1 Valery Alexeev

I will try to give an account of the complete moduli of higher dimensional varieties. Let me begin by giving an overview of what we know about the dimension 1 case. We have a moduli space $\mathcal{M}_g$, introduced 150 years ago by Riemann. There is this wonderful compactification $\overline{\mathcal{M}}_g$ (the Deligne-Mumford compactification, also due to Grothendieck, ...). The two are quite similar. Then there is the space $\overline{\mathcal{M}}_{g,n}$, which again looks bigger, but the differences are quite minor. In particular, there is $\overline{\mathcal{M}}_{0,n}$, which is really very easy. It is a very explicit combinatoria object, some blowup of $\mathbb{P}^{n-3}$. There is also the moduli space $\overline{\mathcal{M}}_{g,n,\beta}$, where you add some weights between zero and 1. Again, we have the special case $\overline{\mathcal{M}}_{0,n,\beta}$. Then we have the Kontsevich maps; the moduli space of stable maps $\overline{\mathcal{M}}_{g,n}(V)$. There are many papers about these first of all because of the importance of applications (e.g. Gromov-Witten theory), and secondly because you can compute things.

I will speak about the dimension $n > 1$ case. The analogue of $\mathcal{M}_g$ is the moduli space of surfaces of general type $\mathcal{M}_{c_1^2,c_2}$. This space is already very hard and very complicated. $\mathcal{M}_g$ is mysterious, but at least it is smooth as a stack. $\mathcal{M}_{c_1^2,c_2}$ is not even equi-dimensional, and even describing its irreducible components is hard. As I said, the difference between $\mathcal{M}_g$ and $\overline{\mathcal{M}}_g$ is very minor, so maybe we can still go somewhere. Also, even if the general case is hopeless, there may be some examples we can work with. In particular, there are analogues of $\overline{\mathcal{M}}_{0,n}$ and $\overline{\mathcal{M}}_{0,n,\beta}$ that can be described in complete detail. Another special case is the case of abelian varieties; stable abelian varieties are quite nice and can be described quite explicitly.

The plan for the course is this course

1. Complete moduli and MMP
2. Stable toric varieties
3. Hyperplane arrangements
4. Abelian varieties
5. Surfaces

The first lecture is introductory. The first four lectures should be quite explicit. The last lecture is the case of the moduli space of surfaces of general type.

The plan for today:
The very first example: degrees of curves
Redo for surfaces (KSB (Kollar? Sheppard? Bard?) 1989).
Redo for n-dimensional pairs; for stable maps.
Sings of MMP: k, klt, dlt, slc (sklt?, sdlt?).
Ex: curves, hyy arrs, toric vars.
Polytopes and toric vars.
$(X, B_1 + \varepsilon B_2)$ lc $\Leftrightarrow B_2 \not\supset T$-orbits
$\overline{\mathcal{M}}_{g,n,\beta}$ after Hassett
Exs: surfaces

The very first example. Suppose you have a 1-dimensional family X of curves of genus g over some base S which is not complete. How do you complete it. First you apply the stable reduction theorem, which says that after some base change on S, th fiber can be made into a curve with simple normal crossings. This may not be stable. How do we make it stable? If there are (-1)-curves, you can contract them, leaving the surface smooth, so you contract them all. If there is a (-2)-curve, you can contract it to a singular point, but the singularities are rational double points of type A_n. After that, X_0 is a stable curve which is nodal with canonical class $K_{X_0} > 0$ ample if and only if $|\operatorname{Aut}(X_0)| < \infty$. For a curve E in the central fiber, $K_{X_0} \cdot E < 0$ if and only if $E \cong \mathbb{P}^1$ and $E^2 = -1$ and $K_{X_0} \cdot E = 0$ if and only if $E \cong \mathbb{P}^1$ and $E^2 = -2$.

Theorem 1.1. *For every $X \to S^0$, there is a finite base change $S' \to S$ and a completion such that $X' \to S'$ is a flat family of stable curves.*

The condition that K was ample means that $K_{X'/S'} + X'_0$ is ample and X'_0 nodal if and only if (X', X'_0) has log canonical (lc) singularities. The nice thing about curves is that you still have a reduction theorem in mixed characteristic.

So what do we do in dimension n? You have to give a label to everything. We started with $X' \to S'$ and you did some stuff, ending up with the *log canonical model* $X'_{can} \to S'$ of the pair (X', X'_0). You know that there is a theory of minimal models in all dimensions, so we can repeat the procedure in higher dimensions.

Suppose we have a family of surfaces. Then after base change, we get a surface with normal crossings. Instead of contracting this and that, you just go straight to the canonical model. $K_{X'/S'} + X'_0$ ample and (X', X'_0) lc, then we say that X'_0 is *semi-log canonical* (slc). $X' = \operatorname{Proj}_{S'} \bigoplus_{d\geq 0} \pi_* \mathcal{O}_{X'}(d(K_{X'} + X'_0))$. Compare to $\operatorname{Proj} \bigoplus_{d\geq 0} H^0(\mathcal{O}_X(d(K_{X'} + X'_0)))$; there is very little difference.

You have to prove existence and uniqueness of the model. uniqueness is easy, and existence is a recent result. The outcome is that for any 1-parameter family of varieties of general type, there is a unique limit with ample canonical class and something slc.

Where do we go from here? You can try to construct this moduli space in general, or you can look at the special cases. You can use the theorem to guess the answer, and then construct your moduli space by various other methods (in the cases of toric, hyperplane arrangements, abelian varieties).

You can redo this for n-dimensional pairs. So we have a family of surfaces with divisors. The stable reduction theorem still works. The log canonical model is not for (X', X'_0) but for $(X', X'_0) + \sum b_i B_i$.

When you do this stuff carefully, you run into hard technical problems for surfaces. Q: does the formation of the log canonical ring commute with base change. That depends on the moduli functor. If you do it carefully, you run into problems (not in the special cases), and I will try to delay them until Saturday.

You can redo this for stable maps. Suppose you have a variety (say a curve) X and a stable map $X \to V$ (parts of X collapse). We say X is stable if $K_X > 0$ and X nodal, and we say the map is stable if $K_{X/V} > 0$ and X nodal. So you just think of families $X \to S^0 \times V$, and in the construction, you do everything over $S' \times V$. All the same general machinery works to give you the unique limit of any family of stable maps. This higher dimensional moduli should exist in this case as well.

Singularities of MMP: lc, klt, dlt, slc. You know the first three (from the pre-reading). slc will be the generalization to the "nodal case."

For a pair (X, B) to be lc, X should be a normal variety over $k = \bar{k}$, and $B = \sum b_i B_i$, where $0 < b_i \leq 1$ and B_i are (not necessarily distinct) Weil divisors. There should exist a log resolution $f\colon Y \to X$ (i.e. Y smooth and the exceptional set of f union $f^{-1}_{strict\ trans}\operatorname{Supp}(B)$ has simple normal crossings). We need $K_X + B$ to be a $\mathbb{Q}$-Cartier divisor, so $N(K_X + B)$ is Cartier. In this case, we can write $K_Y = f^*(K_X + B) + \sum_{E_j\ irr\ divs} a_j E_j$. Then lc means that all $a_j \geq -1$ (which implies $b_i \leq 1$), klt means that all $a_j > -1$ (which implies $b_i < 1$), and dlt means that there is a Y with $Exc(f)$ is a union of divisors and all $a_j > -1$ for all exceptional divisors. Note that dlt depends on the resolution; if you keep going, you might get some -1's. Some finite generation result for klt which can be pushed to dlt.

Example 1.2 (Curves)**.** Let X be a curve, with some divisors B_i (may not be distinct). What does it mean for the pair (X, B) to be lc? It means that X is smooth and whenever B_i coincide for $i \in I$, then $\sum_{i \in I} b_i \leq 1$. It is klt if for every such colletion, $\sum b_i < 1$ (in particular, this implies all $b_i < 1$). In this case, dlt is the same as lc. ◇

Example 1.3 (Hyperplane arrangements)**.** You have hyperplanes B_i intersecting in $\mathbb{P}^{r-1}$. What does it mean for $(\mathbb{P}^{r-1}, B)$ to be lc? It means that for every $I \subset \{1, \dots, n\}$, $\sum_{i \in I} b_i \leq \operatorname{codim} B_i$ (if the intersection is non-empty. klt means that this inequality is strict. ◇

Example 1.4 (Toric varieties)**.** Suppose X is a toric variety with a torus T acting on it. Let $B_1 = X \setminus T$. Then toric geometry tells us two standard facts: (1) $K_X + B_1 = 0$ in a canonical way, and (2) (X, B_1) is lc (this follows from the first fact because a toric variety always has a toric resolution; pull back $K_X + B_1 = 0$ to get $0 = K_Y + f^{-1}B$+exceptional divisors with $a_j = -1$). If you add another divisor B_2, then $(X, B_1 + \varepsilon B_2)$ for $0 < \varepsilon \ll 1$ is lc if and only if $B_2 \not\supset T$-orbits. The reason is that when you resolve, you add exceptional divisors with $a_j = -1$, so you are maxxed out. This already tells you that if you work with coeffiecients 1 and ε, then you are in the toric situation. ◇

I cannot teach you about polytopes and toric varieties in 5 minutes; I hope you already know how to see a variety if I show you a polytope.

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Fix a weight $\beta = (b_1, \dots, b_n)$ (rational numbers $0 < b_i \leq 1$).

Definition 2.1. A stable pair is $(X, B = \sum b_i B)$ where X is projective connected reduced, and B_i are Weil divisors such that

1. (on singularities) slc
2. (numberical) $K_X + B$ ample

A stable map is $f\colon (X, B) \to Z$ satisfying two conditions; the first are the same, and the second is changed to saying that $K_X + B$ is ample over Z (e.g. if the map is finite, this is a non-condition). ◇

Ideal theorem: Fix a dimension, β, and some other invariants, then there exists a projective moduli space $\overline{\mathcal{M}}$ of stable maps.

Example 2.2. The weighted moduli spaces $\overline{\mathcal{M}}_{g,\beta}$ due to Hassett. This is indeed a projective smooth stack. If $g = 0$, it is a fine moduli space (i.e. it is a smooth projective variety). ◇

The goal is to generalize to higher dimensions. We'll sometimes use this dream theorem for inspiration.

When you talk about moduli spaces, you need a functor. The minimal condition is that you look at flat families. Here you have to be more careful because the B_i are only Weil divisors, not Cartier divisiors. We'll be more careful about this later.

Let's review the dimension 1 case again. What does the mysterious condition slc mean for curves? (X, B) is a curve with points, then

1. (slc) when $\{B_i | i \in I\}$ coincide, $\sum_{i\in I} b_i \leq 1$.
2. (numerical) for all irreducible components $E \subseteq X$, $\deg(K_X + B)|_E > 0$. This degree is $2p_0(E) - 2\sum_{B_i \subseteq E} 1 + E(X - E)$.

What is the definition of slc in higher dimensions? () You should require that in codimenison 1, it is at worst nodal. This already implies that it is Gorenstein in codimension 1, so you have the notion of ω_X. () You also require that the B_i do not contain the components of the double locus. () We ask that $[\omega_X^{\otimes N}(N \sum b_i B_i)]^{\vee\vee}$ be invertible. This allows us to talk about $K_X + B$; it will be a $\mathbb{Q}$-Cartier divisor. Next, you can take a normalization, in which you will have the divisors B_i and the double locus. () We would like to require that $(X^\nu, B^\nu + \text{(double locus)})$ is lc. We're almost done. (4) We ask that X is (S2) (Serre condition 2, which is normal minus R1).

The other condition is dlt, which is better than lc because it implies Cohen-Macaulay, and lc only implies normal. Similarly we may want sdlt, which would imply Cohen-Macaulay, whereas slc only implies S2. I will not give a definition of sdlt, but there is a reasonable candidate.

Stable toric varieties

I use the word stable by analogy with stable curves; some people use the word "broken" toric varieties, which kind of gives you an idea of what they are.

(TV) the segment is the polytope for $\mathbb{P}^1$. (STV) two intervals glued at ends should be two $\mathbb{P}^1$'s meeting at a point. (TV) square is $\mathbb{P}^1 \times \mathbb{P}^1$, triangle is $\mathbb{P}^2$. Triangle with corner cut is $Bl_{pt}\mathbb{P}^2$. With two corners cut is is the blow-up at two points; one curve can be blown down to get $\mathbb{P}^1 \times \mathbb{P}^1$. (STV) [[★★★ picture]] If we glue two triangles to two adjacent edges of a square, that is two $\mathbb{P}^2$'s glued to a $\mathbb{P}^1 \times \mathbb{P}^1$ along a couple of $\mathbb{P}^1$'s, and all three of these intersect at a point. There is a 1-parameter family where this guy is a limit of $\mathbb{P}^2$'s . . . you "break" two corners of the triangle and leave them hinged.

(TV) In toric geometry, there is a correspondence. Fix a lattice $\Lambda \cong \mathbb{Z}^r$ and a torus $T = (\mathbb{C}^\times)^r$ (you don't have to work over $\mathbb{C}$, but I will for simplicity). Then there is a correspondence between {integral polytopes with vertices in Λ} and $\{(X, L)$ polarized linearized toric variety$\}$ (X normal projective toric variety and L is an ample line bundle with T action). Q: does toric variety mean normal. VA: yes, I do require normal.

(STV) To $\Delta = \bigcup P^\alpha \in$ {coplex of integral polytopes} we associate an element of $\{$family of (X, L) polarized STV$\}$. Stanley-Riesner varieties are the ones that come from breaking a polytope, but you can have two triangles joined at a vertex or multiple edges between two vertices. $H^0(\Delta, \underline{\mathrm{Aut}}) = \mathrm{Aut}(X)$. Something is parameterized by $H^1(\Delta, \underline{\mathrm{Aut}})$. We have $C^0 \to C^1 \xrightarrow{\partial} C^2$. In our example, $C^1 = \mathbb{C}^\times \oplus \mathbb{C}^\times$, $C^0 = (\mathbb{C}^\times)^2 \oplus (\mathbb{C}^\times)^2 \oplus (\mathbb{C}^\times)^2$, $C^2 = 0$, and the first homology is zero.

Example 2.3. [[★★★ picture: triangle in a triangle; corresponding vertices joined]] Here, we will get $H^1 = \mathbb{C}^\times$. ◇

In a huge class of examples, the varieties I get are slc.

Our first example [[★★★ two triangles on a square]] the topological space $|\Delta|$ is a manifold with boundary. This implies that X is Cohen-Macaulay.

Example 2.4. [[★★★ picture: two triangles glued at a vertex]] is not S2 and not CM. ⋄

Consider the variety [[★★★ first example; two triangles on a square]], with the three boundary edges are B_1, B_2, and B_3.

Lemma 2.5. *$(X, \sum B_i)$ is slc and $K_X + \sum B_i = 0$.*

Proof. It is S2 because it is CM. The next condition is to look at the normalization which is [[★★★ picture: break off the hinged parts]] and check that you get lc. □

Now condier adding an additional divisor [[★★★ picture]]. This is a tropical picture

Lemma 2.6. *$(X, \sum B_i + \varepsilon B_{n+1})$ is slc if and only if B_{n+1} does not contain any T-orbits.*

The proof is the same. When you break off the hinged parts, the extra divisor is a line that intersects each of the other lines (on the boundary) at one point each.

We either work with all weights 1, or with a bunch of 1's and an ε. The 1's correspond to the boundary and the ε corresponds to an ample Cartier divisor.

Definition 2.7. A stable toric variety *over Z* is as follows. We have a torus $T = (\mathbb{C}^\times)^r$ acting on $\mathbb{P}^N$ $(r < N)$, and in $\mathbb{P}^N$, we have a closed T-invariant subvariety Z. We define a stable toric variety over Z to be a finite morphism $f \colon X \to Z$ from a stable toric variety X. ⋄

Theorem 2.8 (main theorem)**.** *Fix Δ. Then there exists a projective moduli space (which is a scheme) of stable toric varieties over Z, $M^T(Z, \Delta)$.*

The theorem is much more general; this corresponds to the multiplicity-free case. Doesn't have to be a torus; you can do it for spherical varieties. The proof is in one of my papers with . . . . I won't try to give the proof.

How is this different from the toric Hilbert scheme? In that case, you'd look at subschemes of Z. These would normally be non-reduced and non-normal. I insist that we work with nice normal S2 toric varieties. Why can't I just restrict to the reduced case?

Example 2.9. $\{t x_0 x_2 - x_1^2 = 0\}$ is a family (the parameter is t). As $t \to 0$, in Hilb, we get $x_1^2 = 0$, a double line. In M^T, you break the $\mathbb{P}^1$ segment by removing a point. This somehow says that the map to $\mathbb{P}^2$ is a 2-to-1 map, not an embedding. ⋄

1-parameter degenerations

I have the curves X_t from the previous example, with maps $X_t \to \mathbb{P}^2$, and I have $f^*\mathcal{O}(1) = L_t$, and a morphism $\bigoplus H^0(\mathbb{P}^2, \mathcal{O}(d)) \to \bigoplus H^0(X_t, L_t^d)$. $H^0(\mathcal{O}_{\mathbb{P}^2}(1))$ has a basis x_0, x_1, x_2 (corresponding to the two endpoints of the segement and the one in the middle). Fix isomorphisms $(X_t, L_t) \cong (\mathbb{P}^1, \mathcal{O}(2))$. Then the map $H^0(\mathcal{O}_{\mathbb{P}^2}(1)) \to H^0(X_t, L_t)$ is given by $x_i \mapsto t^{\delta_{i,1}} e_i$. To compute the limit (how to break the picture), you take the lower convex hull of the height function and project the linear parts down. This is how you prove that every family has a unique limit point.

You have this height function h. You can take the discrete Laplace dual. Every face will correspond to a point in the dual space. Dimension 1 is too small, so let's do a 2-dimensional picture.

[[★★★ picture big triangle breaking into our first example]] This should be the projection of some height function. The height function is defined on the six lattice points. If you take the discrete Laplace dual, you get [[★★★ tropical picture: trivalent tree to depth 2]]. This is $\mathrm{trop}(f_t \colon (X, L)_t \to Z)$. If you take a different family, you may still get the same limit, but the tropical thing will change.

So far, I have stable toric varieties over Z. The MMP interpretation is that $(X, \sum B_i) \to Z$ is stable map. I proved that this is slc and the other condition. Here, we have all weights are 1. Stable toric pairs (X, D) will have weights all 1's and an ε. This is a special case of the previous on. If we have $L = \mathcal{O}_X(D)$, we get $\phi_{|H^0(X,L)|^*} \colon X \to \mathbb{P}^N = Z \supset H = \{x_0 + \cdots + x_n = 0\}$, then $D = f^*H$.

Conclusion for today: there is a moduli space of stable toric varieties. I used MMP here for motivation, but then independently on constructs this moduli space. I will use this in the later lectures in two ways. Tomorrow I'll describe higher dimensional generalizations of one of the guys. On Friday, I'll consider the compactification of the moduli space of abelian varieties, and this will correspond to stable toric pairs.

When you look at 1-parameter degenerations, they are described by height functions. How many height functions do you have (if you allow real heights)? It looks like $\mathbb{R}^m$. You can say that two heights are equivalent if they give you the same subdivision. This gives a fan on $\mathbb{R}^m$. This defines a secondary toric variety. the main component of $M^T(\mathbb{P}^n, D)$ is a possibly non-normal toric variety and its normalization is this guy.

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Quiz for today: Suppose you have a family of [[★★★ triangle]] $X_t \cong \mathbb{P}^2 \to Z$ degenerating to [[★★★ that breaking of the triangle you always draw for proper non-projective]] $X_0 = \lim_{t\to 0} \to Z$. Can this happen? If you recall, you have to have some height function that the broken picture is a projection of. This should remind you of that famous picture of Escher. If you didn't know, that picture is not really possible; it is an optical trick. However, there is a point in the moduli space for this stable toric variety. The result is that the moduli space of stable toric varieties has multiple components; not everything can be written as a limit of things in the main component.

I will state three main theorems and try to give as many examples as possible. Fix positive integers $r, n \in \mathbb{N}$ (n divisors in $\mathbb{P}^{r-1}$), and a weight $\beta = (b_1, \dots, b_n)$, with b_i rational $0 < b_i \le 1$.

Theorem 3.1. *There exists a family $(\mathcal{X}, B_1, \dots, B_n) \to \overline{M}_\beta(r, n)$ such that every fiber $(X, \sum b_i B_i)$ is a stable curve. Moreover, there is an open subset $M_\beta(r,n) \subseteq \overline{M}_\beta(r,n)$ such that the restriction of the family is a family of lc pairs $(\mathbb{P}^{r-1}, B_i)$. Furthermore, all fibers are non-isomorphic.*

The *weight domain* (the possible values of β) is $D = \{\beta = (b_i) | 0 < b_i \le 1, \sum b_i > r\}$. It looks like a cube, with a corner cut off by the inequality. We take a chamber decomposition, where the walls are $\sum_{i\in I} b_i = k$ for all $I \subseteq \{1, \dots, n\}$ and for all $1 \le k \le n-1$. For example, something on the boundary lies on a different chamber from something in the interior. [[★★★ picture]]

Theorem 3.2. *(1) If $Ch(\beta) = Ch(\beta')$, then $\overline{M}_\beta = \overline{M}_{\beta'}$ and $(\mathcal{X}, B_i)_\beta = (\mathcal{X}, B_i)_{\beta'}$. (2) if $\beta' \in \overline{Ch(\beta)}$, then we get a commutative (not cartesian) diagram*

$$\begin{array}{ccc} \mathcal{X}_\beta & \longrightarrow & \mathcal{X}_{\beta'} \\ \downarrow & & \downarrow \\ \overline{M}_\beta & \longrightarrow & \overline{M}_{\beta'} \end{array}$$

Moreover, if $\beta' > \beta$ (in every coordinate), then $\overline{M}_\beta \xrightarrow{\sim} \overline{M}_{\beta'}$ and $\mathcal{X}_\beta \to \mathcal{X}_{\beta'}$ is birational (when you go down, it doesn't have to be birational). (3) For all $\beta > \beta'$, we have morphisms (dashed is rational map)

$$\begin{array}{ccc} \mathcal{X}_\beta & \dashrightarrow & \mathcal{X}_{\beta'} \\ \downarrow & & \downarrow \\ \overline{M}_\beta & \longrightarrow & \overline{M}_{\beta'} \end{array}$$

and on fibers, X' is the log canonical model for $(X, \sum b_i' B_i)$ (in particular, the model exists).[1]

Theorem 3.3. *(1) Every X is Cohen-MaCaulay, and $X \setminus \bigcup B_i$ is Gorenstein. (2) for β in the maximal chambers, then X is Gorenstien and B_i are Cartier.*

Example 3.4. Suppose we have $(\mathbb{P}^2 + 5 \text{ lines})_t$ in general position, and as $t \to 0$, the lines converge to two triple points [[★★★ picture, with numbered lines]]. Let's take $\beta = (1,1,1,1,1)$. If the sum of the weights is less than one, the lines can all coincide. If the sum of weights is less than 2, then three lines can go through the same point. Here the sum is 5, so we get general position. What happens in the limit? In the central fiber, you'd blow up the two points. You get the blowup of $\mathbb{P}^2$ at two points. You've blown up a 3-fold, so you get two extra $\mathbb{P}^2$'s. The lines will break up [[★★★ picture with ears + picture with triangle ears (tropical?)]]. You would think that this is the limit, but it's not. If you try this with weights $(1,1,1,1,1-\varepsilon)$, then this is indeed a stable pair ($K + B$ is ample). If you compute $(K+B)\cdot C$, where C is a piece of line number 5, you get ε. So so long as $\varepsilon > 0$, you're stable, but for $\varepsilon = 0$, that curve has to be contracted, so the actual picture is [[★★★ picture + picture with triangles]]. Let's call the weight $\beta' = (1, 1, \dots, 1)$; this is in the closure. There is another way to go to the closure; consider the weights $(\frac{1+\varepsilon}{2}, \frac{1+\varepsilon}{2}, 1, 1, 1-\varepsilon)$. Now there was no reason to blow up the first point; you only had to blow up the second point, so the picture is [[★★★ picture]]. We have morphisms [[★★★ triangle picture]]; one is birational, but the other is not; we lost a whole $\mathbb{P}^2$. ⋄

In addition to producing progressively cooler pictures, I'd like to tell you about how to construct these things.

I will start with the Grassmanian of r-dimensional subspaces of $\mathbb{C}^n$ with the Plüker embedding $G(r,n) \hookrightarrow \mathbb{P}(\bigwedge^r \mathbb{C}^n)$. We have the torus $\widetilde{T} = (\mathbb{C}^\times)^n$ acts

[1] The map to the log canonical model is only rational. For surfaces, it is usually an acutal map if you work with normal surfaces.

on the Grassmanian, $T = \widetilde{I}$ the diagonal copy of $\mathbb{C}^\times$. there are $\binom{n}{r}$ Plüker coordinates $p_{i_1,\dots,i_r}$ with $i_1 < \cdots < i_r$, which have characters in $\mathbb{Z}^n$ (each entry is zero or 1). There is a hypersimplex $\Delta(r,n)$, which is (1) the convex hull of these (2) $\{(x_i) \in \mathbb{R}^n \underline{| 0 \le} x_i \le 1, \sum x_i = r\}$. $[V \subset \mathbb{C}^n] \in G(r,n)$ has an embedded toric variety $\overline{T[V]}$ (over $G(r,n)$). There is a moment polytope P_V (called the metroid polytope). P_V is the convex hull of the $V(p_I)$ such that $p_I(V) \neq 0$. It is also $\{(x_i) \in \mathbb{R}^n | K_{\mathbb{P}^{r-1}} + \sum x_i B_i = 0, (\mathbb{P}^{r-1}, \sum x_i B_i \text{ lc}\}$. $V^r \hookrightarrow \mathbb{C}^n$ fixed, so we get $\mathbb{P}^{r-1} \cong \mathbb{P}V \hookrightarrow \mathbb{P}^{n-1}$, with $B_i = \mathbb{P}V \cap H_i$ where $H_i = \{z_i = 0\}$. Note that the definition still works if something is contained in something.

Example 3.5. Begin with a hyperplane arrangement $(\mathbb{P}^{r-1}, \sum b_i B_i) = (\mathbb{P}V, \sum b_i B_i)$ which is lc. Over $G(r,n)$, I have the universal family $U \subseteq \mathbb{P}^{n-1} \times G(r,n)$, $\pi\colon U \to G(r,n)$. I have the point $[\mathbb{P}V \subseteq \mathbb{P}^{r-1}] \in G(r,n)$. I take the orbit $T \cdot [\mathbb{P}V \subseteq \mathbb{P}^{r-1}]$. I claim that the stabilizer is trivial, so the orbit is isomorphic to T. Take the preimage of the orbit $\pi^{-1}(U)$. I can take the quotient $\pi^{-1}(U)/T$, which will recover the pair I started with. ⋄

Example 3.6. Take $r = 2$ and $n = 4$, four points on $\mathbb{P}^1$. The easiest degeneration is where you break the $\mathbb{P}^1$ to get points 1 and 2 on one piece and 3 and 4 on the other piece. Start with $\Delta(2,4)$, a hypersimplex (looks like an octahedron) with vertices labelled by distinct pairs of numbers between 1 and 4. What is the configuration where the first two points coincide. What is the metroid polytope of this arrangement? You see that the Plüker coordinate $p_{12} = 0$ and $p_{ij} \neq 0$ for $(i,j) \neq (i,j)$. So we get the lower pyramid (the top vertex is 12). What is the condition for this to be log canonical? it is that $P_V = \{x_1 + x_2 \le 1\}$, which is the lower pyramid. I am working with $\beta = (1,1,1,1)$. What is the locus where the pairs are log canonical? They are the places where 1 and 2 do not coincide. I'm looking at a certain open subset (given by GIT) $\pi^{-1}(U)^{ss}_\beta$ where the pair is log canonical. When I divide by the torus action, I get the line with points 3 and 4 and a point missing. I can now redo this for the configuration with 3 and 4 coincide. Then I redo it where 1 and 2 coincide and 3 and 4 coincide. Then the torus action downstairs is not free, but the action upstairs is free. You end up with a line with two points missing, which you have to divide by $\mathbb{C}^\times$, which gives you a point. When you stick these together, you get the two lines with 1 and 2 on one side and 3 and 4 on the other.

Somehow, the base is a stable toric variety, and I throw away its boundary to get $Y \to G(r,n)$. The GIT quotient is $X = \pi^{-1}(Y) /\!\!/ T$.

If you study GIT, you know there is a choice of line bundle and linearization of it. In this case, we need an ample line bundle on $U \subset \mathbb{P}^{r-1} \times G(r,n) \hookrightarrow \mathbb{P}^{r-1} \times$Plüker and a linearization. It turns out that this information is equivalent to the weight β. If I have a weight, then the line bundle is $p_1^*\mathcal{O}(\sum b_i - r) \otimes p_2^*\mathcal{O}(1)$. If $\sum b_i - r \to 0$, then the first factor will disappear. This shows that $\overline{M}_\beta$ will be the GIT quotient $G(r,n) /\!\!/_\beta T$ for generic β. It is well-known that this is also the GIT quotient $\mathbb{P}^{r-1} /\!\!/_\beta PGL(r)$. ⋄

When the weights are $\beta = (1,\dots,1,\varepsilon,\dots,\varepsilon)$, with $K + B > 0$ by $K + B \approx 0$, then this is the toric case.

Example 3.7. $(\mathbb{P}^2, B_1, \dots, B_n)$ a configuration of lines, so I have $(1,1,1,\varepsilon,\dots,\varepsilon)$ ($n-3$ ε's). Then all X's are stable toric varieties. If $n = 5$ you get pictures of triangles where the three sides have coefficient 1 and there are two more divisors. [[★★★ picture]] These are described by puzzles like this, where the pieces are either triangles or rhombuses. Here are some examples: [[★★★ pictures]] Your homework is to count these puzzles. I think you can get the staircase with 6 ε's, showing that that moduli space is not irreducible. ⋄

4 Valery Alexeev

Nobody turned in the homework. The Quiz for today is for you to stare at these two pictures [[★★★ pictures]] and see that they are basically the same. They are both stairways to heaven, going up and up and up. The second one is seven lines in $\mathbb{P}^2$, so $r = 3$ and $n = 7$, with $\beta = (1, 1, 1, \varepsilon, \varepsilon, \varepsilon, \varepsilon)$. You complete the puzzle by adding the divisors like this [[★★★ picture]]. The implies that $\overline{\mathcal{M}}_\beta(3, 7)$ is not irreducible, which implies that $\overline{\mathcal{M}}_{\mathbb{1}}(3, 7)$ is not irreducible either. Last time I told you you need 9 lines, but you can do it with 7.

Abelian varieties

Abelian varieties are of the form $A = \mathbb{C}^g/\mathbb{Z}^{2g} = (\mathbb{C}^\times)^g/\mathbb{Z}^g$ (the later form is the more general version).

Ideal Theorem: If you fix $\beta = (b_1, \ldots, b_n)$, a dimension, and some other stuff, then there exists a projective $\overline{\mathcal{M}}$, the moduli space of stable pairs $(X, B = \sum b_i B_i)$ satisfying (1) (X, B) slc, and (2) $K_X + B > 0$.

I will attempt to give a more complete picture for surfaces tomorrow, but for now we look at special cases, taking inspiration from this thing we wish were a theorem.

What is a polarization on an abelian variety A? A polarization λ is an ample divisor Θ, modulo algebraic equivalence. If a polarization is principal, then Θ is unique up to translation. Of course, $K_A = 0$, so $K_A + \varepsilon\Theta$ will be ample. If we pick ε very small, then singularities of the pair $(A, \varepsilon\Theta)$ will be essentially the same as those of A.

When the weights are 1 or ε, and $K + B \approx 0$ (but positive), then we are in the toric situation (i.e. X has to be something like a stable toric variety). The ideal theorem says that there has to be a compact moduli space here, of toroidal nature. If the polarization is principal, then the divisor is essentially unique, so it is the same as the moduli space of polarized varieties: $(A, \varepsilon\Theta) \leftrightarrow (A, \lambda)$. You have to be careful about the divisors matching up; in the ideal theorem, B has to be an actual divisor, not a divisor up to linear equivalence.

So one has to switch somehow to another variety $(A, \lambda) \leftrightarrow (X, \Theta)$. In doing so, we give up the notion of $0 \in A$. So X is a torsor under A, but Θ is an actual divisor. One instance of this is very familiar. $(Pic^0 C, \lambda) \leftrightarrow (Pic^{g-1}, \Theta_{g-1})$. If you work with one thing, you hardly see the difference, but in families, these things behave differently. Theorem: there is an equivalence of categories between principally polarized abelian varieties and torsors with divisor. The bad news is that this only works for principal polarizations. For the non-principal case, Martin Olsson suggested a solution with log structures. It is not in the spirit of the ideal theorem. I believe it can be done without log structures as well.

Consider [[★★★ broken interval, labelled $(0, 1, 1)$]]. I have a family X_t, where for $t \neq 0$, $(X_t, \varepsilon B_t) = (\mathbb{P}^1, \varepsilon(2\text{pts})$. In the limit, where $t = 0$, I have (X_0, B_0) a stable toric pair, a couple of $\mathbb{P}^1$'s joined at a point, where B_0 is one point on each $\mathbb{P}^1$.

If I have a toric variety X and an ample divisor B, let $L = \mathcal{O}_X(B)$. Then $H^0(X, L) = \bigoplus \mathbb{C}e_i$, where the e_i are the lattice points in the polytope. We have $\theta \in H^0(X, L)$, with $(\theta) = B$, $\theta = \sum c_i e_i$. In a family, $\mathbb{C}[\![t]\!][1/t]$ or meromorphic functions on $\{0 < |z| < \varepsilon\}$. Then $c_i(t) = c_i' t^{h_i}$, where c_i' are invertible and h_i are the heights of the lattice points. The projection of the lower convex hull of the height function gives the paving of the polytope that gives you the limit.

In tropical geometry, you look at some tropical polynomials like "$h_0 x^0 + h_1 x^1 + h_2 x^2$"$= \max(0 \cdot x + h_0, 1 \cdot x + h_1, 2 \cdot x + h_2)$. This gives a piecewise linear thing. Looking at the points where it breaks, you get the associated tropical variety.[[★★★ picture]] This is related to the toric picture by the Laplace transform. The tropical picture is like N space and the toric picture is like M space. At every point on the piecewise linear function gives you a slope in the dual space. You take some difference to get the values. The transform of the picture is the function with heights (h_1, h_2, h_3).

Let's do this with a slightly more complicated 2-dimensional picture [[★★★ picture triangle with two ears]], then the tropical picture is [[★★★ same, with dualish lines on it]].

Now I'm going to do something like this for families of abelian varieties. I'll start with the simplest picture. Start with $\lambda = \mathbb{Z}^g$ (the pictures are for $g = 1$). In the dual space $\Lambda^* \otimes \mathbb{R} = \mathbb{R}^g$, you'll get something tropical. For the height function, I'll take a non-homogeneous quadratic form, $h = q +$ linear, and I'll require that the quadratic form q is positive definite. If I take the lower convex envelope and project down, I'll get some sub-division. From this, I can construct some graded algebra R, and $\operatorname{Proj} R \to \operatorname{Spec} \mathbb{C}[\![t]\!][1/t]$ is a family (or use the meromorphic functions on $\{0 < |z| < \varepsilon\}$ as a base). When you do the construction, you get that for $t \neq 0$, $X_t = \mathbb{C}^{*g}/\mathbb{Z}^g$, and for $t = 0$, X_0 is the stable toric variety for the periodic decomposition, quotiented by $\mathbb{Z}^g$. Each of the intervals is a $\mathbb{P}^1$, and the periodic decomposition is an infinite chain of

$\mathbb{P}^1$'s. When you quotient by $\mathbb{Z}^g$, you get (X_0, D_0) [[★★★ picture like nodal cubic, with D_0 a point on it]]. What is different from the previous case is that $\mathbb{C}^{*g}/\mathbb{Z}^g$ is half way to algebraic, but it is not algebraic; you can't make sense of this quotient algebraically. You have to do something; there are three ways to solve the problem. One way is to work in the complex analytic topology, so you have a family of complex analytic varieties. The nice thing is that once you quotient by the action, you get an algebraic variety. Then there is the approach of Tate and Mumford. Mumford's approach is purely algebraic. You look at the central fiber first, where you get this infinte chain of $\mathbb{P}^1$'s. Though this is not a variety, it is a scheme, and it is locally of finite type. On such a thing, you can still define an ample line bundle and an ample divisor. Then this action by $\mathbb{Z}$ is properly discontinuous in the Zariski topology (it makes perfect sense in the algebraic category). Then after you quotient, you can descend the line bundle. That's only for the central fiber. You can replace the central point by some artinian ring to thicken it up; you can then get a thickening of the central fiber. After you do it for all such artinian rings, you can use Grothendieck's algebraization theorem to extend to a family. Mumford got his Fields medal for this stuff. There is a third solution, which is to use rigid algebraic geometry.

Now let's understand the tropical side of things. The Laplace transform of this picture is again a piecewise linear quadratic function, which you can project down. The corner locus will be some tropical variety, which will be periodic (you'll still have an action of $\mathbb{Z}^g$). If you vary the heights, the sub-division will change abruptly. One picture lives in Λ and the other lives in Λ^*, but we can identify them using the quadratic function q. There is an associated bilinear form $q\colon \Lambda \times \Lambda \to \mathbb{R}$, which gives us an isomorphism $\Lambda \xrightarrow{\sim} \Lambda^*_{\mathbb{R}}$.

A higher-dimensional picture is [[★★★ picture of two interlaced square lattices]]. [[★★★ In the tropical side?]] the 4-gons will become hexagons and the other 4-gons will become triangles, giving [[★★★ picture with hexagons dual to triangles]]. These decompositions have names. The square one (white) is called the Delanay decomposision (1920s), and the other one is called the [[★★★]] decomposition (1908). In 2007, something called the tropical theta divisor ("Tropical Jacobians"). It tells you that a tropical variety is something which describes a 1-parameter degeneration. Let's see what the result of the degeneration is in this case (the triangle tiling). Each triangle is a $\mathbb{P}^2$, and modulo the period, there are only two of them. When you divide, you'll get the two $\mathbb{P}^2$'s glued to eachother along three $\mathbb{P}^1$'s, and this is a degeneration of abelian surfaces. There is a divisor on it; algebraically, you have a line in each plane, and they intersect at three points (one on each of the three shared lines). Some people draw the divisor like this $ and call it a dollar sign.

The picture with the squares. Each square is a $\mathbb{P}^1 \times \mathbb{P}^1$, and modulo the period, there is only one copy. So when you quotient, you get a $\mathbb{P}^1 \times \mathbb{P}^1$ glued to itself along two lines (you can introduce a twist (shift) in the gluing). The degenerations are described by $H^1(\underline{P}, \mathrm{Aut})$. There is a $\mathbb{C}^\times$ of abelian varieties here.

Start with an abelian variety $\mathbb{C}^\times/\mathbb{Z}$. On this, there is a divisor Θ. Then we go to a $\mathbb{Z}$-cover, which is $\mathbb{C}^\times$, on which we have a periodic divisor, given by function $\theta = \sum_{i\in\mathbb{Z}} c_i z^i$, where the c_i are quadratic non-homogeneous (in i). Now we repeat it in a family. Then $\theta = \sum_{i\in\mathbb{Z}} c_i(t) z^i$, where $c_i = c_i' t^{h_i}$, where $h_i\colon \mathbb{Z} \to \mathbb{Z}$ is quadratic non-homogeneous: $h_i = q +$ linear, with $q \geq 0$.

Theorem 4.1. *There exists a space $\overline{AP}_g$, the moduli space of stabil semiabelic ($\leftrightarrow$ toric) pairs, with an open (but possibly not dense) subspace $AP_g = A_g$, the moduli space of principally polarized abelian varieties. The normalization of the main irreducible component of this space is $\overline{A}_g^{vor}$, a toroidal compactification of A_g for the second Voronai fan.*

Back to the first picture, where you have a polytope (broken interval) and a height function. If you look at all possible height function, it is a vector space, and it is broken into cones depending on the decomposition of the polytope that they give you. This gives you a fan, called the secondary fan, which gives the secondary toric variety. Now consider the height functions $q\colon \mathbb{Z}^g \to \mathbb{R}$ which are quadratic. Projecting the lower convex hull, you get a periodic decomposition. Breaking up the vector space of such q by the decomposition they give, you get the second Voronai fan.

5 Valery Alexeev - Moduli of surfaces

Today I'm going to talk about surfaces, and I'll try not to skip technical details. The reason is that for the previous lectures, there are papers with proofs. For surfaces, there is no definite source. There is supposed to be a book, but it has four authors, so it is delayed.

Let $\pi\colon X \to S$ be a flat family of slc surfaces, and let $N \in \mathbb{N}$ such that $Nb_i \in \mathbb{Z}$. So each fiber X_s is slc, in particular is S_2. Let $Z \subseteq X$ be a subset so that for all $s \in S$, $\operatorname{codim}(Z_s, X_s) \geq 2$. On $X \setminus Z$, $\omega_{X/S}$ and $\mathcal{O}_X(N\sum b_iB_i)$ are invertible. So the bad set, where these sheaves are possibly not invertible, is contained in Z. Let $j\colon X \setminus Z \hookrightarrow X$.

Definition 5.1. $L^N_{X\to S} := j_*\big(\omega^{\otimes N}_{X/S} \otimes \mathcal{O}_X(N\sum b_iB_i)|_{X\setminus Z}\big) =$"$N(K+B)$". ⋄

Remark 5.2. Note that formation of $L^N_{X\to S}$ does *not* commute with base change. In particular, for the (key) base change $s \to S$, the construction does not commute. In particular, the value of K^2 jumps. ⋄

Definition 5.3. Fix $\beta = (b_1, \dots, b_n)$, $V \subseteq \mathbb{P}$ (some projective scheme; for stable pairs, $V = pt$), and coefficients c_1, c_2, and c_3. Define $\mathcal{M}_N(S) = \Big\{$flat projective families $f\colon (X, \sum b_iB_i) \to S\times V$ such that (1) X, B_i are flat over S, (2) $(X_s, \sum b_iB_i)_s \to V$ is a stable map, (3) $L^N_{X\to S}$ is invertible, ample over $S\times V$, and $(L^N_{X\to S})_s = L^N_{X_s\to s}$, and (4) $(K_{X_s}+B_s)^2 = c_1, (K_{X_s}+B_s)H_s = c_2, H_s^2 = c_3$ for $H_s = f_s^*\mathcal{O}_V(1)\Big\}$. ⋄

$\mathcal{M}_N$ definitely depends on N. In characteristic p, the moduli space definitly depends on N. In charactaristic zero, I think it may not.

Definition 5.4. $\mathcal{M}^K(S)$ is the same as $\mathcal{M}^N$, but the K stands for "Kollár" and condition (3) is replaced by (3′) for all m such that $m\beta \in \mathbb{Z}^n$, $L^m_{X\to S}$ is flat over S, and $(L^m_{X\to S})_s = L^m_{X_s\to s}$, and some $L^N_{X\to S}$ is invertible. ⋄

Remark 5.5. Under assumptions (3,3′), formation of the sheaf $L^N_{X\to S}$ *does* commute with base change $S' \to S$. ⋄

Note that $L^N_{X\to S}$ does not depend on the "nice" $Z \subset X$. It should be called "the caturation in codimension 2 relative over S"; it can be defined as $\lim j_{Z,*}(\dots)$. Under conditions (3) or (3'), $L^N_{X\to S}$ is called the *hull* of $\omega^{\otimes N}_{X/S} \otimes \mathcal{O}_X(N\sum b_iB_i)$. The reference is Kollár, Hulls and husks.

Problem 1: embedded components of B.

Example 5.6 (Hacking, Hassett)**.** Consider the surface $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ as the fiber. Take the base $S = \mathbb{A}^1$. In the central fiber, take the divisor $s_0 + 2f$ in $|FF_2$, and blow it up. We get s_4 in $\mathbb{F}_4$. So you have two glued surfaces in the central fiber. Take $2s_0$ on the $\mathbb{F}_0$ and $4f + 4(s_4+4f)$ on the $\mathbb{F}_4$, which intersect the curve along which the two surfaces are glued four times. [[★★★ picture]] The key thing is that on the $\mathbb{F}_4$, you have a nodal curve, which we will smooth in the generic fiber (explicitly, it is something with genus 35; you can find this in my paper on stable limits of surfaces). Now, contract the $\mathbb{F}_0$ component to a point. So in the central fiber, we'll have $\mathbb{F}_4$, with the exceptional fiber contracted. It will be a cone on a quartic. The genus of the curve in the central fiber is one higher than the (arithmetic) genus of the curve in the generic fiber. You can see this; when you contract all four intersection points to a point, the genus jumps up by one. $p_a(C_{0,red}) = g+1$. But in flat families, arithmetic genus is constant. We can conclude that $B_0 \subseteq X_0$ is not reduced. But you can compute that $K_S + \frac{1}{2}B$ is $\mathbb{Q}$-Cartier ($2K_X + B$ is Cartier) and ample over $S = \mathbb{A}^1$, so this is the log canonical model for .... So I have a divisor on the 3-fold, but when I restrict to the central fiber, it is not a divisor, it is only a closed subscheme.

The problem here is that B is not $\mathbb{Q}$-Cartier, but $(B_0)_{red}$ is $\mathbb{Q}$-Cartier. In this situation, you necessarily aquire an embedded component. If B_0 were Cartier, we would just lift that divisor in the family. You should expect this. $K + B$ is Cartier in the log canonical model, but K and B need not be Cartier. ⋄

What are we supposed to do now? Work with a subscheme instead of a divisor? What are the definitions of lc and slc in that case? This seems like a very serious problem, but there are several solutions:

1A. This problem does not happen if $b_i = 1$. This is proven in my paper on limits of stable pairs. This is not so good; $\frac{1}{2}$ is a perfectly good coefficient.

1B. Work with subschemes $B_i \subseteq X$ which are closed and flat over S. That is, require $(X, \sum b_i(B_i)^{div})$ to be a stable pair. I don't like this solution. Once you allow these embedded points to be there, then you have a surface, with a divisor, and these embedded points can crawl everywhere, and that is just unnatural. Maybe you could allow them in the bad fiber, but it seems like you have to allow them everywhere.

1C. Replace divisors by finite maps $B_i \to X$, where the B_i are reduced and codimension 1. I call these *branch divisors*. So when you form the divisor, you just take the divisor given by the image. How does this cure the example? It will look like this [[★★★ picture with two nodes]]; this is really a branch divisor, because at one point it is 2-to-1.

1D. (Kollár) "$B = (K+B)-K = L-K$" where L is an ample $\mathbb{Q}$-Cartier divisor. On a smooth surface, you can interpret NB as a morphism $\phi\colon \omega_{X/S}^{\otimes N} \to L_N^{X\to S}$.

1E. Only work with coefficients $(b_i + \varepsilon_i)$ and 1. This means that in the definition, you insist that these divisors are $\mathbb{Q}$-Cartier. This is a cheap way out, because $\frac{1}{2}$ is a perfectly good coefficient.

I like solutions 1C and 1D.

Problem 2: The properness criterion of MMP for a non-normal generic fiber. We started with a normal 3-fold in the picture I described before. What if you start with some family of surfaces where the generic fiber is not normal? Do we have a MMP for such things?

Example 5.7 (Kollár)**.** Start with the surface $\mathbb{F}_n$ and you attach to it an $\mathbb{F}_m$. You attach a divisor, which is the simplest thing you could have: s_n with s_m and $s_n + nf$. [[★★★ picture]] $K + B$ is slc and big, but $\bigoplus_{d\geq 0} H^0(d(K+B)$ is not finitely generated. Kollár has a more sophisticated example where the surface is irreducible. ◇

This example is really not a problem. You take the normalization and run MMP for all the pieces. [[★★★ picture]] Then you want to say that this glues uniquely together. The solution is to require $(K_X + B)|_E$ matches on the "left" and "right". This should be treated in the étale topology (the normalization could be connected, so you take a cover where it breaks into pieces). With this condition, everything glues nicely and the triple points are not a problem. In higher dimensions, there would be more trouble because you'd have things of higher codimension. So the surface in the example does not appear as a limit if you impose this condition.

Construction of moduli

The construction is standard once you have good properties of the stack $\mathcal{M}$. The properties are

1. properness, which is ok by MMP and above
2. boundedness, which is ok (V.A. 1994)
3. local closedness, which is Problem 3.

I could have $(X, \sum b_i B_i, L)$, where L is relatively ample invertible. If you make a base change, to get (X_T, B_T, L_T), would this be in $\mathcal{M}(T)$ (i.e. would it be an admissible family)? Local closedness means that for $S^u = \bigsqcup S_i$ with the S_i closed, then $T \to S^u$.

Fix some N. By boundedness, you can fix it so that L^N is very ample. Then L gives you an embedding into some projective space of fixed dimension. So you are in some Hilbert scheme. You cut out .... The problem begins with the fact that $L_N^{X\to S}$ does not commute with base change (if I don't start with a good family). If you know local closedness, then you know that you can form an admissible family. The only thing that is different from what we want is the embedding. You quotient out by the embedding and you get a quotient stack $\mathcal{M} = U/PGL$. The properness implies that this is algebraic with finite stabilizers.

Problem 3 has been solved. The statement is true, but one has to prove it. HK did it in dimension 2 with $B = \varnothing$. Kollár Husks and hulls gives a comprehensive treatment. According to me, this moduli space exists in complete generality (at least with contastant coefficients).