# 1 Kai Behrend - Foundations of Donaldson-Thomas theory

These foundations on Donaldson-Thomas theory are undergoing some change, so some of what I'll say is work in progress with Mike Rose and I. Ciocon-Fontanine, inspired by a paper by Kapranov and Ciocon-Fontanine RQuot.

Everything will happen over $\mathbb{C}$. $Y$ will be a Calabi-Yau 3-fold (i.e. a connected, projective, smooth $\mathbb{C}$-scheme of dimension 3 with a chosen isomorphism $\omega_Y = \wedge^3\Omega_Y \cong \mathcal{O}_Y$). For example, $Y$ could be the generic quintic hypersurface in $\mathbb{P}^4$. $X$ will be a moduli space of coherent sheaves on $Y$ (with trivialized determinant), assumed to be compact (e.g. stable of Hilbert polynomial $p(n)$). To $X$, we associate a number $\#^{vir}X$, the *virtual count* of points in $X$. Goal of Donaldson-Thomas theory: (1) define $\#^{vir}X$, and (2) compute $\#^{vir}X$.

**Example 1.1.** $X = \operatorname{Hilb}^n Y$, the moduli scheme of ideal sheaves $\mathcal{I} \subseteq \mathcal{O}_Y$ such that $\mathcal{O}_Y/\mathcal{I}$ is a skyscraper sheaf (could be supported at many points) of length $n$. Then

$$\sum_{n=0}^{\infty} \#^{vir}(\operatorname{Hilb}^n Y)t^n = \left(\prod_{n=1}^{\infty} \frac{1}{(1-(-t)^n)^n}\right)^{\chi(Y)}$$

was conjectured by Maulik-Nekrasov-Okounkov-Panthenpole in 2003. ◇

Outline of the lectures. Roughly, I hope to get through two subjects per lecture.

1. moduli space
2. deformation theory
3. virtual fundamental class (which gives rise to the definition of $\#^{vir}X$ from deformation theory, when $X$ is compact)
4. symmetric obstruction theory (this is where the assumption of Calabi-Yau 3-fold comes in
5. obstruction cone is Lagrangian
6. the microlocal function $\nu\colon X \to \mathbb{Z}$. Define $\chi(X,\nu)$, the weighted Euler characteristic (breaking up $X$ into strata where $\nu$ is constant; for this, you don't need $X$ compact).
7. The main theorem: $\#^{vir}X = \chi(X,\nu)$ in the compact case
8. equivariant case if there is a $\mathbb{C}^\times$ action on $Y$
9. Hilbert scheme
10. conics on the quintic.

## The moduli space

All sheaves I'm going to be interested in on $Y$ are torsion-free. For a sheaf $E$, $p(n) = \chi(Y, E(n))$ is the Hilbert polynomial of $E$ (it is a polynomial of degree 3).

**Definition 1.2.** $E$ is *stable* if for every proper subsheaf $0 \subsetneq E' \subsetneq E$, $p_{E'}(n)/\operatorname{rk} E' < p_E(n)/\operatorname{rk} E$ for $n \gg 0$. ◇

We assume $p(n)$ is chosen so that there exist no semi-stable sheaves which are not stable of the Hilbert polynomial $p(n)$. For example, we can assume rank is 1. If you want to learn more about stability of sheaves, look at Huybrechts-Lehn.

Determinant: If you have a torsion-free sheaf $E$, you can resolve it by locally free sheaves

$$\cdots \to E^{-1} \to E^0 \to E \to 0$$

and you can make it finite by the syzygy theorem (the third guy is automatically locally free). Then $\det E = \wedge^{\operatorname{rk} E^0} E^0 \otimes (\wedge^{\operatorname{rk} E^{-1}} E^{-1})^{-1} \otimes \cdots$. Trivial determinant means $\det E \cong \mathcal{O}_Y$.

**Remark 1.3.** If $E$ is rank 1, then $\det E = E^{\vee\vee}$. This is an exercise. If you know a proof or know one, let me know. Somebody: there is a proof in the book by Okenek.

There is always a canonical map $E \to E^{\vee\vee} = \mathcal{O}_Y$ ideal sheaf. ◇

**Theorem 1.4** (See a paper by Simpson or Huybrechts-Lehn)**.** *There exists a fine moduli scheme of ideal sheaves (I use "ideal sheaf" to mean torsion-free rank 1 sheaves with trivialized determinant). For higher rank, it is a Deligne-Mumford stack (always assuming the degree and rank are coprime so that I don't have to worry about the strictly semi-stable thing).*

**Remark 1.5** (the construction)**.** For a given $p(n)$, choose $q \gg p \gg 0$. $V = V_{[p,q]}$ is a graded vector space with $\dim V_n = p(n)$ for all $p \leq n \leq q$. Let $A = \bigoplus_{n\geq 0} \Gamma(Y, \mathcal{O}(n))$ and $A' = \bigoplus_{n>0} \Gamma(Y, \mathcal{O}(n))$, $G = GL(V)^{gr} = \prod_{n=p}^{q} GL(V_n)$. Take $L^i = \operatorname{Hom}_{\mathbb{C}}(A'^{\otimes i}, \operatorname{End}_{\mathbb{C}} V)^{gr}$ for $i \geq 0$, and $L = \bigoplus_{i\geq 0} L^i$ (this is finite dimensional). Make $L$ into a differential graded Lie algebra by defining a differential. If $\mu \in L^r$ and $\nu \in L^s$ are elements, then

$$\begin{aligned} d\mu(a_1, \ldots, a_{r+1}) &:= \sum_{i=1}^{n-1} (-1)^i \mu(a_1, \ldots, a_i a_{i+1}, \ldots, a_{r+1}) \\ [\mu, \nu](a_1, \ldots, a_{r+s} &:= \mu(a_1, \ldots, a_r) \circ \nu(a_{r+1}, \ldots, a_{r+s}) \\ &\quad - (-1)^{rs} \nu(a_1, \ldots, a_s) \circ \mu(a_{s+1}, \ldots, a_{r+s}) \end{aligned}$$

$G$ acts on $L$ by conjugation. Then $\mathfrak{g} = L^0$. The derivative of $G \to GL(L^n)$ is $\mathfrak{g} = L^0 \to \mathfrak{gl}(L^n)$, given by $x \mapsto [x, -]$.

Now define $F \colon L^1 \to L^2$ by $\mu \mapsto d\mu + \frac{1}{2}[\mu, \mu] = d\mu + \mu \circ \mu$. This is a quadratic function. The zero scheme of $F$ (the subscheme of $L^1$ cut out by $F$) is $Z(F) = \{\mu \in L^1 | d\mu + \frac{1}{2}[\mu, \mu] = 0\} = MC(L) \xhookrightarrow{\text{closed}} L^1$. The equation $d\mu + \frac{1}{2}[\mu, \mu] = 0$ is called the Mourer-Cartan equation.

**Exercise.** *Prove that $\mu \colon A \otimes V \to V$ satisfies the MC equation if and only if it is an action (if and only if it makes $V$ into a graded $A$-module).*

Thus, $\mathcal{X} = [Z(F)/G]$ is the quotient stack, the stack of graded $A$-modules such that the underlying $\mathbb{C}$ vector space is isomorphic to $V$, modulo isomorphisms as graded $A$-modules.

Stability: You want to count the stable points under the $G$ action. You modify slightly. Consider the torus $T = (\mathbb{C}^\times)^{q \cdot p}$ acting by rescaling on each $V_n$ for $p \leq n \leq q$. Then $\mathbb{P}(L^1) = L^1/T$. The group $\tilde{G}$ is $\prod_{n=p}^{q} PGL(V_n)$ and $[Z(F)/G] \supset \big[\big(Z(F) \subset \mathbb{P}(L^1)\big)/\tilde{G}\big]$ is an open substack. GIT stability of $\tilde{G}$ on $\mathbb{P}(L^1)$ gives you a notion of stable points $Z(F)^{\text{Stab}}$. Then $X = [Z(F)^{\text{Stab}}/G]$ is a projective scheme (if there are semi-stable points, it is quasi-projective). ◇

**Exercise.** *Note that we have {stable sheavs with Hilbert polynomial $p(n)$}$/\cong \xrightarrow{\Gamma_*}$ {$A$-modules $V$ with HP $p(n)$} $\xrightarrow{truncate} X$. Check that Giesecker stability corresponds to GIT stability.*

## Deformation Theory

If $\mu \in Z(F)$, then $T_{Z(F)}(\mu) = \{\nu \in L^1, d(\mu + \varepsilon\nu) + \frac{1}{2}[\mu + \varepsilon\nu, \mu + \varepsilon\nu] = 0\}$.

**Exercise.** $T_{Z(F)}(\mu) = \{\nu \in L^1 | d\nu + [\mu, \nu] = 0\} = \{\mu \in L^1 | d^\mu \nu = 0\}$ *where* $d^\mu = d + [\mu, -]$.

**Exercise.** *Because $\mu$ satisfies the MC equation,* $(d^\mu)^2 = 0$.

Over $Z(F)$, we have the trivial graded vector bundle $L$, with differential $d^\mu$. Let's denote this by $\mathcal{E}$, a "perfect complex" on $Z(F)$. This $\mathcal{E}$ descends to $X$. On the quotient $X$, $T_X(\mu) = \ker(d^\mu \colon L^1 \to L^2)/\operatorname{im}(d^\mu \colon L^0 \to L^1) = H^1(L, d^\mu)$.

**Definition 1.6.** The *higher tangent* spaces are $T_X^i(\mu) = H^{i+1}(\mathcal{E}|\mu) = H^{i+1}(L, d^\mu)$. ◇

If $\mu \in Z(F)$ (this means that $\mu$ makes $V$ into a graded $A$-module), it makes $\operatorname{End}_{\mathbb{C}} V$ into an $A$-bimodule by $(a, \mu b)(x) = a\mu(bx)$. Then $(L, d^\mu)$ is a well-known thing; it is the Hochschild complex of $A'$ with values in $\operatorname{End} V$.

Fact: If $E$ and $F$ are stable sheaves, then $\operatorname{Ext}^i_{\mathcal{O}_Y}(E, F) = \operatorname{Ext}^i_A(\Gamma_* E_{\geq p}, \Gamma_* F_{\geq p})^{gr}$. This is basically Serre's theorem.

Another fact: $\operatorname{Ext}^i_A(\Gamma_* E_{\geq p}, \Gamma_* F_{\geq p})^{gr} = \operatorname{Ext}^i_A(\Gamma_* E_{[p,q]}, \Gamma_* F_{[p,q]})^{gr}$.

Another fact: The Hochschild cohomology $H^i\big(HC^\bullet(A, \operatorname{End}_{\mathbb{C}} V)^{gr}\big) = \operatorname{Ext}^i_A(V, V)^{gr}$.

Putting these facts together, we get

**Corollary 1.7.** $T_X^i(\mu) = \operatorname{Ext}^{i+1}_{\mathcal{O}_Y}(E, E)$ *if $E$ is the stable sheaf corresponding to $\mu$.*

**Remark 1.8.** $l_{\geq 1}[1]$ with $G$-action is a dg scheme, and $\mathcal{E}$ is the tangent complex of the dg scheme. It is not the tangent complex of $X$. $\mathcal{E}^\vee$ has a canonical map to the cotangent complex $L_X$ by obstruction theory. I will not talk much about this dg scheme structure. ◇

There was a question. I think $F = df$ is not true.

## 2 Kai Behrend

Remember last time I constructed the moduli space $X$ of stable sheaves on a Calabi-Yau 3-fold. It was $MC(L)^{st}/G$. I want to give some justfications for going through that construction:

1. It is a direct construction, avoiding the Quot scheme.
2. It gives $X$ as a differential graded scheme. Let $W = L_{\geq 1}[1]$, and let $\mathcal{A} = \operatorname{Sym} W^*$ (this is the *graded* symmetric algebra). Define a derivation $Q\colon \mathcal{A} \to \mathcal{A}$ by defining it on generators: $Q\colon W^* \to \operatorname{Sym} W^*$. Make it by summing two parts, $Q_1\colon W^* \to W^*$, the dual of $d\colon L \to L$, and $Q_2\colon W^* \to \operatorname{Sym}^2 W^*$, the dual of the Lie bracket.

   **Exercise.** *$d^2 = 0$, $d$ is a derivation on the Lie bracket, and Jacobi identity are equivalent to the single condition that $Q^2 = 0$.*

   $(\mathcal{A}, Q)$ is a differential graded algebra, and $G$ acts on this. $\mathcal{A}^0 = \operatorname{Sym} L^{1*}$ and $\mathcal{A}$ is an $\mathcal{A}^0$-module, with $Q$ $\mathcal{A}^0$-linear. Then $\mathcal{A}$ is a sheaf of differential graded algebras on $L^1$. $G$ acts on it, so it descends to a sheaf on $M = L^{1\,\mathrm{Stab}}/G$. You have $(\cdots \xrightarrow{Q} \mathcal{A}^{-1} \xrightarrow{Q} \mathcal{A}^0)$, and you get that $h^0(\mathcal{A}, Q) = \mathcal{A}^0/Q\mathcal{A}^{-1}$. You can check that $\operatorname{Spec}(\mathcal{A}^0/Q\mathcal{A}^{-1}) = MC^{\mathrm{Stab}}(L) = Z(F)$. Then $X = \operatorname{Spec} h^0(\mathcal{A})/G$.

   We will need this differential graded stuff for "categorification", which has not yet been worked out.

   It is not easy to write down a category of dg schemes, for which you need some heavy duty homotopy theory. The underlying classical scheme has a universal property and it has a tangent complex that gives the right deformation theory.
3. Indoctrination. These days, differential graded Lie algebras show up a lot in deformation theory in characteristic zero (Manetti). I'm convinced that eventually, differential graded Lie algebras will take over moduli theory. If you find a finite dimensional dg algebra, you get a global moduli space. You really should always construct moduli spaces as $MC(L)/G$ for some dg Lie algebra $L$. One very popular differential graded Lie algebra is the Dolbeout Lie algebra. Take $\Omega^{0,*}(Y, \operatorname{End}_{\mathcal{O}_Y}(E))$, so $L^1$ are $\bar\partial$ operators, $MC$ is integrability, and $G$ is the guage group. To construct stuff this way, you have to do infinite dimensional stuff which is not very algebraic.
4. Replace $V$ by a graded vector space (different from the grading it already has) $V = V^{-1} \oplus V^0$. $L^n = \bigoplus_{i+j=n} \operatorname{Hom}(A^{\otimes i}, \operatorname{End}^j_{\mathbb{C}}(V))$. You get a doubly graded dg Lie algebra.

   Maybe you get a moduli of complexes over derived category objects. I have no idea if this works, but if you're looking for a research problem, I think this might be promising. If you do, let me know about it so that we avoid duplication of research.

### 3. The virtual fundamental class

The basic setup that is called the toy model is as follows. $M$ is smooth variety over $\mathbb{C}$. $E/M$ is a vector bundle, $s \in \Gamma(M, E)$, and $X = Z(s)$ (we hope that this is our moduli space). We can draw this as a cartesian square

$$\begin{array}{ccc} X & \longrightarrow & M \\ \big\downarrow & \ulcorner & \big\downarrow s \\ M & \xrightarrow{0} & E \end{array}$$

In this case, $[X]^{vir}$ is what Fulton calls the localized top chern class of $E$, $0^!_{E|X}[C_{X/M}] \in A_{\dim M - \operatorname{rk} E}(X)$, defined as follows. If $\mathcal{I}$ is the ideal sheaf of $X$ in $M$, $c_{X/M} = \operatorname{Spec} \bigoplus I^n/I^{n+1}$. $C_{X/M} \hookrightarrow E|_X \hookrightarrow E$ is a scheme of cones over $X$. Something $\mathbb{C}^\times$-action flow to $\infty$. You can multiply a section by an element of $\mathbb{C}^\times$ and you can let it go to $\infty$ to get a cone on $X$ [[★★★ picture getting vertical lines on the zeros of the section $s$]]. All of this is explained in Fulton's book on intersection theory. $C_{X/M}$ is pure dimension of the same dimension as $M$.

$E^\vee \xrightarrow{s^\vee} \mathcal{O}_M$, and by definition, the image is the ideal sheaf $\mathcal{I}$ of $X$ in $M$. Restricting to $X$, we have the diagram of sheaves on $X$

$$\begin{array}{ccccc} E^\vee|_X & \longrightarrow & \Omega_M|_X & = & \mathcal{F} \in \mathsf{D}(\mathcal{O}_X) \\ {\scriptstyle s^\vee}\big\downarrow & & \big\| & & \\ I/I^2 & \xrightarrow{\text{“}d\text{”}} & \Omega_M|_X & = & \tau_{\geq 1} L_X \in \mathsf{D}(\mathcal{O}_X) \end{array}$$

$\Omega_M|_X$ in the derived category is perfect of amplitude in $[-1,0]$. The bottom row in the derived category is completely intrinsic to $X$, $\tau_{\geq 1}L_X$. Let $\mathcal{F}$ be the top row., $\phi\colon \mathcal{F} \to \tau_{\geq 1}L_X$ in $\mathsf{D}(\mathcal{O}_X)$ has properties: (1) $h^0(\phi)$ is an isomorphism (2) $h^{-1}(\phi)$ is an epimorphism. This is called a "perfect obstruction theory" on $X$.

**Theorem 2.1.** *(1) $[X]^{vir}$ depends only on the perfect obstruction theory $\mathcal{F} \to \tau_{\geq 1}L_X$. (2) $\mathcal{F} \to \tau_{\geq 1}L_X$ defines $[X]^{vir}$.*

Now I'll try to explain how this fits into our example. Remember we have $MC^{st}(L) \hookrightarrow (L^1)^{st}$ and $G$ acts. I'm going to define $G' = G/\mathbb{C}^\times$. The $\mathbb{C}^\times$ acts trvially, so the quotient will be a stack even if I pass to the stable locus, so I get rid of it to get a scheme by taking the quotient by $G'$. So we have

$$\begin{array}{ccc} MC^{st}(L)/G' & \hookrightarrow & (L^1)^{st}/G' \\ \parallel & & \parallel \\ X & \hookrightarrow & M \end{array}$$

Graded trivial vector bundle on $L^1$: $L^0 \to L^1 \xrightarrow{d^\mu} L^2 \to L^3$, where $d^\mu = d + [\mu, -]$ for $\mu \in L^1$. It is easy to check that $(d^\mu)^2 = 0$ if and only if $\mu \in MC$. This descends to $M$ to give a vector bundle on $M$ and $d^\mu$. At $\mu \in X$, we get a complex, the Hochschild chain complex $HC^\bullet(A, \operatorname{End}^\mu_{\mathbb{C}}(V))$, which computes $\operatorname{Ext}^i_{\mathcal{O}_Y}(E_\mu, E_\mu)$.

Now use Serre duality and CY3 condition (for the first time) to get a perfect pairing

$$\operatorname{Ext}^i_Y(E,E) \otimes \operatorname{Ext}^{3-i}(E,E) \to \mathbb{C}.$$

You also use stability to conclude that $\operatorname{Ext}^0_Y(E,E) = \mathbb{C}$. This implies that $\operatorname{Ext}^3_Y(E,E) = \mathbb{C}$. It also says that $\operatorname{Ext}^2_Y(E,E) = \operatorname{Ext}^1_Y(E,E)^\vee$ and everything else vanishes.

Now assume also that $H^1(Y,\mathcal{O}) = H^2(Y,\mathcal{O}) = 0$. Many people put this in the definition of a Calabi-Yau manifold. Under this assumption, $H^1(Y,\mathcal{O}) = 0$ means that there are no deformations of line bundles, so if you have the correct Hilbert polynomial, your determinant is automatically trivial. Of course, there is still a choice of isomorphism, given by $\mathbb{C}^\times$, which we've already made up for. So in this case, our $X$ is the moduli of sheaves with trivial determinant. Basically, by changing the group slightly, we killed $\operatorname{Ext}^0$. To preserve symmetry, we will kill $\operatorname{Ext}^3$. Replace $(L^\bullet, d^\mu)$ on $X$ by $\tau_{[1,2]}(L^\bullet, d^\mu)$. Before truncation, $(L^\bullet, d^\mu) = R\operatorname{Hom}(E,E)$. Once I truncate, I get $\tau_{[1,2]}(L^\bullet, d^\mu) = R\operatorname{Hom}(E,E)_0$, the so-called "traceless ext". We've replaced $(L, d^\mu)$ by $\big(L^1/dL^0 \to \ker(L^2 \to L^3)\big)$.

**Lemma 2.2.** *$W^0 = L^1/dL^0$ and $W^1 = \ker(L^2 \to L^3)$ are vector bundles on $X$.*

So my perfect object is $\mathcal{F} = (W^0 \xrightarrow{d^\mu} W^1)$. It is actually not difficult to see that $W^0 \cong T_M|_X$.

$$\begin{array}{ccc} W^0 & \xrightarrow{d^\mu} & W^1 \\ \wr\uparrow & & \uparrow \\ T_M|_X & \longrightarrow & \mathcal{N}_{X/M} \end{array} \qquad \begin{array}{ccc} W^{1*} & \longrightarrow & W^{0*} \\ \downarrow & & \downarrow = \\ I/I^2 & \longrightarrow & \Omega_{A/X} \end{array}$$

The map $\mathcal{N}_{X/M} \to W^1$ is induced by $\mathcal{F}$. $f$ section of $L^2$ over $L^1$ is a section of $W^1$. The diagram is dual to the one on the right, which is the map $\mathcal{F}$ down to $\tau_{\geq 1}L_X$. $C_{X/M} \hookrightarrow \mathcal{N}_{X/M} \hookrightarrow W^1$. Check that $[X]^{vir} = 0^!_{W^1}[C_{X/M}] \in A_0(X)$. So $\#^{vir}X$ is the proper pushforward to $*$.

# 3 Kai Behrend

## Microlocal geometry

If you want to learn more about microlocal geometry, there are some notes by MacPherson from Park City. There is also a book that contains all the results and all the details: Kashiwara-Shapira, Sheaves over manifolds.

Everything will be over $\mathbb{C}$. Let $X$ be a singular scheme, embedded in a smooth scheme $M$. The content of microlocal geometry is that you can study $X$ by means of the symplectic geometry of the cotangent bundle $\Omega_M$ of $M$.

**Remark 3.1.** If $V \subset M$ is a closed subvariety and $V^0$ is the smooth locus of $V$. Then define $\ell(V) = \overline{\mathcal{N}^\vee}_{V^0/M} \subseteq \Omega_M$ to be the closure conormal bundle in the cotangent bundle of $M$. This is a Lagrangian cone. ⋄

A conic Lagrangian subvariety of $\Omega_M$ is a closed subvariety of dimension of $\dim M$ such that the restriction of the symplectic form $\sigma = d\alpha = \sum dp_i \wedge dx_i$ (where $\alpha = \sum p_i dx_i$) vanishes and invariant with respect to the $\mathbb{C}^\times$ action on $\Omega_M$ (given by scaling the fibers). The property of being Lagrangian is a generic property, so we can just require that $\alpha$ vanishes at the generic point. We have coordinates $(x_i, p_i)$. We may as well assume $V^0 = \{x_i = 0 | i \le k\}$. Then $\mathcal{N}^\vee_{V^0/M} = \{x_i = 0, p_j = 0 | i \le k < j\}$.

**Exercise.** *All conic Lagrangian prime cycles (i.e. irreducible) on $\Omega_M$ are obtained in this way. That is, every such cycle is the closure of the conormal bundle of a variety on the base.*

So $L\colon Z_*(M) \to \mathcal{L}(\Omega_M)$, where $Z_*(M)$ are the algebraic cycles on $M$ and $\mathcal{L}(\Omega_M)$ are the conic Lagrangian cycles, given by $V \mapsto (-1)^{\dim V}\ell(V)$, is an isomorphism of groups. We can restrict this to an isomorphism $L\colon Z_*(X) \xrightarrow{\sim} \mathcal{L}_X(\Omega_M)$ (everything supported on $X$).

**Example 3.2** ("distinguished cycle"). $c_X = \sum_{c'} (-1)^{\dim \pi(c')} mult(c')\pi(c')$, where the sum is over all $c'$ irreducible components of $c_{X/M}$ and $\pi\colon c_{X/M} \to X$. It turns out that this cycle is independent of the choice of embedding of $X$ in $M$. $L(c_X)$ could be called the "distinguished Lagrangian cycle of $X$ in $\Omega_M$". Of course, this does depend on the embedding of $X$ in $M$. ⋄

Q: are there any assumptions on $X$; can it be reducible of various dimensions? KB: $X$ can be any scheme which is embeddable into something of finite type; it can be any closed subscheme of $M$.

We get the commutative diagram

$$\begin{array}{ccc} Z_*(X) & \xrightarrow{L} & \mathcal{L}_X(\Omega_M) \\ & \searrow^{c_0^M} & \downarrow 0^!_{\Omega_M|X} \\ & & A_0(X) \end{array}$$

where $c_0^M$ is the degree zero part of the Chern-Mather class.

The isomorphism $L$ factors as $Z_*(X) \xrightarrow{Eu} \mathrm{Con}(X) \xrightarrow{Char} \mathcal{L}_X(\Omega_M)$, where $\mathrm{Con}(X)$ is the constructible functions $X \to \mathbb{Z}$. $Eu$ is MacPherson's local Euler obstruction (you can find this in Fulton's book on intersection theory), and $Char$ is the characteristic cycle. both of these are isomorphisms. The easiest thing to define is $Char^{-1}$, and this was first done by Ginzburg. $Char^{-1}\colon \mathcal{L}_X(\Omega_M) \to \mathrm{Con}(X)$ is given by $[c] \mapsto \big(P \mapsto I_P([c],[\Delta])\big)$, where $\Delta$ is the graph $d\rho$ of the square of a Euclidean distance function $\rho\colon M \to \mathbb{R}$ from $P$ (basically, $\rho(x) = \sum x_i^2$, $d\rho = \sum 2x_i dx_i$). Ginzburg proves that $P$ is an isolated point of the intersection, so the intersection multiplicity $I_P([c],[\Delta])$ makes sense. [[★★★ picture: symplectic manifold $\Omega_M$ with zero section $M$. Lagrangian cone looks like a collection of vertical lines.]] At every point in $M$, you compute the intersection number of the graph with the cone.

Really, we're intersecting the cone $[c]$ (vertical lines) with $[0]$. The dimensions are complementary, so I should be getting a cycle of degree zero. Every point of $M$ has a well-defined contribution, which is that intersection number. This kind of vague statement is justified by this theorem.

**Theorem 3.3** (MacPherson/Kashiwara, 1970s)**.** *If $X$ is compact, then for every cycle $c \in Z_*(X)$, $\chi(X, Eu(c))$ (this means chop $X$ into pieces where the function is constant; compute their Euler characteristics, and add them up with weight given by the function) is equal to $\int_X c_0^M$. Kashiwara's formulation is $\chi(X, Char^{-1}(c)) = \int_X 0^! c$.*

I think of this theorem as a generalization of the Gauß-Bonnet theorem to singular schemes. If $X$ is smooth, $(-1)^{\dim X}\chi(X) = \int_X e(\Omega_X)$.

**Example 3.4** (distinguished cycle)**.** Let $\nu_X := Eu(c_X)$ be the "distinguished constructible function on $X$". This is intrinsic to $X$. It is the microlocal function in the story. If you read some recent paper, some joker started calling it $\chi^B$, but I won't use that notation.

In this case, the theorem says $\boxed{\int_X 0^! L(c_X) = \chi(X, \nu_X).}$ ◇

## Symmetric obstruction theories

An obstruction theory for the scheme $X$ is an arrow in $\mathsf{D}(\mathcal{O}_X)$, $\mathcal{F} \to \tau_{\geq -1} L_X$, with certain properties which I won't repeat. The obstruction theory is *symmetric* if it is endowed with $\beta \colon \mathcal{F} \otimes \mathcal{F} \to \mathcal{O}[1]$ (this is of course derived tensor product and this is a morphism in $\mathsf{D}(\mathcal{O}_X)$) such that the induced map $\alpha \colon \mathcal{F} \to \mathcal{F}^\vee[1]$ is (1) an isomorphism in $\mathsf{D}(\mathcal{O}_X)$, and (2) symmetric in the sense that $\alpha^\vee[1] = \alpha$ in $\mathsf{D}(\mathcal{O}_X)$. Of course, there are signs involved, but I thought about it for months and determined that that was the right sign ... but that was years ago.

In our case, we're interested in $X$, the moduli space of sheaves on the Calabi-Yau $Y$. We have $\pi \colon X \times Y \to X$, with $\mathcal{F}^\vee = R\pi_* R\mathcal{H}om(E, E)_0$, and $\beta$ comes from Serre duality.

**Remark 3.5.** You can write $\mathcal{F}$ much more explicitly in my construction of the moduli space: $\mathcal{F} = [W^0 \to W^1]$ was the trunction of the obstruction complex I constructed on $X$. I explained that $X$ came with an embedding into some $M$, and $W^0 = T_M|_X$. What would have been nice, but I cannot prove, is for $W^1 = \Omega_M|_X$. It is a vector bundle of the right dimension, but it is not straightforward to prove. So I really wanted that $\mathcal{F} = [T_M|_X \to \Omega_M|_X]$, and the map is self-dual. If you manage to prove this, let me know. In the absence of that, I have to throw all this homological algebra at you. ◇

On $X$ itself (not on $M$), I do have the exact sequence

$$0 \to T_X \to W^0 \to W^1 \to ob \to 0.$$

The kernel is the tangent bundle and the cokernel is defined to be the obstruction sheaf. If you take the dual of this obstruction theory, you're supposed to get the same thing back. There is an isomorphism (in the derived category) to $[W^{1*} \to W^{0*} \to ob \to 0]$. So $T_X = ob^\vee$.

Last time I did construct $c_{X/M} \hookrightarrow W^1$, and $[X]^{vir} = 0^![c_{X/M}]$.

$$\begin{array}{ccccc} W^1 & \longrightarrow & ob & \twoheadleftarrow & \Omega_M|_X \\ \uparrow & & \uparrow & & \uparrow \\ c_{X/M} & \longrightarrow & cv & \longleftarrow & C \end{array}$$

Where $cv$ is the subsheaf of cones (big sheaves on some big site; you can also do it with stacks, but you don't have to), it stands for "curvilinear obstructions". The big thing is that $\boxed{ob = \Omega_X \text{ and } T_X = ob^\vee.}$ Forming the pullback $C$ gives a scheme of cones in $\Omega_M|_X$. Q: what is the property of $X$ that allows you to understand the dual of $T_X$? KB: it is the other way around $T_X = ob^\vee$, but $ob \neq T_X^\vee$.

**Proposition 3.6.** $[X]^{vir} = 0^!_{\Omega_M|_X}[C]$.

Now I'm in the situation where I can apply microlocal geometry.

## Toy models for symmetric obstruction theories

$\omega$ is a 1-form on $M$ and $X = Z(\omega) \subset M$. Then (repeating what I said last time) how do you get the obstruction theory?

$$\mathcal{F} \;=\; \begin{array}{ccc} [T_M|_X & \longrightarrow & \Omega_M|_X] \\ \downarrow & & \| \\ [\mathcal{I}/\mathcal{I}^2 & \longrightarrow & \Omega_M|_X] \end{array} \qquad \begin{array}{ccc} [T_M|_X & \longrightarrow & \Omega_M|_X] \\ \| & & \downarrow \\ [T_M|_X & \longrightarrow & \mathcal{N}_{X/M}] \end{array}$$

(the diagram on the right is the dual) If $\omega$ is closed, then the obstruction theory is symmetric. $\Gamma_\omega \subset \Omega_M$ is Lagrangian, and $\omega \colon M \hookrightarrow \Omega_M$ with $\omega^*(\sum dp_i \wedge dx_i) = \sum df_i \wedge dx_i = d\omega = 0$. Here $\Gamma_\omega$ is the graph of $\omega$. In the limit (rescaling), $c_{X/M} \hookrightarrow \Omega_M$ is the Lagrangian cone.

**Theorem 3.7.** *You need only* $\frac{\partial f_i}{\partial x_j} \equiv \frac{\partial f_j}{\partial x_i}(f_1, \ldots, f_n)$ *to get the result that* $c_{X/M} \hookrightarrow \Omega_M$ *is Lagrangian and that the obstruction theory is symmetric. In this case we say that* $\omega$ *is* almost closed.

**Proposition 3.8.** *If* $X$ *has a symmetric obstruction theory and* $X \hookrightarrow M$ *embedded, then étale locally in* $M$ *there exists a closed 1-form cutting out* $X$ *and giving rise to the given symmetric obstruction theory.*

Q: what if $X$ is smooth of the wrong dimension? KB: If $X$ is smooth and $M = X$, with $\omega = 0$. Then the obstruction theory is

$$\begin{array}{ccc} [T_M|_X & \xrightarrow{0} & \Omega_M|_X] \\ \downarrow & & \| \\ [0 & \longrightarrow & \Omega_X] \end{array}$$

You check locally that $C \hookrightarrow \Omega_M$ is Lagrangian. Because $C$ is locally isomorphic to the normal cone, we get the corollary.

**Corollary 3.9.** *The underlying cycle $[C]$ is the distinguished Lagrangian cycle, so $\#^{vir} X = \int_X 0^! L[c_X] = \chi(X, \nu_X)$.*

# 4 Kai Behrend

Let me start today by going over the argument from the end of last lecture.

(1) Recall that $X \hookrightarrow M$ is an arbitrary scheme embedded into a smooth scheme. There is no canonicalway that the normal cone $C_{X/M}$ is embedded into the cotangent bundle $\Omega_M$. If $X$ were smooth, the conormal bundle would be embedded in $\Omega_M$, not the normal bundle. But $C_{X/M}$ defines $c_X \in Z_*X$. For every cycle, I pass to it's smooth locus and then I get an embedding into $\Omega_M$. And hence I get $L(c_X) \in \mathcal{L}_X(\Omega_M)$. The mirolocal index theorem (reviewd last time) says that if $X$ is compact, then $\int_X 0^! L(c_X) = \chi(X, \nu_X)$, where $\nu_X = Char^{-1}(L(c_X)) = Eu(c_X)$. In the case where $X$ is smooth, we get $\int_X 0^![N^\vee_{X/M}]$, but because I have a short exact sequence of vector bundles on $X$

$$0 \to N^\vee_{X/M} \to \Omega_M|_X \to \Omega_X \to 0$$

this is $\int_X e(\Omega_X) = \chi(X, Eu[X]) = \chi(X, (-1)^{\dim X}) = (-1)^{\dim X}\chi(X)$. This is the Gauss-Bonnet theorem.

(2) If $\omega \in \Gamma(M, \Omega_M)$ such that $X = Z(\omega)$, then we get $\omega^\vee : T_M \to I$, where $I$ is the ideal sheaf of $X$. I can restrict this to $X$, $T_M|_X \to I/I^2$, and take duals to get

$$C_{X/M} \hookrightarrow \mathcal{N}_{X/M} \hookrightarrow \Omega_M|_X \hookrightarrow \Omega_M$$

so I get the cone $C_{X/M}$ as a closed subscheme of $\Omega_M$. If $\omega$ is almost closed, then $\omega\colon C_{X/M} \to \Omega_M$ is Lagrangian. Then it follows that $\omega_*[C_{X/M}] = L(c_X)$.

(3) If $X$ has an obstruction theory $\mathcal{F} \to \tau_{\geq -1}L_X$, then taking duals we get $(\tau_{\geq -1}L_X)^\vee \to \mathcal{F}^\vee$, which induces a subsheaf of cones $cv \hookrightarrow ob = h^1(\mathcal{F}^\vee)$ as we saw last time. If $F \twoheadrightarrow ob$ is an epimorphism from a vector bundle $F$, then the pullback $C = cv \times_{ob} F$ is a cone scheme $C \hookrightarrow F$ such that $[X]^{vir} = 0^![C]$. If the obstruction theory is symmetric, then $ob = \Omega_X$ canonically and the embedding $X \hookrightarrow M$ defines $\Omega_M|_X \twoheadrightarrow \Omega_X$. So we get $C \hookrightarrow \Omega_M|_X \hookrightarrow \Omega_M$, and $[X]^{vir} = 0^![C]$. Then $[C] = L(c_X)$. This can be checked locally, where you can put yourself in the situation explained in (2) where you have an almost closed 1-form, so locally $(C \hookrightarrow \Omega_M) = (\omega\colon C_{X/M} \hookrightarrow \Omega_M)$. Q: when did you prove that last equality? KB: I didn't go into details about why that is true. If the obstruction theory is given by an almost closed 1-form, then I can take [[★★★ something something]]. Then $[X]^{vir} = 0^! L(c_X)$, so $\#^{vir}X = \int_X 0^! L(c_X) = \chi(X, \nu_X)$.

**Remark 4.1** (A few remarks on derived geometry)**.** We have $W = L_{\geq 1}[1]$, where $L$ is a dg Lie algebra. We have the guage group $G$ acting (this is the Lie group with Lie algebra given by $L_0$, which we threw away in $L_{\geq 1}$). $W$ is a graded linear manifold. $\mathcal{A} = \operatorname{Sym} W^*$, the graded commutative algebra of functions on $W$, had this derivation of degree 1 $Q\colon \mathcal{A}^i \to \mathcal{A}^{i+1}$. This is a vector field of degree 1 on $W$. You should think of the moduli space $X$ as the zero locus $Z(Q)$. ◇

**Remark 4.2** (Speculation)**.** Using CY3, we get Serre duality, which gives rise to an inner product (analogue of the Killing form) $\kappa\colon L \otimes L \to \mathbb{C}[-3]$. I have $\operatorname{Ext}^1(E,E) \otimes \operatorname{Ext}^2(E,E) \to \mathbb{C}$ because we are on a Calabi-Yau 3-fold. This $\kappa$ is a product of degree $-3$, by we pass to $W$ and then we have $\kappa\colon L[1] \otimes L[1] \to \mathbb{C}[-1]$, so $\kappa$ is really of degree $-1$ on $W$: we have $\kappa\colon W \otimes W \to \mathbb{C}[-1]$, or $\kappa\colon W \to W^*[-1]$. Think of $\kappa$ as a differential 1-form on $W$. Then $\sigma := d\kappa$ is a symplectic form of degree $-1$ on $W$. "Cyclicity"[1] translates into the fact that $Q$ is a Hamiltonian vector field for $\sigma$. I get a function $f = \langle Q, \kappa\rangle$ of degree 0 (since $\deg Q = 1$ and $\deg \kappa = -1$), so it is in $\mathcal{A}^0 = \operatorname{Sym} L^{1*} = \mathcal{O}_{L^1}$. Contracting, I get $Q \lrcorner \sigma = df$, where $f$ is the Hamiltonian. To make this precise, you have to allow that $\kappa$ is non-constant of $Q$ has higher degree terms.

The main point I want to make is that the extra geometry that comes from CY 3-fold is that your moduli space is a symplectic manifold with symplectic form of degree -1. $X = Z(Q)$, so it looks like $X = Z(df)$, but this is only true if $\sigma$ is really non-degenerate. $X = Z(Q) \subsetneq Z(df)$ because $\sigma$ is only non-degenerate on cohomology, not on $W$. Anyway, eventually I think some good will come out of all this. The (symmetric) obstruction theory $(L, d + [\mu, -], \beta)$ on $X$ is just the shadow of the degree $-1$ symplectic structure on the classical scheme $X$. ◇

By the way, $M = \operatorname{Hom}(A \otimes V, V)^{\mathrm{Stab}}/G$ is space of the tensor algebra $TA$-module structure ($TA$ is non-commutative) on $V$. So you can think of $M$ as a moduli space of sheaves on a non-commutative scheme $Y$. I think this is an important principle.

Take infinite dimensional model such as $\Omega^{0*}(Y, \operatorname{End}_{\mathcal{O}} E)_0 = L$ (Delbeout something). Here cyclicity holds $\kappa([x,y],z) = \kappa(x,[y,z])$. This more or less

[1] The two main properties of cyclicity are (1) $\kappa(dx, y) \pm \kappa(x, dy) = 0$ and (2) $\kappa([x,y],z) = \kappa(x,[y,z])$.

directly follows from the usual linear algebra that the trace satisfies that equation.

There is the Transfer Theorem for cyclic $L_\infty$-algebras, which says that as complexes of vector spaces with inner product, $L \cong \mathrm{Ext}^1(E,E)_0 \oplus \mathrm{Ext}^2(E,E)_0 = W^0 \oplus W^1$. Of course, this is very non-canonical. So if I have a dg Lie algebra structure on $L$, it transfers to an $L_\infty$ structure on $W = W^0 \oplus W^1$, which is a derivation $Q$ (which can have terms of arbitrarly high order, not just linear and quadratic) on $\mathcal{A} = \mathrm{Sym}\, W^*$ such that $Q^2 = 0$. If you really think about what this means, you get infinitely many operations $\mu_n \colon \mathrm{Ext}^1_Y(E,E)_0^{\otimes n} \to \mathrm{Ext}^2_Y(E,E)$. If the $L_\infty$-algebra only has components in degree 1 and 2, then this only amounts to operations in degrees 1 and 2. Then you write down $f(X) = \sum_{n=-2}^{\infty} \frac{(-1)^{n(n-1)/2}}{(n+1)!}\kappa(\mu_n(x,\dots,x),x) \colon \mathrm{Ext}^1(E,E) \to \mathbb{C}$, which is a formal function such that $Z(df) \subset \mathrm{Ext}^1(E,E)$ is isomorphic to the completion of $X$ at $E$. One of the big questions is to understand the radius of convergence (it is bigger than 0?).

$X \subseteq M$ is locally $Z(\omega)$, where $\omega$ is almost closed. We'd like to have $\omega$ closed, or even exact, so $X = Z(df)$. I can't prove that, but this works in a formal neighborhood at every point.

---

In the end, I also want to explain how to do some computations. I'll start by explaining some properties of the microlocal function $\nu_X$. Some properties:

1. If $X$ is smooth of dimension $n$, then $\nu_X = (-1)^n$. This follows directly from the definition.

2. $\nu_{X\times Y} = \nu \boxdot \nu_Y$.

3. If $f \colon X \to Y$ smooth of relative dimension $n$, then $\nu_X = (-1)^n f^*\nu_Y$.

4. In particular, if $f \colon X \to Y$ is étale, then $\nu_X = f^*\nu_Y$.

5. If $X = Z(df)$ for $f \colon M \to \mathbb{C}$, and $P \in X$, then $\nu_X(P) = (-1)^{\dim M}(1 - \chi(F_P))$ = "Milnor number", where $F_P$ is the Milnor fibre of $f$ at $P$.More generally, $\nu_X$ is the fiberwise Euler characteristic $(\Phi_f)$, where $\Phi_f \in \mathrm{Perv}(X)$ is the perverse sheaf of vanishing cycles of $f$. A fact from microlocal geometry is that "the characteristic variety of $\Phi_f$" is equal to $C_{X/M} \overset{df}{\hookrightarrow} \Omega_M$ (due to Lê and Mebkhout)

$$\begin{array}{ccc} \mathrm{Perv}(X) & \xleftrightarrow[\text{correspondence}]{\text{Riemann-Hilbert}} & \text{reg. holonomic } \mathcal{D}\text{-mods} \\ \Big\downarrow{\scriptstyle \text{fiberwise } \chi} & & \Big\downarrow{\scriptstyle \text{Char}} \\ Con(X) & \longrightarrow & \mathcal{L}_X(\Omega_X) \end{array}$$

going down is decategorification and going up is categorification.

In the case $X = Z(df)$, $\chi(X, \nu_X) = \sum_i (-1)^i \dim_{\mathbb{C}} \mathbb{H}^i(X, \Phi_f)$. "categorified Donaldson-Thomas invariants".

6. $\nu_X(P)$ depends only on analytic neighborhood of $X$ at $P$.

7. Conjecture: $\nu_X(P)$ depends only on formal neighborhood of $X$ at $P$.

# 5 Kai Behrend

**Remark 5.1** (Heuristics: why vanishing cycles)**.** Suppose $f\colon X \to \mathbb{C}$ is a 1-parameter family of projective varieties. Suppose that the generic fibers are non-singular, and the singularities are all in the special fiber: $Z(df) = Z(f)$. $X_\eta$ the generic fiber is smooth. I explained that

$$\#^{vir} X_\eta = \int_X e(\Omega_X) = (-1)^{\dim X_\eta}\chi(X_\eta) = \sum (-1)^{\dim X_\eta - i}\dim H^i(X_\eta).$$

$\#^{vir}$ should be invariant. There is a spectral sequence $H^p(X_0, \Psi_f^q) \Rightarrow H^{p+q}(X_\eta, \mathbb{C})$. This is a Leray spectral sequence of the embedding $X_\eta \subset X$ [[★★★ should be special fiber $X_0 \subset X$?]]. $\mu_f'$: fiberwise Euler characteristic of $\Psi_f$: constructible function on $X_0$. $\chi(X_0, \mu_f) = \chi(X_\eta)$. This function is supported at singularities. Q: if $X$ projective, it can have a lot of cohomology, but the vanishing cycle around a singular point is a tiny little thing. A: it's all the cycles including the vanishing cycles. KB: this is copied from SGA7 Exp. I. I should probably call this "nearby cycles"; there is always this confusion between vanishing cycles and nearby cycles. $\mu_f' = 1 \pm \mu_f$. I keep saying that the moduli space should be the singular locus in the special fiber, but if I deform .... I don't remember what the relavence was supposed to be. ⋄

Now $X$ is back to what it was the whole time (a scheme with a symmetric obstruction theory). One more property of the microlocal function $\nu_X$. Suppose $\mathbb{C}^\times$ acts on $X$ with an isolated fixed point $P$. This corresponds to a sheaf $E$ on the Calabi-Yau $Y$. Suppose the $\mathbb{C}^\times$ action preserves the obstruction theory:

$$\begin{array}{ccccc} \mathcal{F} & \xrightarrow{\sim} & g^*\mathcal{F} & \longrightarrow & \mathcal{F} \\ & & \downarrow & & \downarrow \\ & & X & \xrightarrow{g} & X \end{array}$$

and the isomorphism should satisfy a cocycle condition. Then $\mathbb{C}^\times$ acts on the Zariski tangent space $T_X(P) = \mathcal{E}xt^1(E,E)_0$. Suppose all the weights of this action are non-zero (this is what I mean by "$P$ is an isolated fixed point").

**Theorem 5.2** (Behrend-Fantechi)**.** *$\nu_X(P) = (-1)^{\dim T_X(P)} = \frac{n_1\cdots n_d}{(-n_1)\cdots(-n_d)}$ if $n_1, \ldots, n_d$ re the weights of $\mathbb{C}^\times$ on $\mathcal{E}xt^1$ and $-n_i$ are the weights of $\mathbb{C}^\times$ on* $\mathrm{Ext}^\vee = \mathrm{Ext}^2$.

*"Proof"*. Let me do this in the case $X = Z(df)$, where $f\colon M \to \mathbb{C}$ is holomorphic and $M$ is smooth. Suppose $\mathbb{C}^\times$ acts and $f$ is homogeneous of degree 0. Say $M = \mathbb{C}^n$. Then $\mathbb{C}^\times$ acts on the induced obstruction theory in the required way. Then the nodal fiber $F_P = \{x \in M | f(x) = \delta, \|x\| < \varepsilon\}$. Then $S^1 \subset \mathbb{C}^\times$. acts on the Milnor Fiber $F_p$. Since all the weights are non-zero, you can prove that there are fixed points so $\chi(F_P) = 0$. So $\nu_X(P) = (-1)^{\dim M}(1-0)$. The proof in the general case is similar, but you have to change [[★★★ something]] and find a replacement for the Milnor fiber. □

Mac Mahon function: $M(q) = \prod_{n\geq 1}\frac{1}{(1-q^n)^n} = \sum_{n\geq 0} p(n)q^n$, where $p(n)$ is the number of 3-dimensional partitions of length $n$. $\mathrm{Hilb}^n Y$, where $Y$ is Calabi-Yau 3-fol (compact or not), then

1. ($Y$ compact) $\sum_{n\geq 0} \#^{vir}\,\mathrm{Hilb}^n Y\, q^n = M(-q)^{\int_Y e(T_Y \otimes \omega_Y)}$. (Li Levine-Pandhaupale)
2. ($Y$ arbitrary) $\sum_{n\geq 0}\chi(\mathrm{Hilb}^n Y, \nu)q^n = M(-q)^{\chi(Y)}$. (Behrend-Fantechi)

These formulas are identical if $Y$ is compact and Calabi-Yau.

Let $C \subset Y$ be a super rigid rational curve (i.e. $C \cong \mathbb{P}^1$ and $N_{C/Y} = \mathcal{O}(-1)\oplus\mathcal{O}(-1)$) with $Y$ a compact Calabi-Yau 3-fold. It is known that in degrees 1 and 2 these are the only kind of curves on the generic .... Let $X_n(C,Y)$ be the moduli of ideal sheaves $\mathcal{I} \subset \mathcal{O}_Y$ such that $\mathcal{O}_Y/\mathcal{I}$ defines a subscheme of $Y$ whose associated 1-cycle is $C$ and $\chi(\mathcal{O}_Y/\mathcal{I}) = n$. So it is really $Y$ with $C$ and $n-1$ points floating around (which can be on or off of $C$). This an open and closed subscheme of the moduli with the same Hilbert polynomial (because of the super rigidity). [[Q: It's not of finite type, is it? KB: it is; $n$ is fixed and I'm fixing the Hilbert polynomial, and the associated 1-cycle is one copy of $C$ (no multiplicities).]] Let $N_n(C,Y)$ be the virtual count, contribution of $C$ to the Donaldson-Thomas invariant with this Hilbert polynomial. The generating function for this thing turns out to be

$$\sum_{n\geq 0} N_n(C,Y)q^n = M(-q)^{\chi(Y)}\frac{q}{(1+q)^2}.$$

Now we can stratify the moduli space $X_n(C,Y)$. $Z_{n,0}(C,Y) \subset X_n(C,Y)$ closed. $Z_{n-i,i}(C,Y)$ is where exactly $i$ points (with multiplicity) are off $C$. $Z_{1,n-1}(C,Y) = \mathrm{Hilb}^{n-1}(Y \setminus C)$ is the open stratum. $X_n(C,Y) = \bigsqcup_{i=0}^{n-1} Z_{n-i,i}(C,Y)$.

$\mathrm{Hilb}^i(Y \smallsetminus C) \times Z_{n-i,0}(C,Y) = Z_{n-i,i}(C,Y)$ is contained as a closed subscheme of $U$, the open subset of $\mathrm{Hilb}^i(Y \smallsetminus C) \times X_{n-i}(C,Y)$ defined as the open subset where the two subschemes have disjoint support.

$$
\begin{array}{ccc}
\mathrm{Hilb}^i(Y \smallsetminus C) \times Z_{n-i,0}(C,Y) & = & Z_{n-i,i} \\
\quad\downarrow{\scriptstyle \text{closed}} & & \downarrow{\scriptstyle \text{incl. of stratum}} \\
U & \xrightarrow{\text{étale}} & X_n(C,Y) \\
\quad\downarrow{\scriptstyle open} & & \\
\mathrm{Hilb}^i(Y \smallsetminus C) \times X_{n-i}(C,Y) & &
\end{array}
$$

Notation: if $f\colon X \to Y$ is a morphism of schemes, then $\tilde{\chi}(X,Y) = \chi(X, f^*\nu_Y)$, and $\tilde{\chi}(X) = \tilde{\chi}(X,X) = \chi(X,\nu_X)$.

I want to compute $\tilde{\chi}\big(X_n(C,Y)\big) = \sum_{i=0}^{n-1} \tilde{\chi}\big(Z_{n-i,i}(C,Y), X_n(C,Y)\big)$.

$$
\begin{aligned}
\tilde{\chi}\big(Z_{n-i,i}(C,Y), X_n(C,Y)\big) = \\
&= \tilde{\chi}\big(\mathrm{Hilb}^i(Y \smallsetminus C) \times Z_{n-1,0}(C,Y), \mathrm{Hilb}^i(Y \smallsetminus C) \times X_{n-i}(C,Y)\big) \\
&= \tilde{\chi}(\mathrm{Hilb}^i(Y \smallsetminus C)) \cdot \tilde{\chi}\big(Z_{n-i,0}(C,Y), X_{n-i}(C,Y)\big) \\
&= \text{known}
\end{aligned}
$$

$\nu_{X_n(C,Y)} = \nu_U = \nu_{\mathrm{Hilb}^i(Y \smallsetminus C) \times X_{n-i}(C,Y)}$. Closed stratum

$$
\tilde{\chi}\big(Z_{n-i,0}(C,Y), X_{n-i}(C,Y)\big) = \tilde{\chi}\big(Z_{n-i,0}(\mathbb{P}^1, N), X_{n-i}(\mathbb{P}^1, N)\big)
$$

$N = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$. [[Q: [[★★★ something]] is very lucky. KB: I could stratify further; I don't think the full power of this method has been exploited.]] Now we are on $N = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$. We have $\mathbb{C}^\times$ action. We get a $\mathbb{C}^\times$ action preserving the CY, with isolated fixed points. We know $\nu_X(P)$ if $P$ is fixed; it is $(-1)^{n-1}$ (MNOP1), and we can count the fixed points, which means you're piling boxes in two corners of the room, not just one corner. In this case, the two corners are connected by an (infinite) row of boxes, so the two piles never meet.

I don't have to worry about the value of the function at any other point because the $\mathbb{C}^\times$ action is something so their stuff cancels out. So I get the formula from earlier.