1 Martin Olsson

I want to talk about log geometry in the sense of Fontaine, Illusie, and Kato. The history of the subject makes it a little inaccessable sometimes. It came out of trying to prove some conjectures of Fontaine. I want to start by explaining an old example.

Let Δ be a smooth curve over $\mathbb{C}$ and $o \in \Delta$, with $\Delta^\times = \Delta \smallsetminus \{o\}$. Let $t \in \Gamma(\Delta, \mathcal{O}_\Delta)$ a uniformizer at o. Choose $f\colon X \to \Delta$ proper and semi-stable (i.e. étale locally $\mathcal{O}_\Delta[x_1, \ldots, x_n]/(x_1, \ldots, x_n - t)$).[1] We then get a local system $V = (R^i f_*^{an}\mathbb{C})/\Delta^\times$. This is the same thing as a representation of $\pi_1(\Delta^\times)$.

There is an algebraic construction which tells you more. Consider $X^\times := f^{-1}(\Delta^\times) \xrightarrow{j} X \xleftarrow{i} X_0$. We have $\Omega^1_{X/\Delta}(\log)$, the subsheaf of $j_*\Omega^1_{X^\times/\Delta^\times}$ which in local coordinates is generated by $\Omega^1_{X/\Delta}$ and the dx_i/x_i. This is a locally free sheaf. You get a complex

$$\Omega^\bullet_{X/\Delta}(\log) : \mathcal{O}_X \to \Omega^1_{X/\Delta}(\log) \to \Omega^2_{X/\Delta}(\log) \to \cdots$$

You can then define $E = R^i f_* \Omega^\bullet_{X/\Delta}(\log)$. This is a locally free sheaf on Δ. It has more structure, namely the Gauss-Manin connection $\nabla\colon E \to E \otimes \Omega^1_\Delta(\log)$. This is all in the algebraic category, but I can think of it as an analytic vector bundle, so we get the local system $V = \ker\bigl(\nabla^{an}\colon E_{an} \to E_{an} \otimes \Omega^1_\Delta(\log)\bigr)|_{\Delta^\times}$. We have $i_\Delta\colon \operatorname{Spec}\mathbb{C} \overset{0}{\hookrightarrow} \Delta$, and $\mathbb{C} \cong i^*_\Delta \Omega^1_\Delta(\log)$, given by $1 \mapsto dt/t$. You get $N\colon E(0) \to E(0)$. You have $\nabla(tE) \subseteq tE$.

Let $D \subseteq \Delta_{an}$ be a disk around 0 and $s \in D^\times = D \smallsetminus \{0\}$. Then $\pi_1(D^\times) = \mathbb{Z}$, generated by a loop around zero. Then I have an action of $\mathbb{Z}$ on the vector space V_s, with 1 acting by $T\colon V_s \to V_s$. It is a theorem that this T is a unipotent operator, so I can take it's log to get a nilpotent matrix acting on this vector space.

Theorem 1.1. *The conjugacy class of* $\log T$ *is* N.

That stuff is very old. The question it begs is the following. This E lives in the closed fiber. Do you really need the whole family over the disk to get this N. So the question is, "what extra structure do you need in addition to X_0 to recover $E(0)$ and N?"

[1] I'll assume you know about the étale topology, but if you aren't too familiar with it, think of it as the analytic topology.

Let me pose another question that is closer in spirit to this meeting. This second problem concerns main components/deformation theory. This is an experimental science; you go example by example. It is just a fact of life that when you have a moduli space of higher dimensional things, you get lots of irreducible components.

Example 1.2. Let (E, e) be an elliptic curve over $k = \bar{k}$. It is an exercise to check that you can find an embedding $j\colon E \hookrightarrow P$, where P is a rational surface, with $E \in |-K_P|$ and $j^* I_E \simeq \mathcal{O}_E$. Let $X_0 = P \cup_E P$. This is called a log K3. If you wanted to study a moduli space of K3 surfaces, this is the kind of thing you'd stick at the boundary. An old paper of Friedman showed that the versal deformation space (we're looking at the complete local ring at the point in the moduli space corresponding to X_0) looks like $V_1 \cup V_2$ where

1. V_1 and V_2 are smooth,
2. $\dim V_2 = 20$ and $V_2 \smallsetminus (V_2 \cap V_1)$ correspond to smoothings of X_0, and
3. V_1 classifies locally trivial deformations.

Let me say what this last thing means. You have two components, V_1 (singular deformations) and V_2 (smooth deformations). The question is, "how do you isolate V_2?" This is perhaps the most important question for this series of lectures. You can answer it with log geometry. ◇

There is a third question, which is the connection with stacks (e.g. orbicurves). Q: what is a locally trivial deformation? MO: X_0 locally looks like $\mathbb{C}[x, y, z]/xy$. Locally trivial means that the local ring after deforming still looks like that. The smoothings look like $\mathbb{C}[t][x, y, z]/(xy - t)$.

Now I'll start with foundations. You'll have to bear with me for a lecture and a half or so.

Monoids

You have to be very careful with monoids. You want them to be like groups, but they are very subtle. We will usually write the composition additively.

Definition 1.3. A *monoid* is a commutative semi-group with unit. Morphisms of monoids preserve the unit. ◇

Abelian groups are monoids, so $\mathsf{Ab} \subseteq \mathsf{Mon}$. This inclusion has a left adjoint $M \mapsto M^{gp} = \{(a,b) | (a,b) \sim (c,d) \text{ if there is an } s \text{ such that } s+a+d = s+b+c\}$. In particular, any map from M to an abelian group factors uniquely through $M \to M^{gp}$.

Definition 1.4. M is *integral* if for all $m \in M$, $+m\colon M \to M$ is injective (equivalently, if $M \to M^{gp}$ is injective). It is called *saturated* if it is integral and if $M = \{m \in M^{gp} | \exists n > 0 \text{ such that } nm \in M\}$ ◇

If you know about toric varieties, when you dualize a cone, you always get a saturated monoid.

Definition 1.5. M is *fine* if it is integral and finitely generated. It is fs if it is fine and saturated. ◇

Definition 1.6. A *prelog structure* on a scheme X is a sheaf of monoids M and a map of sheaves of monoids $\alpha\colon M \to (\mathcal{O}_X, \cdot)$.[2] A prelog structure is a *log structure* if $\alpha\colon \alpha^{-1}(\mathcal{O}_X^\times) \to \mathcal{O}_X^\times$ is an isomorphism. ◇

Example 1.7. Let k be a field, let X be smooth over k, and let $D \subseteq X$ be a divisor with normal crossings (it could look like components *étale* locally; it could be a nodal cubic, for example). Let $M = \{f \in \mathcal{O}_X | f|_{X \smallsetminus D} \in \mathcal{O}_{X \smallsetminus D}^\times\}$. Here, M is a subsheaf of $\mathcal{O}_X$, but it need not be in general.

If I have an étale morphism $\pi\colon X \to \mathbb{A}^n$ such that $D = \pi^{-1}(V(x_1 \cdots x_r))$ (the first r hyperplanes). Then M is the subsheaf of $\mathcal{O}_X$ generated by $\mathcal{O}_X^\times$ and $x_1, \ldots, x_r$. ◇

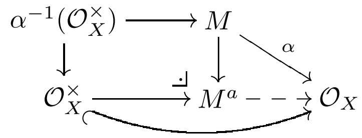
The next example looks stupid but is very important

Example 1.8. Let $X = \operatorname{Spec} k$, and $M = k^\times \oplus \mathbb{N}$, with $k^\times \oplus \mathbb{N} \to k$ given by $(u,n) \mapsto u(0)^n$ (where $0^0 = 1$ and $O^n = 0$ for $n \neq 0$). This arises from $0 \in \Delta$. Example 1.7 gives M_Δ on Δ. $i_\Delta^* M_\Delta$ on $\operatorname{Spec} \mathbb{C}$ is this example. ◇

Lemma 1.9. *The inclusion (log structures on X) $\hookrightarrow$ (prelog structures on X) has a left adjoint $M \mapsto M^a$.*

[2] This is the only case were we'll use multiplicative notation for a monoid.

Proof. Define M^a as the pushout in the category of sheaves of monoids



□

Example 1.10. Let X be a scheme, P a monoid, and $\beta\colon P \to \Gamma(X, \mathcal{O}_X)$ a morphism of monoids (e.g. Example 1.7 with $\mathbb{N}^r \to k[x_1, \ldots, x_n]$ given by $e_i \mapsto x_i$). β corresponds to $P \to \mathcal{O}_X$. This leads to a log structure $P^a \to \mathcal{O}_X$. ◇

Notation: Let P be a monoid, and R a ring. Write $\operatorname{Spec}(P \to R[P])$ for $\operatorname{Spec} R[P]$ with the log structure associated to the natural map $P \to R[P]$.

Definition 1.11. A *log scheme* is a pair (X, M_X), where X is a scheme and M_X is a log structure (the α is omitted from the notation). ◇

Considering these pairs gives you a good category with deformation theory.

Definition 1.12. Let $f\colon Y \to X$ be a morphism of schemes and let M be a log structure on X. Then the composite $f^{-1}M \to f^{-1}\mathcal{O}_X \to \mathcal{O}_Y$ is a prelog structure. We define the pullback f^*M to be the associated log structure. ◇

I have to tell you what morphisms of log schemes are. If (X, M_X) and (Y, M_Y) are log schemes, then a morphism $(Y, M_Y) \to (X, M_X)$ is a pair $(f, f^\flat)$ where $f\colon Y \to X$ is a morphism of schemes and $f^\flat\colon f^*M_X \to M_Y$ is a morphism of log structures (i.e. a morphism over $\mathcal{O}_X$).

Exercise. *Say (X, M_X) is a log scheme and P is a monoid. Then* $\operatorname{Hom}_{\mathsf{logSch}}((X, M_X), \operatorname{Spec}(P \to \mathbb{Z}[P]) \simeq \operatorname{Hom}_{\mathsf{Mon}}(P, \Gamma(X, M_X))$. *This is an exercise in adjoints.*

Exercise. *$f\colon Y \to X$ and P a monoid, and $\beta\colon P \to \Gamma(X, \mathcal{O}_X)$. Then $f^*(P^a) = (P \to \Gamma(X, \mathcal{O}_X) \to \Gamma(Y, \mathcal{O}_Y))^a$.*

Definition 1.13. A log structure M on X is called *fine* if étale locally there is a fine monoid P and a map $\beta\colon P \to \Gamma(X, \mathcal{O}_X)$ such that $M \simeq P^a$. ◇

Remark 1.14. In the Zariski topology, the plane with the nodal cubic divisor is not right. You really want to use the étale topology. $\diamond$

A *chart* for a fine log structure M on X is a fine monoid P and a map $P \to \Gamma(X, M)$ such that $\big(P \to \Gamma(X, M) \to \Gamma(X, \mathcal{O}_X)\big)^a \to M$ is an isomorphism. This is the same thing as a map $(f, f^\flat)\colon (X, M) \to \mathrm{Spec}(P \to \mathbb{Z}[P])$ such that $f^\flat$ is an isomorphism.

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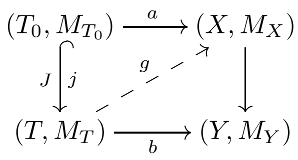
Recall that we have the category of log schemes, logSch, whose objects are pairs (X, M_X), where X is a scheme and M_X is a sheaf of monoids with a map of sheaves of monoids $\alpha\colon M_X \to \mathcal{O}_X$ such that $\alpha^{-1}(\mathcal{O}_X^\times) \to \mathcal{O}_X^\times$ is an isomorphism. A morphism $(X, M_X) \to (Y, M_Y)$ is a morphism of schemes $f\colon X \to Y$ and a map $f^\flat\colon f^* M_Y \to M_X$.

The goal for today is to say something about differentials. I want to explain how to do algebraic differential geometry in this category.

Definition 2.1. A morphism $(f, f^\flat)\colon (X, M_X) \to (Y, M_Y)$ is *strict* if $f^\flat\colon f^* M_Y \to M_X$ is an isomorphism. $\diamond$

You should think of strictness as the anology of being a closed immersion.

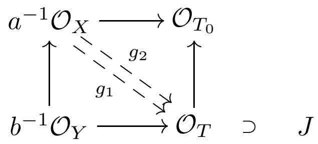
Consider the diagram (with all log structures integral, to be careful)



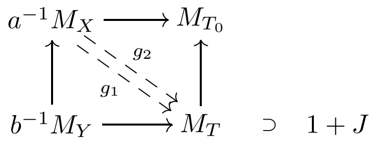
$(*)$

If $T_0 \hookrightarrow T$ is a closed immersion defined by ideal J, with $J^2 = 0$ and j is strict (sometimes called a log closed immersion with $J^2 = 0$), we're interested in filler arrows.

First let's fix two filler arrows g_1 and g_2. The difference should correspond to a derivation. Let me remind you how that goes. T_0 and T have the same "étale topological space" (by which I mean that they have equivalent categories of étale sheaves). I have the diagram of sheaves of algebras



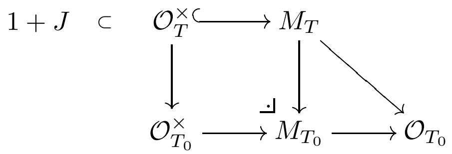
Then $g_1 - g_2\colon a^{-1}\mathcal{O}_X \to J$ is a derivation $\partial_{g_1 - g_2}\colon a^{-1}\mathcal{O}_X \to J$. In our situation, we also have the log structures



with $(1+a)(1-a)=1$.

Lemma 2.2. $M_T|_{1+J} \xrightarrow{\sim} M_{T_0}$

Proof. □



I get a map $D_{g_1-g_2}\colon a^{-1}M_X \to J$ such that for every section $m \in a^{-1}M_X$, we have $g_1(m) = (1+D_{g_1-g_2}(m)) + g_2(m)$. You have to check that this is actually additive. Passing to the associated groups, we get $D_{g_1-g_2}\colon a^{-1}M_X^{gp} \to J$. There is a map of diagrams going from the log structures to the sheaves of rings. (1) That means that for every local section $m \in a^{-1}M_X$,

$$a^{\#}(m)D_{g_1-g_2}(m) = \partial_{g_1-g_2}(\alpha(m))$$

where $a^{\#}\colon a^{-1}M_X \xrightarrow{\alpha} a^{-1}\mathcal{O}_X \to \mathcal{O}_{T_0}$. That is, we want to say $D_{g_1-g_2}(m) =$"$d\log(\alpha m)$". (2) $D_{g_1-g_2}|_{b^{-1}M_Y} = 0$.

Remark 2.3. $D_{g_1-g_2}$ determines $\partial_{g_1-g_2}$. ◇

Define $\Omega^1_{(X,M_X)/(Y,M_Y)} := \frac{\Omega^1_{X/Y} \oplus (\mathcal{O}_X \otimes_{\mathbb{Z}} M_X^{gp})}{I}$ where I is the $\mathcal{O}_X$-submodule generated by

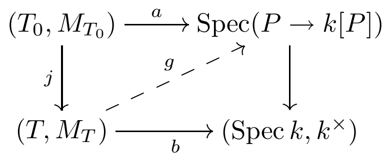
(i) $(d\alpha(m), 0) - (0, \alpha(m)\otimes m)$

(ii) $(0, 1\otimes m)$ where $m \in \operatorname{im}(f^{-1}M_Y \to M_X)$.

Summary: if a dotted arrow filling in $(*)$ exists, then the set of such g is a torsor under $\operatorname{Hom}(a^*\Omega^1_{(X,M_X)/(Y,M_Y)}, J)$. I explained one direction (that any two maps differ by an element of this Hom). You have to check that if you add an element of the Hom to a given filler, then you get another filler.

There are two approaches you could take. I wanted to be very explicit and write down the formula for this Ω^1, but it is completely characterized by this property (you do the construction to prove existence, but then you never care about the formula again). The point is that you have a nice sheaf with this nice property.

Example 2.4. Say P is a fine monoid, k is a field. Let's compute $\Omega^1_{\operatorname{Spec}(P\to k[P])/k}$, where k means $\operatorname{Spec} k$ with the trivial log structure (meaning the monoid $\mathcal{O}^\times$ with the inclusion; this is the initial object in the category of log structures). we have



From last time, a corresponds to a map $\gamma_0\colon P \to \Gamma(T_0, M_{T_0})$. A g would correspond to $\gamma\colon P \to \Gamma(T, M_T)$. If we fix one such γ, another map g' would correspond to a map γ' of the form $\gamma+\rho$, where $\rho\colon P \to \Gamma(T, 1+J)$, which is exactly the same as a map $\rho\colon P^{gp} \to \Gamma(T_0, J)$. The universal property implies that $\Omega^1_{\operatorname{Spec}(P\to k[P])/k} \cong \mathcal{O}_{\operatorname{Spec} k[P]} \otimes_{\mathbb{Z}} P^{gp}$ (using the universal property of tensor product). ◇

In general, there is a derivation $d\colon \mathcal{O}_X \to \Omega^1_{(X,M_X)/(Y,M_Y)}$. One way to see that is that there is a map $\mathcal{O}_X \to \Omega^1_{X/Y} \to \Omega^1_{(X,M_X)/(Y,M_Y)}$. [[★★★ I missed the other explanation]]

Example 2.5. $P = \mathbb{N}^r$, so $\operatorname{Spec} k[P]$ is $\mathbb{A}^r_k$. We have that $\Omega^1_{\operatorname{Spec}(P\to k[P])/k} = \mathcal{O}_{\mathbb{A}^r} \otimes_{\mathbb{Z}} \mathbb{Z}^r$. Maybe that's not so interesting, but let's think about what is d. You have the standard generators for $\mathbb{N}^r$; call the corresponding variables $x_1, \dots, x_r$. Then

$$d\colon k[x_1,\dots,x_r] \to k[\underline{x}](1\otimes e_1) \oplus \cdots \oplus k[\underline{x}](1\otimes e_r)$$

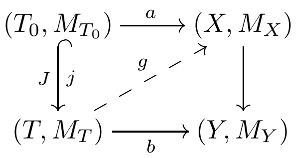
◇

Exercise. $d(x_i) = x_i(1 \otimes e_i)$. *That is, we can think of* $1 \otimes e_i$ *as* dx_i/x_i.

Fact 1: if $(X, M_X) \to (Y, M_Y)$ is strict, then $\Omega^1_{(X,M_X)/(Y,M_Y)} = \Omega^1_{X/Y}$. Fact 2: if (X, M_X) and (Y, M_Y) are fine (which means that étale locally, they are the associated log structure to the prelog structure coming from some fine monoid), then $\Omega^1_{(X,M_X)/(Y,M_Y)}$ is quasi-coherent and is coherent if locally noetherian and f is of finite type.

Depending on how you learned algebraic geometry, the whole theory of differentials either goes through, or it seems very mysterious. Hopefully, you learned by following SGA1 or EGA. The point is that this lifting property is what you need to develop most of the theory of differentials.

Definition 2.6. A morphism $f\colon (X, M_X) \to (Y, M_Y)$ is *log smooth* (or *smooth*, if there is no confusion about what category we're in) if $X \to Y$ is locally of finite presentation and for every diagram $(*)$ (reproduced below) of solid arrows

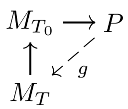


$(*)$

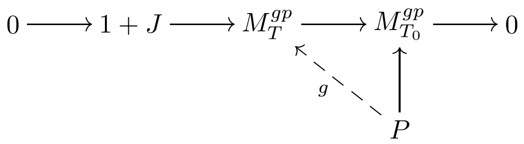
there exists étale locally on T a dashed arrow. ◇

Remark 2.7. The definition of log étale can be obtained by requiring that the dashed arrow is unique. This is equivalent to log smooth plus $\Omega^1_{(X,M_X)/(Y,M_Y)} = 0$. ◇

Example 2.8. $(Y, M_Y) = (\operatorname{Spec} k, k^\times)$ and $(X, M_X) = \operatorname{Spec}(P \to k[P])$, where P is fine.



I have



pull back to get an exact sequence

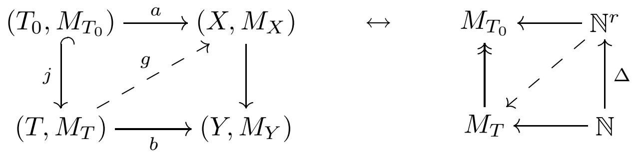
$$0 \to 1 + J \to E \to P^{gp} \to 0$$

If P^{gp} is a free group, then I can split this. If the torsion of P is invertible in k, then something. The upshot is that $\operatorname{Spec}(P \to k[P]) \to (\operatorname{Spec} k, k^\times)$ is log smooth if and only if the order of $(P^{gp})_{tors}$ is invertible in k. ◇

Example 2.9. Take $X = \operatorname{Spec} k[x_1, \ldots, x_n]/(x_1 \ldots x_r) = \operatorname{Spec} k[\mathbb{N}^r][x_{r+1}, \ldots, x_n] \otimes_{\Delta, k[\mathbb{N}], \beta} k$, where $\Delta\colon \mathbb{N} \to \mathbb{N}^r$ is the diagonal map and $\beta\colon \mathbb{N} \to k$ is given by $n \mapsto 0^n$. M_X is the log structure associated to $\mathbb{N}^r \to \mathcal{O}_X$ and $M_k = k^\times \oplus N \to k$ given by $(u, n) \mapsto u \cdot \beta(n)$.

The claim is that $(X, M_X) \to (\operatorname{Spec} k, M_k)$ is log smooth. This is good news from the point of view of moduli because it means that we will get log smooth things on the boundary.

Why is the claim true? Consider the case $n = r$ to not be too confusing



you pick any lift on the right, then the diagram doesn't commute, but it commutes up to a unit, so you change something a little bit.

The same argument show that $\Omega^1_{(X,M_X)/(Y,M_Y)}$ is a free module on generators dx_i/x_i modulo the relations $\sum_{i=1}^r dx_i/x_i = 0$. ◇

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Last time, I introduced the notion of log smoothness. A morphism $(X, M_X) \to (Y, M_Y)$ is log smooth if it is locally of finite presentation and for every $T_0 \subseteq T$ defined by a square zero ideal with j strict, there is a filler arrow g.

$$\begin{array}{ccc} (T_0, M_{T_0}) & \xrightarrow{a} & (X, M_X) \\ {\scriptstyle J}\big\downarrow{\scriptstyle j} & \overset{g}{\nearrow} & \big\downarrow \\ (T, M_T) & \xrightarrow[b]{} & (Y, M_Y) \end{array}$$

Classically, to be smooth, it is the same as saying that the morphism is étale locally affine space over the base.

Theorem 3.1 (Kato's structure theorem)**.** *Let $f\colon (X, M_X) \to (Y, M_Y)$ is a morphism of fine log structures and assume $\beta\colon Q \to \Gamma(Y, M_Y)$ is a chart. Then the following are equivalent:*

1. *f is log smooth.*
2. *étale locally on X, there is a chart $P \to \Gamma(X, M_X)$ and a map of monoids $\theta\colon Q \to P$ so that*

$$\begin{array}{ccc} P & \longrightarrow & M_X \\ \big\uparrow{\scriptstyle \theta} & & \big\uparrow \\ Q & \longrightarrow & M_Y \end{array}$$

such that (i) $\ker \theta^{gp}$ and the torsion part of $\operatorname{coker}(\theta^{gp})$ have order invertible on X, and (ii) the natural map $X \to Y \times_{\operatorname{Spec}\mathbb{Z}[Q]} \operatorname{Spec}\mathbb{Z}[P]$ is étale.

Exercise. *(2)⇒(1). The other direction is harder.*

Loosly speaking, being log smooth means you're toric.

Corollary 3.2. *Suppose $(B_0, M_{B_0}) \hookrightarrow (B, M_B)$ is a strict closed immersion (of fine log schemes) defined by a nilpotent ideal, and $(X_0, M_{X_0}) \to (B_0, M_{B_0})$ is log smooth. Then étale locally on X_0, there exists a log smooth lifting $(X, M_X) \to (B, M_B)$ (i.e. this morphism is log smooth) such that*

$$\begin{array}{ccc} (X_0, M_{X_0}) & \overset{strict}{\hookrightarrow} & (X, M_X) \\ \big\downarrow & \ulcorner & \big\downarrow \\ (B_0, M_{B_0}) & \hookrightarrow & (B, M_B) \end{array}$$

If you know about deformation theory of smooth schemes, this is very promising.

Remark 3.3. In general, when you do deformation theory, you really want the underlying morphisms of schemes to be flat. The underlying morphisms of log smooth morphisms need not be flat. This can be problematic. ⋄

Definition 3.4. A map of fine monoids $\theta\colon Q \to P$ is called *integral* if the map on monoid algebras $\mathbb{Z}[Q] \to \mathbb{Z}[P]$ is flat. ⋄

If you write out the equational condition for flatness of a map of rings, this is the following condition. For every $p_1, p_2 \in P$ and $q_1, q_2 \in Q$ such that $p_1 + \theta(q_1) = p_2 + \theta(q_2)$, there exists $p \in P$ and $q_3, q_4 \in Q$ such that $p_1 = p + \theta(q_3)$, $p_2 = p + \theta(q_4)$, and $q_1 + q_3 = q_2 + q_4$.

A map of fine log schemes $f\colon (X, M_X) \to (Y, M_Y)$ is integral if for every geometric point $\overline{x} \to X$, the map on monoids $M_{Y,f(\overline{x})}/\mathcal{O}^\times =: \overline{M}_{Y,f(\overline{x})} \to \overline{M}_{X,\overline{x}} := M_{X,\overline{x}}/\mathcal{O}^\times$.

Being integral has nothing to do with being an integral scheme. It means it is universally integral in the category of monoids (i.e. any pushout remains an integral monoid).

Fact: If $(X, M_X) \to (Y, M_Y)$ is log smooth and integral, then $X \to Y$ is flat, and in (2), you can take $Q \to P$ to be an integral morphism.

Remark 3.5. In the corollary, if $(X_0, M_{X_0}) \to (B_0, M_{B_0})$ is integral, then any lifing $(X, M_X) \to (B, M_B)$ is also integral. $M_{X_0} = M_X/1 + J$, so when you quotient out by $\mathcal{O}^\times$, they are the same. So if you have a log smooth integral morphism, then its log smooth deformations are automatically integral. ⋄

Q:The property of being integral is stbale under pullback? MO: yes.

Setup: Start with a strict closed immersion defined by square zero ideal J and $(X_0, M_{X_0}) \to (B_0, M_{B_0})$ log smooth integral.

$$\begin{array}{ccc} (X_0, M_{X_0}) & \hookrightarrow\dashrightarrow & (X, M_X) \\ {\scriptstyle f_0}\big\downarrow & \ulcorner & \big\downarrow \\ (B_0, M_{B_0}) & \hookrightarrow & (B, M_B) \end{array}$$

We will call an extension a log smooth deformation.

1. étale locally on X_0, there is a log smooth deforamtion,
2. Given an (X, M_X), the automorphism group is given by $\mathrm{Hom}(\Omega^1_{(X_0,M_{X_0})/(B_0,M_{B_0})}, J \otimes \mathcal{O}_X) = T_{X_0/B_0}(log) \otimes J$. The point is that any dashed arrow

$$\begin{array}{ccc} (X_0, M_{X_0}) & \longrightarrow & (X, M_X) \\ \downarrow & \nearrow & \downarrow \\ (X, M_X) & \hookrightarrow & (B, M_B) \end{array}$$

   Then $0 \to J \otimes \mathcal{O}_{X_0} \to \mathcal{O}_X \to \mathcal{O}_{X_0} \to 0$. Any morphism is an isomorphism and ...
3. Any two log smooth liftings are étale locally isomorphic

$$\begin{array}{ccc} (X_0, M_{X_0}) & \longrightarrow & (X', M_{X'}) \\ \downarrow & \nearrow & \downarrow \\ (X, M_X) & \longrightarrow & (B, M_B) \end{array}$$

   So the stack of log smooth deformations is a gerbe.

Theorem 3.6. *(1) there is a canonical obstruction $\eta \in H^2(X_0, T_{X_0/B_0}(\log) \otimes J)$ such that $\eta = 0$ if and only if there exists a log smooth deformation. (2) if $\eta = 0$, then the set of log smooth deformations form a torsor under $H^1(X_0, T_{X_0/B_0}(\log) \otimes J)$. (3) the automorphism group of any deformation is isomorphic to $H^0(X_0, T_{X_0/B_0}(\log) \otimes J)$.*

Part (3) is the universal property of differentials. Q: is this false if the map is not flat? MO: I think you run into trouble; the kernel won't be $J \otimes \mathcal{O}_X$. You can make some statement if you're over the dual numbers.

Let me tell you what the obstruction is. It's exactly how it is if you read about ordinary smooth deformations in SGA1. For simplicity, let's assume that X_0 is separated. First, choose an (étale) covering $\mathcal{U} = \{U_i\}$ of X_0 and choose liftings $(\tilde{U}_i, M_{\tilde{U}_i}) \to (B, M_B)$ of (U_i, M_{U_i}). Now we try to patch them together. We have $(U_{ij}, M_{U_{ij}}) \hookrightarrow (\tilde{U}_i|_{U_{ij}}, M_{\tilde{U}_i}|_{U_{ij}})$ and $(U_{ij}, M_{U_{ij}}) \hookrightarrow (\tilde{U}_j|_{U_{ij}}, M_{\tilde{U}_j}|_{U_{ij}})$. By the comment and cohomology of a quasi-coherent sheaf vanishes, we know that there is an isomorphism $\theta_{ij} \colon (\tilde{U}_i|_{U_{ij}}, M_{\tilde{U}_i}|_{U_{ij}}) \to (\tilde{U}_j|_{U_{ij}}, M_{\tilde{U}_j}|_{U_{ij}})$. But now we we need to satisfy a cocycle condition $\partial_{ijk} = \theta_{ij} + \theta_{jk} - \theta_{ik} \in T_{X_0/B_0}(\log) \otimes J$ (everything restricted to U_{ijk}).

Exercise. *$\{\partial_{ijk}\}$ is a Cech 2-cocycle.*

η is the corresponding cohomology class. You can check that η is a boundary if and only if we could have chosen our θ's better so that $\partial_{ijk} = 0$.

Now let's apply this to some examples. Probably, you really just care about schemes, so let's just start with a scheme. Suppose k is a field, and $X_0 \to \operatorname{Spec} k$ is some scheme we're interested in. Suppose that étale locally, $X_0 = \operatorname{Spec} k[x_1, \ldots, x_n]/x_1 \cdots x_r$. Let M_k be the log structure on k given by $k^\times \oplus \mathbb{N} \to k$ given by $(u, n) \mapsto u \cdot 0^n$. Question: When does there exist a log structure M_{X_0} and a morphism $(X, M_{X_0}) \to (\operatorname{Spec} k, M_k)$ which locally is "the standard one" (one of the examples from before)? You view your X_0 as $(k \otimes_{k[\mathbb{N}]} k[\mathbb{N}^r])[x_{r+1}, \ldots, x_n]$ and you get a natural log structure which we call the standard one. Answer: d-semistability: when the line bundle $\mathcal{E}xt^1(\Omega^1_{X_0/k}, \mathcal{O}_{X_0})$ on $D = X_0^{sing}$ is trivial. In fact, there is more you can say; the log structure is unique up to something.

Example 3.7. (1) nodal curve. (2) Exercise where you blow up the 3-torsion points on $E \hookrightarrow \mathbb{P}^2$ and take X_0 to be the gluing of two copies. $H^2(X_0, T_{X_0} \otimes J) = 0$. The Hodge diamond $H^i(X_0, \Omega^j_{X_0/k}(\log))$ is as on the exercises:

$$\begin{array}{ccc} 1 & 0 & 1 \\ 0 & 20 & 0 \\ 1 & 0 & 1 \end{array}$$

$\Omega^2(\log) = \mathcal{O}_{X_0}$ is the dualizing sheaf implies X_0 is smoothable. ⋄

4 Martin Olsson

Today I want to discuss Alexeev's moduli stack of broken toric varieties from the point of view of log geometry.

X will be a free abelian group of finite rank. $Q \subseteq X_{\mathbb{R}}$ will be an integral polytope. $T = \mathrm{Hom}_{\mathsf{Gp}}(X, \mathbb{G}_m)$ will be the torus.

Example 4.1. $X = \mathbb{Z}$, $Q = [-1, 1] \subseteq \mathbb{R}$. The quiz is: what is the moduli space of polarized toric varieties for this polytope: [[★★★ picture interval broken at 0]] ⋄

We have the associated toric variet. Take M to be the integral points of the $Cone(1, Q) \subseteq \mathbb{R} \times X_{\mathbb{R}}$, which form a graded monoid, and take $\mathrm{Proj}\,\mathbb{Z}[M]$. I can write this as $\big(\mathrm{Spec}\,\mathbb{Z}[M] \smallsetminus \{0\}\big)/\mathbb{G}_m$. Consider the log scheme $\big(\mathrm{Spec}(M \to \mathbb{Z}[M]) \smallsetminus \{0\}\big)/\mathbb{G}_m$, which is the same scheme, but with a log structure.

Now we want to degenerate this guy, so we should be thinking of functions on integral points. Say $Z \subseteq Q$ are the integral points. Let $\psi\colon Z \to \mathbb{R}$ be a function. Then we are supposed to consider the set $G_\psi = \{(h, x) \in \mathbb{R} \times X_{\mathbb{R}} | x \in Z, h \geq \psi(x)\}$. The lower boundary of G_ψ is a piece-wise linear function on Q, which gives me a *paving* of Q (which is what you think it is; you break your polytope into sub-polytopes with some expected properties). Associated to this we're supposed to get a degeneration. I want to actually degenerate it as a log scheme.

Example 4.2. On the polytope $[-1, 1]$ I could consider the function $(7, 9, 2)$, which gives me the interval, or $(3, 2, 4)$, which gives the broken interval. ⋄

Definition 4.3. A *paving* S of Q is a collection of integral sub-polytopes of Q such that (1) if $\omega, \eta \in S$, $\omega \cap \eta \in S$, (2) any face of $\omega \in S$ is in S, and (3) $Q = \bigcup_{\omega \in S} \omega$ and the ω have disjoint interior. ⋄

Define an associated monoid H_S in as follows. For all ω, define N_ω to be the integral points of $Cone(1, \omega) \subseteq \mathbb{R} \times X_{\mathbb{R}}$. Let $N_S^{gp} := \mathrm{colim}_{\omega \in S} N_\omega^{gp}$.

Example 4.4. Consider the broken interval (with points x, z, y), then I get the diagram of groups

$$\begin{array}{ccc} \mathbb{Z}x & \hookrightarrow & \mathbb{Z}x \oplus \mathbb{Z}z \\ \mathbb{Z}z & \hookrightarrow & \mathbb{Z}x \oplus \mathbb{Z}z,\ \mathbb{Z}z \oplus \mathbb{Z}y \\ \mathbb{Z}y & \hookrightarrow & \mathbb{Z}z \oplus \mathbb{Z}y \end{array}$$

Thus, $N_S^{gp} = \mathbb{Z}x \oplus \mathbb{Z}y \oplus \mathbb{Z}z$. ⋄

There is a set map $\rho\colon M \to N_S^{gp}$. Because of how we did the limit, ρ is well-defined. This is not a monoid map. Define H_S to be the submonoid of N_S^{gp} generated by things of the form $p * q = \rho(p) + \rho(q) - \rho(p + q)$. In our example, $H_S \cong \mathbb{N}$ is generated by $x + y - 2z$.

Facts: (1) H_S is finitely generated; (2) $H_S^\times = \{0\}$ if and only if the paving comes from a height function as Valery explained.

Define $M \rtimes H_S$ to have elements pairs (m, h), where $m \in M$ and $h \in H_S$, where $(m, h) + (m', h') := (m + m', h + h' + m * m')$. In our example, $M = \langle x, y, z\rangle/(x + y = 2z)$. Then $M \rtimes H_S = \langle x, y, z, t\rangle/(x + y = t + 2z)$. There is a map of monoid algebras $\mathbb{Z}[H_S] \to \mathbb{Z}[M \rtimes H_S]$.

In our example, this map is $\mathbb{Z}[t] \to \mathbb{Z}[t][x, y, z]/(xy = tz^2)$. If I take Proj, I get a family over the affine line: $\mathrm{Proj}\,\mathbb{Z}[t][x, y, z]/(xy = tz^2) \to \mathbb{A}^1_t$. This is a degeneration of $\mathbb{P}^1$ corresponding to $\mathcal{O}(2)$ (the interval had length 2) with the paving we had. This has a natural log structure because it arose as a monoid algebra.

In general, $\mathrm{Proj}(M \rtimes H_S \to \mathbb{Z}[M \rtimes H_S]) \to \mathrm{Spec}(H_S \to \mathbb{Z}[H_S])$. This is a degeneration of the toric variety $\mathrm{Proj}(M \to \mathbb{Z}[M])$ as a log scheme. Note that the log structure on the base could be quite complicated.

Definition 4.5 ("sort of a cop out")**.** A *standard object* over $k = \bar{k}$ is the data $\big(M_k, f\colon (X, M_X) \to (\mathrm{Spec}\,k, M_k)$, T-action, line bundle L with T-action$\big)$, where M_k is a log structure on k, f is a log smooth proper map, and (X, M_X) isomorphic to the closed fiber of a family coming from a convex paving S (of Q) as above. ⋄

$\mathcal{K}_Q$ is a stack over Z which to any scheme B associates the groupoid data $\big(M_B, f\colon (X, M_X) \to (B, M_B), L, \theta, \rho\big)$, where M_B is a log structure on B, f is log smooth with $X \to B$ proper, L is a relatively ample line bundle on X, ρ is an action of T on (X, M_X, L) over (B, M_B), and $\theta \in f_*L$ such that

- for every geometric point $\bar{s} \to B$, the zero locus of $\theta_{\bar{s}}$ in $X_{\bar{s}}$ does not contain any T-orbit, and
- $\big(M_{\bar{s}}, (X_{\bar{s}}, M_{X_{\bar{s}}}) \to (\bar{s}, M_{\bar{s}}), L_{\bar{s}}\big)$ is a standard object.

Theorem 4.6. *$\mathcal{K}_Q$ is an algebraic stack with finite diagonal and toric singularities (i.e. is log smooth), and is equal to the main component in Alexeev's moduli space.*

We have a natural log structure $(\mathcal{K}_Q, M_{\mathcal{K}_Q})$ and toric singularities means log smooth over $\mathbb{Z}$ with trivial log structure (this was Kato's theorem).

Exercise. *$X = \mathbb{Z}$ and $Q = [-1, 1]$. I think $\mathcal{K}_Q \cong \mathbb{P}^{2,1}$. The coordinate is t; there is a μ_2 at 0 and the log structure $M_{\mathcal{K}_Q}$ is defined by the divisor at ∞.*

Let's try to verify that this guy is log smooth. What does it mean to say that $(\mathcal{K}_Q, M_{\mathcal{K}_Q})$ is log smooth (I can verify this even if I don't yet know it is algebraic).

$$\begin{array}{ccc} (T_0, M_{T_0}) & \xrightarrow{a} & (\mathcal{K}_Q, M_{\mathcal{K}_Q}) \\ {\scriptstyle J}\big\downarrow & \overset{?}{\nearrow} & \\ (T, M_T) & & \end{array}$$

Lemma 4.7. *To check that something is smooth, it is enough to consider the case where a is strict.*

So I need to fine a lifting

$$\begin{array}{ccc} (X_0, M_{X_0}, L_0, \theta_0, \rho_0) & \dashrightarrow & (X, M_X, L, \theta, \rho) \\ {\scriptstyle \mathcal{K}_Q \ni}\big\downarrow & & \big\downarrow \\ (T_0, M_{T_0}) & \hookrightarrow & (T, M_T) \end{array}$$

Exercise. *$\Omega^1_{(X_0, M_{X_0})/(T_0, M_{T_0})} \cong Lie(torus) \otimes \mathcal{O}_{X_0}$, so to prove log smoothness, it is enough to show that $H^2(X_0, \mathcal{O}_{X_0}) = 0$ because the obstruction to lifting the log scheme is .... In fact, by standard reduction, it is enough to consider the case where T_0 is a field k.*

how do you compute it? You go back to your picture of the paving. You get an exact sequence (where $m = \dim Q$)

$$\mathcal{O}_{X_0} \to \prod_{\dim \omega = m} \mathcal{O}_{X_0, \omega} \to \prod_{\dim \eta = m-1} \mathcal{O}_{X_0, \eta} \to \cdots$$

This implies that the cohomology of $\mathcal{O}_{X_0}$ is computed by

$$\prod_{\dim \omega = m} k \to \prod_{\eta} k \to \cdots$$

which just computes $H^*(|Q|, k) = k$. This proves that you can always lift the scheme. The line bundle and the section are not so bad. Lifting the torus action is a little more complicated, so I won't talk about it.

Example 4.8. Recall the embedding $E \hookrightarrow P$ obtained from blowing up the 3-torsion, and X_0 the gluing of two of them along E. We have $(X_0, M_{X_0}) \to (\operatorname{Spec} k, M_k)$, where M_k is given by $k^\times \oplus \mathbb{N} \to k$. Consider the function $F\colon$ (artinian local k-algebras) $\to$ Set given by $A \mapsto \{$log smooth deformations of (X_0, M_{X_0}) to $\operatorname{Spec} A$ with log structure associated to $\mathbb{N} \to A$ given by $1 \mapsto$ image of $t\}$.

We know that F is unobstructed (because $H^2(X_0, T_{X_0}(\log)) = 0$). We also know that the tangent space is 20-dimensional because $h^1(X_0, T_{X_0}(\log)) = 20$. That means that F is (pro)represented by $k[\![t]\!][\![s_1, \ldots, s_{20}]\!]$, which is 21-dimensional. We were expecting 20-dimensional, so what is the extra dimension?

The extra dimension comes from $\operatorname{Aut}(M_k) = k^\times$. Part of the data of $(X_0, M_{X_0}) \to (k, M_k)$ is $f^\flat\colon M_k \to M_{X_0}$. ◇

Q: could you say something about how you compute that 20? MO: first you show that the dualizing sheaf is trivial. Then by Serre duality, it is easy to fill in all the other parts of the Hogde diamond. Then there is some argument that the Euler characteristic should be 24.

If I fix the log structure on the base, I get the wrong tangent space. You really have to allow different log structures on the base, and allow isomorphisms of those as part of the structure.

5 Martin Olsson

This is the last lecture about log geometry. I want to switch gears a bit and talk about the connection between log geometry and algebraic stacks, and do a bunch of examples.

Warm-up. Let X be a scheme and let $r \geq 1$ be a integer. Consider the category C whose objects are pairs (M, β), where M is a fine log structure on X and $\beta\colon \mathbb{N}^r \to \overline{M} = M/\mathcal{O}^\times$ such that β locally on X lifts to a chart for M.[1] The morphisms $(M, \beta) \to (M', \beta')$ are isomorphisms of log structures $\sigma\colon M \to M'$ such that $\beta' = \overline{\sigma} \circ \beta$.

Claim. C *is equivalent to the category* D*, defined as follows. The objects are collections* $(\gamma_1 \colon L_1 \to \mathcal{O}_X, \ldots, \gamma_r \colon L_r \to \mathcal{O}_X)$*, where the* L_i *are line bundles and the* γ_i *are* $\mathcal{O}_X$*-module morphisms (need not be isomorphisms; could all be zero). The morphisms are isomosphisms of such data (i.e. isomorphisms of the* L_i *over* $\mathcal{O}_X$*).*

Proof. I'll sketch one direction. Given (M, β), construct the (L_i, γ_i) as follows. We have

$$\begin{array}{ccccccc} & & & & \mathbb{N}^r & & e_i \\ & & & & \downarrow{\scriptstyle\beta} & & \downarrow \\ M & \xrightarrow{\pi} & M/\mathcal{O}_X^\times & = & \overline{M} & & \beta(e_i) \end{array}$$

Take L_i to be the line bundle associated to the $\mathcal{O}_X^\times$-torsor $\pi^{-1}(\beta(e_i))$. There is a map of sheaves $\pi^{-1}(\beta(e_i)) \to \mathcal{O}_X$ induced by the given map $M \to \mathcal{O}_X$.

I'll leave it to you to go the other way. □

On the other hand, the groupoid D is equivalent to the groupoid of maps $X \to [\mathbb{A}^r/\mathbb{G}_m^r] = [\mathbb{A}^1/\mathbb{G}_m]^r$, maps from X to the stack quotient. The stack $[\mathbb{A}^1/\mathbb{G}_m]$ parameterizes line bundles with maps to $\mathcal{O}_X$.

More generally, if P is a fine monoid, let $\mathcal{S}_P = \big[\operatorname{Spec}\mathbb{Z}[P]/D(P^{gp}) = \operatorname{Hom}(P^{gp}, \mathbb{G}_m) = \operatorname{Spec}\mathbb{Z}[P^{gp}]\big]$. This classifies pairs (M, β) where M is a fine log structure and $\beta\colon P \to \overline{M}$ which locally lifts to a chart.

[1] $$\begin{array}{ccc} & & M \\ & {\scriptstyle\text{locally}}\nearrow & \downarrow \\ \mathbb{N}^r & \longrightarrow & \overline{M} \end{array}$$

Example 5.1. $\mathbb{A}^1_t \to [\mathbb{A}^1/\mathbb{G}_m]$, in this dictionary, is a line bundle with a map $(L \to \mathcal{O}_{\mathbb{A}^1})$. It is given by $L = (t)$ with the map $(t) \hookrightarrow \mathcal{O}_{\mathbb{A}^1}$. $\Omega^1_{\mathbb{A}^1/[\mathbb{A}^1/\mathbb{G}_m]}$ is computed as the ideal of the diagonal mod its square.

$$\begin{array}{ccccc} & & (x,1) & \longleftarrow & x \\ \mathbb{A}^1 & \xleftarrow{pr_1} & \mathbb{A}^1 \times \mathbb{G}_m & \xleftarrow{\Delta} & \mathbb{A}^1 \\ \downarrow & & \downarrow{\scriptstyle\rho} & & \\ [\mathbb{A}^1/\mathbb{G}_m] & \longleftarrow & \mathbb{A}^1 & & \end{array}$$

So $\Omega^1_{\mathbb{A}^1/[\mathbb{A}^1/\mathbb{G}_m]} = k[t]\Delta^*(u-1)$. You should think of $d\colon k[t] \to k[t]\Delta^*(u-1)$ given by $t \mapsto ut - t$. That is, "dt/t"$= \Delta^*(u-1)$. ◇

Example 5.2. Let X be a toric variety over a field k (so it is normal, with an action of the torus T, which is dense in X). Then you can consider the stack quotient $\Omega^1_{X/[X/T]}$, which will be a subsheaf of $j_*\Omega^1_T$. It is exactly the subsheaf $\Omega^1_X(\text{log along } \partial X)$. If you write $X = \operatorname{Spec} k[P]$, then this is our old friend $\Omega^1_X(\log)$.

Q: if you take $[\mathbb{P}^1/\mathbb{G}_m]$, what is it? MO: it is two guys glued together, so two log structures marked by a color. Whenever you have an algebraic stack, you can make a space out of if (for $\mathbb{A}^1/\mathbb{G}_m$, there is a closed point and a generic point), which will be the fan of the toric variety.

$$\begin{array}{ccccccc} X & \xrightarrow{g} & \operatorname{Spec} k & \longleftarrow & T & \hookleftarrow & \operatorname{Spec} k \\ \downarrow & & \downarrow & & \downarrow & & \\ [X/T] & \longrightarrow & [\operatorname{Spec} k/T] & \longleftarrow & \operatorname{Spec} k & & \end{array}$$

So $\Omega^1_{X/[X/T]} = g^*\Omega^1_{\operatorname{Spec} k/[\operatorname{Spec} k/T]} = \mathit{Lie}(T)^\vee \otimes_k \mathcal{O}_X$. ◇

Example 5.3. Say X is a scheme, $D \subset X$ is a Cartier divisor, and $r \geq 1$ is an integer. Let's construct the universal r^{th} root of D. Let L be the ideal of D, which has a map $\gamma\colon L \to \mathcal{O}_X$. We want to classify line bundles $\mathcal{K}$ with maps

$\delta\colon \mathcal{K} \to \mathcal{O}$ and $\iota\colon \mathcal{K}^{\otimes r} \xrightarrow{\sim} L$ such that the diagram below commutes

$$\begin{array}{ccc} \mathcal{K}^{\otimes r} & \xrightarrow{\sim} & L \\ & {\scriptstyle \delta^{\otimes r}} \searrow \quad \swarrow {\scriptstyle \gamma} & \\ & \mathcal{O} & \end{array} \qquad \begin{array}{ccc} \mathcal{X}_{D,r} & \longrightarrow & [\mathbb{A}^1/\mathbb{G}_m] \\ \downarrow & \ulcorner & \downarrow {\scriptstyle t\mapsto t^r} \\ X & \xrightarrow{(L,\gamma)} & [\mathbb{A}^1/\mathbb{G}_m] \end{array}$$

This $\mathcal{X}_{D,r}$ defines $(\mathcal{K}, \delta, \iota)$. If $D = (f) \subset \mathcal{O}_X$, then $\mathcal{X}_{D,r} = \left[(\operatorname{Spec} \mathcal{O}_X[z]/z^r = f)/\boldsymbol{\mu}_r\right]$. So this gives you a global construction, which you understand how it looks like locally. ◇

Example 5.4. Let X be a nodal curve over a field k. In the other talks, we've seen something about putting stacky structure at various points, but how do you actually do that? How would I put a $\boldsymbol{\mu}_3$ at the node for example? Here is one way to do it. First, consider the case of a single node. You can't write a Zariski local neighborhood where the node looks like two axes. You have to do it étale locally, and then you have to descend a stack in the étale topology, which it's not so clear how to do.

Give it the canonical log structure, $(X, M_X) \to (\operatorname{Spec} k, k^\times \oplus \mathbb{N} \to k)$, where the $\mathbb{N}$ is a local parameter at the node. You want $k[t][x,y]/(xy - t^3)$. You have $\left[\frac{k[t,z,w]}{(zw-t)}/\boldsymbol{\mu}_3\right] \to k[t][x,y]/(xy-t^3)$, where $\zeta \in \boldsymbol{\mu}_3$ acts by $z \mapsto \zeta z$ and $w \mapsto \zeta^{-1} w$, $x \mapsto z^3$ and $y \mapsto w^3$. We have

$$\begin{array}{ccc} M_X & & \\ \uparrow & & \\ k^\times \oplus \mathbb{N} & \longrightarrow & k^\times \oplus \mathbb{N} \\ (u,n) & \longmapsto & (u,3n) \end{array} \qquad \begin{array}{ccc} M_X & \longrightarrow & \mathcal{N} \\ \uparrow & & \uparrow \\ k^\times \oplus \mathbb{N} & \longrightarrow & k^\times \oplus \mathbb{N} \\ (u,n) & \longmapsto & (u,3n) \end{array}$$

Let $\mathcal{X} \to X$ be the stack over X classifying diagrams of fine log structures on the right, such that for every geometric point $\overline{x} \to X$, the diagram on the left is isomorphic to the middle one if $\overline{x}$ hits the node and the one on the right otherwise

$$\begin{array}{ccc} \overline{M}_{X,x} & \longrightarrow & \overline{\mathcal{N}}_{\overline{x}} \\ \uparrow & & \uparrow \\ \mathbb{N} & \xrightarrow{\cdot 3} & \mathbb{N} \end{array} \qquad \begin{array}{ccc} \mathbb{N} & \longrightarrow & \mathbb{N} \\ {\scriptstyle \wr}\uparrow & & \uparrow{\scriptstyle \wr} \\ \mathbb{N} & \xrightarrow{\cdot 3} & \mathbb{N} \end{array} \qquad \begin{array}{ccc} \mathbb{N}^2 & \xrightarrow{\cdot 3} & \mathbb{N}^2 \\ {\scriptstyle \Delta}\uparrow & & \uparrow{\scriptstyle \Delta} \\ \mathbb{N} & \xrightarrow{\cdot 3} & \mathbb{N} \end{array}$$

◇

Proposition 5.5. *If $X = \operatorname{Spec}(k[x,y]/xy)$, then $\mathcal{X} = [\operatorname{Spec}(k[z,w]/zw)/\boldsymbol{\mu}_3]$.*

Remark 5.6. This construction between "twisted curves" and nodal curves with extra log data. ◇

The general story is this. Consider a finite category (i.e. a directed graph) D, for example $(\bullet \to \bullet \to \bullet)$. For any scheme X, define $\mathcal{L}og^D(X)$ to be the category of functors from D to log structures on X. For example, $\mathcal{L}og^{(\bullet\to\bullet\to\bullet)}(X)$ is the set of diagrams of log structures $M_1 \to M_2 \to M_3$ on X. Morphisms in $\mathcal{L}og^D(X)$ are isomorphisms of functors.

Theorem 5.7. *$\mathcal{L}og^D$ is an algebraic stack.*

<ins>Back to deformation theory.</ins> We started with some diagram on the left

$$\begin{array}{ccc} (X_0, M_{X_0}) & \dashrightarrow & (X, M_X) \\ {\scriptstyle \text{log smooth}}\downarrow & & \vdots \\ (T_0, M_{T_0}) & \overset{J}{\hookrightarrow} & (T, M_T) \end{array} \qquad \begin{array}{ccc} & \mathcal{L}og^{(\bullet\to\bullet)} & (M_1 \to M_2) \\ & \downarrow & \downarrow \\ Y & \xrightarrow{M_Y} \mathcal{L}og^{\bullet} & M_1 \end{array}$$

For any log scheme (Y, M_Y), define $\mathcal{L}og_{(Y,M_Y)}$ to be the fiber product of the diagram on the right. So $\mathcal{L}og_{(Y,M_Y)}(f\colon X \to Y)$ is the set of pairs $(M, f^\flat)$ M a log structure on X and $f^\flat\colon f^*M_Y \to M$, so we're upgrading a morphism to a morphism of log structures.

This is equivalent to

$$\begin{array}{ccc} X_0 & \dashrightarrow & X \\ {\scriptstyle \mathcal{L}og(f_0)}\downarrow & & \downarrow{\scriptstyle \text{smooth}} \\ \mathcal{L}og_{(T_0,M_{T_0})} & \overset{\tilde{J}}{\hookrightarrow} & \mathcal{L}og_{(T,M_T)} \\ \downarrow & & \downarrow \\ T_0 & \overset{J}{\hookrightarrow} & T \end{array}$$

log smoothness is equivalent to the map $X_0 \to \mathcal{L}og_{(T_0,M_{T_0})}$ being smooth. Deformation theory should be governed by $H^i(X_0, T_{X_0/\mathcal{L}og_{(T_0,M_{T_0})}} \otimes \tilde{J})$.

$(X_0, M_{X_0}) \to (T_0, M_{T_0})$ is integral if and only if $\mathcal{L}og(f_0)$ has image in flat locus of $\mathcal{L}og_{(T_0, M_{T_0})} \to T_0$. This implies the theorem from before.

Example 5.8. $\mathrm{Proj}\big(M \rtimes H_S \to \mathbb{Z}[M \rtimes H_S]\big) \to \mathrm{Spec}(H_S \to \mathbb{Z}[H_S])$, and we have an action of the torus on $\mathbb{Z}[M \rtimes H_S]$. This is equivalent to

$$\begin{array}{ccccc}
\mathrm{Proj}(\mathbb{Z}[M \rtimes H_S]) & \subsetneq & [\mathrm{Spec}\,\mathbb{Z}[M]/\mathbb{G}_m] & \longrightarrow & B\mathbb{G}_m \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{L}og_{\mathrm{Spec}(H_S \to \mathbb{Z}[H_S])} & \xrightarrow{\text{étale}} & [\mathrm{Spec}\,\mathbb{Z}[M \rtimes H_S]/D(M^{gp})] & \longrightarrow & BD(M^{gp})
\end{array}$$

it can be shown that the bottom map is étale. [[★★★ some stuff]] ◇