes of C. This may be a set even if C is not small. We can also define $\pi_1(\mathsf{C},x)$ to be the monoid $\mathsf{C}(x,x)$.

What happens if C is a monoidal category? Note that $\otimes$ induces a monoid structure on $\pi_0(\mathsf{C})$.

Definition 18.4. A category $\mathcal{S}$ is *skeletal* if there is only one object in each isomorphism class. That is, $Ob(\mathcal{S}) = \pi_0(\mathcal{S})$. ◇

Given a category C, we may choose representatives for $\pi_0(\mathsf{C})$ (by the axiom of choice). Define $\mathcal{S}$ to be the full subcategory of C with these objects. There is a natural inclusion functor $i\colon\mathcal{S}\to\mathsf{C}$, and the claim is that i is an equivalence of categories. To see this, note that i is essentially surjective by construction, and because it is defined as a full subcategory, it is fully faithful. It's good to see at least once why an essentially surjective fully faithful functor is an equivalence. Define a functor $T\colon\mathsf{C}\to\mathcal{S}$ by taking $T(c)$ to be the representative x of $\pi_0(\mathsf{C})$ in the same isoclass as c, and how do we define T on morphisms? For all objects $c\in\mathsf{C}$, choose an isomorphism $\theta_c\colon c\xrightarrow{\sim} Tc$ (again using choice). We want this θ to be a natural isomorphism from id_{C} to $T\circ i$, so for $f\colon c\to c'$ just define $Tf = \theta_{c'}\circ f\circ\theta_c^{-1}$.

If C is a monoidal category, then we get an induced monoidal structure on a skeleton $\mathcal{S}$ (so that the inclusion $i\colon\mathcal{S}\to\mathsf{C}$ is a morphism of monoidal cateogies). However, you'll end up with crazy associators. That is, even if C starts off being strict, the monoidal structure on $\mathcal{S}$ may not be strict!

[[break]]

Ok, what is the relationship with homological algebra? Let's assume (for lack of time) that C is a groupoid with a monoidal structure, and assume that π_0 is a group.

Definition 18.5. A *Picard groupoid* is a monoidal groupoid C so that for all $x \in \mathsf{C}$, there exists $\overline{x} \in \mathsf{C}$ such that $x \otimes \overline{x} \cong 1 \cong \overline{x} \otimes x$. ◇

Example 18.6. If A is a monoidal category, it contains a maximal groupoid (the subcategory of all isomorphisms), which contains some maximal Picard groupoid (the full subgroupoid of "invertible objects"). ◇

Example 18.7. If R is a ring, consider (R, R)-bimodules with monoidal structure $\otimes_R$. Inside of this category, we have $Pic(R)$, whose objects are "invertible" bimodules ${}_RM_R$ and whose morphisms are isomorphisms of bimodules. In this case, $\pi_0(Pic(R)) = Pic$ is a group, and you can check, for example, that $\pi_1(Pic(R), 1) \cong Z(R)^\times$. ◇

Example 18.8. Taking $R = \mathbb{C}$, the previous example gives $Pic(\mathbb{C})$, the category of "lines". ◇

Example 18.9. Let $Y \subseteq X$ be a CW pair, with $* \in Y$. Define the relative Picard groupoid to have objects homotopy classes of maps $(D^1, \partial D^1) \to (Y, *)$, and whose morphisms are homotopy classes of maps $(D^1 \times D^1, \partial(D^1 \times D^1), \partial D^1 \times D^1) \to (X, Y, *)$, with the usual composition. Moreover, there is a monoidal structure (given by gluing the objects like in π_1 and gluing squares appropriately). This is a Picard groupoid. It turns out it tells you something about the relative homotopy groups $\pi_n(X, Y)$. ◇

Theorem 18.10. *Any skeletal Picard groupoid C are of the following form:*

- *$\pi_0(\mathsf{C}) = Q$ is a group, $\pi_1(\mathsf{C}, 1) = A$ is an abelian group, $\pi_1(\mathsf{C}, x) \cong A$ for any $x \in \mathsf{C}$, Q acts on A,*
- *all the morphisms together form a group under $\otimes$, and this group is $Q \ltimes A$, but the associator is not trivial.*

The objects are Q and the morphisms are $Q \times A$ (there are no morphisms between objects which are not equal, because skeletal)

Proof. HW 9. □

What is the associator a? At the object level, if we plug in $x, y, z \in Q$, then a gives us an automorphism of $xyz \in Q$, so it is some $a(x, y, z) \in A$. If we have morphisms f, g, h, then we get that $fgh \circ a(x, y, z) = a(x, y, z) \circ fgh$, which is always going to be satisfied because A is abelian. So the associator is of the form $a \colon Q \times Q \times Q \to A$. But it cannot be any map because it has to satisfy the pentagon axiom. That is, for $w, x, y, z \in Q$, we must have that

$$a(x, y, z) - a(wx, y, z) + a(w, xy, z) - a(w, x, yz) + a(w, x, y) = 0$$

That is, a must be a cocycle in $Z^3(Q, A)$. Actually, I've cheated when talking about what a natural transformation is (I assumed the action of Q on A is trivial), but you can fix it. You can check that if you change a by a coboundary, you get an isomorphic monoidal category.

So we started with Picard groupoids with $\pi_0 = Q$ and $\pi_1 = A$ (with a given action), gone to skeletal Picard groupoids, and then gone to $H^3(Q; A)$. Last time we saw that $H^3(Q; A)$ parameterizes crossed modules. Recall that a crossed module is a morphism $\partial \colon K \to E$ and a morphism $\rho \colon E \to \operatorname{Aut}(K)$ with some conditions such that $E/K \cong Q$ and $\ker \partial = A$.

There is another way to see this relationship. Suppose we were only interested in strict Picard groupoids. Then the objects form a group G_0 (you need strictness for this), and the morphisms (taken all together) form a group G_1. You have the source and target morphisms $G_1 \rightrightarrows G_0$ (which must be group homomorphisms). Additionally, you have identity maps, so you have a group homomorphism $G_0 \to G_1$, which splits the source map s, so we can take $E = G_0$, then $G_1 = K \rtimes E$. Then the target map induces $t \colon K \to E$, which has to satisfy some conditions (exactly the conditions of a crossed module). It turns out that an equivalence of crossed modules induces a map on groupoids. If you only allow strict monoidal functors, this need not be an equivalence, but if you allow all monoidal functors, this is an equivalence.

19 Dold-Kan Correspondence

Definition 19.1. A is an *abelian* category if

1. A is enriched[1] over $(\mathsf{Ab}, \otimes)$ (in general, such a category is called additive).
2. A has all finite limits and colimits. In particular, kernels and cokernels (equalizers and coequalizers with the zero map) exist. You can check that, in fact, if you have kernels, cokernels, and finite products and coproducts, then you get all finite limits and colimits.
3. For all $f\colon X \to Y$, the natural map from $\operatorname{coim}(f) := \operatorname{coker}(\ker(f) \to X)$ to $\operatorname{im}(f) := \ker(Y \to \operatorname{coker}(f))$ is an isomorphism. [[★★★ in particular, this tells you that the natural map from the initial object to the terminal object is an isomorphism.]] ◇

Example 19.2. Ab, R-mod (R a ring), $\mathsf{Ab}(X)$ (sheaves of abelian groups on a topological space X), and $\mathcal{R}$-mod (for a sheaf of rings $\mathcal{R}$) are abelian categories. ◇

Example 19.3 (non-examples)**.** Filtered abelian groups and topological abelian groups are not abelian categories (they don't satisfy the last conditions). ◇

In an abelian category A, you can talk about two maps composing to zero, so you can talk about $\mathsf{Chain}(\mathsf{A})$. Because of the third condition on abelian categories, the homology of a chain complex is well-defined.

Recall that for $A_\bullet \in s\mathsf{Ab}$, $N_n(A_\bullet) = \bigcap_{i>0} \ker d_i$, with $d\colon N_n(A_\bullet) \to N_{n-1}(A_\bullet)$ given by d_0. If $A_\bullet \in s\mathsf{A}$ for some abelian category A, we can define $N_n(A_\bullet) = \ker(\prod_{i>0} d_i)$, and d_0 induces a differential. Remember that we also had a functor $Alt_*\colon ss\mathsf{Ab} \to \mathsf{Chain}$; you can also do this for an arbitrary abelian category, but we don't want to use it. By the way, there is a functor [[★★★ maybe an adjoint to Alt_*, but it doesn't look like it.]] $F_\bullet$, given by $F_n(C_*) = C_n$, with $d_i\colon F_n(C_*) \to F_{n-1}(C_*)$ given by $d_i = 0$ for $i > 0$ and $d_0 = d$. You can easily check the simplicial identities (all the compositions are zero).

Theorem 19.4. *For an abelian category A, the Moore complex functor $N_*\colon s\mathsf{A} \to \mathsf{Chain}(\mathsf{A})$ is an equivalence of categories.*

This is the linear analogue of the statement that simplicial sets and topological spaces have the same homotopy category.

Proof. We'll do the case $\mathsf{A} = \mathsf{Ab}$, but you usually care about the theorem in the case of modules on a ringed space. The proof works for any abelian category.

We claim that the inverse to N_* is given by $K_\bullet := L \circ F_\bullet\colon \mathsf{Chain} \to ss\mathsf{Ab} \to s\mathsf{Ab}$. We'll verify this after the break.

[[break]]

Some observations:

(a) $Alt_* \circ F_\bullet = \mathrm{id}_{\mathsf{Chain}}$. This is pretty clear.

(b) $N_* \circ L \cong Alt_*$ (by HW7, problem 3).

(c) $N_*(A_\bullet) \cong Alt_*(A_\bullet)/D_*(A_\bullet)$ (by HW7, problem 3), where D_* is the subcomplex of Alt_* consisting of degenerate simplicies.

It follows (from b and a) that $N_* \circ L \circ F_\bullet \cong Alt_* \circ F \cong \mathrm{id}_{\mathsf{Chain}}$. Now we claim that there are natural isomorphisms $K_\bullet N_*(A_\bullet) \xrightarrow{\alpha} A_\bullet$. Well,

$$K_m N_*(A_\bullet) = \bigoplus_{\sigma\colon [m]\twoheadrightarrow[n]} N_n^{(\sigma)}(A_\bullet)$$

The map α_m is given by sending $x \in N_n^{(\sigma)}(A_\bullet)$ to $A(\sigma)(x) \in A_m$. We have to check that α is a morphism in $s\mathsf{Ab}$ and that it is an isomorphism. We have to check that $A(f)\alpha_m = \alpha_{m'} K_\bullet N_*(f)$ for any $f\colon [m'] \to [m]$ in Δ. Well, any such f can be factored canonically as a surjection $\sigma'\colon [m'] \twoheadrightarrow [n']$ followed by an injection $f'\colon [n'] \hookrightarrow [n]$. But you may

[1] Given a monoidal category $(\mathsf{C}, \otimes, \dots)$, an *enriched category over* $(\mathsf{C}, \otimes)$, A, has objects $Ob(\mathsf{A})$ and *morphism objects* (need not be sets!) $\mathsf{A}(X, Y)$ for every pair of objects $X, Y \in Ob(\mathsf{A})$, with composition morphisms $c_{X,Y,Z}\colon \mathsf{A}(Y, Z) \otimes \mathsf{A}(X, Y) \to \mathsf{A}(X, Z)$, with an "identity" $1 \to \mathsf{A}(X, X)$ for each $X \in Ob(\mathsf{A})$, such that composition is associative (figure out what this means), and the identity is an identity. If $\mathsf{C} = \mathsf{Set}$ with $\otimes = \times$, this is the notion of a category. If C has a forgetful functor to Set, it is easier to think about enriched categories, but you have to be careful about what the monoidal structure is. For example, Ab has two nice monoidal structures, given by $\otimes$ and $\oplus$.

recall that $K_\bullet N_*(f)(x) = A(f')(x) \in N_{n'}^{(\sigma')}(A_\bullet)$, which is sent by α to $A(\sigma')(A(f')(x)) = A(f)(A(\sigma)(x))$ (by $f'\sigma' = \sigma f$), which is exactly what $A(\sigma)(x)$ is sent to. So we get the commutativity we wanted, so α is a morphism in $s\mathsf{Ab}$.

Now we want to show that α is an isomorphism. Note that $N_*(\alpha)\colon N(LF_\bullet N_*(A_\bullet)) \to N_*(A_\bullet)$ is the identity (rather, the isomorphism we had before).

Finally, we need a lemma: if $f: A_\bullet \to B_\bullet$ is a morphism in $s\mathsf{Ab}$ such that $N_*(f)$ is an isomorphism, then f is an isomorphism. Once we have this, we're clearly done.

The proof the lemma (that $f_n\colon A_n \to B_n$ is an isomorphism) is done by induction on n. If $n = 0$, then $N_0(A_\bullet) = A_0$, so it is easy. Now assume we've proven it up to n (for all $A_\bullet$, $B_\bullet$, and f). Define the *Path-Loop-Space* $PA_\bullet$ of a simplicial abelian group $A_\bullet : \Delta^\circ \to \mathsf{Ab}$ to be the composition of $A_\bullet$ with $P\colon \Delta \to \Delta$, given by $[n] \mapsto [n+1] = [n] \cup \{\infty\}$ (all maps send ∞, the largest element, to ∞). I claim that there is a morphism $\pi\colon PA_\bullet \to A_\bullet$ given by $PA_n = A_{n+1} \xrightarrow{d_{n+1}} A_n$. Define the *loop space* $\Lambda A_\bullet$ to be the kernel of $PA_\bullet \to A_\bullet$ (which happens to be onto).

$$\begin{tikzcd}
 & & A_{n+1} \arrow[d, equal] & & \\
0 \arrow[r] & \Lambda A_n \arrow[r] \arrow[d, "(\Lambda f)_n"] & PA_n \arrow[r, "\pi"] \arrow[d, "(Pf)_n = f_{n+1}"] & A_n \arrow[r] \arrow[d, "f_n"] & 0 \\
0 \arrow[r] & \Lambda B_n \arrow[r] & PB_n \arrow[r] & B_n \arrow[r] & 0 \\
 & & B_{n+1} \arrow[u, equal] & &
\end{tikzcd}$$

By induction, the last downward map is an isomorphism. We'll show that $(\Lambda f)_n$ is an isomorphism, so by the 5-Lemma, f_{n+1} is an isomorphism. $\square$

20 ???

One more class on simplicial stuff, then we'll move on to spectral sequences.

We had the following diagram for any abelian category A.

$$\begin{tikzcd} \mathsf{Chain}^{\geq 0}(\mathsf{A}) \arrow[dr, "F_\bullet"'] & & s\mathsf{A} \arrow[ll, "N_*"'] \\ & ss\mathsf{A} \arrow[ur, "L"'] & \end{tikzcd}$$

There is a point in the argument where you have to use the last axiom of an abelian category. The argument went like this. We checked that $N_*LF \cong \mathrm{id}_{\mathsf{Chain}}$. For the other direction, we need to show that $LF_\bullet N_*(A_\bullet) \cong A_\bullet$. We want to get a morphism in $s\mathsf{A}(LF_\bullet N_*(A_\bullet), A_\bullet) \cong ss\mathsf{A}(F_\bullet N_*(A_\bullet), UA_\bullet)$ (where $UA_\bullet$ is just $A_\bullet$ thought of as a semi-simplicial set).

Lemma 20.1. $ss\mathsf{A}(F_\bullet C_*, B_\bullet) \cong \mathsf{Chain}(C_*, N_*B_\bullet)$.

Once we have that lemma, the identity map on $N_*A_\bullet$ induces a map in $\mathsf{Chain}(N_*A_\bullet, N_*A_\bullet) \cong s\mathsf{A}(LF_\bullet N_*A_\bullet, A_\bullet)$.

Proof. Suppose we have a map $\phi\colon F_\bullet C_* \to B_\bullet$, so maps $\phi_n\colon F_nC_* = C_n \to B_n$ such that $\phi_n d_i = d_i\phi_{n+1}$. Since all the differentials (except the 0-th one) are zero, so such a ϕ consists of maps so that $d_i\phi_{n+1} = 0$ for $i > 0$ (image of ϕ lies in the kernels of all the d_i, $i > 0$) and so that $d_0\phi_{n+1} = \phi_n d$, which just says that the differentials agree, so this is just a chain map from C_* to $N_*B_\bullet$. □

So we have this map α. It has the property that $N_*\alpha$ is an isomorphism (it's the identity map). The final step is to prove that this α is an isomorphism. We did this with the following lemma.

Lemma 20.2. *If $f\colon A_\bullet \to B_\bullet$ is a morphism so that N_*f is an isomorphism, then f is an isomorphism.*

The proof uses the path construction. Given a simplicial object $X_\bullet\colon \Delta^\circ \to \mathsf{A}$. Let $P\colon \Delta \to \Delta$ be the functor given by $[n] \mapsto [n+1] = [n] \cup \{\infty\}$. Then we call the composition $\Delta^\circ \xrightarrow{P} \Delta^\circ \xrightarrow{X_\bullet} \mathsf{A}$ the *path object* $PX_\bullet$ of $X_\bullet$. We have map $PX_\bullet \to X_\bullet$, induced by the natural transformation of endofunctors $\mathrm{id}_\Delta \to P$ (given by $d_{n+1}\colon [n] \to [n+1]$).

If A is abelian, we can define the *loop object* $\Lambda X_\bullet := \ker(PX_\bullet \to X_\bullet)$. Note that this only makes sense for an abelian category A. For what follows, we write chain complexes as $C_* = (0 \leftarrow C_0 \leftarrow C_1 \leftarrow \cdots)$. We define $C_*[1]$ to be $(0 \leftarrow C_1 \leftarrow C_2 \leftarrow \cdots)$. This is not invertible because you lose C_0.

Lemma 20.3. $N_*(\Lambda A_\bullet) \cong N_*(A_\bullet)[1]$

Proof.

$$N_n(\Lambda A_\bullet) = N_n\big(\ker(A_{n+1} \xrightarrow{d_{n+1}} A_n)\big) = \bigcap_{i=1}^{n+1} \ker d_i$$

$$= N_{n+1}(A_\bullet) = (N_*(A_\bullet)[1])_n$$

and the differentials agreeing (both are d_0) □

Proof of Lemma 20.2. We do it by induction on n. For $n = 0$, it is clear, now assume the result up to n. Then we get

$$\begin{tikzcd} & & A_{n+1} \arrow[d, equal] & & \\ 0 \arrow[r] & \Lambda A_n \arrow[r] \arrow[d, "(\Lambda f)_n"] & PA_n \arrow[r, "\pi"] \arrow[d, dashed, "(Pf)_n = f_{n+1}"] & A_n \arrow[r] \arrow[d, "f_n", "\wr"'] & 0 \\ 0 \arrow[r] & \Lambda B_n \arrow[r] & PB_n \arrow[r] \arrow[d, equal] & B_n \arrow[r] & 0 \\ & & B_{n+1} & & \end{tikzcd}$$

By induction, $(\Lambda f)_n$ is an isomorphism, so by the 5-Lemma, we're done. □

Remark 20.4. The 5-Lemma requires the third axiom of an abelian category, so we really need to have an abelian category. In a category without the third axiom, the diagram chase gives us that $\ker f_{n+1} = 0$ and $\operatorname{coker} f_{n+1} = 0$. ◇

Lemma 20.5. *If you're in an additive category with all finite limits and colimits and $f\colon A \to B$, with $\ker f = \operatorname{coker} f = 0$ and $\operatorname{im} f = \operatorname{coim} f$, then f is an isomorphism.*

Every map f factors as

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d] & B \\
\operatorname{coker}(\ker f \to A) =: \operatorname{coim} f \arrow[r] & \operatorname{im} f := \ker(B \to \operatorname{coker} f) \arrow[u]
\end{tikzcd}$$

We have that the two vertical maps are isomorphisms because $\ker f = \operatorname{coker} f = 0$, and the bottom arrow is an isomorphism by assumption.

[[break]]

$$\begin{tikzcd}
\mathsf{Chain} \arrow[r, shift left, "LF_\bullet"] & s\mathsf{Ab} \arrow[l, shift left, "N_*"] \arrow[dr, "U"'] & s\,\mathsf{Set} \arrow[l, "Free"'] \arrow[r, "|\cdot|"] & \mathsf{Top} \arrow[dl, "\Delta_\bullet"] \\
& & \cup \\
& & \mathsf{Kan}
\end{tikzcd}$$

$$\begin{tikzcd}
H_n \arrow[r, leftrightarrow] & \pi_n \arrow[r, "U"] & \pi_n & \pi_n \arrow[l] \\
& & H_n \arrow[ul, "Free"] & H_n \arrow[l]
\end{tikzcd}$$

$\pi_n(X_\bullet, *)$ is defined for any Kan simplicial set. For example, $\Delta_\bullet(X)$ is Kan for any topological space X (you use a retraction of Δ^n onto any horn $\wedge^k$). In HW8, we showed that $\pi_n(\Delta_\bullet X) \cong \pi_n(X)$.

Another example of a Kan semi-simplicial set is a simplicial group. In HW8, we showed that $H_n(N_* A_\bullet) \cong \pi_n(A_\bullet, 0)$.

Definition 20.6. If $X_\bullet \in s\,\mathsf{Set}$, then $H_n(X) := \pi_n(Free(X_\bullet))$. ◇

Corollary 20.7. *Let (A, n) be the chain complex A concentrated in degree n. Then $X(A, n) := |ULF_\bullet(A, n)|$ is a $K(A, n)$. That is, $\pi_i(X(A, n)) = 0$ for $i \neq n$ and $\pi_n(X(A, n)) = A$.*

An isomorphism that we left out of HW8 is that if $X_\bullet$ is Kan, then $\pi_n(X_\bullet) \cong \pi_n(|X_\bullet|)$. There is a canonical map $\pi_n(X_\bullet) \to \pi_n(|X_\bullet|)$ basically given by $|\cdot|$. You prove that it is injective and surjective using simplicial approximation theorems.

Let Top_n be the full subcategory of Top consisting of spaces with $\pi_i = 0$ for $i \neq n$. We have a functor $F_n = ULF_\bullet[n]\colon \mathsf{Ab} \to \mathsf{Top}_n$ and a functor $\pi_n\colon \mathsf{Top}_n \to \mathsf{Ab}$ (assuming $n \geq 2$). Is this an equivalence of categories? Certainly not, because there are lots of maps between topological spaces, not just coming from group homomorphisms.

If C is a category (like Top or Chain) and W (weak equivalences) is a collection of morphisms, then the localization $\mathsf{C}[W^{-1}]$ is a category with the following universal property

$$\begin{tikzcd}
\mathsf{C} \arrow[rr, "i"] \arrow[dr, "\forall j"'] & & \mathsf{C}[W^{-1}] \\
& \mathsf{D} \arrow[ur, dashed, "\exists!"'] &
\end{tikzcd}$$

(the uniqueness is up to natural isomorphism)

Remark 20.8. $\mathsf{C}[W^{-1}]$ doesn't always exists, but when it exists, it is unique. ◇

Definition 20.9. Define the derived categories $d\mathsf{Top} = \mathsf{Top}[W^{-1}]$ (W usual weak equivalences) and $d\mathsf{Chain} = \mathsf{Chain}[W^{-1}]$ (W quasi-isomorphisms). ◇

More generally, for a functor F, you can try to formally invert all the morphisms that become isomorphisms under F. You can't always do it, but you can try.

Theorem 20.10. *For $n \geq 2$, the functor $F_n\colon \mathsf{Ab} \to d\mathsf{Top}_n = h\mathsf{CW}_n$ is an equivalence of categories.*

The proof is by obstruction theory.

Theorem 20.11. $\mathsf{Set} \xrightarrow{\sim} d\mathsf{Top}_0$ *and* $\mathsf{Gp} \xrightarrow{\sim} d\mathsf{Top}_1^{pt}$ *(given by* $G \mapsto BG = |N_\bullet(\mathsf{C}_G)|$*). Moreover* $h\mathsf{Gpoid} \xrightarrow{\sim} d\mathsf{Top}_{\leq 1}$ *(given by* $G \mapsto |N_\bullet(G)|$*, where a weak equivalence of groupoids is an equivalence).*

Chris took small Picard groupoids, and showed that the homotopy category is equivalent to $d\mathsf{Top}_{\leq 2}^{pt,conn}$, with $\pi_0 G \leftrightarrow \pi_1 X$ and $\pi_1 G \leftrightarrow \pi_2 X$. X connected, with $\pi_i(X, x_0) = 0$ for $i > 2$. Then I get a fibration $K(\pi_2 X, 2) \to X \xrightarrow{\pi} K(\pi_1 X, 1)$ (more generally, you get a Postnikov tower). The k-invariant is the obstruction to finding a section of π, and it lives in $H^3(\pi_1 X; \pi_2 X)$.

Theorem 20.12. $h(2-\mathsf{Gpoid}) \xrightarrow{\sim} d\mathsf{Top}_{\leq 2}$ *(weak equivalences on 2-groupoids are functors inducing isos on* π_0*,* π_1*,* π_2*.*

A Picard groupoid is a 2-groupoid with only one object.

21 Spectral sequences

I predict that all of you will need spectral sequences at some point, so you shouldn't be afraid of them. The goal is to be able to do computations. So far, we can't compute group homology except for some special cases. This tool will allow you to compute them for a lot of groups.

Definition 21.1. A *spectral sequence* in an abelian category A consists of the following data:

1. objects $E^r_{p,q} \in \mathsf{A}$ ($p, q \in \mathbb{Z}$, $r \in \mathbb{Z} \geq a$ (usually $a = 0, 1$, or 2) with differentials $d_r \colon E^r_{p,q} \to E^r_{p-r,q+r-1}$ (with $d_r^2 = 0$),
2. "turning the page" isomorphisms $E^{r+1}_{p,q} \cong H(E^r_{p,q}, d_r)$.

$\diamond$

Think of r as fixed, then the collection $E^r_{p,q}$ is called the "r-th page" of the spectral sequence. It is a grid of objects, with diagonal differentials. Suppose $r = 1$, then the page looks like the thing on the left

$$\begin{tikzcd}
{} & E^1_{0,2} \arrow[l, "d_1"'] & E^1_{1,2} \arrow[l, "d_1"'] & E^1_{2,2} \arrow[l, "d_1"'] \\
{} & E^1_{0,1} \arrow[l, "d_1"'] & E^1_{1,1} \arrow[l, "d_1"'] & E^1_{2,1} \arrow[l, "d_1"'] \\
{} & E^1_{0,0} \arrow[l, "d_1"'] & E^1_{1,0} \arrow[l, "d_1"'] & E^1_{2,0} \arrow[l, "d_1"']
\end{tikzcd}
\qquad
\begin{tikzcd}
{} & E^2_{0,2} & E^2_{1,2} & E^2_{2,2} & {} \\
{} & E^2_{0,1} & E^2_{1,1} \arrow[ull, "d_2"] & E^2_{2,1} \arrow[ull, "d_2"] & {} \arrow[ull, "d_2"] \\
{} & E^2_{0,0} & E^2_{1,0} \arrow[ull, "d_2"] & E^2_{2,0} \arrow[ull, "d_2"] & {} \arrow[ull, "d_2"]
\end{tikzcd}$$

When you turn the page (by taking homology), you get the second page (on the right). Then you keep turning the pages and the differentials go further to the left and up.

Definition 21.2. The *total degree* of $E^r_{p,q}$ is $p+q$. Then each d_r *decreases* the total degree by 1. $\diamond$

The things of a fixed total degree are all the things on a fixed anti-diagonal.

If we ignore the bigrading (so take $E^r = \bigoplus E^r_{p,q}$), we get cycles $\ker d_r = B^r \subseteq E^r$ and boundaries $\operatorname{im} d_r = Z^r \subseteq E^r$. Then $E^1 = Z^0/B^0$, and the kernel of d_1 is Z^1/B^0 and the image is B^1/B^0. You can keep going, writing everything as a subquotient of E^0. We know that $E^r \cong Z^r/B^r$. We can define $Z^\infty = \bigcap_{r\geq 0} Z^r$ and $B^\infty = \bigcup_{r\geq 0} B^r$ (assume for the moment that these exist in A), and then define the ∞-*page* $E^\infty = Z^\infty/B^\infty$.

Definition 21.3. A spectral sequence $\{E^r_{p,q}, d_r\}$ *(weakly) converges* to H_* (written $E^a_{p,q} \Rightarrow H_{p+q}$) if we have

1. objects $H_n \in A$ ($n \in \mathbb{Z}$) that are filtered (increasingly), so $0 \subseteq \cdots F_p H_n \subseteq F_{p+1} H_n \cdots \subseteq H_n$,[1] and
2. isomorphisms $E^\infty_{p,q} \cong F_p H_{p+q}/F_{p-1} H_{p+q}$. $\diamond$

Q: if the spectral sequence converges to zero (and we don't require the extra condition), then it looks like it converges to anything with the trivial filtration.

Definition 21.4. A spectral sequence $\{E^r_{p,q}, d_r\}$ is *bounded* if for all r and n, all but finitely many of the $E^r_{p,q}$ of total degree n are zero. $\diamond$

Example 21.5 (first quadrant spectral sequences)**.** If $E^r_{p,q} = 0$ whenever $p < 0$ or $q < 0$, then the spectral sequence is bounded). $\diamond$

Note that (in any spectral sequence) once you know $E^r_{p,q} = 0$, then $E^{r+1}_{p,q} = E^\infty_{p,q} = 0$. In a bounded spectral sequence, for each p and q, there is an r such that $E^r_{p,q} \cong E^{r+1}_{p,q} \cong E^\infty_{p,q}$. This is because the differentials eventually go to and come from zero.

Definition 21.6. A bounded spectral sequence $\{E^r_{p,q}, d_r\}$ *converges* to H_* (written $E^a_{p,q} \Rightarrow H_{p+q}$) if we have

1. objects $H_n \in A$ ($n \in \mathbb{Z}$) that are finitely filtered (increasingly), so $0 = F_s H_n \subseteq \cdots F_p H_n \subseteq F_{p+1} H_n \cdots \subseteq F_t H_n = H_n$, and
2. isomorphisms $E^\infty_{p,q} \cong F_p H_{p+q}/F_{p-1} H_{p+q}$. $\diamond$

[1] Some books require $\bigcup F_p H_n = H_n$ or $\bigcap F_p H_n = 0$. We should probably require this.

Example 21.7 (Hochschild-Serre spectral sequence)**.** For any group extension

$$1 \to N \to G \to Q \to 1$$

there is a spectral sequence $E^2_{p,q} = H_p(Q; H_q(N)) \Rightarrow H_{p+q}(G)$. This is obviously first quadrant, so it is bounded. The main unknown in this game are the differentials. Notice also that there is an extension problem to be solved ($H_{p+q}(G)$ is only given by a series of extensions of $E^\infty_{p,q}$).

What is the action of Q on $H_q(N)$? You get a morphism $\rho\colon Q \to Out(N)$. There is a lemma that if you have a group N, the inner automorphisms act trivially on $H_q(N)$. It follows that you get an action of $Out(N)$ on $H_q(N)$, which induces an action of Q on $H_q(N)$. $\diamond$

[[break]] Consider

$$\begin{tikzcd} & \langle (1\ 2\ 3) \rangle \arrow[d, equal] & & & \\ 1 \arrow[r] & \mathbb{Z}/3 \arrow[r] & S_3 \arrow[r, "\sigma"] & \mathbb{Z}/2 \arrow[r] & 1 \end{tikzcd}$$

Then the Hochschild-Serre spectral sequence is $H_p(\mathbb{Z}/2, H_q(\mathbb{Z}/3)) \Rightarrow H_{p+q}(S_3)$. Recall that $H_0(G; M) = M_G$ and $H^0(G; M) = M^G$. So the E^2 page is

7	$\mathbb{Z}/3$						
6							
5							
4	0	0	0	0	0	$\cdots$	
3	$\mathbb{Z}/3$						
2	0	0	0	0	0	$\cdots$	
1	$0 = (\mathbb{Z}/3)_{\mathbb{Z}/2}$						
0	$\mathbb{Z}$	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	$\cdots$
	0	1	2	3			

If $\mathbb{Z}/2 \to \mathrm{Aut}(\mathbb{Z}/3)$ were trivial, the extension would be a direct product. I claim that for any action, $H^2(\mathbb{Z}/2; \mathbb{Z}/3) = 0$ (this means that any extension splits). What is the map $\mathrm{Aut}(\mathbb{Z}/3) \to \mathrm{Aut}(H_q(\mathbb{Z}/3))$. Since $H_1(\mathbb{Z}/3) = \mathbb{Z}/3$, the action of $\mathbb{Z}/2$ is nontrivial, so we get $E^2_{0,1} = (\mathbb{Z}/3)_{\mathbb{Z}/2} = (\mathbb{Z}/3)/\langle a - (-a) \rangle = 0$. $H_{even}(\mathbb{Z}/3) = 0$, so the coinvariants are also zero (and $H_p(\mathbb{Z}/2; H_q(\mathbb{Z}/3)) = 0$ for all p). I claim that $H_{1+4n}(\mathbb{Z}/3)$ has trivial $\mathbb{Z}/2$-action and $H_{3+4n}(\mathbb{Z}/3)$ has non-trivial $\mathbb{Z}/2$-action (we'll prove this lemma soon), so the first column is 4-periodic

Lemma 21.8. *$H_p(G; \mathbb{Z}/n) = 0$ if $|G| < \infty$ and $gcd(n, |G|) = 1$.*

Proof. (1) multiplication by $|G|$ annihilates $H_p(G; M)$ for all $p > 0$. This uses the *transfer map.* If $\pi\colon (E, e_0) \to (B, b_0)$ is an k-sheeted cover ($k < \infty$), then you have an induced map $\pi_*\colon H_n(E, e_0) \to H_n(B, b_0)$, but you also get a map the other way, $tr\colon H_n(B, b_0) \to H_n(E, e_0)$. If you have a map $\sigma\colon \Delta^n \to B$, then there are k lifts of this map to E; call them $\sigma_1, \dots, \sigma_k$. The transfer map is induced by the chain map in singular complexes $[\sigma] \mapsto \sum_{i=1}^k [\sigma_i]$. You can check that $\pi_* \circ tr$ is multiplication by k.

Now we can apply this transfer map to $EG \to BG$, which is a $|G|$-sheeted covering map. Then multiplication by $|G|$ on $H_n(BG) = H_n(G)$ factors through $H_n(EG) = 0$.

(2) If $gcd(m, n) = 1$, then $m\colon H_p(X, \mathbb{Z}/n) \to H_p(X, \mathbb{Z}/n)$ is an isomorphism (this is because it is an isomorphism on $\mathbb{Z}/n$ and H_p is a functor).

The only way both of these can be true if $H_p(|G|; \mathbb{Z}/n) = 0$. $\square$

The upshot is that everything else in our E^2 page is zero. If you look at this E^2 page, then you see that none of the differentials (for any r) can be nontrivial! This is basically because $E^{even>0} = 0$. So $E^\infty = E^2$. So we get

i	$H_i(S_3)$
0	$\mathbb{Z}$
1	$\mathbb{Z}/2$
2	0
3	$\mathbb{Z}/6$
4	0
5	4-periodic from here on

Note that there are no non-trivial abelian extensions of $\mathbb{Z}/2$ by $\mathbb{Z}/3$.

$$\begin{array}{ccc} E_{0,2} & E_{1,2} & E_{2,2} \\ E_{0,1} & E_{1,1} & E_{2,1} \\ E_{0,0} & E_{1,0} & E_{2,0} \end{array}$$

22 Leray-Serre spectral sequence

The lemma left over from last time was on HW1. [[★★★]]

Theorem 22.1. *If $F \to E \xrightarrow{\pi} B$ is a (Serre) fibration, then there exists a spectral sequence $E^2_{p,q} = H_p(B; H_q(F)) \Rightarrow H_{p+q}(E)$.*

Recall that a fibration $F \to E \xrightarrow{\pi} B$ is a map $\pi\colon E \to B$, where B is a path connected space with a point b_0, and $F = \pi^{-1}(b_0)$ such that for any homotopy $D^n \times I \to B$ and any lift of $D \times \{0\}$ to E, there exists a lift of the homotopy:

$$\begin{tikzcd} D^n \arrow[r] \arrow[d, hook, "i_0"'] & E \arrow[d, "\pi"] \\ D^n \times I \arrow[r] \arrow[ur, dashed, "\exists"] & B \end{tikzcd} \qquad \begin{tikzcd} |\Lambda^{n+1}_0| \arrow[r] \arrow[d, hook] & E \arrow[d, "\pi"] \\ |\Delta^{n+1}| \arrow[r] \arrow[ur, dashed, "\exists"] & B \end{tikzcd}$$

Since $(D^n \times I, D^n \times 0) \cong (|\Delta^{n+1}|, |\Lambda^{n+1}_0|)$, this is equivalent to the condition on the right. If B is a point, then all spaces E are "Kan" because there is a retraction $|\Delta^{n+1}| \to |\Lambda^{n+1}_0|$, but if B is not a point, then this is a real condition. In $s\mathsf{Set}$, a *Kan fibration* is a map $E_\bullet \to B_\bullet$ satisfying

$$\begin{tikzcd} \Lambda^{n+1}_{0,\bullet} \arrow[r] \arrow[d, hook] & E_\bullet \arrow[d] \\ \Delta^{n+1}_\bullet \arrow[r] \arrow[ur, dashed, "\exists"] & B_\bullet \end{tikzcd}$$

Such things realize to Serre fibrations.

Remark 22.2. The lift is unique if and only if F is discrete. In this case, you call $E \to B$ a covering map. ◇

Another way to say that $\pi\colon E \to B$ is a Serre fibration is that for all CW complexes X, homtopies of maps from X to B lift to E.[1] This implies that $\pi_1(B, b_0)$ acts on F (up to homtopy). For $[\alpha] \in \pi_1(B, b_0)$, I have

$$\begin{tikzcd} F \times 0 \arrow[r, hook] \arrow[d, hook] & E \arrow[d, "\pi"] \\ F \times I \arrow[r, "\alpha\circ p_2"'] \arrow[ur, dashed, "h"] & B \end{tikzcd}$$

I define $a(\alpha)\colon F \to F$ to be $a(\alpha) := h(-, 1)$. F may not be a CW complex, but since we're only trying to define an action up to homotopy (and F is weakly equivalent to a CW complex), we don't run into trouble. We have to show that this is independent of α and independent of h. Suppose $\alpha' \in [\alpha]$, with $\alpha \simeq_\beta \alpha'$ and h' a lift of α', then

$$\begin{tikzcd} F \times \sqcup \arrow[r] \arrow[d, hook] & E \arrow[d, "\pi"] \\ F \times \square \arrow[r, "\beta\times p_2"'] \arrow[ur, dashed, "H"] & B \end{tikzcd}$$

($\sqcup = I \times 0 \cup \partial I \times I \subseteq I \times I = \square$) Define the map $F \times \sqcup \to E$ to be h and h' on $F \times (I \times 0)$ and $F \times (I \times 1)$ respectively, and take the standard inclusion on $F \times (0 \times I)$. Check that the outer square commutes, so we can fill in H. Then $H|_{I\times 1}$ gives me a homotopy from $a(\alpha)$ to $a(\alpha')$.

In the theorem, the coefficients $H_q(F)$ are twisted by the action of $\pi_1(B, b_0)$. If I have a space B and a $\pi_1(B)$-module M, then I can define $H_p(B; M)$ as $H_p\big(C_*(\tilde{B}) \otimes_{\mathbb{Z}[\pi_1 B]} M\big)$, where $\tilde{B}$ is the universal cover of B (assuming it exists[2])

Claim. *The Hochschild-Serre spectral sequence is a special case of the Leray-Serre spectral sequence.*

Proof. If $1 \to N \to G \to Q \to 1$ is an extension of groups, then there is a fibration $K(N, 1) \to K(G, 1) \to K(Q, 1)$ inducing the correct action of Q on $H_q(N)$. You can use this to show that inner automorphisms act trivially on $H_q(N)$ because conjugation in π_1 is change of basepoint, and the homology doesn't care about basepoint. You take $K(G, 1) = |\mathsf{C}_G|$ to

[1] If you take X to be any space, this is the definition of (non-Serre) fibrations.

[2] Otherwise, you can use a flat bundle or a locally constant sheaf. [[★★★ how do you define sheaf *homology*?]]

get the fibration; this makes surjectivity $K(G,1) \to K(Q,1)$ obvious, and it obvious it is a fibration (because $\mathsf{C}_G \to \mathsf{C}_Q$ is a Kan fibration), but it is not obvious that the fiber is $K(N,1)$. Suppose it is $F \to K(G,1) \to K(Q,1)$ and look at the long exact sequence in homotopy groups to see that F must be a $K(N,1)$. Then you have to check that the action of Q on $H_p(N)$ is the right action. □

Ok, what does the theorem mean? First of all, it means you have $E^r_{p,q}$ and differentials d_r. Second of all, you have convergence (convergence is automatic because this is first quadrant). This means that there is a filtration $F^pH_{p+q}(E)$ such that $F^pH_{p+q}(E)/F^{p-1}H_{p+q}(E) \cong E^\infty_{p,q}$.

Example 22.3 (Gysin sequence)**.** Let $S^n \to E \xrightarrow{\pi} B$ be a fibration. Then $H_q(S^n)$ is very easy to understand. We get a picture of E^2:

$$\begin{array}{ccccc}
\vdots & 0 & 0 & 0 & \cdots \\
n & H_{0,\rho}B & H_{1,\rho}B & H_{2,\rho}B & \cdots \\
\vdots & 0 & 0 & 0 & \cdots \\
0 & H_0B & H_1B & H_2B & \cdots \\
 & 0 & 1 & 2 & \cdots
\end{array}$$

Let $\rho\colon \pi_1B \to \mathrm{Aut}(H_nS^n) = \{pm1\}$, then the $H_{i,\rho}B$ are the homologies of B with twisted coefficients. We turn pages for a while, and nothing happens until we get to E^{n+1}, where we get $d_{n+1}\colon H_{n+j}B \to H_{j,\rho}B$. After that, all the differentials are zero. The conclusion is that $E^n = E^2$, and $E^\infty = E^{n+2}$.

It happens that if a spectral sequence is so sparse (if it has only two rows or only two columns), then it contains exactly the same information as some long exact sequence. So you can think of spectral sequences as generalizations of long exact sequeces.

[[break]]

The two-row spectral sequence tells us that we get an exact sequence

$$\begin{tikzcd}
 & E^\infty_{p,0} \arrow[d, equal] & E^2_{p,0} \arrow[d, equal] & E^2_{p-n-1,n} \arrow[d, equal] & E^\infty_{p-n-1,n} \arrow[d, equal] & \\
0 \arrow[r] & E^{n+2}_{p,0} \arrow[r] & E^{n+1}_{p,0} \arrow[r, "d_{n+1}"] & E^{n+1}_{p-n-1,n} \arrow[r] & E^{n+2}_{p-n-1,q} \arrow[r] & 0
\end{tikzcd}$$

and that there is a short exact sequence (at the bottom)

$$\begin{tikzcd}
 & H_{p-n,\rho}(B) \arrow[d, equal] & & & H_p(B) \arrow[d, equal] & \\
{} \arrow[r, dashed, "d_{n+1}"] & E^2_{p-n,n} \arrow[dr, two heads] \arrow[drr, dashed] & & & E^2_{p,0} \arrow[r, dashed, "d_{n+1}"] & {} \\
 & 0 \arrow[r] & E^\infty_{p-n,n} \arrow[r] & H_p(E) \arrow[r] \arrow[ur, dashed] & E^\infty_{p,0} \arrow[r] \arrow[u, hook] & 0
\end{tikzcd}$$

Then the dashed sequence is exact, so we get a long exact sequence. Finally, we always have an *edge homomorphism*, an isomorphism between $H_p(B)$ and $E^2_{p,0}$

All together, we get a long exact sequence

$$H_{p+1}E \to H_{p+1}B \to H_{p-n,\rho}B \to H_pE \xrightarrow{\pi_*} H_pB \xrightarrow{d_{n+1}} H_{p-n-1,\rho}B$$

◇

Remark 22.4. Such a fibration $S^n \to E \to B$ has an Euler class $e(\pi) \in H^{n+1,\rho}(B)$, and d_{n+1} is given by cap product with $e(\pi)$. ◇

Example 22.5 (Wang sequence)**.** Let $F \to E \to S^n$ be a fibration. This works like the Gysin sequence, but you end up with only two non-zero columns. Playing around, you get a similar long exact sequence. The case $n = 1$ is somehow special, so let $F \to E \to S^1$ be a fibration on the circle. This is usually an easy way to construct an interesting space. If you remove a point of S^1, you get a fibration over the interval, which must be trivial, so a fibration over the circle is given by monodromy (a self homotopy equivalence) $\alpha\colon F \to F$. Then the fibration is given by $F \times I/\sim$ where $(0,f) \sim (1,\alpha(f))$.

You can go back. Using that $S^1 \simeq K(\mathbb{Z},1)$, we know that a fibration is the same thing as an element of $\mathrm{Hom}(\pi_1 E, \mathbb{Z})$. The sequence $0 \to \pi_1 F \to \pi_1 E \to \pi_1 S^1 \cong \mathbb{Z}$ splits, so you know that $\pi_1 E \cong \pi_1 F \rtimes_{\alpha_*} \mathbb{Z}$

In this case, the spectral sequence is

$$\begin{array}{ccccc}
2 & (H_2F)_{\mathbb{Z}} & (H_2F)^{\mathbb{Z}} & 0 & 0 \\
\\
1 & (H_1F)_{\mathbb{Z}} & (H_1F)^{\mathbb{Z}} & 0 & 0 \\
\\
0 & H_0F & H_0F & 0 & 0 \\
\\
 & 0 & 1 & 2 & \cdots
\end{array}$$

Since the non-zero columns are adjacent, the differentials already have to be zero, so $E^\infty = E^2$! $H_p(S^1; M) = H_p(\mathbb{Z}; M)$ because $S^1 = K(\mathbb{Z},1)$, and we have

$$H_p(\mathbb{Z}; M) = \begin{cases} M_{\mathbb{Z}} & p = 0 \\ M^{\mathbb{Z}} & p = 1 \\ 0 & p > 1 \end{cases}$$

You can see that $H_1(S^1; M) \cong H^0(S^1; M) M^{\mathbb{Z}}$ using Poincaré duality. Later, we'll see another way to do this using a universal coefficients theorem.

So we get short exact sequences

$$0 \to (H_pF)_{\mathbb{Z}} \to H_p(E) \to (H_{p-1}F)^{\mathbb{Z}} \to 0$$

This is equivalent to a long exact sequence (the Wang sequence)

$$\begin{tikzcd}
H_{p+1}E \arrow[r] & H_pF \arrow[r, "\mathrm{id} - \alpha_*"] & H_pF \arrow[rr] \arrow[dr] & & H_pE \arrow[rr] \arrow[dr] & & H_{p-1}F \arrow[r, "\mathrm{id} - \alpha_*"] & H_{p-1}F \\
 & & & (H_pF)_{\mathbb{Z}} \arrow[ur, hook] & & (H_{p-1}F)^{\mathbb{Z}} \arrow[ur, hook] & &
\end{tikzcd}$$

◇

23 Edge homomorphisms (in Leray-Serre spectral sequence)

For HW10 problem 1, we had

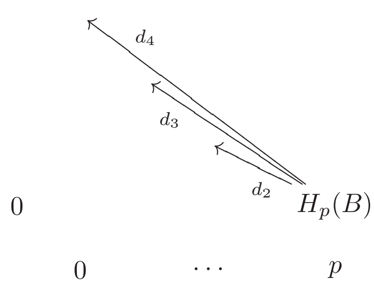
$$1 \to \mathbb{Z}/n \to D_{2n} \to \mathbb{Z}/2 \to 1$$

The problem was to calculate $H_2(D_{2n})$. You had to figure out the action of $\mathbb{Z}/2$ on $H_q(\mathbb{Z}/n)$, among other things. The main problem was that you also had to figure out one d_2 differential (in the case n even; in the odd case, there was no problem): $E^2_{p,q} = H_p(\mathbb{Z}/2; H_q(\mathbb{Z}/n))$

$$\begin{array}{ccccccc}
2 & 0 & 0 & 0 & 0 & 0 & \cdots \\
1 & \mathbb{Z}/2 & \mathbb{Z}/2 & \mathbb{Z}/2 & \mathbb{Z}/2 & \mathbb{Z}/2 & \cdots \\
0 & \mathbb{Z} & \mathbb{Z}/2 & 0 & \mathbb{Z}/2 & 0 & \cdots \\
 & 0 & 1 & 2 & 3 & 4 & 5
\end{array}$$

We had $(\mathbb{Z}/n)_{\mathbb{Z}/2} \cong \mathbb{Z}/2$. The goal was to figure out that the d_2 differential above is zero (from which you can conclude that $H_2(D_{2n}) \cong \mathbb{Z}/2$).

Remember the Leray-Serre spectral sequence is, for a given fibration $F \xrightarrow{i} E \xrightarrow{\pi} B$, there is a spectral sequence $E^2_{p,q} = H_p(B; H_q(F)) \Rightarrow H_{p+q}(E)$, where there is an action (up to homotopy) of $\pi_1(B)$ on F.



We have that $H_p(B) = E^2_{p,0} \supseteq E^3_{p,0} = \ker d_2 \supseteq E^4_{p,q} = \ker d_3 \supseteq \cdots \supseteq E^\infty_{p,0}$.

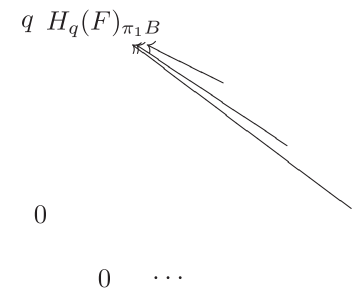
Lemma 23.1. *The induced map $\pi_*\colon H_p(E) \to H_p(B)$ is given by the composition $H_p(E) \to H_p(E)/F^{p-1}H_pE = E^\infty_{p,0} \subseteq E^2_{p,0} = H_pB$.*

This will follow from the construction of the spectral sequence.

Corollary 23.2. $\operatorname{im}\pi_* = E^\infty_{p,0} = \bigcap_{r\geq 2}\ker d_r$. *In particular, π_* is onto if and only if $d_r = 0$ for all $r \geq 2$ (e.g. if π has a section).*

In the problem on HW10, we had a section of π because D_{2n} is a semi-direct product.

There is another edge of the spectral sequence



Here we have $H_q \twoheadrightarrow (H_qF)_{\pi_1B} = E^2_{0,q} \twoheadrightarrow E^3_{0,q} = E^2_{0,q}/\operatorname{im} d_2 \twoheadrightarrow \cdots \twoheadrightarrow E^\infty_{0,q} \subseteq H_q(E)$.

Lemma 23.3. *This is the induced map $i_*\colon H_q(F) \to H_q(E)$. In particular, $i_*(g_*(x) - x) = 0$ for $g \in \pi_1B$ and $x \in H_q(F)$.*

The last part of the lemma follows from $i \circ g \simeq i$:

$$\begin{tikzcd}
F \times \{0\} \arrow[r, hook, "i"] \arrow[d, hook] & E \arrow[d, "\pi"] \\
F \times I \arrow[r, "g\circ p_2"'] \arrow[ur, dashed, "h"] & B
\end{tikzcd}$$

We get $h_t\colon F \to E$, a homotopy between $i = h_0$ and $i \circ g = h_1$ (this is how the action of g was defined).

Corollary 23.4. $\ker i_* = \langle \operatorname{im} d_r, g(x) - x\rangle$ *where* g *varies over* $\pi_1 B$ *and* x *varies over* $H_q F$*. In particular,* i_* *is injective if and only if the* $\pi_1 B$ *action is trivial and the* $d_r = 0$ *for* $r \geq 2$ *(e.g. if* i *has a section).*

In the group extension situation, the only way there can be a section of i is if the extension is trivial:

$$\begin{tikzcd}
1 \arrow[r] & N \arrow[r] \arrow[d, equal] & N \times Q \arrow[r] & Q \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & N \arrow[r, "i"] & G \arrow[r, "\pi"] \arrow[u, "s\times\pi"] \arrow[l, bend left, "s"] & Q \arrow[r] & 1
\end{tikzcd}$$

5-term exact sequence

Considering our conclusions about edge homomorphisms near the lower left corner, what do we get? In the lower leftmost box, we already have no non-zero differentials. The conclusion is that $H_0(F)_{\pi_1 B} \cong H_0(B; H_0(F)) = E^2_{0,0} = E^\infty_{0,0} = H_0(E)$. [[★★★ any time we said the action is obviously trivial, we were assuming connected fiber]] In particular, if E is connected, then the action of $\pi_1 B$ on the components of F must be transitive.

$$\begin{tikzcd}
1 & H_1(F)_{\pi_1 B} & & \\
0 & \checkmark & H_1(B; H_0 F) & H_2(B; H_0 F) \arrow[ull, "d_2"'] \\
 & 0 & 1 & 2
\end{tikzcd}$$

We get the exact sequence

$$H_2 E \xrightarrow{\pi_*} H_2(B; H_0 F) \xrightarrow{d_2} (H_1 F)_{\pi_1 B} \xrightarrow{i_*} H_1 E \xrightarrow{\pi_*} H_1(B; H_0 F) \to 0$$

This works for any spectral sequence. For example, for any group extension, you get this nice 5-term sequence.

Now assume that $\pi_i F = 0$ for $1 \leq i \leq q-1$. Then we get

$$\begin{tikzcd}
q & (H_q F)_{\pi_1 B} & & & \\
\vdots & 0 & 0 & 0 & \\
0 & \checkmark & & & H_{q+1}(B; H_0 F) \arrow[ulll] \\
 & 0 & \cdots & & q+1
\end{tikzcd}$$

So $H_i E \cong H_i(B; H_0 F)$. If $q = 1$, then there is no assumption.
[[break]]

Lemma 23.5. *If* X *is a CW complex, then there exists a fibration* $\tilde{X} \xrightarrow{i} X \xrightarrow{\pi} K(G, 1)$*, where* $G = \pi_1(X, x_0)$ *and* $\tilde{X}$ *is the universal covering space of* X*.*

General fact: any map $f\colon X \to K$ can be "turned into" a fibration. If $k_0 \in K$ is base point (thought of as an object in the fundamental groupoid of K), then the *homotopy fiber* of f over k_0 is the under category $(k_0 \downarrow f) = \{(x, \gamma) | x \in X, \gamma\colon I \to K, \gamma(0) = k_0, \gamma(1) = f(x)\}$. Take $X' = \{(x, \gamma) | x \in X, \gamma\colon I \to K, \gamma(1) = f(x)\} \subseteq X \times PK$. There are nice maps $f'\colon X' \to K$ and $\pi_1\colon X' \to X$, given by $f'\colon (x, \gamma) \mapsto \gamma(0)$ and $\pi_1\colon (x, \gamma) \mapsto x$, with $f \circ \pi_1 \simeq f'$ (not equal, but good enough for any homotopy purposes).

$$\begin{tikzcd}
F \arrow[r, equal] \arrow[d] & \text{ho-fiber}(f) \arrow[d] \\
X' \arrow[r, shift left, "\pi_1"] \arrow[dr, bend right, "f'"'] & X \arrow[l, shift left, "\text{const}"] \arrow[d] \\
 & K
\end{tikzcd}$$

The claim is the π_1 is a homotopy equivalence, with homotopy inverse $x \mapsto (x, \text{const}_x)$.

Proof. We have a map $X \xrightarrow{f} K = K(G,1) := X \cup 3\text{-cells} \cup 4\text{-cells} \cup \cdots$. This gives an associated fibration $F \to X' \xrightarrow{f'} K(G,1)$, whose fiber is the homotopy fiber of f. We have the usual covering map $i\colon \tilde{X} \to X$. I claim that this map lifts to a map to the fiber:

$$\begin{tikzcd}
F \arrow[r] & X' \arrow[r, "f'"] & K(G,1) \\
\tilde{X} \arrow[u, "\simeq"] \arrow[r, "i"] & X \arrow[u, "\simeq"] \arrow[ur, "f"'] &
\end{tikzcd}$$

You can see that $F \simeq \tilde{X}$ by looking at the long exact sequence in homotopy groups and using that the square commutes. This gives that it is a weak homotopy equivalence. It is not trivial to check that if K is a CW complex, then PK has the homotopy type of a CW complex . . . eventually F is a CW complex, so weak equivalence implies homotopy equivalence by Whitehead's theorem. $\square$

Now use the 5-term sequence where $B = K(G,1)$, $E = X$, and $F = \tilde{X}$. Here, we're using the general case with $q = 2$ ($\pi_1 F = 0$). The boring information from $E^2_{1,0} = E^\infty_{1,0}$ is $H_1 X \cong H_1 G$. We get

$$H_3 X \to H_3 G \to (H_2 \tilde{X}_G) \to H_2 X \to H_2 G \to 0$$

24 Constructing spectral seqeunces

This week, you can submit the homework a little later than usual

For HW11, problem 2, you have to use the Leray-Serre *cohomology* spectral sequence. In that case, you have one extra thing going for you, which is that the differentials are derivations (for some product structure). This is enough to do the problem (at least rationally; if you try to do it integrally, you get lots of torsion having to do with stable homotopy groups of spheres).

For problem 1, I also didn't give you enough information; I haven't told you the spectral sequence associated to a double complex so that you'd try to figure it out yourself. In principal, you could have worked this out with everything you know.

Today, we'll construct the spectral sequence associated to a filtered chain complex. The input is a chain complex C, which you'd like to know the homology of. In the Leray-Serre case, where you have $F \xrightarrow{i} E \xrightarrow{\pi} B$, and $C = C_*(E)$. We also have a filtration of chain complexes

$$0 \subseteq \cdots \subseteq F_pC \subseteq F_{p+1}C \subseteq \cdots C.$$

In the Leray-Serre case, we may assume B is a CW complex because we can pull the fibration back along the weak equivalence $|\Delta_\bullet B| \to B$. Then the fiber doesn't change, and the resulting map from the new total space to E is a weak equivalence:

$$\begin{tikzcd} F \arrow[r, "i"] \arrow[d, equal] & E \arrow[r, "\pi"] & B \\ F \arrow[r] & E' \arrow[r] \arrow[u, "\wr"] \arrow[ur, phantom, "\ulcorner"] & {|\Delta_\bullet B|} \arrow[u, "\wr"] \end{tikzcd} \qquad \begin{tikzcd} F \arrow[r, "i"] \arrow[d, equal] & E \arrow[r, "\pi"] & B \\ F \arrow[r] & \pi^{-1}(B^{(p)}) \arrow[r] \arrow[u] \arrow[ur, phantom, "\ulcorner"] & B^{(p)} \arrow[u] \end{tikzcd}$$

Then we have a filtration of B (it is a CW complex). We can pull back the fibration along the inclusions $B^{(p)} \to B$. Define $F^pC_*E = \operatorname{im}\big(C_*(\pi^{-1}(B^{(p)}))\big)$

Theorem 24.1. *Let C be a bounded below ($F_{p_0}C = 0$ for some p_0)*[1] *exhaustive ($\bigcup F_pC = C$) filtered chain complex [[★★★ such that C is also bounded below as a chain complex]]. Then there exists a spectral sequence with $E^0_{p,q} = F_pC_{p+q}/F_{p-1}C_{p+q}$ and $d_0 = [d]$ converging to $H_{p+q}(C)$ (which has the filtration induced by the original filtration).*

If you have a chain complex, you can only take the homology. If you have a filtered object in an abelian category, you can only take its associated graded. If you have a filtered chain complex, then you can do both, and they don't quite commute. We have that $E^1_{p,q} = H_{p+q}(F_pC/F_{p-1}C, [d])$, which is one direction, and $E^\infty_{p,q} = F_p(H_{p+q}(C))$, which is the other direction.

$$\begin{tikzcd} & \text{filtered } C \arrow[dl, "\mathrm{Gr}"'] \arrow[dr, "\text{homology}"] & \\ \text{chain cx } F_pC/F_{p-1}C \arrow[d, "\text{homology}"'] & & \text{filtered } H(C) \arrow[d, "\mathrm{Gr}"] \\ E^1 \arrow[rr, squiggly, "\text{use } d_r\ r\geq 1"] & & E^\infty \end{tikzcd}$$

One example is going to be the Leray-Serre spectral sequence. Another one is the total complex of a double complex $D_{p,q}$. $\mathrm{Tot}(D_{*,*})$ is filtered by columns (or rows). The nice thing about this case is that you can explicitly write d_r in terms of the horizontal and vertical differentials (tic-tac-toe).

Proof of Theorem. I'll ignore the q index; it just goes along for the ride (in fact, the proof works for a filtered differential object, which need not be a chain complex). The key is to define the *r-approximate cycles* $A^r_p = \{x \in F_pC | dx \in F_{p-r}C\}$. Note that $A^{r-1}_p \subseteq A^r_p$, and $A^{r-1}_{p-1} \subseteq A^r_p$. We can define $A^\infty_p = \bigcap A^r_p$ (this is actually a finite intersection because of bounded below). Note that the differential maps A^r_p to A^r_{p-r}. The miracle formula is

$$E^r_p := \frac{A^r_p}{dA^{r-1}_{p+r-1} + A^r_{p-1}}$$

The differential induces a map $d^r \colon E^r_p \to E^r_{p-r}$, which is clearly a differential (because d is).

Lemma: $H_p(E^r_*, d^r) = E^{r+1}_p$. To prove this, just write it out. By definition, $E^{r+1}_p = \frac{A^{r+1}_p}{dA^r_{p+r} A^r_{p-1}}$. Let's write a surjection $A^{r+1}_p \twoheadrightarrow \ker d^r$.

[1] It is not enough to assume $\bigcap F_pC = 0$, because then you don't get $\bigcap F_pH(C) = 0$.

By definition, d takes A_p^{r+1} to A_{p-r-1}^{r-1}, which is in the kernel of d^r. Now let's show that, modulo A_{p-1}^{r-1}, you get everything in the kernel of d^r. If something maps to zero, it is a sum of elements in dA_{p-1}^{r-1} and A_{p-r-1}^{r-1} (some chasing around). Now we have a surjection $A_p^{r+1} \twoheadrightarrow \ker d^r \twoheadrightarrow H_p(E_*^r, d^r)$. Check for yourself that the kernel of this surjection is $dA_{p+r}^r + A_{p-1}^r$.

Now we have to check that the spectral sequence converges under our assumptions. Define $A_p^\infty = \bigcap A_p^r$. This is a finite intersection because of our bounded below assumption. The key observation is that $A_p^\infty = \ker(d\colon F_pC \to F_pC)$. The other fact is that $d\big(\bigcup A_p^r\big) = \operatorname{im} d \cap F_pC$. The point is that

$$\frac{F_pH(C)}{F_{p-1}H(C)} \cong \frac{A_p^\infty}{A_{p-1}^\infty + d\big(\bigcup_r A_{p+r}^r\big)} \cong E_p^\infty$$

This follows from the observation that for a fixed p, E_p^r stablilizes for some finite r. Actually, the dA_{p+r-1}^{r-1} doesn't stabilize. You have to put a union. You have to remember what we meant by E_p^∞, it is $\bigcap_r Z_p^r / \bigcup_r B_p^r$. Then you get the right E^∞.

It is clear that $F_{p_0}H(C) = 0$. It is also clear that $\bigcup_p F_pH(C) = H(C)$. This is what we needed for convergence. □

[[break]]

In the Leray-Serre spectral sequence, we defined $F_pC(E)$ to be the image of $C_*\big(\pi^{-1}(B^{(p)})\big)$. The final thing you need to prove the Leray-Serre spectral sequence is the following lemma, which I'll skip.

Lemma 24.2. *$E_{p,q}^1 = C_p^{\text{cell}}(B; H_qF)$ and $E_{p,q}^2 = H_p(B; H_qF)$.*

Cohomology spectral sequence

Cohomology spectral sequences work like this. For a filtered cochain complex C, the terms are $E_r^{p,q}$. The filtration is decreasing, so $F^{p+1}C \subseteq F^pC$. The differentials are $d_r\colon E_r^{p,q} \to E_r^{p+r,q-r+1}$. Convergence means that $E_\infty^{p,q} \cong F^pH_{p+q}(C)/F^{p+1}H_{p+q}(C)$.

Example 24.3 (cohomology Leray-Serre)**.** We have a fibration $F \to E \to B$, then there is a cohomology spectral sequence $E_2^{p,q} = H^p(B; H^qF) \Rightarrow H^{p+q}(E)$. The main point is that this is a spectral sequence of graded rings! ◇

Given a filtered differential graded algebra C^*, a chain complex with an associative multiplication $C^p \otimes C^q \xrightarrow{\mu} C^{p+q}$ with the compatibility $d(ab) = da \cdot b + (-1)^{|a|} a \cdot db$, with a filtration respecting all this. In this situation, $H(C)$ is again a differential graded algebra.

Theorem 24.4. *(E_r^*, d_r) is again a differential graded algebra and $E_{r+1}^* \cong H^*(E_r^*, d_r)$ is an isomorphism of differential graded algebras.*

In the Leray-Serre case, the differential graded algebra structure on $E_2^{p,q} = H^p(B; H^qF)$ is given by using the cup product for both cohomologies. The convergence is an isomorphism of differential graded algebras, and the differentials in the spectral sequence are derivations.

For problem 2, $A^* = \operatorname{Hom}_{\mathbb{Z}}(A, \mathbb{Z})$, and the free algebra on a vector space V of degree n is the symmetric algebra when n is even and the exterior algebra when n is odd.

25 Universal coefficients

Today: some more spectral sequences and applications. Thursday: Derived categories and derived functors.

So far, I've only shown you the Leray-Serre spectral sequence, but in the homework, you've seen the spectral sequence associated to a double complex. Recall that if $(D_{p,q}, d^v, d^h)_{p,q\geq 0}$ is a double complex, then you get two spectral sequences converging to the homology of the total complex $\mathrm{Tot}_{p+q}(D_{*,*})$ (but with different filtrations!):

$$E^0_{p,q} = (D_{p,q}, d^v) \tag{I}$$
$$E^0_{p,q} = (D_{q,p}, d^h) \tag{II}$$

The second spectral sequence is not a cohomological spectral sequence, it is a homological spectral sequence transposed.

Example 25.1. Say $P_* \twoheadrightarrow M$ is a projective resolution of some (right) R-module M, and Q_* is some other chain complex of projective (left) R-modules. Let $D_{*,*} = (P_* \otimes_R Q_*, d_P, d_Q)$. The first spectral sequence gives us (using that Q_* has projective terms to get the E^2 term)

$$E^1_{p,q} = P_p \otimes H_q Q_* \qquad E^2_{p,q} = \mathrm{Tor}^R_p(M, H_q Q_*) \tag{I}$$

$$E^1_{p,q} = H_q(P_*)\otimes Q_p = \begin{cases} M\otimes Q_p & p=0 \\ 0 & p>0 \end{cases} \qquad E^2_{p,q} = E^\infty_{p,q} = H_p(M\otimes Q_*) \tag{II}$$

So we know that $\mathrm{Tor}^R_p(M, H_q Q_*) \Rightarrow H_{p+q}(\mathrm{Tot}) = H_{p+q}(M\otimes Q_*)$. ◇

In the case $R=\mathbb{Z}$, $Q = C_*(X)$ for a topological space X, and M some abelian group, then we get (we can flip our tensor products) the *homological universal coefficients spectral sequence*

$$E^2_{p,q} = \mathrm{Tor}^{\mathbb{Z}}_p(H_q X, M) \Rightarrow H_{p+q}(C_*X\otimes M).$$

The spectral sequence is concentrated at $p=0,1$ because $\mathbb{Z}$ is a PID, so we are in the first two columns. [[★★★ But the differentials change p by 2, so we have that $E^2 = E^\infty$. The conclusion is that there is a short exact sequence

$$0 \to \mathrm{Tor}^{\mathbb{Z}}_1(H_{q-1}X, M) \to H_q(X;M) \to H_q X\otimes_{\mathbb{Z}} M \to 0$$

]] The edge homomorphism at $p=0$ is $E^2_{0,q} = M\otimes_R H_q Q_* \to H_q(M\otimes_R Q_*)$

When will I use the more general spectral sequence? Suppose $G=\pi_1 X$ and M is a G-modules. Then you may want to compute $H_*(X;M) := H_*(C_*\tilde{X}\otimes_{\mathbb{Z}G} M)$. Now we have the spectral sequence $\mathrm{Tor}^{\mathbb{Z}G}_p(H_q\tilde{X}, M) \Rightarrow H_{p+q}(C_*\tilde{X}\otimes_{\mathbb{Z}G} M)$. You might call this the universal coefficients spectral sequence for twisted coefficients.

You know another universal coefficient theorem, for cohomology. It can be expressed as a spectral sequence

$$\mathrm{Ext}^p_{\mathbb{Z}}(H^q X, M) \Rightarrow H^{p+q}(X;M)$$

Again, the spectral sequence is concentrated in the columns $p=0,1$. The more general *cohomological universal coefficients spectral sequence* is

$$E^{p,q}_2 = \mathrm{Ext}^p_R(H_q Q_*, M) \Rightarrow H^{p+q}(Q_*;M) := H_{p+q}(\mathrm{Hom}_R(Q_*, M))$$

The proof is done by considering $D_{p,q} = \mathrm{Hom}_R(Q_q, P_p)$, where $M\to P_*$ is an injective resolution and Q_* is a complex of projectives. The edge homomorphism at $p=0$ is

$$H^q(Q_*, M) = H_q(\mathrm{Hom}_R(Q_*, M)) \to E^2_{0,q} = \mathrm{Hom}_R(H_q Q_*, M)$$

If $Q_* = C^* X$ for some topological space X and M is an interesting $\pi_1 X$-module, then you really need the big spectral sequence.

In HW12, you'll do the Künneth spectral sequence, which generalizes the Künneth theorem in the same way that the universal coefficients spectral sequences generalize the universal coefficients theorems.

After the break, we'll calculate $\pi_{n+1}S^n$. In the next homework, you'll compute all homotopy groups of spheres rationally ($\pi_k S^n\otimes_{\mathbb{Z}}\mathbb{Q}$). This stuff is due to Serre.

[[break]]

$\pi_{n+1}(S^n)$

Theorem 25.2 (Serre)**.** $\pi_{n+1}(S^n) \cong H_{n+2}(K(\mathbb{Z},n)) \cong \begin{cases} 0 & n=0,1 \\ \mathbb{Z} & n=2 \\ \mathbb{Z}/2 & n\geq 3 \end{cases}$

It's true in general that $\pi_{n+k}(S^n)$ stabilizes (more generally, $\pi_{n+k}(\Sigma^n X)$ stabilizes) for large n (Fruedenthal's suspension theorem, in fact, it is stable by $n = 2\dim X - 1$). This can also be proven easily by induction with spectral sequences using the path-loop fibration.

Proof. Assume $n > 1$ (we know the result for $n = 0, 1$). We know that $\pi_i(S^n) = 0$ for $i < n$ and that $\pi_n(S^n) \cong \mathbb{Z}$. By adding cells of dimenions at least $n+2$ we construct an inclusion map $S^n \to K(\mathbb{Z}, n)$ inducing an isomorphism on π_n. We can turn this into a fibration (changing S^n to something homotopy equivalent; we won't change the notation). Let F be the fiber. Using the long exact sequence of homotopy groups, we get $\pi_{n+1}(S^n) \cong \pi_{n+1}(F)$. Note that F is n-connected because of the long exact sequence in homotopy groups. So by the Hurewicz theorem, $\pi_{n+1}(F) \cong H_{n+1}(F)$. The Leray-Serre spectral sequence tells us that $H_{n+1}(F) \cong H_{n+2}(K(\mathbb{Z}, n))$.

$$
\begin{array}{ccccccccc}
n+1 & H_{n+1}F & & & & & & & \\
n & 0 & & & & & & & \\
 & \vdots & & & & & & & \\
 & 0 & & 0 & & & & & \\
0 & \mathbb{Z} & 0 & \cdots & 0 & \mathbb{Z} & 0 & & H_{n+2}(K(\mathbb{Z},n)) \\
 & 0 & & & & n & & n+2 &
\end{array}
$$

Note that $H_{n+1}(K(\mathbb{Z}, n)) = 0$ because there are no $(n+1)$-cells. All the differentials are zero except for the one shown, which must be an isomorphism because we know the homology of S^n.

Now we have to compute $H_{n+2}(K(\mathbb{Z}, n))$. The special case is $n = 2$. This "easy way" is to observe that $\mathbb{C}P^\infty$ has the right homtopy groups (you have to know this). The other way is to take the path space fibration $\Omega_k K \to P_k K \to K$ for some base point $k \in K$. Applying the Leray-Serre spectral sequence to the case $K = K(\mathbb{Z}, n)$, where $\Omega_k K = K(\mathbb{Z}, n-1)$, we can do an induction.

The first fibration is $S^1 \cong K(\mathbb{Z}, 1) \to P \simeq 1 \to K(\mathbb{Z}, 2)$. The spectral sequence is

$$
\begin{array}{ccccccc}
\vdots & & & & & & \\
2 & 0 & 0 & 0 & \cdots & & \\
1 & \mathbb{Z} & 0 & \mathbb{Z} & 0 & H_4K & \cdots \\
0 & \mathbb{Z} & 0 & \mathbb{Z} & 0 & H_4K & \\
 & 0 & 1 & 2 & & &
\end{array}
$$

We know that the two rows have to be equal (they are both $H_n(K; \mathbb{Z})$), and the E^∞ page has to be full of zeros, so all the E^2 differentials have to be isomorphisms, so $H_{2n}(K(\mathbb{Z}, 2)) \cong \mathbb{Z}$.

Now consider the fibration $K(\mathbb{Z}, 2) \to * \to K(\mathbb{Z}, 3)$. Then we get the spectral sequence

$$
\begin{array}{ccccccc}
4 & \mathbb{Z} & 0 & 0 & \mathbb{Z} & 0 & H_5K \\
3 & 0 & 0 \; d_3 & 0 & 0 & 0 & 0 \\
2 & \mathbb{Z} & 0 & 0 & \mathbb{Z} & 0 & H_5K \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \mathbb{Z} & 0 & 0 & \mathbb{Z} & 0 & H_5K \\
 & 0 & 1 & 2 & 3 & 4 & 5
\end{array}
\qquad
\begin{array}{ccccccc}
4 & ? & 0 & 0 & \mathbb{Z} & 0 & H_5K \\
3 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & E^4 & 0 & H_5K \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \mathbb{Z} & 0 & 0 & 0 & 0 & H_5K \\
 & 0 & 1 & 2 & 3 & 4 & 5
\end{array}
$$

The non-zero rows have to be equal ($\pi_1(K(\mathbb{Z}, 3)) = 0$, so there cannot be a non-trivial action, so they are all just homology with coefficients in $\mathbb{Z}$). The goal is to compute that d_3 is multiplication by 2. The key fact will be the Pontrjagin product on $H_*(K(\mathbb{Z}, n))$. Serre did it with cohomology an used cup products, but this is a good approach to keep in mind. $\square$

26 Derived categories

We were in the middle of proving the following theorem.

Theorem 26.1 (Serre). $\pi_{n+1}(S^n) \cong H_{n+2}(K(\mathbb{Z},n)) \cong \begin{cases} 0 & n=0,1 \\ \mathbb{Z} & n=2 \\ \mathbb{Z}/2 & n \geq 3 \end{cases}$

Remark 26.2. $\pi_k^{\text{stable}} := \pi_{k+r}S^r$ for sufficiently large r, is known only for small k (up to about $k=200$). Serre computed the first few of these using spectral sequences. For example,

k	0	1	2	3	4
π_k^{st}	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0

HW3 implies that $|\pi_k^{st}| < \infty$ for all $k > 0$.

It is known that π_*^{st} is a graded commutative ring under smash product, and all the non-zero graded elements are nilpotent. It is known that the prime p cannot appear as an exponent until $k = p-1$ or something like that. ⋄

Definition 26.3. If a topological space K has a homotopy associative multiplication $\mu\colon K \times K \to K$, then $H_*(K)$ is a graded ring via the *Pontrjagen product*: we always have the edge homomorphism $\times\colon H_*(K) \times H_*(K) \to H_*(K \times K)$, which we compose with μ_* to get a map $H_pK \times H_qK \to H_{p+q}K$. ⋄

Example 26.4. If K is a topological group, then it has an actually associative multiplication, so we get this ring structure. For example, we know that $K(\mathbb{Z},n)$ is a topological group. We had

$$\begin{tikzcd} H_k \arrow[r, mapsto] & \pi_k \arrow[rr, mapsto] & & \pi_k \\ \mathsf{Chain} \arrow[r] & s\mathsf{Ab} \arrow[r] & s\,\mathsf{Set} \arrow[r, "|\cdot|"] & \mathsf{Top} \\ (\cdots \to \overset{n}{\mathbb{Z}} \to \cdots) \arrow[rrr, mapsto] & & & K(\mathbb{Z},n) \end{tikzcd}$$

But a simplicial abelian group is the same thing as an abelian group object in $s\,\mathsf{Set}$, so the image of the composition $s\mathsf{Ab} \to s\,\mathsf{Set} \to \mathsf{Top}$ lands in group objects of Top; the important thing is that the geometric realization respects products.

I claim that the fibration $K(\mathbb{Z},n-1) \to * \to K(\mathbb{Z},n)$ is a fibration of topological groups. This means that we'll get a spectral sequence of rings. To see that this is a fibration of topological groups, it is enough to get the fibration from Chain:

$$\begin{tikzcd} & n & n-1 & & & \\ 0 \arrow[r] & \mathbb{Z} \arrow[r] & 0 \arrow[r] & 0 & & K(\mathbb{Z},n) \\ 0 \arrow[r] \arrow[u] & \mathbb{Z} \arrow[r, "\mathrm{id}"] \arrow[u] & \mathbb{Z} \arrow[r] \arrow[u] & 0 \arrow[u] & & P \simeq * \arrow[u] \\ 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & \mathbb{Z} \arrow[r] \arrow[u] & 0 \arrow[u] & & K(\mathbb{Z},n-1) \arrow[u] \end{tikzcd}$$

The image of the middle guy in Top is contractible because it's homotopy groups are the homology groups of the chain complex (which are zero) and it is a CW complex, so all homotopy groups being zero implies contractible by Whitehead's theorem. Because of something [[★★★]], the image in Top is a fibration. ⋄

Lemma 26.5. *If $Z \in H_2(K(\mathbb{Z},2)) \cong \mathbb{Z}$ is a generator, then z^2 is twice a generator in $H_4(K(\mathbb{Z},2)) \cong \mathbb{Z}$.*

Proof. Remember that the even homologies of $K(\mathbb{Z},2)$ are $\mathbb{Z}$ and the odd homologies are zero (you did something similar on the homework, computing the rational homology or cohomology of $K(A,n)$'s).

$$\begin{tikzcd} 0 & 0 & 0 & 0 & 0 & \\ \mathbb{Z} & 0 & \mathbb{Z} & 0 & \mathbb{Z} & 0 \\ \mathbb{Z} & 0 & \mathbb{Z} \arrow[ull, "\sim"] & 0 & \mathbb{Z} & 0 \\ & & z & & z^2 & \end{tikzcd}$$

We have that $d_2(z^2) = 2zd(z)$. We know that the product structure on $E^2_{2,1} = H_2(K(\mathbb{Z},2)) \otimes H_1S^1$. Something about the product structure, so

z^2 is twice the generator. I'll post an exercise which is a better proof, which shows that something is a divided polynomial ring. □

Corollary 26.6. $H_5(K(\mathbb{Z},3)) \cong \mathbb{Z}/2$

Proof.

$$\begin{array}{ccccccc}
 & \mathbb{Z} & & & & & \\
 & 0 & 0 & 0 & 0 & 0 & 0 \\
z & \mathbb{Z} & 0 & 0 & \mathbb{Z} & 0 & \\
 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & \mathbb{Z} & 0 & 0 & \mathbb{Z} & 0 & H_5 \\
 & & & & x & &
\end{array}$$

x is a generator. We have that $d_3x = z \in H_2(K(\mathbb{Z},2))$, so $d_3(zx) = d_3(z)x + zd_3x = 0 + z^2$, which is twice the generator. The H_5 has to kill off what is left, which is a $\mathbb{Z}/2$. □

Now we're poised to finish the theorem. If we're doing the Leray-Serre spectral sequence for $K(\mathbb{Z},3) \to * \to K(\mathbb{Z},4)$, then we get

$$\begin{array}{cccccccc}
 & \mathbb{Z}/2 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & \mathbb{Z} & 0 & 0 & 0 & \mathbb{Z} & 0 & 0 \\
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & \mathbb{Z} & 0 & 0 & 0 & \mathbb{Z} & 0 & H_6 \\
 & & & & & 4 & &
\end{array}$$

Note that the $\mathbb{Z}$ in the $(4,3)$ spot doesn't hit the $\mathbb{Z}/2$, so it doesn't cause a problem; we get that $H_6(K(\mathbb{Z},4) \cong H_5(K(\mathbb{Z},3)) \cong \mathbb{Z}/2$. In general, the higher stable homotopy group you're trying to understand, the more differentials you have to understand.

[[break]]

Derived categories and derived functors

Rather than just explain the derived category of an abelian category, I'll explain some more general stuff, so put of your abstract hat. First, let's talk about the localization of categories. I hope you already know about localization of commutative rings and modules.

Let C be a category, and let $\mathcal{E}$ be a subcategory of "equivalences". If $X \to Y$ is a morphism in $\mathcal{E}$, we write is as $X \xrightarrow{\sim} Y$.

Definition 26.7. A functor $\varepsilon\colon \mathsf{C} \to \mathcal{E}^{-1}\mathsf{C}$ is an *$\mathcal{E}$-localization of* C if $\varepsilon(\mathcal{E})$ has only isomorphisms and is initial among functors from C satisfying this property. That is, for any $F\colon \mathsf{C} \to \mathsf{D}$ such that $F(\mathcal{E})$ has only isomorphisms, there is a unique (really unique!) functor $\tilde{F}\colon \mathcal{E}^{-1}\mathsf{C} \to \mathsf{D}$ such that $F = \tilde{F} \circ \varepsilon$. ⋄

Note that $\mathcal{E}^{-1}\mathsf{C}$ need not exist. The idea in general is to construct $\mathcal{E}^{-1}\mathsf{C}$ by taking all the objects of C, and make the morphisms zig-zags $(X \xleftarrow{\sim} X_0 \to X_1 \xleftarrow{\sim} \cdots \xleftarrow{\sim} X_n \to Y)$ modulo some kind of equivalence.

There are examples in algebra and topology that are not that different.

Example 26.8. If R is a commutative ring and S is a multiplicative subset, then there exists a localized ring $S^{-1}R$. This ring is called the classical localization. You can think of R as an additive category with one object whose morephisms are elements of R and whose composition is multiplication in R. ⋄

Example 26.9. Let $\mathsf{C} = R\text{-}\mathsf{mod}$, let S be a multiplicative subset, and let $\mathcal{E}_S(M,N) = \{f\colon M \to N \mid \ker f \text{ and } \operatorname{coker} f \text{ are } S\text{-torsion}^1\}$. I claim that $\mathcal{E}_S^{-1}\mathsf{C} \simeq S^{-1}R\text{-}\mathsf{mod}$. We'll see this later. It is very handy that

[1] $M \in R\text{-}\mathsf{mod}$ is S-torsion if for all $m \in M$, there is some $s \in S$ such that $sm = 0$.

you can construct the category of representations of something without constructing the something first. ◇

Example 26.10. Let $\mathsf{C} = \mathsf{Chain}(\mathsf{A})$ for A an abelian category, and let $\mathcal{E}$ consist of quasi-isomorphisms. Then we define the *derived category of* A to be $\mathsf{D}(\mathsf{A}) := \mathcal{E}^{-1}\mathsf{C}$. Later, we'll see how to think about morphisms in this category. ◇

Example 26.11. Let $\mathsf{C} = \mathsf{Top}$ and let $\mathcal{E}$ consist of all weak equivalences. Then we'll see that $\mathcal{E}^{-1}\mathsf{C} \simeq h\mathsf{CW}$. ◇

Example 26.12. Let $\mathsf{C} = \mathsf{Top}$ and let $\mathcal{E}$ consist of maps that induce isomorphisms on some generalized homology theory h_* (if you don't know about generalized homology theories, imagine h as your favorite homotopy functor). Amazingly, you get some really interesting things. For example, if you're interested in the homotopy type of a simply connected space, then it is enough to understand it's image under the localization of $\mathbb{Z}/p$ homology and rational homology. ◇

How to construct $\mathcal{E}^{-1}\mathsf{C}$?

Definition 26.13. $X \in \mathsf{C}$ is $\mathcal{E}$*-local* if $f^* \colon \mathsf{C}(B, X) \to \mathsf{C}(A, X)$ is a bijection for all $f \colon A \xrightarrow{\sim} B$ (i.e. for all $f \in \mathcal{E}$). ◇

In Example 26.8, the single object is local if and only if $\cdot s \colon R \to R$ is a bijection for all $s \in S$. That is, the object is local if and only if S consists of units.

In Example 26.9, a module M is local if and only if M is uniquely S-divisible (i.e. for all $m \in M$ and $s \in S$, there exists a unique $n \in M$ such that $sn = m$).

In Example 26.10, it is really hard to tell if an object is local, but there is a trick to avoid the problem.

Lemma 26.14. $\mathcal{E}^{-1}\mathsf{Chain}(\mathsf{A}) \simeq \mathcal{E}_h^{-1}(h\mathsf{Chain}(\mathsf{A}))$.

We'll skip the proof of this lemma, but it's easy. On HW13: If $\mathsf{C} = h\mathsf{Chain}^-(\mathsf{A})$ (bounded above complexes), then chain complexes of injectives are local. This is a generalization of the acyclic to injective lemma.

In Example 26.11, it turns out that only the point is local, and in Example 26.12, it turns out there are more local objects.

Definition 26.15. An $\mathcal{E}$-localization of $A \in \mathsf{C}$ is an equivalence $A \xrightarrow{\sim} L_A$, where L_A is local. ◇

In Example 26.9, a the map $M \to S^{-1}M$ is a localization. In Example 26.10, a localization is an injective resolution.

There is a little lemma that L_A is unique up to unique isomorphism. If $B \xrightarrow{\sim} L_B$ is a localization, then there is a unique $L_f \colon L_A \to L_B$. Because of the definition of local, a map from L_A to L_B is the same thing as a map from A to L_B, which we already have.

Theorem 26.16. *If C has all localizations (every object has a localization), then the functor from the full subcategory of local objects $Loc_{\mathcal{E}}(\mathsf{C}) \hookrightarrow \mathsf{C} \xrightarrow{\varepsilon} \mathcal{E}^{-1}\mathsf{C}$, is an equivalence of (large) categories.*

27 ???

Organizing principle: let $\mathcal{E} \subseteq \mathsf{C}$ be a subcategory (of "equivalences", written $A \xrightarrow{\sim} B$). Assume that C has all $\mathcal{E}$-localizations (this is a really strong assumption).

(a) The derived category exists; it's an initial object $a\colon \mathsf{C} \to \mathcal{E}^{-1}\mathsf{C}$ in the category of $\mathcal{E}$-inverting functors from C.

(b) For any functor (no exactness assumptions) $F\colon \mathsf{C} \to \mathsf{D}$, the (right) derived functor exists; it's an initial object in the category of triangles

$$\begin{tikzcd} \mathsf{C} \arrow[r, "F"] \arrow[d, "a"'] & \mathsf{D} \arrow[dl, phantom, "\Uparrow\alpha"] \\ \mathcal{E}^{-1}\mathsf{C} \arrow[ur, "RF"'] & \end{tikzcd}$$

Recall that $X \in \mathsf{C}$ is $\mathcal{E}$-local if $f^*\colon \mathsf{C}(B,X) \to \mathsf{C}(A,X)$ is a bijection whenever $f\colon A \xrightarrow{\sim} B$ is an equivalence. A localization of A is an equivalence $A \xrightarrow{\sim} I_A$, where I_A is local. Having all localizations means that every object has a localization.

Example 27.1. We has three sorts of examples last time:

1. Classical localization of k-modules (k any commutative ring). Here, all localizations exist ($M \to S^{-1}M$ always exists).

2. $\mathsf{C} = h\mathsf{Chain}(\mathsf{A})$, where A is abelian and $\mathcal{E}$ the category of quasi-isomorphisms. If A is the category of sheaves of modules on a ringed space $(X, \mathcal{O}_X)$. This is a theorem of Spaltenstein (the precise reference is on the homework). On the homework, you'll prove the easy case, where $\mathsf{C} = h\mathsf{Chain}^-(\mathsf{A})$ and A has enough injectives. In this case, complexes of injectives are local and the equivalence is a generalization of injective resolutions.

3. $\mathsf{C} = h\mathsf{Top}$, with $\mathcal{E}$ the category of weak equivalences.

It is really important that we took the homotopy categories in 2; otherwise, localization do not exist in general. In example 3, localization don't always exist (consider a continuous bijection $\mathbb{Z} \to \mathbb{Q}$). The way out is to use colocalizations. The following lemma says that you may as well consider the homotopy category if you're only interested in the derived category. ◇

Lemma 27.2. *If* $\mathsf{C} = \mathsf{Chain}(\mathsf{A})$ *or* Top, *then*

$$\begin{tikzcd} C \arrow[r, "a"] \arrow[d, two heads] & \mathcal{E}^{-1}\mathsf{C} \arrow[d, shift left, "\cong"] \\ h\mathsf{C} \arrow[r] \arrow[ur, dashed, "\exists!"] & \mathcal{E}^{-1}(h\mathsf{C}) \arrow[u, shift left] \end{tikzcd}$$

Proof of (a). By the axiom of choice, for each object A, we may choose a localization $i_A\colon A \xrightarrow{\sim} I_A$ (we have assumed that localizations exist for all objects). This automatically (see the diagram on the right) gives us a functor $\mathsf{C} \to Loc_{\mathcal{E}}\mathsf{C}$, the full subcategory of local objects, and i is a natural transformation

$$\begin{tikzcd} \mathsf{C} \arrow[r, "I"] \arrow[dr, "\mathrm{id}"'] & Loc_{\mathcal{E}}\mathsf{C} \arrow[d, phantom, "\cap"] \arrow[dl, phantom, "\Uparrow i"] \\ & \mathsf{C} \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "i_A", "\sim"'] \arrow[d, "f"'] & I_A \arrow[d, "\exists! I_f"] \\ B \arrow[r, "i_B", "\sim"'] & I_B \end{tikzcd}$$

Note that if f is an equivalence, then I_f is an isomorphism; you construct the inverse by observing that $\mathsf{C}(A, I_A) \cong \mathsf{C}(B, I_A) \cong \mathsf{C}(I_B, I_A)$, so i_A corresponds to some $I_B \to I_A$ [[★★★ show this inverts I_f.]]. Thus, I is $\mathcal{E}$-inverting.

Define $a\colon \mathsf{C} \to \mathcal{E}^{-1}\mathsf{C}$ by taking objects to be objects in C, and by taking $\mathcal{E}^{-1}\mathsf{C}(X,Y) = \mathsf{C}(I_X, I_Y)$, with the functor a taking X to X and f to I_f. By the previous paragraph, a is $\mathcal{E}$-inverting. Now assume $F\colon \mathsf{C} \to \mathsf{D}$ is $\mathcal{E}$-inverting. We want to find a unique $\tilde{F}\colon \mathcal{E}^{-1}\mathsf{C} \to \mathsf{D}$ such that $\tilde{F} \circ a = F$. On objects, we must have $\tilde{F}(X) = F(X)$, and given $\phi \in \mathsf{C}(I_X, I_Y)$, we have $F\phi\colon F(I_X) \to F(I_Y)$. But we have a beautiful natural transformation

$$\begin{tikzcd} F(X) \arrow[r] \arrow[d, "Fi_X"', "\cong"] & F(Y) \arrow[d, "Fi_Y", "\cong"'] \\ F(I_X) \arrow[r, "F\phi"] & F(I_Y) \end{tikzcd}$$

Note that Fi_X and Fi_Y must be isomorphisms since F is $\mathcal{E}$-inverting, and since $\tilde{F}$ must be a functor, $\tilde{F}\phi$ must be equal to $(Fi_Y)^{-1} \circ F\phi \circ Fi_X$. □

Remark 27.3 (Notation)**.** If A is an abelian category, then the *derived category* is defined as $\mathsf{Der}\,\mathsf{A} := \mathcal{E}^{-1}h\mathsf{Chain}(A)$. Similarly, $\mathsf{Der}^{\pm}\,\mathsf{A} = \mathcal{E}^{-1}h\mathsf{Chain}^{\pm}(A)$ and $\mathsf{Der}^{b}\,\mathsf{A} = \mathcal{E}^{-1}h\mathsf{Chain}^{b}(A)$. ◇

Lemma 27.4. *$Loc_{\mathcal{E}}\mathsf{C} \subseteq \mathsf{C} \xrightarrow{a} \mathcal{E}^{-1}\mathsf{C}$ is an equivalence of categories.*

Proof. We have

$$\begin{tikzcd} \mathsf{C} \arrow[r, "a"] \arrow[d, "I"'] & \mathcal{E}^{-1}\mathsf{C} \arrow[dl, dashed, "\exists!\tilde{I}"] \\ Loc_{\mathcal{E}}\mathsf{C} & \end{tikzcd}$$

It is easy to check that $\tilde{I}$ is an inverse. □

Now suppose you have *any* functor $F\colon \mathsf{C} \to \mathsf{D}$. Then the claim is there there exists a universal pair (RF, α)

$$\begin{tikzcd} \mathsf{C} \arrow[r, "a"] \arrow[d, "F"'] & \mathcal{E}^{-1}\mathsf{C} \arrow[dl, dashed, "RF"] \arrow[dl, phantom, near end, "\overset{\alpha}{\Longrightarrow}" description] \\ \mathsf{D} & \end{tikzcd}$$

That is, for any other functor $G\colon \mathcal{E}^{-1}\mathsf{C} \to \mathsf{D}$ and any natural transformation $\beta\colon F \Rightarrow G \circ a$, then there exists a unique natural transformation $\eta\colon RF \Rightarrow G$ such that $\beta = \eta a \circ \alpha$.

Example 27.5. Consider $M \otimes_R -\colon \mathsf{Chain}(R\text{-}\mathsf{mod}) \to \mathsf{Chain}(\mathsf{Ab})$. This is not $\mathcal{E}$-inverting (i.e. not exact).

$$\begin{tikzcd} \mathsf{Chain}(R\text{-}\mathsf{mod}) \arrow[r] \arrow[d, "M\otimes -"] & \mathsf{Der}(R\text{-}\mathsf{mod}) \arrow[d, dashed, "?"] \\ \mathsf{Chain}(\mathsf{Ab}) \arrow[r, "\alpha"] & \mathsf{Der}(\mathsf{Ab}) \end{tikzcd}$$

◇

[[break]]

Proof of (b). We just define $RF = F \circ \tilde{I}$ and $\alpha = Fi$

$$\begin{tikzcd} \mathsf{C} \arrow[rr, "a"] \arrow[dr, "I"'] \arrow[ddr, bend right, "\mathrm{id}"'] & \arrow[d, phantom, "\overset{\mathrm{id}}{\Longrightarrow}"] & \mathcal{E}^{-1}\mathsf{C} \arrow[dl, "\tilde{I}"] \arrow[dd, "RF"] \\ & Loc_{\mathcal{E}}\mathsf{C} \arrow[d, phantom, "\cap"] \arrow[l, phantom, "\overset{i}{\Longleftarrow}"] \arrow[dr, phantom, "\circlearrowleft"] & \\ & \mathsf{C} \arrow[r, "F"] & \mathsf{D} \end{tikzcd}$$

Now suppose we have $G\colon \mathcal{E}^{-1}\mathsf{C} \to \mathsf{D}$ and a natural transformation $\beta\colon F \to Ga$. Then we want to show there is a unique $\eta\colon RF \to G$ such that $\beta = \eta a \circ \alpha$. We have that $RF(X) = F(I_X)$, and for $\phi \in \mathcal{E}^{-1}\mathsf{C}(X, Y) := \mathsf{C}(I_X, I_Y)$, $RF\phi = F\phi$.

$$\begin{tikzcd} FX \arrow[r, "Fi_X = \alpha_X"] \arrow[d, "\beta_X"'] & RFa(X) = F(I_X) \arrow[d, "\beta_{I_X}"] \arrow[dl, dashed, "\eta a?"] \\ GaX \arrow[r, "Gai_X"'] & GaI_X \end{tikzcd} \qquad \begin{tikzcd} \mathsf{C} \arrow[r, "I"] \arrow[rrr, bend right, "a"'] & Loc_{\mathcal{E}}\mathsf{C} \arrow[r, phantom, "\subseteq"] \arrow[d, phantom, "\Uparrow ia"] & \mathsf{C} \arrow[r, "a"] & \mathcal{E}^{-1}\mathsf{C} \\ & {} & & \end{tikzcd}$$

We have that $i_X\colon X \to I_X$ is an equivalence, so ai_X is an isomoprhism, so Gai_X is an isomorphism, so we must define $\eta_X\colon RFX \to GX$ to be $(Gai_X)^{-1} \circ \beta_{I_X}$ [[★★★ wait, that doesn't go between the right things]]. □

Corollary 27.6. *If A is an abelian category with enough injectives, then $\mathsf{Der}^{-}(\mathsf{A}) = \mathcal{E}^{-1}\mathsf{Chain}^{-}(\mathsf{A})$ and for $A, B \in \mathsf{A}$, $\mathsf{Der}^{-}(\mathsf{A})(A[n], B) = \mathrm{Ext}^n_{\mathsf{A}}(A, B)$. [[★★★ well, it should be $A[-n]$, but since in our complexes differential goes down, maybe $A[1]$ should be defined as A in degree -1.]]*

Proof.

$$\begin{aligned} \mathsf{Der}^{-}(\mathsf{A})(A[n], B) &= h\mathsf{Chain}^{-}(\mathsf{A})(A[n], I_B) && \text{(HW13)} \\ &= H_n(\mathsf{A}(A, I_B)) && \text{(next time)} \\ &=: \mathrm{Ext}^n_{\mathsf{A}}(A, B) \end{aligned}$$

□

Remark 27.7. The R means "right derived", coming from the fact that we require $\alpha\colon F \to RFa$. If we require α to go the other way and you want the derived functor to be terminal, you get LF, but then you have to uses colocalizations. $\diamond$

28 ???

Lemma 28.1. *Let A be an object in an abelian category A, and let $\Sigma^n A \in \mathsf{Chain}(\mathsf{A})$ be A concentrated in degree n.*

(a) $[\Sigma^n A, C_*] \cong H_n(\mathsf{A}(A, C_*))$*, which is isomorphic to* $\mathsf{A}(A, H_n C_*)$ *if A is projective or C_* is a complex of injectives.*

(b) $[C_*, \Sigma^n A] \cong H_n(\mathsf{A}(C_*, A)) =: H^n(C_*; A)$*, which is isomorphic to* $\mathsf{A}(H_n C_*, A)$ *if A is injective or C_* is a complex of projectives.*

You can think of $\Sigma^n A$ as the "A-n-sphere" by part (a) (interpreting H_n as homotopy groups), or you can think of it as a $K(A, n)$ (which should represent cohomology) by part (b).

Proof. (a) A chain map from $\Sigma^n A$ to C_* is a map $f\colon A \to C_n$ such that $d \circ f = 0$:

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "f"] & 0 \arrow[d] \\ C_{n+1} \arrow[r] & C_n \arrow[r, "d"] & C_{n-1} \end{tikzcd}$$

Thus, a chain map is exactly an element of $Z_n(\mathsf{A}(A, C_*))$. A nulhomotopy of f is a map $h\colon A \to C_{n+1}$ such that $f = dh$. So nulhomotopies are exactly elements of $B_n(\mathsf{A}(A, C_*))$. Thus, we get part (a).

Part (b) is just as easy. □

Corollary 28.2. $\mathsf{Der}^-(\mathsf{A})(\Sigma^{-n} A, B) \cong \mathsf{Der}^-(\mathsf{A})(A, \Sigma^n B) \cong \mathrm{Ext}^n_{\mathsf{A}}(A, B)$.

Proof. $\mathsf{Der}^-(\mathsf{A})(\Sigma^{-n} A, B) = h\mathsf{Chain}^-(\mathsf{A})(\Sigma^{-n}, I_*) =:$ [1]$[\Sigma^{-n} A, I_*]$, where I_* is an injective resolution (i.e a localization) of B. By the Lemma, this is $H_{-n}(\mathsf{A}(A, I_*)) =: \mathrm{Ext}^n_{\mathsf{A}}(A, B)$. Note that the injective resolution is number in a slightly funny way to conform to our convension of the differentials always decrease degree: $0 \to B \to I_0 \to I_{-1} \to \cdots$. □

Remark 28.3. Recall that we had the Yoneda product on Ext given by splicing (if you think of Ext as parameterizing extensions). This just corresponds to composition in the derived category: $\Sigma^{-n-m} A \xrightarrow{\Sigma^{-m} f} \Sigma^{-m} B \xrightarrow{g} C$.

Recall that $\mathrm{Ext}^n(A, B)$ is generated by Ext^1 (because any long exact sequence in an abelian category can be "unspliced" into a bunch of short exact sequences.). This implies that $\mathsf{Der}^-(\mathsf{A})^n(A, B) := \mathsf{Der}^-(\mathsf{A})(\Sigma^{-n} A, B)$ is generated by morphisms of degree 1.

If you have an arbitrary triangulated category, a necessary criterion for it to be the derived category of some abelian category is for the graded Hom to be generated by things of degree 1. Sometimes, this condition is sufficent. ◇

After the break, we'll switch from local objects to colocal objects. This will show us why CW complexes are so special. But first, let's review group homology.

I keep changing HW13. I also changed HW12 to fix the Hurewicz problem. It is actually now more general; it shows that homotopy groups of finite simply connected CW complexes are finitely generated.

Review of group (co)homology

(a) We always assumed G is discrete; later, I'll talk about what happens if G has a topology. We defined $H_n(G) = H_n(K(G, 1))$, but this was not so good because it wasn't functorial, so we used the bar complex. Then we actually find a functorial model of $K(G, 1)$, namely BG. Recall that $BG := |N_\bullet \mathsf{C}_G|$ was defined as the geometric realization of the nerve of the category associated to G.

Remark 28.4. You can cicumvent topological spaces; everything is totally combinatorial.

$$\begin{aligned} H_n(G) &\cong H_n(|N_\bullet \mathsf{C}_G|) \\ &\cong H_n(Alt_* Free(N_\bullet \mathsf{C}_G)) \\ &\cong H_n(N_* Free(N_\bullet \mathsf{C}_G)) \end{aligned}$$ ◇

[1] In the derived category, this is really supposed to be $h\mathsf{Chain}(\mathsf{A})(I_A, I_B)$, but we only need to resolve B because I_B is local, so $h\mathsf{Chain}(\mathsf{A})(I_A, I_B) \cong h\mathsf{Chain}(\mathsf{A})(A, I_B)$.

Recall that $N_n\mathsf{C}$ are n-tuples of composible morphisms, with degeneracy maps given by interting identities and face maps given by composing two consecutive maps. Thus, $N_n\mathsf{C}_G \cong G^{n+1}$.

Now suppose A is a trivial G-module, then how would we compute $H^n(G;A)$? We would look at the complex $\mathsf{Ab}(\mathbb{Z}[G^n],A) \cong \mathsf{Set}(G^n,A)$. This gives us the usual cocyle/coboundary definition of group cohomology.

If A is a non-trivial G-module, there is a totally analogous analysis. You have to know that the universal cover of BG is $EG = |N_\bullet \mathsf{D}_G|$, where D_G is the transport category on G (more generally, if G acts on X, we get a category whose objects are X and whose morphisms are $G \times X$). Recall from Hochschild cohomlogy that we had a simplicial G-bimodule, and we quotiented on one side by the G action to get the Bar complex. We need to compute the homology group $H^n(G;A) \cong H_n(G\text{-}\mathsf{mod}(\mathbb{Z}[G^{*+1}],A) \cong H_n(\mathsf{Set}(G^*,A))$. The last step works because $\mathbb{Z}[G^{*+1}]$ is a free $\mathbb{Z}[G]$-module. Again, you get n-cocycles in A modulo n-coboundaries in A. The only difference is that the alternating sums of things depend on the action, where they didn't before.

Remark 28.5. This approach genralizes to groupoid homology, category homology (just replace C_G by your favorite small category), n-groupoids (and n-categories), and topological groups. If you have a topology on the objects and morphisms of a category, then the nerve is a simplicial topological space (not just a simplicial set). But if you think about geometric realization, $|X_\bullet| = \coprod \Delta^k \times X_k/\sim$, you may as well use the topology on X_k in the definition.

If G is any topological group, you get BG and EG as before, and it still turns out that $EG \cong *$ and $EG \to BG$ is a G-principal bundle. A lemma shows that this is the universal G-bundle (i.e. isomorphism classes of G-principal bundles on a CW complex X are parameterized by $[X, BG]$). If you apply this to the discrete case, you see that maps to a $K(G,1)$ (i.e. BG) parameterize covering spaces with fiber G. ◇

Warning 28.6. If G is not discrete, BG is *not* a $K(G,1)$, because $\pi_n(BG) \cong \pi_{n-1}(G)$ (because EG is contractible). Thus, the homology of BG doesn't agree with the homology of G. ⌟

If G is a topological group and if A is a G-module (with a topology), then you can define continuous cohomology. Similarly, you can define smooth cohomology or Borel cohomology (if you have measures).

[[break]]

Survey on colocalizations

Recall that we had a subcategory of equivalences $\mathcal{E} \subseteq \mathsf{C}$. We had a nice construction if we have all localizations. There is an obvious concept of *colocal* objects: X is colocal if $\mathsf{C}(X,A) \xrightarrow{\cong} \mathsf{C}(X,B)$ for any equivalence $A \xrightarrow{\sim} B$. We say that C has all cololcaizations if for every object X, there is an equivalence $C_X \to X$ where C_X is colocal. If you have all colocalizations, you get the same story. The principle is that if C has all $\mathcal{E}$-colocalizations, then $\mathcal{E}^{-1}\mathsf{C}$ exists and the composition $CoLoc_{\mathcal{E}}\mathsf{C} \subseteq \mathsf{C} \to \mathcal{E}^{-1}\mathsf{C}$ is an equivalence of categories. You define the morphisms $\mathcal{E}^{-1}\mathsf{C}(X,Y) = \mathsf{C}(C_X, C_Y)$. Furthermore, for any functor $F\colon F \to D$, there exists a left derived functor LF:

$$\begin{tikzcd} \mathsf{C} \arrow[r] \arrow[d, "F"'] & \mathcal{E}^{-1}\mathsf{C} \arrow[dl, "LF"] \arrow[dl, phantom, shift right=3, "\Leftarrow"] \\ \mathsf{D} & \end{tikzcd}$$

Theorem 28.7. *If $\mathsf{C} = h\mathsf{Top}$ and $\mathcal{E}$ consists of weak equivalences, then CW complexes are colocal and all colocalizations exist!*

Proof. By definition of weak equivalence, spheres a colocal. $X^{(n)}$ is the mapping cone on the attaching map $\coprod S^{n-1} \to X^{(n-1)}$. Then use the long exact sequence for mapping cones, then by induction on the n-skeleton, you get that it is local.

Weak equivalence exist because $|\Delta_\bullet X| \to X$ is a weak equivalence from a CW complex. □

Corollary 28.8. *The derived category $\mathsf{Der}(h\mathsf{Top}) \cong \mathsf{Der}(\mathsf{Top})$ of topological spaces is equivalent to $h\mathsf{CW}$.*

29 More about derived functors

[1]

This is our last week. There is no homework/office hours/discussion this week. Today, we'll do more on derived functors, and on Thursday, we'll talk about triangulated categories and functors.

Last time, we started with a category C and a subcategory of "equivalences" $\mathcal{E}$. We proved that if there are enough $\mathcal{E}$-localizations, the map $Loc_{\mathcal{E}}\mathsf{C} \hookrightarrow \mathsf{C} \to \mathcal{E}^{-1}\mathsf{C}$ is an equivalence of categories. If we had enough localizations, we could choose a localizing functor $I\colon \mathsf{C} \to Loc_{\mathcal{E}}\mathsf{C}$. For an arbitrary functor $F\colon \mathsf{C} \to \mathsf{D}$, we got an induced derived functor $RF\colon \mathcal{E}^{-1}\mathsf{C} \to \mathsf{D}$.

$$\begin{tikzcd}
\mathsf{C} \arrow[rr, "a"] \arrow[dr, shift left, "I"] \arrow[ddr, bend right, "\mathrm{id}"'] & & \mathcal{E}^{-1}\mathsf{C} \arrow[dl, shift left, "\tilde{I}"] \arrow[dd, ":=RF"] \\
& Loc_{\mathcal{E}}\mathsf{C} \arrow[ul, hook, shift left] \arrow[ur, "\cong"] \arrow[d, hook] & \\
& \mathsf{C} \arrow[r, "F"] & \mathsf{D}
\end{tikzcd}$$

We take $\mathsf{C} = h\mathsf{Chain}(\mathsf{A})$ and $\mathcal{E}$ to be quasi-isomorphisms. If A has enough injectives, then for any chain complex X_*, we can choose a quasi-isomorphism to a complex of injectives $X_* \xrightarrow{\sim} I_X$. We saw in HW13 that complexes of injectives are local; we'll see this in a little more generality later.

Now suppose we have some functor $F\colon h\mathsf{Chain}(\mathsf{A}) \to \mathsf{D}$. Then RF is applied by replacing the chain complex with a complex of injectives and applying F. Given an object $A \in \mathsf{A}$, we can think of it as a complex concentrated in degree zero. Then a localization is exactly the same thing as an injective resolution $A \to I_A$. Then $H_nRF(A) = H_n(FI_A)$ (I guess we're assuming $\mathsf{D} = h\mathsf{Chain}(\mathsf{B})$). One nice thing is that you don't have to take homology. As you saw on HW13, taking homology loses some information about the complex.

Example 29.1 (The case of an exact functor between abelian categories)**.** Let $f\colon R \to S$ be a ring homomorphism. Then we get an induced functor $f_*\colon S\text{-}\mathsf{mod} \to R\text{-}\mathsf{mod}$, which is exact. This extends to a functor $f_*\colon h\mathsf{Chain}(S\text{-}\mathsf{mod}) \to h\mathsf{Chain}(R\text{-}\mathsf{mod}) \xrightarrow{a} \mathsf{Der}(R\text{-}\mathsf{mod})$ (this is the functor you usually derive. I claim that in this case, becuase we have an exact functor, $Rf_* = f_*$.

f_* is exact means that when you apply f_* to an acyclic (exact) complex, it remains acyclic. Recall from HW12 that there is a short exact sequence $X \xrightarrow{g} Y \to Cg$ [[★★★ up to homotopy?]], and if g is a quasi-isomorphism, then Cg is acyclic. Applying f_*, we get a short exact sequence $0 \to f_*X \to f_*Y \to f_*Cg \to 0$ because f_* is exact, and f_*Cg is still acyclic, so $f_*X \to f_*Y$ must be a quasi-isomorphism. Thus, f_* respects quasi-isomophisms (which are isomorphisms in $\mathsf{Der}(R\text{-}\mathsf{mod})$. This means that we get a factorization (by the universal property of $\mathcal{E}^{-1}\mathsf{C}$)

$$\begin{tikzcd}
h\mathsf{Chain}(S\text{-}\mathsf{mod}) = \mathsf{C} \arrow[r, "a"] \arrow[dr, "f_*"'] & \mathcal{E}^{-1}\mathsf{C} \arrow[d, "Rf_* = f_*"] \\
& \mathsf{Der}(R\text{-}\mathsf{mod})
\end{tikzcd}$$

So we have that "$Rf_* = f_* = Lf_*$" (the natural transformation is basically the identity, so it can go both ways). ◇

In general, all you have is a natural transformation $F \to RF$, but it isn't clear what they have to do with eachother.

Remark 29.2. Usually we derive functors between abelian categories $F\colon \mathsf{A} \to \mathsf{B}$ by replacing F by the induced functor $F\colon h\mathsf{Chain}(\mathsf{A}) \to \mathsf{Der}(\mathsf{B})$ to get $RF\colon \mathsf{Der}(\mathsf{A}) \to \mathsf{Der}(\mathsf{B})$ (all of these derived categories are bounded above). We can define $R_nF(X) := H_n(RF(X))$, which is what people usually mean when they say "the derived functors of F". ◇

Remark 29.3. If $F\colon \mathsf{A} \to B$ is left exact, then $R_0F(X) \cong FX$. If $X \to I_*$ is an injective resolution, then $0 \to FX \to FI_0 \to FI_1$ is exact (further down, it is not exact because F is only left exact), so $H_0(FI_*) \cong FX$. You can derived functors that aren't left exact, but you lose this property. We'll see Thursday that under some weaker conditions, you get some long exact sequences. ◇

[1] Brought to you by Chris Schommer-Pries.

Example 29.4. Given our ring map $f\colon R \to S$, we can also define the functor $f^*\colon R\text{-}\mathsf{mod} \to S\text{-}\mathsf{mod}$, given by ${}_RM \mapsto S \otimes_R M$. We have enough projectives, so you can construct the left derived functor $Lf^*\colon \mathsf{Der}^-(R\text{-}\mathsf{mod}) \to \mathsf{Der}^-(S\text{-}\mathsf{mod})$. Since f^* is a right exact functor, we get an isomorphism $L_0 f^* X \cong f^* X$. ⋄

Fix a ring R and $A \in \mathsf{Chain}(\mathsf{mod}\text{-}R)$. Then we can construct a functor $h\mathsf{Chain}(R\text{-}\mathsf{mod}) \to h\mathsf{Chain}(\mathsf{Ab})$, given by ${}_RM \mapsto \mathrm{Tot}^{\oplus}(A \otimes_R M)$. We have enough projectives, so we can left derive to get $A \overset{L}{\otimes} -\colon \mathsf{Der}^-(R\text{-}\mathsf{mod}) \to \mathsf{Der}^-(\mathsf{Ab})$.

Claim. *If $A \to A'$ is a quasi-isomorphism, then $A \overset{L}{\otimes}_R - \cong A' \overset{L}{\otimes}_R -$.*

We have a natural transformation $A \overset{L}{\otimes} B \to A' \overset{L}{\otimes} B$ coming from the quasi-isomorphism. We want to show that this induces quasi-isomorphisms. It is enough to check this when B is a complex of projectives (because the first thing you do to apply $\overset{L}{\otimes}$ is replace B by a complex of projectives). In particular, projective things are flat. Now we want to compute the homology of some total complexes. For this, we use spectral sequences. We actually have a map of double complexes before we take total complexes. This induces a map of spectral sequences (i.e. we have maps $E^r_{p,q} \to E'^r_{p,q}$ for each r, p, q). We have $E^1_{p,q} = H_{-q}(A) \otimes_R B_p$ and $E'^1_{p,q} = H_{-q}(A') \otimes_R B_p$ because B_p is flat. The induced map on E^1 pages is an isomorphism (because we assumed the map $A \to A'$ induces isomorphisms on all homologies). This implies that the induced map on E^∞ pages is an isomorphism. In fact, it implies that the homologies of the total complexes are isomorphic [[★★★ you have to use the fact that you have a map on total complexes respecting the filtration and use the 5-lemma repeatedly]].

Corollary 29.5. $\overset{L}{\otimes}_R\colon \mathsf{Der}^-(\mathsf{mod}\text{-}R) \times \mathsf{Der}^-(R\text{-}\mathsf{mod}) \to \mathsf{Der}^-(\mathsf{Ab})$ *(we need the bounded below stuff to make sure the spectral sequence converges).*

Let A be an abelian category with enough injectives. Define $\mathsf{Chain}(\mathsf{A})^\circ \times \mathsf{Chain}(\mathsf{A}) \to \mathsf{Chain}(\mathsf{Ab})$ by $\mathrm{Hom}(P, Q) = \mathrm{Tot}(\mathrm{Hom}(P_p, Q_q))$, where the double Hom complex has differentials $d^h f = f \circ d_P$ and $d^v f = (-1)^{p+q} d_q \circ f$. Fix P. then we have $\mathrm{Hom}(P, -)\colon h\mathsf{Chain}(\mathsf{A}) \to h\mathsf{Chain}(\mathsf{Ab}) \to \mathsf{Der}(\mathsf{Ab})$. We can right derive to get $R\,\mathrm{Hom}(P, -)\colon \mathsf{Der}^+(\mathsf{A}) \to \mathsf{Der}^+(\mathsf{Ab})$. If $P \to P'$ is a quasi-isomorphism, then we get an induced isomorphism $R\,\mathrm{Hom}(P, -) \cong R\,\mathrm{Hom}(P', -)$. Peter mentioned that $\mathrm{Ext}^n(A, B) = H_{-n} R\,\mathrm{Hom}(A, B)$; it requires a bit of argument.

All this works great if you have enough localizations or colocalizations. If you don't have enough, can the derived functor still exist?

[[break]]

Remark 29.6. In HW12, you saw the Cartan-Eilenberg resolution, a resolution $P_{*,*} \to X_*$ which was very special. In particular, you had a quasi-isomorphism $\mathrm{Tot}(P_{*,*}) \to X_*$, so we get colocalizations of all complexes this way. Then $LF(X) = \mathrm{Tot}(FP_{*,*})$. You then get a nice spectral sequence converging to the homomlogy of this guy. This can be useful. ⋄

Now suppose you don't have enough projectives. For example, fix a topological space X and a sheaf of rings $\mathcal{O}_X$. Then the category $\mathcal{O}_X\text{-}\mathsf{mod}$ (sheaf of $\mathcal{O}_X$-modules) is an abelian category. It (always?) has enough injectives, but usually does not have enough projectives!

There is a fix. You have $\mathcal{E} \subseteq \mathsf{C}$. Suppose there exists a full subcategory $\mathsf{B} \subseteq \mathsf{C}$ such that

$$\text{for all } X \in \mathsf{C}, \text{ there is a } Y \in \mathsf{B} \text{ and an equivalence } Y \xrightarrow{\sim} X. \qquad (*)$$

then you can define $\mathcal{E}_{\mathsf{B}} = \mathcal{E} \cap \mathsf{B}$, and you have a functor $\mathcal{E}_{\mathsf{B}}^{-1}\mathsf{B} \to \mathcal{E}^{-1}\mathsf{C}$ (assuming both exist, which they do if you do some set theory arguments). The condition implies that this functor is essentially surjective.

Definition 29.7. Let $F\colon \mathsf{A} \to \bar{\mathsf{A}}$ be a functor between abelian categories. A complex X is *F-acyclic* if $F(X)$ is acyclic. ⋄

Next time, we'll see that if F is a triangulated functor (to be defined next time) from $h\mathsf{Chain}(\mathsf{A})$ to $h\mathsf{Chain}(\bar{\mathsf{A}})$ and if $\mathsf{B} \subseteq h\mathsf{Chain}(\mathsf{A})$ satisfies $(*)$ and all acyclics in B are F-acyclic, then $\mathcal{E}_{\mathsf{B}}^{-1}\mathsf{B} \to \mathcal{E}^{-1}h\mathsf{Chain}(\mathsf{A})$ is fully faithful. Given this, for all $X, Y \in \mathsf{B}$ such that $X \xrightarrow{\sim} Y$, then $FX \to FY$ is an equivalence. Then we can define the left derived functor LF as the left derived functor $LF\colon \mathcal{E}_{\mathsf{B}}^{-1}\mathsf{B} \to \mathsf{Der}(\mathsf{A})$.

One last thing we don't really have enough time to cover is composition of derived functors. If $\mathsf{A} \xrightarrow{G} \mathsf{B} \xrightarrow{F} \mathsf{C}$, then for free, you get a natural transformation $R(FG) \overset{\xi}{\Rightarrow} RF \circ RG$. When you apply G to a complex of injectives in A, you usually don't get a complex of injectives, but if you did, you would get that ξ is an isomorphism. It turns out that you only need enough G-acyclics which are sent to F-acyclics to get that ξ is an isomorphism. You do this by running a spectral sequence in two different ways. One way, it collapses (because you build a nice Cartan-Eilenberg type resolution), and the other way, you get something else. The upshot is that you get $E^2_{p,q} = R_pF(R_qG(A)) \Rightarrow R_{p+q}(FG)(A)$.

30 Triangulated categories

Today is the last day. We'll finish what we were doing last time and give an overview of triangulated categories.

We had a functor between abelian categories $F\colon \mathsf{A} \to \mathsf{B}$ that we wanted to derive. We got an induced functor $F\colon h\mathsf{Chain}(\mathsf{A}) \to h\mathsf{Chain}(\mathsf{B}) \to \mathsf{Der}(\mathsf{B})$. So our setup is that $\mathsf{C} = h\mathsf{Chain}(\mathsf{A})$, and $\mathcal{E}$ consists of quasi-isomorphisms. In general, if there are enough $\mathcal{E}$-local objects, we can construct a functor $I\colon \mathsf{C} \to Loc_{\mathcal{E}}\mathsf{C}$

$$
\begin{tikzcd}
\mathsf{C} \arrow[rr, "a"] \arrow[dr, "I"] \arrow[ddr, bend right, "\mathrm{id}"'] & & \mathcal{E}^{-1}\mathsf{C} \arrow[dl, "\tilde{I}"'] \arrow[dd, ":=RF"] \\
& Loc_{\mathcal{E}}\mathsf{C} \arrow[d, hook] \arrow[dl, phantom, "\overset{i}{\Rightarrow}"] & \\
& \mathsf{C} \arrow[r, "F"] & \mathsf{D}
\end{tikzcd}
$$

This worked as long as we had enough local objects, and the dual picture worked so long as we had enough colocal objects.

Suppose we have a full subcategory $\mathsf{B} \subseteq \mathsf{C}$. Define $\mathcal{E}_{\mathsf{B}} = \mathcal{E} \cap \mathsf{B}$. The idea is that if we choose a nice enough B, we might still be able to define RF even if we don't have enough local objects. If B has enough $\mathcal{E}_{\mathsf{B}}$-local objects, $\mathcal{E}^{-1}\mathsf{C}$ is equivalent to $\mathcal{E}_{\mathsf{B}}^{-1}\mathsf{B}$, and we get a factorization

$$
\begin{tikzcd}
\mathsf{C} \arrow[rr, "a"] \arrow[ddr, bend right, "\mathrm{id}"'] & & \mathcal{E}^{-1}\mathsf{C} \\
& \mathsf{B} \arrow[ul, hook] \arrow[r, "a"] \arrow[d, hook] & \mathcal{E}_{\mathsf{B}}^{-1}\mathsf{B} \arrow[u, "\cong"] \arrow[d, dashed] \\
& \mathsf{C} \arrow[r, "F"] & \mathsf{D}
\end{tikzcd}
$$

We saw that if F is exact, then it is equal to RF (at least in setup with chains). Thus, "failure of $F = RF$ is equivalent to the failure of F to be exact." That is, you somehow want to say that RF is the best exact approximation of F. The problem is that $h\mathsf{Chain}(\mathsf{A})$ and $\mathsf{Der}(\mathsf{A})$ are *not* abelian categories, so it doesn't make sense to say that RF is exact. So we'd like to understand in what sense RF is exact. The punchline is that both of these categories are actually triangulated categories. This extra structure will allow us to figure out what nice properties B should have.

Recall that in HW13, for a map $f\colon X \to Y$, we defined the cone and we showed that you get $X \xrightarrow{f} Y \to Cf \to \Sigma X$. This had some nice properties (you got some long exact sequences). This is really the structure that we're going to try to abstract.

Definition 30.1 (Verdier)**.** A *triangulated structure* on an additive category C is an automorphism[1] $\Sigma\colon \mathsf{C} \to \mathsf{C}$ and a family of distinguished triangles (or exact triangles) $\{A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} \Sigma A\}$ (this triangle is sometimes abreviated (u, v, w)) satisfying

(T0) any triangle isomorphic to an exact triangle is an exact triangle,[2] and the triangle $(\mathrm{id}_A, 0, 0) = (A \xrightarrow{\mathrm{id}} A \to 0 \to \Sigma A)$ is exact.

(T1) for all $u\colon A \to B$, there exist v and w such that (u, v, w) is an exact triangle.

(T2) (Rotation) If (u, v, w) is an exact triangle, then so are $(v, w, -\Sigma u)$ and $(-\Sigma^{-1} w, u, v)$. [[★★★ I'd rather build the minus signs into the Σ]]

(T3) If you have two of the three morphisms of a morpism of exact triangles, the third one exists:

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, dashed, "\exists h"] & \Sigma A \arrow[d, "\Sigma f"] \\
A' \arrow[r, "u'"] & B' \arrow[r] & C' \arrow[r] & \Sigma A'
\end{tikzcd}$$

(T4) (Octahedron axiom) If we have a commutative triangle $A \xrightarrow{f} B \xrightarrow{g} C$ and exact triangles on g and $g \circ f$, then there exists an exact triangle on f and there exists morphisms making (Δ means exact and $\circlearrowleft$ means commutative)

$$\begin{tikzcd}
& B \arrow[dr] \arrow[d, phantom, "\circlearrowleft"] & \\
A \arrow[ur] \arrow[rr] & {} & C
\end{tikzcd}
\qquad
\begin{tikzcd}
\exists C' \arrow[dr] \arrow[ddrr, dashed, bend right, "\exists"'] & & B \arrow[ll] \arrow[dr] \arrow[dl, phantom, "\Delta"] \arrow[dr, phantom, "\Delta", shift left=4] \arrow[d, phantom, "\circlearrowleft"] & & A' \arrow[ll] \arrow[llll, bend right, "\circlearrowleft"'] \\
& A \arrow[ur] \arrow[rr, "\Delta"'] \arrow[dl, phantom, "\circlearrowleft"] & {} & C \arrow[ur] \arrow[dl] \arrow[dr, phantom, "\circlearrowleft"] & \\
& & B' \arrow[ul] \arrow[uurr, dashed, bend right, "\exists"'] & &
\end{tikzcd}$$

You can think of this as the statement that $(C/A)/(B/A) \cong C/B$. [[★★★ stick in the 5-lemma result to get rid of the exists on the C']]

◇

In the case of chain complexes, the exact triangles are the ones isomorphic to one of the form $X \xrightarrow{f} Y \to Cf \to \Sigma X$. It is an exercise to check that $h\mathsf{Chain}(\mathsf{A})$ satisfies the axioms.

There is a feeling that the definition of a triangulated category isn't quite right. Probably in 20 years or so (maybe less), this will be an obsolete definition. Nevertheless, it is a useful definition in the absence of something better. The trouble is that the morphism produced in (T3) is not a unique morphism. In the example of chian complexes, you can construct a canonical morphism from the homotopy $u'f \simeq gu$. But just from knowing that there exists a homotopy, you don't get a canonical choice of filler. The catch phrase is "cones aren't functorial."

[[break]]

If you have a map $u\colon A \to B$ in $h\mathsf{Chain}(\mathsf{A})$, and suppose you complete this to an exact triangle with Cu. Suppose that you can find an automorphism of B not homotopic to the identity making the leftmost square commute

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, equal] & B \arrow[r, "v"] \arrow[d, "a"] & Cu \arrow[d, dashed] \\
A \arrow[r, "u"] & B \arrow[r, "hva^{-1}"] & Cu
\end{tikzcd}$$

The upshot is that given the data $(A \xrightarrow{u} B)$, the map $B \to Cu$ is not canonically determined.

[1] An actual invertible functor, not just an autoequivalence. In a category of chain complexes, this will be shifting the chain complexes up or down.

[2] A morphism/isomorphism of triangles is what you think it is: three morphisms making three squares commute.

Definition 30.2. If C is triangulated, and $F\colon \mathsf{C} \to \mathsf{B}$ is a functor to an abelian category, then F is *homological* if for every exact triangle (u, v, w), the sequence

$$FA \xrightarrow{Fu} FB \xrightarrow{Fv} FC \xrightarrow{Fw} F\Sigma A$$

is exact. We also define $F_i(X) = F(\Sigma^{-i}X)$. ◇

In HW13, we saw that homology (i.e. H_0) and $[X, -]$ are homological functors. There is an analogous notion of a cohomological functor, of which $[-, X]$ was an example.

If we had a functor of abelian categories $F\colon \mathsf{A} \to \mathsf{B}$, then we get an induced functor $h\mathsf{Chain}(\mathsf{A}) \to h\mathsf{Chain}(\mathsf{B}) \xrightarrow{H_0} \mathsf{Ab}$ which is homological.

Let C be a triangulated category, and let $\mathcal{E}$ be the equivalences given by a homological functor F (i.e. a morphism g is an equivalence if $F_i g$ is an isomorphism for all i). Then

1. The Ore condition is satisfied.
2. For $\mathcal{E}^{-1}\mathsf{C}$, it is enough to use "roofs."
3. $\mathcal{E}^{-1}\mathsf{C}$ is triangulated.

1. We want to show that if $X' \to Y \xleftarrow{\sim} Y'$, then there exists some W and morphisms such that such that

$$\begin{tikzcd}
& W \arrow[dl, "\sim"'] \arrow[dr] & \\
X' \arrow[dr] & & Y' \arrow[dl, "\sim"', "s"] \\
& Y &
\end{tikzcd}
\qquad
\begin{tikzcd}
& & & W \arrow[dl] \arrow[dr, dashed] & \\
& & X' \arrow[dl] \arrow[dr] & & Y' \arrow[dl, "\sim"', "s"] \\
& C \arrow[dl] \arrow[dr, equal] & & Y \arrow[dl] & \\
\Sigma W & & C & &
\end{tikzcd}$$

Since s is an equivalence, $F_i(C) = 0$ for all i. It follows that the map $W \to X$ is an equivalence. The arrow from W to Y is given by the axiom (T3).

2. It is enough to show that if you compose two roofs, you get another roof, but this follows from the Ore condition immediately by "popping" the middle valley in $X \xleftarrow{\sim} X' \to Y \xleftarrow{\sim} Y' \to Z$.

3. We need to say when $X \xleftarrow{\sim} X' \to Y \xleftarrow{\sim} Y' \to Z \xleftarrow{\sim} Z' \to \Sigma X$ is an exact triangle. Popping everything up, we have

$$\begin{tikzcd}
& & & X'' \arrow[ddll, "\sim"'] \arrow[dr] & & & \\
& & & & Y'' \arrow[dl, "\sim"'] \arrow[dr] & & \\
& X' \arrow[dl, "\sim"'] \arrow[dr] & & Y' \arrow[dl, "\sim"'] \arrow[dr] & & Z' \arrow[dl, "\sim"'] \arrow[dr] & \\
X & & Y & & Z & & \Sigma X
\end{tikzcd}$$

We declare the triangle to be exact if $X'' \to Y' \to Z' \to \Sigma X$ is exact.

Proposition 30.3. *Suppose B is a full subcategory of $\mathsf{C} = h\mathsf{Chain}(\mathsf{A})$, with $\mathcal{E}$ weak equivalences, and $F\colon \mathsf{A} \to \mathsf{D}$ such that $F\colon \mathsf{C} \to \mathsf{Der}(\mathsf{D}) \xrightarrow{H_0} \mathsf{D}$ is homological. Suppose further that any acyclics in B are F-acyclic (they remain acyclic upon applying F) and for all $X \in \mathsf{C}$, there is some $Y \in \mathsf{B}$ and an equivalence $Y \xrightarrow{\sim} X$. Then $\mathcal{E}_B^{-1}\mathsf{B} \to \mathcal{E}^{-1}\mathsf{C}$ is an equivalence. Thus, we may use $\mathcal{E}_B$-local objects to define/compute RF.*

Given any $A \to Z \xleftarrow{\sim} B$ with $A, B \in \mathsf{B}$, we'd like to find a roof in B giving it. Well, we can pop it to get $A \xleftarrow{\sim} Z' \to B$. Then we can find some $Y \in \mathsf{B}$ equivalent to Z. Then we get a roof $A \xleftarrow{\sim} Y \to B$ giving the same morphism.