1 Tom Bridgeland - Stability conditions

The general idea is to start with a triangulated category D (e.g. the bounded derived category of coherent sheaves on a variety). To this, we associate a complex manifold $\mathrm{Stab}(\mathsf{D})$, the *space of stablity conditions*. Each point $\sigma \in \mathrm{Stab}(\mathsf{D})$ defines a subcategory of *semi-stable objects* $\mathsf{P} \subseteq \mathsf{D}$.

Motivations:

1. String theory (this is really where this stuff came from). The goal is to understand the "stringy Kähler moduli space" $\mathcal{M}_K(X)$. By mirror symmetry, this is supposed to be $\mathcal{M}_{\mathcal{C}}(\check{X})$. This came from Mike Douglas' work on Π-stability for D-branes). We won't talk about any string theory here. There are no examples $\mathsf{D}(CY_3)$ (though this may change very soon).

2. $\mathrm{Stab}(\mathsf{D})$ helps us to understand the structure of D (e.g. gives a space on which $\mathrm{Aut}(\mathsf{D})$ acts).

3. To try to define classes of objects in D which have nice moduli spaces. It would be really useful to find moduli spaces parameterizing complexes (not just sheaves). There are some situations where we can do this (e.g. that's how you show equivalence of derived categories under 3-fold flops). There has been some work by Abramovich and Polishchuk.

4. Try to understand wall-crossing for Donaldson-Thomas invariants. The relevant names are Joyce, Toda, and Kontsevich-Soibelman.

These are fine motivations, but they haven't really born fruit yet.

The plan of these talks is roughly as follows:

1. Stability conditions on abelian categories (this may be a bit boring)

2. Hall algebras

3. Triangulated case

4. Examples (in particular, the conifold)

5. Counting invariants (Seandroi's product formula)

Abelian case

Let A be an abelian category (e.g. coherent sheaves, or modules over a ring). $K(\mathsf{A})$ is the Grothendieck group, the free abelian group on isomorphism classes of objects of A, modulo the relation that $[B] = [A] + [C]$ whenever there is a short exact sequence

$$0 \to A \to B \to C \to 0.$$

Definition 1.1. A *stability function* on A is a homomorphism of abelian groups $Z\colon K(\mathsf{A}) \to \mathbb{C}$ such that for $E \neq 0$, $Z(E) \in \mathcal{H} = \{re^{i\pi\theta} | r > 0, 0 < \theta \leq 1\}$ (notice that this is half-closed). ⋄

Example 1.2. Take $\mathsf{A} = \mathsf{Coh}(X)$, where X is some smooth projective curve over $\mathbb{C}$, and take $Z(E) = -\deg(E) + i\,\mathrm{rk}(E) = i(\mathrm{rk}(E) + i\deg(E))$ (notice that this lands in this sends non-zero things into the upper half plane because you either have positive rank, or you are torision, so you have positive degree). ⋄

Example 1.3. Take $\mathsf{A} = R\text{-}\mathsf{mod}$, the category of finitely generated modules over some finitely generated $\mathbb{C}$-algebra R. In this case, $K(\mathsf{A}) = \mathbb{Z}^{\oplus N} = \mathbb{Z}[S_1] \oplus \cdots \oplus \mathbb{Z}[S_n]$ where S_i are the simple modules up to isomophism. As long as we make sure that the S_i are mapped into the upper half plane, then everything else will be. So the set of stability conditions is $\mathcal{H}^N$. We tend to think of stability conditions in algebraic geometry as essentially unique, but in algebra, there are lots of choices. ⋄

Example 1.4. Let $X = \mathbb{P}^2$ and $\mathsf{A} = \mathsf{Coh}(X)$. The obvious thing to do is to set $Z(E) = -\deg(E) + i\,\mathrm{rk}(E)$, but that doesn't work, because a sky-scraper sheaf has rank and degree zero. In fact, there are no stability functions on A. This might make you think that this is a very bad definition. It turns out that the derived category has some interesting stability conditions, but you have to use some other t-structure. This also somehow agrees with what you would expect from physics. So although this looks very bad, I claim it is actually correct. There are generalizations of this definition which would allow from something else, but if you use those, you don't get a complex manifold. There are millions of definitions you could use, but the interesting thing is that you get a complex manifold with this definition. ⋄

Let $Z\colon K(\mathsf{A}) \to \mathbb{C}$ be a stability function. Then every non-zero object has a phase $\phi(E) = \frac{1}{\pi}\arg(Z(E)) \in (0,1]$. A non-zero object E is called *semi-stable* if for every non-zero sub-object $A \subseteq E$, $\phi(A) \leq \phi(E)$. Equivalently, for every non-zero quotient $E \twoheadrightarrow Q$, $\phi(E) \leq \phi(Q)$. This is sometimes called the "see-saw property." For $\phi \in (0,1]$, define $\mathsf{P}(\phi)$ to be the full subcategory of A consisting of semi-stable objects E with $\phi(E) = \phi$ (and the zero object).

Remark 1.5. In Example 1.2, this corresponds to Mumford stability. $\mathsf{P}(1) = \{\text{torsion sheaves}\}$ and $\mathsf{P}(\phi) = \{\mu\text{-stable for } \mu = \cot\text{an}(\pi\phi)\}$. In Example 1.3, a result of A. King implies that there exists a projective scheme which is a coarse moduli space for semi-stable object of a fixed class $\alpha \in K(\mathsf{A})$. The functor send S to isomorphism classes of bundles $E \to S$ with an action of the algebra R on E. ⋄

Lemma 1.6. *If $\phi_1 > \phi_2$ and $E_i \in \mathsf{P}(\phi_i)$, then $\mathrm{Hom}_{\mathsf{A}}(E_1, E_2) = 0$.*

Proof. If $f\colon E_1 \to E_2$ is a non-zero map, then we get two short exact sequences

$$\ker f \to E_1 \to \operatorname{im} f \qquad \operatorname{im} f \to E_2 \to \operatorname{coker} f$$

This implies that $\phi_1 = \phi(E_1) \leq \phi(\operatorname{im} f) \leq \phi(E_2) = \phi_2$, a contradiction. □

Definition 1.7. A *Harder-Narasimhan filtration* for an object $E \in \mathsf{A}$ is a filtration

$$0 = E_1 \subset E_1 \subset \cdots \subset E_n = E$$

such that $F_i = E_i/E_{i+1}$ is semi-stable and $\phi(F_1) > \cdots > \phi(F_n)$. ⋄

Lemma 1.8. *If such a filtration exists, it is unique.*

Proof. Suppose you have two filtrations. Look at the last bits

$$\begin{array}{ccccc} E_{n-1} & \hookrightarrow & E & \twoheadrightarrow & F_n \\ \downarrow & & \| & & \downarrow \\ E'_{n'-1} & \hookrightarrow & E & \twoheadrightarrow & F'_{n'} \end{array}$$

Assume that $\phi(F'_{n'})$ is the smallest phase of any filtration factor. This implies that $\mathrm{Hom}(E_{n-1}, F'_{n'}) = 0$. This means that you can fill in the vertical maps. Thus, we know that $\phi(F_n) = \phi(F'_{n'})$. Now we can do the argument the other way and get some maps "up". A standard arugment shows that the compositions have to be the identity (because the map $E_{n-1} \to E$ is an inclution and $E = E$ is the identity), so the vertical maps are isomorphisms. □

Definition 1.9. A *stability condition* on A is a stability function $Z\colon K(\mathsf{A}) \to \mathbb{C}$ such that every object has a Harder-Narasimhan filtration. ⋄

This is not always easy to check. You often have to do some work to show that the filtrations exist.

Suffient condition: (is it necessary?) for existence of a HN filtration. First of all, there should be no infinite chains of quotients $E_1 \twoheadrightarrow E_2 \twoheadrightarrow \cdots$ with descending phases $\phi(E_1) > \phi(E_2) > \cdots$ (weakly noetherian). Secondly, there must be no infinite chains of subobjects $\cdots \subset E_2 \subset E_1$ with $\cdots > \phi(E_2) > \phi(E_1)$ (weakly artinian). You should try to prove this yourself (it is a bit tricky), and if you get stuck, look at my paper, "Stability conditions on triangulated categories" Prop. 2.4.

Examples 1.2 and 1.3 satisfy this. Example 1.2 (coherent sheaves on a curve over $\mathbb{C}$) is weakly noetherian for free because we get actual noetherian-ness. To show weakly artinian, we note that eventually, $\operatorname{rk} E_i = \operatorname{rk} E_{i-1}$, so the degrees must keep getting bigger (degree means first chern class). Then $\phi(E_i) > \phi(E_{i-1})$, which implies $d(E_i) > d(E_{i-1})$, which implies E_{i-1}/E_i has rank 0 and degree negative [[★★★]].

We've gotten some categorical stuff out of this. We start with a big category, and broken it up into small categories so that each object in the original category can be built up in a unique way from objects in the smaller categories.

The Harder-Narasimhan filtration is extremely important.

Suppose $\mathsf{A} = R\text{-}\mathsf{mod}_{fd}$, where R is a finite-dimensional algebra over a finite field $k = \mathbb{F}_q$ (bear with me, we'll eventually do it in characteristic zero with coherent sheaves). Define $\hat{H}(\mathsf{A}) = \{f\colon (\mathsf{A}/\cong) \to \mathbb{C}\}$, the set of functions on isomorphism classes of A. And define $H(\mathsf{A}) \subseteq \hat{H}(\mathsf{A})$ as the functions with finite support. Define the convolution product

$$(f * g)(M) = \sum_{A \subset M} f(A)g(M/A)$$

(note that M has finitely many sub-objects, so it is a finite sum).

Theorem 1.10. *Under the convolution product, $H(\mathsf{A})$ and $\hat{H}(\mathsf{A})$ become associative algebras with unit $1(0) = 1$ and $1(M) = 0$ for $M \ncong 0$.*

2 Tom Bridgeland - Hall Algebras

You'll have to take my word for it that this is an interesting example to think about. It looks funny, but that is to make it very explicit.

Let $\mathsf{A} = R\text{-}\mathsf{mod}$, where R is a finite dimensional algebra (if it is not finite dimensional, you can take nilpotent modules) over $\mathbb{F}_q$ (this is to make only finitely many isomorphism classes of module in each class of K-theory). Last time, we introduced $\hat{H}(\mathsf{A})$, the set of all $\mathbb{C}$-valued functions on isomorphism classes, and $H(\mathsf{A})$, the finitely supported ones.

We introduced the convolution product

$$(f * g)(M) = \sum_{A \subset M} f(A)g(M/A).$$

Lemma 2.1. *Under this multiplication, $\hat{H}(\mathsf{A})$ is associative, with unit $1 = 1_0$ (the zero module).*

Proof. It is clear that $1 * f = f * 1 = f$ because 0 only has itself as a submodule. Next, associativity:

$$\begin{aligned}
[(f * g) * h](M) &= \sum_{B \subset M} (f * g)(B)h(M/B) \\
&= \sum_{A \subset B \subset M} f(A)g(B/A)h(M/B) \\
[f * (g * h)](M) &= \sum_{A \subset M} f(A)(g * h)(M/A) \\
&= \sum_{A \subset M, C \subset M/A} f(A)g(C)h\big((M/A)/C\big)
\end{aligned}$$

These agree because submodule $C \subset M/A$ are in bijection with modules B such that $A \subset B \subset M$, and $(M/A)/C \cong M/B$. □

Hence, we see that

$$(f_1 * \cdots * f_n)(M) = \sum_{0 = M_0 \subset \cdots \subset M_n = M} f_1(M_1/M_0) \cdots f_n(M_n/M_{n-1}). \qquad (\dagger)$$

Now suppose we have a stability condition $Z \colon K(\mathsf{A}) \to \mathbb{C}$. For $0 < \phi \le 1$, we can define elements of the Hall algebra

$$1_{ss}^{\phi}(M) = \begin{cases} 1 & M \in \mathsf{P}(\phi) \\ 0 & \text{else} \end{cases} \qquad 1_{\mathsf{A}}(M) = 1$$

Lemma 2.2 (Reneke)**.** *The Harder-Narasimhan property implies that $1_{\mathsf{A}} = \prod_{\phi}^{\rightarrow} 1_{ss}^{\phi}$.*

Note that this is an infinite product, but it will be finite on any module. The point is that 1_{ss}^{ϕ} will only give something non-zero if the filtration factor is semi-stable with phase ϕ, so there is only one filtration that contributes to the sum (†).

$H(\mathsf{A}) = \oplus_{\alpha \in K_{\ge 0}(\mathsf{A})} H_\alpha(\mathsf{A})$, and $\hat{H}(\mathsf{A})$ is the completion with respect to this filtration. If you're interested, look at a paper of Schiffman "Intro to Hall algebras."

Integration map

Write $(f * g)(M) = \sum_{A, B \subset \mathsf{A}/\cong} n_{AB}^M f(A)g(B)$, where $n_{AB}^M = \big|\{A' \subset M | A' \cong A, M/A' \cong B\}\big|$.

Lemma 2.3. $n_{AB}^M = \dfrac{|\operatorname{Ext}^1(B,A)_M|}{\operatorname{Hom}(B,A)} = \dfrac{|\operatorname{Aut} M|}{|\operatorname{Aut} A| \cdot |\operatorname{Aut} B|}$, *where $\operatorname{Ext}^1(B,A)_M \subset \operatorname{Ext}^1(B,A)$ is the set of extensions isomorphic to M.*

Proof. Define V_{AB}^M the variety parameterizing

$$0 \to A \xrightarrow{f} M \xrightarrow{g} B \to 0$$

Then $(\alpha, \beta, \gamma) \in \operatorname{Aut}(A) \times \operatorname{Aut}(B) \times \operatorname{Aut}(M)$ acts on V_{AB}^M by $(f, g) \mapsto (\gamma \circ f \circ \alpha, \beta \circ g \circ \gamma^{-1})$. Then $\operatorname{Aut}(A) \times \operatorname{Aut}(B)$ acts freely and $\big|V_{AB}^M / \operatorname{Aut}(A) \times \operatorname{Aut}(B)\big| = n_{AB}^M$. The action of $\operatorname{Aut}(M)$ is not free, and

$$\operatorname{Stab}_{(f,g)} = \{1 + f\eta g | \eta \in \operatorname{Hom}(B,A)\}$$

and $V_{AB}^M / \operatorname{Aut}(M) \cong \operatorname{Ext}^1(B,A)_M$. □

Now I will make a big assumption. Assume that A has global dimension 1 (i.e. $\operatorname{Ext}^p_{\mathsf{A}}(M,N) = 0$ for $p > 1$). By Kontsevich and Soibelman, you get existence of an integration map for Calabi-Yau 3-folds (where you don't have dimension 1). The examples we're left with now are path algebras of quivers with no loops. That will have global dimension 1.

Whenever we have finite global dimension for any category, we can define $\chi(M,N) = \dim \operatorname{Hom}(M,N) - \dim \operatorname{Ext}^1(M,N)$.

Define $\mathbb{C}_q[K_{\geq 0}(\mathsf{A})] = \langle x^\alpha | \alpha \in K_{\geq 0}(\mathsf{A})\rangle / (x^\alpha * x^\beta = q^{-\chi(\beta,\alpha)} x^{\alpha+\beta})$,[1] where q is the size of $\mathbb{F}_q$ (though perhaps secretly you want to think of q as indeterminate).

Lemma 2.4. *$I\colon H(\mathsf{A}) \to \mathbb{C}_q[K_{\geq 0}(\mathsf{A})]$, given by $I(f) = \sum_{M\in \mathsf{A}/\cong} \frac{f(M)}{|\operatorname{Aut} M|} x^{[M]}$, is a ring homomorphism.*

I can complete on both sides to get a map $\hat{H}(\mathsf{A}) \to \mathbb{C}_q[\![K_{\geq 0}(\mathsf{A})]\!]$

Proof. Any function is a sum of things of characteristic functions, so let f and g be characteristic functions on A and B, respectively. Then

$$\begin{aligned} I(f*g) &= I\Big(\sum_{M\in\mathsf{A}/\cong} \frac{|\operatorname{Ext}^1(B,A)_M|}{|\operatorname{Hom}(B,A)|}\cdot \frac{x^{[M]}}{|\operatorname{Aut} A||\operatorname{Aut} B|}\Big) \\ &= \frac{|\operatorname{Ext}^1(B,A)|}{|\operatorname{Hom}(B,A)|}\frac{x^{[A\oplus B]}}{|\operatorname{Aut} A||\operatorname{Aut} B|} \\ &= q^{-\chi(B,A)}\frac{x^{[A\oplus B]}}{|\operatorname{Aut} A||\operatorname{Aut} B|} \end{aligned}$$

We also get $I(f) = \frac{x^{[A]}}{|\operatorname{Aut} A|}$ and $I(g) = \frac{x^{[B]}}{|\operatorname{Aut} B|}$. □

Example 2.5. Let $\mathsf{A} = R\text{-}\mathsf{mod}$, where R is the path algebra on the quiver $(\bullet \to \bullet)$. A module is just an assignment of a vector space to each vertex and a linear map for each arrow, so in this case, a module is a map of vector spaces. The indecomposible modules are $S = (\mathbb{C} \to 0)$, $T = (0 \to \mathbb{C})$ (these two are simple), and $E = (\mathbb{C} \xrightarrow{\text{id}} \mathbb{C})$. There is a short exact sequence

$$0 \to T \to E \to S \to 0$$

So $K(\mathsf{A}) = \mathbb{Z}^{\oplus 2}$.

[1] $K_{\geq 0}(\mathsf{A})$ means the cone spanned by actual isomorphism classes in A.

The space $\operatorname{Stab}(\mathsf{A})$ is $\mathcal{H}^2$, and there is one wall. On one side of the wall, $\phi(S) > \phi(T)$ and on the other side $\phi(T) > \phi(S)$, and the wall is where $\phi(S) = \phi(T)$. On the side where $\phi(T) > \phi(S)$, E is unstable. On the other side, E is stable. On the wall, E is semi-stable. Evaluating on the different sides of the wall, we have $1^{\phi(T)}_{ss} * 1^{\phi(S)}_{ss} = 1_{\mathsf{A}} = 1^{\phi'(S)}_{ss} * 1^{\phi'(E)}_{ss} * 1^{\phi'(T)}_{ss}$.

Assuming we are not on the wall (in which case everything would be semi-stable), $\mathsf{P}(\phi(T)) = \{T^{\oplus n} | n \geq 0\}$. Similarly for S. If we let $\alpha = [S]$ and $\beta = [T]$, we get

$$\Phi(x^\beta) + \Phi(x^\alpha) = \Phi(x^\alpha) + \Phi(x^{\alpha+\beta}) + \Phi(x^\beta) \qquad (\ddagger)$$

where $\Phi(x) = \sum_{n\geq 0} \frac{x^n}{|GL_n(q)|} = \sum_{n\geq 0}\frac{x^n}{(q^n-1)\cdots(q^n-q^{n-1})}$. This function is sometimes called the q-exponential or the quantum dilog. The identity ($\ddagger$) is called the 5-term relation for the quantum dilog. ◇

Another identity: Suppose P is a projective module. Then define $1^P_{\mathsf{A}}(M) := |\operatorname{Hom}(P,M)| = q^{\chi(P,M)}$ (again, remember that q is the size of $\mathbb{F}_q$) and $\operatorname{Quot}^P_{\mathsf{A}}(M) = |\operatorname{Hom}^{\twoheadrightarrow}(P,M)|$ (number of surjections $P \twoheadrightarrow M$).

Lemma 2.6. $1^P_{\mathsf{A}} = \operatorname{Quot}^P_{\mathsf{A}} * 1_{\mathcal{A}}$.

Proof. $|\operatorname{Hom}(P,M)| = \sum_{A\subset M} |\operatorname{Hom}^{\twoheadrightarrow}(P,A)|$. This is just the statement that every map has an image. $1_{\mathsf{A}} = \prod_\phi 1^\phi_{ss}$. Both the outer things have product decompositions and you're interested in the guy in the middle. □

Exercise. *Apply this to $\mathsf{A} = \mathsf{Vect}$ and $P = \mathbb{C}^{\oplus N}$ [[★★★ maybe Vect over a finite field?]]. Use the integration map. What do you get?*

3 Tom Bridgland

I want to start by correcting a mistake in a calculation from yesterday. Remember I wanted to compute $I(1_{ss}^{\phi}) = \sum_{n\geq 0} \frac{x^{[S^{\oplus n}]}}{|GL_n(q)|}$, but it is not true that $x^{[S^{\oplus n}]} = q^{\frac{1}{2}n(n-1)}(x^{[S]})^n$ because $x^\alpha * x^\beta = q^{-\chi(\beta,\alpha)}x^{\alpha+\beta}$. So we actually get $I(1_{ss}^{\phi}) = \Phi(x^{[S]})$, where $\Phi(x) = \sum_{n\geq 0} \frac{q^{n(n-1)/2}}{|GL_n(q)|}x^n = \sum_{n\geq 0} \frac{x^n}{(q-1)\cdots(q^n-1)}$

Triangulated categories

Let D be a triangulated category (e.g. $\mathsf{D} = \mathsf{D}^b\mathsf{Coh}(X)$).

Definition 3.1. A *heart* (of a bounded t-structure) is a full subcategory $\mathsf{A} \subset \mathsf{D}$ such that

(a) $\mathrm{Hom}_{\mathsf{D}}(A_1, A_2[k]) = 0$ for all $A_i \in \mathsf{A}$ and $k > 0$, and

(b) for all $E \in \mathsf{D}$, there exists integers $k_1 > \cdots > k_n$ and triangles $E_i \to E_{i+1} \to F_i \to E_i[1]$ with $E_0 = 0$ and $E_n = E$ with $F_i \in \mathsf{A}[k_i]$.[[★★★ add diagram?]] ◇

Remark 3.2. Condition (a) implies that the “filtration” in (b) is unique. The argument is the same as the argument for uniqueness of the HN filtration. ◇

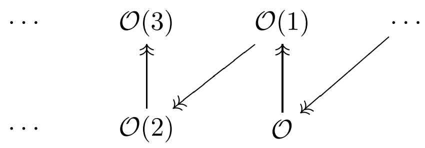
Remark 3.3. If we define $\mathsf{D}^{\leq 0} = \{E \in \mathsf{D} | k_i \geq 0 \text{ for all } i\}$ (this is an unavoidable clash in notation) and $\mathsf{D}^{\geq 1} = \{E \in \mathsf{D} | \text{all } k_i \leq -1\}$. Then $(\mathsf{D}^{\leq 0}, \mathsf{D}^{\geq 1})$ a *bounded t-structure*. Conversely, given a bounded (non-degenerate) t-structure $(\mathsf{D}^{\leq 0}, \mathsf{D}^{\geq 1})$, we can define $\mathsf{A} = \mathsf{D}^{\leq 0} \cap \mathsf{D}^{\geq 1}[1]$. ◇

Example 3.4. The inclusion of an abelian category in it's derived category, $\mathsf{A} \subseteq \mathsf{D}^b(\mathsf{A})$, is a heart. Part (a) comes from the fact that there are no negative Ext's, and part (b) comes from truncations. Define $\tau_{\leq i}(\cdots E_i \xrightarrow{d^i} E_{i+1} \cdots) = (\cdots E_i \to \ker d^i \to 0 \cdots)$. Then you get triangles $\tau_{\leq i-1}(E) \to \tau_{\leq i}(E) \to H^i(E)[-i] \to$, and because we have a bounded derived category, we finish. ◇

So a triangulated category is a floppy thing (it could be the derived category of many different things, for example), and a t-structure rigidifies it.

Remark 3.5. If A is an abelian category which is a heart, then $\mathsf{D} \ncong \mathsf{D}^b(\mathsf{A})$ in general. In fact, you don't even have a map in general. This has to do with the fact that Ext^n is generated by Ext^1 for an abelian category. ◇

What does $\mathsf{D}^b(\mathbb{P}^1)$ look like? The sheave on $\mathbb{P}^1$ look like



But then you get all the shifts of this picture (think of this as a frame in a film strip). But you can take all the $\mathcal{O}(-n)$ and $\mathcal{O}(n)[1]$ as another heart. This category A is equivalent to the category of representations of the quiver $(\bullet \rightrightarrows \bullet)$. In general, your derived category doesn't natually sit in a filmstrip like this and you have to pick a direction to slice it.

Definition 3.6. A *stability condition* on D is a heart $\mathsf{A} \subseteq \mathsf{D}$ together with a stability condition $Z\colon K(\mathsf{A}) \to \mathbb{C}$ on A. ◇

I've just combined two things that don't look like they have anything to do with eachother. They both involve filtrations. I'll give you an alternative definition which is more symmetric.

Remark 3.7. A triangulated category also has a Grothendieck group $K(\mathsf{D})$, the free abelian group on isomorphism classes, modulo the relation $[B] = [A] + [C]$ when there is a triangle $A \to B \to C \to$.

Note that rotating the triangle, we can compute that $[E] = -[E[1]]$. Also note that if A is a heart, then $K(\mathsf{A}) \xrightarrow{\sim} K(\mathsf{D})$. ◇

Definition 3.8 (Alternative)**.** A *stability condition* on D consists of a group homomorphism $Z\colon K(\mathsf{D}) \to \mathbb{C}$ (usually called *central charge*) and full subcategories $\mathsf{P}(\phi) \subseteq \mathsf{D}$ for all $\phi \in \mathbb{R}$ satisfying

(a) $Z(\mathsf{P}(\phi)) \subseteq \mathbb{R}_{>0}e^{i\pi\phi}$

(b) $\mathsf{P}(\phi)[1] = \mathsf{P}(\phi + 1)$

(c) $\phi_1 > \phi_2$ and $A_i \in \mathsf{P}(\phi_i)$ implies $\mathrm{Hom}_{\mathsf{D}}(A_1, A_2) = 0$, and

(d) for all $E \in \mathsf{D}$, there exist $\phi_1 > \cdots > \phi_n$ and triangles $E_{i-1} \to E_i \to F_i \to$ such that $E_0 = 0$ and $E_n = E$ with $F_i \in \mathsf{P}(\phi_i)$. ◇

If you're familiar with t-structures, a t-structure is where you have filtrations like this with ϕ_i integers. This is some kind of more refined thing.

Proof (equivalence of definitions). Given the heart $\mathsf{A} \in \mathsf{D}$ and $Z\colon K(\mathsf{A}) \to \mathbb{C}$, we get $\mathsf{P}(\phi) \subset \mathsf{A}$ for $0 < \phi \leq 1$. Axiom (b) gives $\mathsf{P}(\phi)$ for all $\phi \in \mathbb{R}$. (c) is easy because we know it when $\phi_i \in (0, 1]$ and because of the axioms of a heart. Finally, the filtrations in (d) will come from combining the filtrations from the t-structure with Harder-Narasimhan filtrations. If you want to understand this, you should check it carefully.

To go the other way, how do we define a heart from these data. As usual, (c) tells us that the filtrations in (d) are unique up to isomorphism. For any interval $I \subset \mathbb{R}$, define $\mathsf{P}(I) = \{E \in \mathsf{D} | \phi_i \in I \text{ for all } i\}$. Now set $\mathsf{A} = \mathcal{P}((0, 1])$. (c) and (d) imply that this is a heart. If you think about it, you see that Z is a stability condition on A by (a). □

The later definition is actually much more symmetric. Choosing a heart is a choice of $\mathsf{P}((0, 1])$. But there is no reason to choose 0. Every $\mathsf{P}((\alpha, \alpha + 1])$ defines a heart. Q: in this particular case, is the triangulated category always the derived category of any heart? TB: I don't think so; let's discuss this after lecture.

Technical point: If $I \subseteq \mathbb{R}$ is an interval of length less than 1, then $\mathsf{P}(I)$ is not abelian, but there is still a notion of short exact sequences. A short exact sequence is just a triangle $A \to B \to C \to$ with $A, B, C \in \mathsf{P}(I)$. If you're categorically inclined, you might not like that this is not intrinsic. There is an intrisic version. $\mathsf{P}(I)$ is a quasi-abelian category.

Definition 3.9. A stability condition is *locally finite* if there is an $\varepsilon > 0$ such that for all $\phi \in \mathbb{R}$, $\mathsf{P}(\phi + \varepsilon, \phi - \varepsilon)$ is finite length (i.e. noetherian and artinian). Here a subobject is something that fits into an exact sequence, not just a categorical subobject. We write $\operatorname{Stab} \mathsf{D}$ for the set of locally finite stability conditions on D. ◇

Theorem 3.10. *There is a natural topology on* $\operatorname{Stab} \mathsf{D}$ *such that every connected component* $\operatorname{Stab}^* \mathsf{D} \subseteq \operatorname{Stab} \mathsf{D}$*, there exists a linear subspace* $V \subseteq \operatorname{Hom}_{\mathbb{Z}}(K(\mathsf{D}), \mathbb{C})$ *with a linear topology such that* $\operatorname{Stab}^* \mathsf{D} \to \operatorname{Hom}_{\mathbb{Z}}(K(\mathsf{D}), \mathbb{C})$*, given by* $(Z, \mathsf{P}) \mapsto Z$*, is a local homeomorphism onto an open subset of* V*. In particular,* $\operatorname{Stab} \mathsf{D}$ *is a (possibly infinite dimensional) complex manifold.*

This tells you that deformations of Z lift uniquely to deformations of the whole stability condition. Note that as Z changes, the t structure changes.

Remark 3.11. In practice, we insist that $Z\colon K(\mathsf{D}) \to \mathbb{C}$ factors via a finite dimensional quotient. For example, if $\mathsf{D} = \mathsf{D}^b\mathsf{Coh}(X)$ for X smooth and projective over $\mathbb{C}$, we insist that Z factors through the chern character $ch\colon K(\mathsf{D}) \to H^*(X, \mathbb{Q})$. If we make this constraint, then $\operatorname{Stab} \mathsf{D}$ is finite dimensional. ◇

4 Tom Bridgeland

Last time we were talking about the space of stability conditions. When you combine t-structures and stability conditions on an abelian category, you get this quite interesting thing. Basically, we get that $\operatorname{Stab}\mathsf{D} \to \operatorname{Hom}_{\mathbb{Z}}(K(\mathsf{D}), \mathbb{C})$, given by $(Z, \mathsf{P}) \mapsto Z$, is a local homeomorphism.

Today I want to talk about the example of the conifold. If you really want to know about it, it is in the paper: Tom Bridgeland, "Stability conditions on triangulated categories". Consider $X = \{x_1x_2 - x_3x_4\} \subseteq \mathbb{C}^4$. We have two resolutions $Y^\pm$, which contain curves $C^\pm$ as fibers over the singularity. The common resolution is $f^\pm\colon Z \to Y^\pm$. You can look at Toda's paper "stability conditions and crepand small resolutions".

Theorem 4.1 (Bondal, Orlav, ...)**.** *There exist equivalences* $\Psi = Rf^+_* \circ L(f^-)^*\colon \mathsf{D}^b\mathsf{Coh}(Y^-) \to \mathsf{D}^b\mathsf{Coh}(Y^+)$ *(meaning that there is an equivalence commuting with the pushdown to X), and* $\Phi^\pm\colon \mathsf{D}^b\mathsf{Coh}(Y^\pm) \to \mathsf{D}^b R\text{-}\mathsf{mod}$ *with* $\Phi^- = \Phi^+ \circ \Psi$*, where R is the algebra of the quiver* $\Big(\bullet \underset{b_1,b_2}{\overset{a_1,a_2}{\rightleftarrows}} \bullet\Big)$ *with relations of superpotential* $W = a_1b_1a_2b_2 - a_1b_2a_2b_2$ *(Klebanov-Witten).*

You can do this for any flop; things become a bit more difficult. The black magic rule to interpret the word superpotential is that you think $dW = 0$, which is four relations, one of them being $\partial_{a_2}W = b_2a_1b_1 - b_1a_1b_2 = 0$.

Consider representations of $\mathbb{C}Q$ (the quiver without the relations) of given dimension (d_1, d_2). It is $\big(\operatorname{Hom}(\mathbb{C}^{d_1}, \mathbb{C}^{d_2})^{\otimes 2} \otimes \operatorname{Hom}(\mathbb{C}^{d_2}, \mathbb{C}^{d_1})^{\otimes 2}\big)/GL(d_1) \times GL(d_2)$. Let A_i and B_i be the images of a_i and b_i. Then we get $\Phi = tr(W(A_1, A_2, B_1, B_2))\colon Q\text{-}\mathsf{mod} \to \mathbb{C}$. The moduli stack of representations of $R = \mathbb{C}[Q]/I = \{d\Phi = 0\}$ sits inside the representations of Q. [[★★★ somehow]] the black magic rule I gave before comes from this.

If I write a quiver with some relations, and the algebra you get is CY symmetric, it has to be given by a superpotential (Segal, Bocklandt).

How do you get R in this situation? You get it geometrically by considering $Y^+ = Tot(\mathcal{O}(-1,-1)) \xrightarrow{\pi} C^+ = \mathbb{P}^1$. The tilting object is $E^+ = \pi^*(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$. Then $R = \operatorname{End}_{Y^+}(E^+)$ and $\Phi^+ = R\operatorname{Hom}(E^+, -)$. Similarly, for Φ^-, you take $E^- = \pi^*(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1))$ and $\Phi^- = R\operatorname{Hom}_{Y^-}(E^-, -)$.

Under these equivalences, $\mathcal{O}_{C^-}(-1)$ and $\mathcal{O}_{C^-}(-2)[1]$ go to the simple modules S and T, respectively, where $\dim S = (1,0)$ and $\dim T = (0,1)$. Similarly, S and T correspond to $\mathcal{O}_{C^+}(-1)[1]$ and $\mathcal{O}_{C^+}$, respectively. This should be pretty clear from applying these functors. You get the impression that people spend their time shuffling around adjunctions, but you eventually have to think about specific objects and where they go.

The above equivalences restrict to {objects (topologically) supported on C^-} $\leftrightarrow$ {objects supported on C^-} and {objects supported on $C^\pm$} $\leftrightarrow$ {objects with nilpotent cohomology modules}. $R = \bigoplus_{n\geq 0} R_n$ is a graded algebra. $R_nM = 0$ for all $n \gg 0$ if and only if M is finite length and all simple factors are S and T (the vertex simples); we say such an M is nilpotent. I think you can get a lot by just considering these subcategories. They are all equivalent, so call them D. Then $K(\mathsf{D}) = \mathbb{Z}^{\oplus 2} = \mathbb{Z}[S] \oplus \mathbb{Z}[T]$. Note that the class of a point is $[\mathcal{O}_x] = [S] + [T]$.

Consider "normalized" stability stability conditions, where we assume $Z([S]+[T]) = -1$. I didn't tell you this, but there is always an action of $\mathbb{C}$ on the stability condition, where $\mathbb{C}^\times$ rotates Z. Consider the map $\pi\colon \operatorname{Stab}_n(\mathsf{D}) \to \mathbb{C}$, given by $(Z, \mathsf{P}) \mapsto Z([T])$. This will be a local homeomorphism.

Theorem 4.2 (Toda)**.** *There is a connected combonent* $\operatorname{Stab}^*(\mathsf{D})$ *such that* $\pi\colon \operatorname{Stab}^*(\mathsf{D}) \to \mathbb{C} \smallsetminus \mathbb{Z}$ *is the universal covering.*

Physicists would say the $\mathcal{O}_C[-n]$ are the branes and their mass is not allowed to vanish.

I'll define a point in Stab^* that lies in the upper half plane, and we'll go on a little journey down (between -1 and 0) and see what happens.

We have the heart $\mathsf{Coh}_{C^+}(Y^+)$, the category of coherent sheaves on Y^+ supported on C^+. Note that I'm not claiming that the bounded derived category of this is equal to D; I suspect it's not. The stability function is $Z(E) = ch_2(E)\cdot(\beta + i\omega) - ch_3(E)$, where $\beta, \omega \in H^2(Y^+, \mathbb{R}) \cong \mathbb{R}$, and $\omega > 0$ is ample, and $\beta + i\omega$ is in the complexified Kähler cone, which in this case is just the upper half plane $\mathcal{H}$. You have to check the HN property.

Now let's suppose $\omega \to 0$ with $\beta \cdot C \in (-1, 0)$. We have that $Z(\mathcal{O}_C)$ is in the upper half plane and $Z(\mathcal{O}_x)$ is on the real line. So as $Z(\mathcal{O}_C)$ goes to the real line, it just decides which things end up on which side of the real line. The stuff that ends up on $\mathbb{R}_+$ is no longer in the heart. $\mathsf{P}((0,1])$ changed so that we lose $\mathcal{O}_{C^+}(-k)$ for $k \geq 1$, but we gain $\mathcal{O}_{C^+}(-k)[1]$ for $k \geq 1$. This is like one of those filmstrip pictures from before (in some sense, this is the $\mathbb{P}^1$ diagram I drew before). Call our new heart A. $\mathsf{A} = R\text{-}\mathsf{mod}_{nil} \subset \mathsf{D}$. So you can continuously get across the flop if you've complexified the Kähler class, but in between, you're naturally talking about this non-commutative guy. As an abelian category, it is equivalent, but it is sitting inside the derived category differently.

If we continue our journey a little bit more, we need to tilt again. We lose $\mathcal{O}_{C^+}(k)$ for $k \geq 0$ and we gain $\mathcal{O}_{C^+}(k)[-1]$. I'll leave it to you to check that the new heart is naturally $\mathsf{Coh}_{C^-}(Y^-) \subset \mathsf{D}$. Now you can think about going back along another path (looping around the integer point $k \in \mathbb{Z}$). You get the action of the Seidel-Thomas twist functor $\Phi_{\mathcal{O}_C(k)}$.

Next time I'll come back to Hall algebras and tell you how to do it in characteristic zero. I also want to explain [[★★★ some other stuff]].

5 Tom Bridgeland

It's the last lecture, so I'm allowed to talk about things I don't entirely understand.

Recall the picture from before (it was kind of a baby case with finite fields). You have an abelian category A, to which you associate a Hall algebra $H(\mathsf{A})$. You associate a stability condition to A so that $1_{\mathsf{A}} = \prod_{\phi}^{\rightarrow} 1_{ss}^{\phi}$ (this is basically the Harder-Narasimhan property). In the case of global dimension 1, we had an integration map $I\colon H(\mathsf{A}) \to \mathbb{C}_q[K(\mathsf{A})]$, where the formal identity $1_{\mathsf{A}} = \prod_{\phi}^{\rightarrow} 1_{ss}^{\phi}$ becomes something interesting. A reference is a paper of Kontsevich and Soibelman, which hopefully is coming soon. They say something highly non-trivial, and it works in incredibly general context (they deal with the triangulated case, but I won't). The referecne for this stacky Hall algebra is Joyce's paper "Configurations in abelian categories I, II,...".

Stacky Hall algebras

Let A be an abelian category, equal to $R\text{-}\mathsf{mod}_{fg}$, where R is a finitely generated algebra over $\mathbb{C}$ (for concreteness). You could take $\mathsf{Coh}(X)$, for X a projective variety over $\mathbb{C}$. There is an Artin stack of objects $\mathcal{M} = \bigsqcup_{d\geq 0} \mathcal{M}_d = \bigsqcup_{d\geq 0}[V_d/GL(d)]$, where V_d is some space of matrices (it's an affine variety), and I get rid of the framing by modding out by $GL(d)$. $\mathcal{M}(S)$ is the groupoid of vector bundles $\mathcal{E}$ over S with $R \to \mathcal{E}nd_{\mathcal{O}_S}(\mathcal{E})$.

For $n \geq 1$, $\mathcal{M}^{(n)}$ is the stack of n-flags in A, so $\mathcal{M}^{(n)}(S)$ is the groupoid of flags of vector bundles $0 = \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_n$, with $R \to \mathcal{E}nd_{\mathcal{O}_S}(\mathcal{E}_n)$ preserving the flag, with $\mathcal{F}_i = \mathcal{E}_i/\mathcal{E}_{i-1}$ a vector bundle for all i. Note that $\mathcal{M}^{(1)} = \mathcal{M}$.

We have morphisms $a_i\colon \mathcal{M}^{(n)} \to \mathcal{M}$, given by $(\mathcal{E}_1 \subset \cdots \subset \mathcal{E}_n) \mapsto \mathcal{F}_i = \mathcal{E}_i/\mathcal{E}_{i-1}$, and $b\colon \mathcal{M}^{(n)} \to \mathcal{M}$, given by $(\mathcal{E}_1 \subset \cdots \subset \mathcal{E}_n) \mapsto \mathcal{E}_n$.

Lemma 5.1. *There is a cartesean square*

$$\begin{array}{ccc} \mathcal{M}^{(n+1)} & \xrightarrow{g} & \mathcal{M}^{(2)} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle a_1} \\ \mathcal{M}^{(n)} & \xrightarrow[b]{} & \mathcal{M} \end{array}$$

Proof. Take $f(\mathcal{E}_1 \subset \cdots \subset \mathcal{E}_{n+1}) = (\mathcal{E}_1 \subset \cdots \subset \mathcal{E}_n)$ and $g(\mathcal{E}_1 \subset \cdots \subset \mathcal{E}_{n+1}) = (\mathcal{E}_n \subset \mathcal{E}_{n+1})$. □

Suppose I have a "cohomology theory" for stacks. That is, I have a vector space for each stack $\mathcal{X} \mapsto H(\mathcal{X})$ such that for every (representable, proper) $f\colon \mathcal{X} \to \mathcal{Y}$, I get $f_*\colon H(\mathcal{X}) \to H(\mathcal{Y})$, and for every (finite type) $f\colon \mathcal{X} \to \mathcal{Y}$, I get $f^*\colon H(\mathcal{Y}) \to H(\mathcal{X})$, and these should be functorial (in the correct 2-categorical way). These should have properties:

1. Künneth formula. I want $H(\mathcal{X} \times \mathcal{Y}) \cong H(\mathcal{X}) \otimes H(\mathcal{Y})$. MO: so here H is cohomology with compact support. TB: yeah. I'll give an example of something satisfying these axioms. There may be more interesting examples.

2. Base change. For a cartesean square

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ \mathcal{Z} & \xrightarrow{j} & \mathcal{W} \end{array}$$

   We have $f_* \circ g^* \cong h^* \circ j_*$.

Example 5.2. $H(\mathcal{X})$ is the vector space with basis given by representable maps of finite type $T \to \mathcal{X}$, moduli isomorphism over $\mathcal{X}$. If I have $f\colon \mathcal{X} \to \mathcal{Y}$, and $g\colon T \to X$, I have $f_*(g) = f \circ g$ and if $h\colon T \to \mathcal{Y}$, I have $f_*(h) = (\mathcal{X} \times_{\mathcal{Y}} T \to \mathcal{X})$. It is easy to verify the two axioms. ◇

Example 5.3. You could also quotient by relations $[T \to \mathcal{X}] = [U \to \mathcal{X}] + [(T \setminus U) \to \mathcal{X}]$ when $U \subset T$ is open. Then I think of $H(\mathcal{X})$ as $K_0(St/\mathcal{X})$. This is something more motivic. ◇

So if you have a "cohomology theory" with these properties, then you'll get an associative algebra. Consider

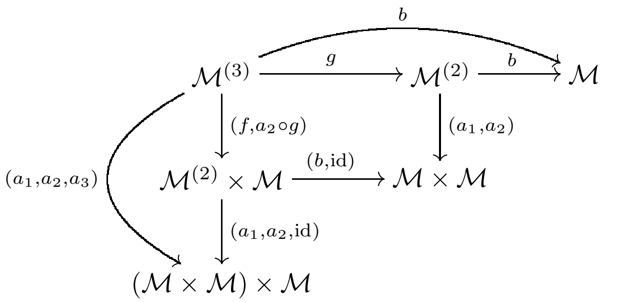
$$\begin{array}{ccc} \mathcal{M}^{(2)} & \xrightarrow{b} & \mathcal{M} \\ {\scriptstyle (a_1,a_2)}\downarrow & & \\ \mathcal{M} \times \mathcal{M} & & \end{array}$$

Then define $m = b_* \circ (a_1, a_2)^*\colon H(\mathcal{M}) \otimes H(\mathcal{M}) \to H(\mathcal{M})$. b is representable because this is the Quot scheme. Q: for this, it doesn't look like you need an isomorphism for Künneth, you just need a map. TB: good point, I just need a map.

Lemma 5.4. *m is associative and unital.*

The unit is $i\colon \operatorname{Spec}\mathbb{C}\to\mathcal{M}$ given by $pt\mapsto 0$, then the unit is $i_*(1)\in H(\mathcal{M})$. Q: the Künneth isomorphisms need to have some properties; fro example, they should be associative. [[★★★ some other stuff]] TB: oh, so maybe you want H to always be a ring. Q: I think the Künneth formula is more or less equivalent. Take something with the diagonal map. TB: I'll have to think about that some more.

Proof. (associativity) fill in the cartesean square



□

Integration map

These ideas are from Kontsevich-Soibelman (if I haven't messed anything up). Assume R has finite global dimension (so the Ext's don't go on forever). Hall algebra: $[f\colon\mathcal{X}\to\mathcal{M}]$ where $\mathcal{X}$ is finite type and f is representable; take $\mu\colon K_0(\text{varieties})\big[[GL(n)]^{-1}|n\geq 1\big]\to\Lambda=\mathbb{Q}(s)$ to be the ring homomorphism given by the Poincaré polynomial. Define $\mathbb{C}_s[K_{\geq 0}(\mathsf{A})]=\Lambda\otimes_{\mathbb{C}}\mathbb{C}[K_{\geq 0}(\mathsf{A})]$ with multiplication $x^\alpha * x^\beta = s^{\chi(\alpha,\beta)}x^{\alpha+\beta}$.[1]

Define an integration map. Given a constructible function $\omega\colon\mathcal{M}\to\Lambda$, define $[f\colon\mathcal{X}\to\mathcal{M}_\alpha]\mapsto\big[\int_{\mathcal{X}}f^*(\omega)d\mu\big]x^\alpha$. So $f^*(\omega)$ is a constructable function; I break up $\mathcal{X}$ according to the values of the function and and add up the pieces with weights. This gives an integration map $I\colon H(\mathsf{A})\to\mathbb{C}_s[K_{\geq 0}(\mathsf{A})]$.

[1] Next we'll assume A is CY_3. then χ is skew symmetric, so we get the same answer as somewhere else.

Lemma 5.5. *I is a ring homomorphism if and only if for $A,B\in\mathsf{A}$, $\int_{\operatorname{Ext}^1(B,A)/\operatorname{Hom}(B,A)}W(E)d\mu = s^{\chi(A,B)}W(A)W(B)$.*

$\int_{\operatorname{Ext}^1(B,A)/\operatorname{Hom}(B,A)}W(E)d\mu = s^{-2\dim\operatorname{Hom}(B,A)}\int_{\operatorname{Ext}^1(B,A)}W(E)d\mu.$

Claim (conjectural)**.** *Suppose A is CY_3 (I'll in fact assume $R=\mathbb{C}[Q]/I$ for some quiver Q and I given by cyclic derivatives of a polynomial superpotential. This implies that framed modules, V_α, sits inside $\mathbb{C}^N$, framed representations of Q with no relations). Then $\omega(E)=s^{[E,E]}MF_W(E)$, where MF_W is the reduced Poincaré polynomial of the Milnor fiber of W at $E\in\mathbb{C}^N$.*

Here $[E,E]$ is the Euler form on the quiver without relations; it is $\dim\operatorname{Hom}(E,E)-N$. In Kontsevich-Soibelman, they have $\dim\operatorname{Hom}(E,E)-\dim\operatorname{Ext}^1(E,E)$, which is something on the quiver with relations. This is not an Euler form because of something.

The claim boils down to the following. If I have a polynomial map of vector spaces $W\colon\mathbb{C}^{n_1}\oplus\mathbb{C}^{n_2}\oplus\mathbb{C}^{n_3}\to\mathbb{C}$, invariant under the $\mathbb{C}^\times$ action with weight $(1,-1,0)$, then $\frac{1}{s^{2n_1}}\int_{X\in\mathbb{C}^n}MF_W(x)\,d\mu = MF_{W|_{\mathbb{C}^{n_3}}}(0)$. I don't know how to prove this or if it is true.