1 Alessio Corti

Most of these lectures will be based on joint work with T(om?) Coates, H(iroshi?) Iritari, and H. H. Tseng. There is a list of problems for this course. References for today: two papers by Abramovich-Graber-Vistoli and the original paper by CR.

Quantum cohomology of stacks

From tomorrow onwards, we will be working with toric stacks, but today we'll be more general. Let $f\colon \Gamma \to \mathcal{X}$ be a stable representable morphism from an orbi-curve to a stack.

An *orbi-curve* is a proper, projective, algebraic curve Γ with some marked points x_i. Each of the marked points has a chart of the form $[\Delta/\mu_{r_i}]$ (where Δ is a disk and μ_r is the group of r-th roots of unity). An orbi-curve has a fundamental group $\pi_1^{orb} = \pi_1(\Gamma \smallsetminus \{x_i\})/\langle \gamma_i^{r_i}\rangle$, where the γ_i is a small loop around x_i.

Orbi-curves have line bundles on them, which are line bundles $\mathcal{L}$, together with an action of μ_{r_i} on $\mathcal{L}_{x_i}$, given by $v \mapsto \zeta^{k_i} v$. Riemann-Roch tells you that

$$\chi(\Gamma, \mathcal{L}) = \deg \mathcal{L} + 1 - g - \sum k_i/r_i$$

I hope you'll accept this stuff without worrying too much about the precise definitions.

Exercise. $\mathbb{P}_{r_1,r_2}$. *Then you can convince yourself that the Picard group is* $\mathbb{Z} \oplus \mathbb{Z}/gcd(r_1, r_2)$.

That's all I'll say about orbi-curves. Now on to stacks.

A stack $\mathcal{X}$ is locally Δ^n/G, where G is a finite group. Stacks have points, and points have stabilizers. A point x has stabilizer G_x. A morphism of stacks $f\colon \mathcal{X} \to \mathcal{Y}$ is *representable* if it induces injections $G_x \to G_{f(x)}$. If G is a finite group, there is a very important stack called BG, which is the quotient $[*/G]$. That is, to give a morphism $\mathcal{X} \to BG$ is the same as to give a principal G-bundle on $\mathcal{X}$. We need this in the case where $\mathcal{X}$ is an orbi-curve.

Example 1.1. If $\mathcal{X} = \Gamma$ is an orbi-curve, then a morphism $\mathcal{X} \to BG$ is a homomorphism $f\colon \pi_1^{orb}\Gamma \to G$ is representable if $f(\gamma_i)$ has order r_i. ⋄

Today I want to discuss orbifold cohomology and orbifold quantum cohomology. For this, we have to introduce the *inertia stack* $I_{\mathcal{X}} = \bigcup_{r\geq 0} \mathrm{Hom}^{rep}(B\mu_r, \mathcal{X})$ (where Hom^{rep} means representable morphisms). Such a morphism is the same as giving a point $x \in \mathcal{X}$ and an injection $\chi\colon \mu_r \hookrightarrow G_x$.

Aside from doing some examples, I'm not really sure how to explain this, so let's do some examples.

Example 1.2. Let $\mathcal{X} = \mathbb{P}^{w_0,\dots,w_n}$ (weighted projective space), which we will think of as $\mathbb{C}(-w_0) \oplus \cdots \oplus \mathbb{C}(-w_n)$ (where $\mathbb{C}(-w)$ is the representation of $\mathbb{C}^\times$ of weight $-w$, so $\lambda\colon x \mapsto \lambda^{-w}x$). In this case, $\mathrm{Box}\mathcal{X} = \{k/w_i | 0 \leq k < w_i\}$ and $I_{\mathcal{X}} = \bigcup_{b\in \mathrm{Box}} \mathbb{P}(V^b)$, where $V^b = \bigoplus_{w_i b\in\mathbb{Z}} \mathbb{C}(-w_i)$.

If we take $\mathcal{X} = \mathbb{P}(1,1,3)$, $I_{\mathcal{X}} = \mathbb{P}(1,1,3) \sqcup \mathbb{P}(3)_{1/3} \sqcup \mathbb{P}(3)_{2/3}$ (subscripts are Box levels). ⋄

Example 1.3. $\mathcal{X} = [M/G]$. In this case, $I_{\mathcal{X}} = \bigsqcup_{g\in C}[M^g/Z(g)]$ (C is conjugacy classes) ⋄

In general, there is a graph of groups $B = \mathrm{Box}\mathcal{X}$, whose elements are injective group homomorphisms χ from μ_r into some stabilizers. The inertia stack is $I\mathcal{X} = \bigsqcup_{\chi\colon \mu_r \to G_\eta} \mathcal{X}_\chi$.

Definition 1.4. $H^\bullet_{orb}\mathcal{X} = H^{\bullet - a(\chi)}(I\mathcal{X})$, where the *age* of χ is $a(\chi)$ (defined below). ⋄

$\chi\colon \mu_r \to G_\eta$, and G_η acts on the tangent space $T_\eta\mathcal{X}$, so we get an induced action given by k_i/r for some $0 \leq k_i < r$. Then we define $a(\chi) = \sum k_i/r$. I still have to tell you what the cup product is.

I have to talk about stable morphisms. For $\beta \in H_2(\mathcal{X})$, let $\mathcal{X}_{0,n,\beta} = \{$stable (no automorphisms) representable morphisms $f\colon (\Gamma, \mu_{r_i}(x_i))_{1\leq i\leq n} \to \mathcal{X}$ of degree β, where Γ is genus zero$\}$. Let me remind you that Γ could be a nodal curve; it doesn't have to be a smooth orbi-curve. The marked points x_i have these little charts $[\Delta/\mu_{r_i}]$. Only the marked points (and sometimes the nodes) have these charts.

Some features:

1. There are evaluation maps $ev_i\colon \mathcal{X}_{0,n,\beta} \to I\mathcal{X}$, given by $ev_i(f) = f(x_i)$. Rather, there aren't, but we can pretend that there are. We have that $\mathcal{X}_{0,n,\beta} = \bigsqcup_{b_1,\dots,b_n\in \mathrm{Box}} X_{0,n,\beta}(b_1, \dots, b_n)$, to be made sense of later.

2. $\mathcal{X}_{0,n,\beta}$ has a virtual dimension

$$\begin{aligned}\mathrm{vdim}_f &= \chi(\Gamma, f^*T_{\mathcal{X}}) + n - 3\\ &= -K_{\mathcal{X}}\cdot\beta + \dim\mathcal{X} - \sum_{i=1}^{n}\sum_{j=1}^{\dim\mathcal{X}} \frac{w_{i,j}}{r_i} + n - 3 \quad \text{(Riemann-Roch)}\end{aligned}$$

and a virtual class $\mathbb{1}_{vir} \in CH_{\mathrm{vdim}}$.

Product on $H^{\bullet}_{orb}(\mathcal{X})$. Consider $\mathcal{X}_{0,3,0} \xrightarrow{e_{1,2,3}} I\mathcal{X}$. Define $a \cup b = \overline{e_3} * (e_1^* a \cup e_2^* b) \cap \mathbb{1}_{vir}$. We have $\iota\colon I\mathcal{X} \to I\mathcal{X}$; if $\chi\colon \mu_r \to G_x$, then ι_χ is χ composed with conjugation.

This is a $\mathbb{P}^1$ with three marked points. Two of them are $|a|$ and $|b|$, and the third one is the intersection of $|a|$ and $|b|$.

Example 1.5. $\mathcal{X} = \mathbb{P}(1,1,3)$, so $I\mathcal{X} = \mathbb{P}(1,1,3) \sqcup \mathbb{P}(3)_{1/3} \sqcup \mathbb{P}(3)_{2/3}$. Then $H^{\bullet}_{orb}$ has a basis $1, \eta, x, \eta', x^2$ (η and η' come from the $\mathbb{P}(3)$'s) of degrees $0, 2/3, 1, 4/3, 2$, respectively (you have to get over the degrees being fractional). These are Chow degrees; if something, then you should double these. We have that $\eta \cup \eta = \eta'$, $\eta' \cup \eta = x^2$, and $\int_{\mathcal{X}} x^2 = 1/3$. Let's explain $\eta \cup \eta' = x^2$. Consider $\mathcal{X}_{0,3,0}(1/3, 2/3, 0)$. In $\mathbb{P}(1,1,3)$, we have the stacky point $\frac{1}{3}(1,1)$, which is a $\mathbb{P}(3)$. There are three marked points; two of them are marked by 1/3 and 2/3. $\eta \cup \eta' = e_3 * \mathbb{1} = \frac{1}{3}\{pt\}$. ◇

Quantum orbifold cohomology. The quantum product is defined (using Poincaré duality) by $(a * b, c) = \sum_\beta \langle a, b, c\rangle_\beta Q^\beta$, where $\langle a, b, c\rangle_\beta = \int_{\mathcal{X}_{0,3,\beta}} (e_1^* a \cup e_2^* b \cup e_3^* c) \cap \mathbb{1}_{vir}$.

Example 1.6. Define $\mathcal{X} = X_3^2 \subset \mathbb{P}(1,1,2)$ be a surface of degree 3. Suppose I blow up $\mathbb{P}^2$ at three colinear points. Then the line containing them is a -2 curve. If I contract that curve, I get this $\mathcal{X}$.

Take $A \subset H^{\bullet}_{orb}(\mathcal{X}, \mathbb{Q})$ with basis $\mathbb{1}, x, u, x^2$, in degrees $0, 1, 1, 2$, respectively. You can figure out that $\int_{\mathcal{X}} x^2 = 3/2$ and $\int_{\mathcal{X}} u^2 = 1/2$.

$$\begin{aligned}\langle x, x, u\rangle_{1/2} &= \frac{1}{4}\langle u\rangle_{1/2} = 3/4\\ \langle x, x, pt\rangle_1 &= 3\\ \langle pt, pt, u\rangle_{3/2} &= 1\end{aligned}$$

Given this, you can write the matrix of quantum multiplication by x (in this basis). It is

$$\begin{pmatrix} 0 & 3Q & 0 & 0\\ 1 & 0 & \frac{1}{2}Q^{1/2} & 3Q\\ 0 & \frac{3}{2}Q^{1/2} & 0 & 0\\ 0 & 1 & 0 & 0\end{pmatrix}$$

Also, $\langle u, u, u\rangle_{1/2} = 3/4$. In this example, $\deg Q = -5/3$. ◇

$D\psi = \psi M$. $\psi\colon \mathbb{C}^\times \to \mathrm{End}\, H^{\bullet}_{orb}(\mathcal{X}, \mathbb{C})$, $D = Qd/dQ$, $\psi = (\psi_0, \ldots, \psi_3)$, and ψ_0 satisfies

$$2D^3(2D-1) - 3Q(3D+2)(3D+1)$$

This is the differential equation you expect for something.

2 Alessio Corti - Toric stacks

Today I want to do an introduction to toric stacks. The references: a paper of BCS and a paper by Fantechi et. al. Unfortunately, if you don't already know something about toric varieties, it will be hard to get much out of this. Toric stacks are a good way to write down examples of stacks, so this is a good way to learn about stacks.

Definition 2.1. A *simplicial stacky fan* is a triple (N, Σ, ρ), where N is a finitely generated abelian group (allowed to have torsion), Σ is a rational simplicial fan in $N_{\mathbb{R}}$, and $\rho\colon \mathbb{Z}^m \to N$ is a homomorphism with finite cokernel such that $\mathbb{R}_+\overline{\rho}_i$ are the the 1-dimensional rays of the fan (where $\overline{\rho}_i$ are the images of the coordinate axes of $\mathbb{R}^m$ in $N_{\mathbb{R}}$). ◇

There is an equivalence of categories between stacky fans and toric stacks. How do you make a stack out of a stacky fan? Let $\mathbb{L} = \ker(\rho\colon \mathbb{Z}^m \to N)$. There is a "Gale dual" sequence

$$0 \to \mathbb{L} \to \mathbb{Z}^m \xrightarrow{\rho} N \qquad \text{(fan sequence)}$$

$$\mathbb{L}^\vee \xleftarrow{D} Z^{\times m} \leftarrow M \leftarrow 0 \qquad \text{(Gale dual)}$$

where $M = \mathrm{Hom}(N, \mathbb{Z})$, and D has finite cokernel. What is $\mathbb{L}^\vee$? It is not too easy; here is the construction. Let $\mathbb{Z}^m \xrightarrow{\rho} N^\bullet \to \mathbb{L}^\bullet \xrightarrow{+1}$ be a mapping cone , so $\mathbb{L} = \mathbb{L}^{-1}$. Dualize and take cohomology gives you

$$0 \to M \to \mathbb{Z}^{m^\bullet} \to H^1(\mathbb{L}^\bullet) =: \mathbb{L}^\vee$$

So $\mathbb{L}^\vee$ is a finitely generated abelian group, and it could have torsion.

Fact: $\mathcal{L}^\vee$ is the Picard group of the corresponding toric stack $\mathcal{X}$.

Think of $\mathbb{L}^\vee$ as the group of characters on an abelian algebraic group $\mathbb{G}$, $\mathrm{Hom}(\mathbb{G}, \mathbb{C}^\times)$. Similarly, $\mathbb{Z}^m$ is the group of characters of $(\mathbb{C}^\times)^m$. Then $\mathcal{X} = [\mathbb{C}^M /\!\!/ \mathbb{T}]$.

That requires a stability condition. If $\sigma \in \Sigma$ is a maximal cone (assume maximal cones are of maximal dimension), then $\bigoplus_{i\in\sigma} \mathbb{Z}e_i \to N$. This leads to $\mathcal{X}_\sigma \subset \mathcal{X}$ and open substack. Q: all these things are simplicial ... one could contemplate non-simplicial toric stacks, right? AC: sure.

Assumptions: I always assume that the natural map $\mathcal{X} \to \mathrm{Spec}\, H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is projective (this is made sense of purely in terms of the coarse moduli space). In particular, the support $|\Sigma| \subseteq N_{\mathbb{R}}$ is convex. Q: is that convexity condition equivalent to saying that the map is proper? AC: yes. I also assume that $\mathcal{X}$ is weak Fano, meaning that $-K_{\mathcal{X}}$ is nef. Equivalently, $\Delta(-K_{\mathcal{X}})$ is weakly convex. Q: why do you want to make these assumptions? AC: there are various issues. The projectivity is needed for the equivariant cohomology is sensible. It is also needed to make sense of Gromov-Witten theory. The weak Fano assumption. . . you'll see.

Example 2.2. $\mathcal{X} = \mathbb{P}^{w_1,w_2}$. Then we have

$$0 \to \mathbb{Z} \xrightarrow{\binom{w_1}{w_2}} \mathbb{Z}^2 \xrightarrow{\rho} \mathbb{Z}$$

$$\mathbb{L}^\vee = \mathbb{Z} \xleftarrow{(w_1\ w_2} \mathbb{Z}^2 \leftarrow M \leftarrow 0$$

So we have the quotient of $(\mathbb{C}^\times)^2$ by the action. $\mathbb{C}^\times \to (\mathbb{C}^\times)^2$, $\lambda \mapsto (\lambda^{w_1}, \lambda^{w_2})$. ◇

Example 2.3. $\mathcal{X} = \mathbb{P}_{w_1,w_2}$. Then

$$0 \to \mathbb{Z} \xrightarrow{\binom{w_1'}{w_2'}} \mathbb{Z}^2 \xrightarrow{(-w_2, w_1)} \mathbb{Z}$$

$\mathbb{L}' = \mathbb{Z} \oplus \mathbb{Z}/gcd(w_1, w_2)$, $\mathbb{P}_{2,2} = \mathbb{P}^/\mu_2$ ◇

Example 2.4. $\mathcal{X} = \frac{1}{3}(1,1)$. Then $\rho\colon \mathbb{Z}^2 \to \mathbb{Z}^2 + \frac{1}{3}(1,1)\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z} \leftarrow \mathbb{Z}^2$ given by $a + b \leftarrow (a, b)$. $\mathcal{X} = \mathbb{C}^2/\mu_3$ where $\mu_3 \to (\mathbb{C}^\times)^2$ is given by $\omega \mapsto (\omega, \omega)$ ◇

Example 2.5. $\mathcal{X} = \mathbb{P}(1, 1, 3)$, $N = \mathbb{Z}^2 + \frac{1}{3}(1,1)\mathbb{Z}$. $\rho\colon \mathbb{Z}^3 \to N$. Here $\rho_1 = (1, 0)$, $\rho_2 = (0, 1)$, and $\rho_3 = -\frac{1}{3}(1, 1)$. Ther are two lattice points of N in the convex hull of these three. They will play an important role later. ◇

Some facts:

- N_{tors} is the generic stabilizer
- the rays determine some divisors; write $N_i = \{v \in N | \overline{v} \in \mathbb{Q}_+\rho_i\}$ [[★★★ what is $\overline{v}$]]. Then $N_i/\langle\rho_i\rangle$ is the stabilizer of D_i.

Enhanced Picard group. Let $\mathcal{X}$ be a stack. Define the enhanced Picard group $\widehat{Pic}(\mathcal{X})$ of $\mathcal{X}$ by the following exact sequence.

$$0 \to \widehat{Pic}(\mathcal{X}) \to Pic(\mathcal{X}) \oplus \mathbb{Z}^{\mathrm{Box}} \to \bigoplus_{b\in \mathrm{Box}} \mathbb{Z}/r_b\mathbb{Z}$$

So $\widehat{Pic}(\mathcal{X}) = \{(L,m) | L \in Pic(\mathcal{X}), m\colon \mathrm{Box} \to \mathbb{Z}$ such that for $\chi\colon B\mu \to \mathcal{X}$, $\chi^* L = m(\chi)\}$.

Remark 2.6. If $f\colon \mathcal{X} \to \mathcal{Y}$ is a representable morphism of stacks, then you get $f^*\colon \widehat{Pic}(\mathcal{Y}) \to \widehat{Pic}(\mathcal{X})$. ◇

If I have a representable morphism from an orbi-curve $f\colon (\Gamma, x_i(r_i)) \to \mathcal{X}$, then f has an enhance degree $\widehat{\deg} f\colon \widehat{Pic}(X) \to \mathbb{Z}$, given by taking (L,m) to $\deg(f^*L) - \sum \frac{m_i}{r_i}$. Q: have you said what the degree of f^*L is? AC: the degree is the thing that makes the Riemann-Roch formula work.

Next I have to tell you how to calculate this for toric stacks. If $\mathcal{X}$ is a toric stack, then $\mathrm{Box} = \bigcup_{\sigma\in\Sigma} \mathrm{Box}(\sigma)$ where $\mathrm{Box}(\sigma) = \{v \in N | \overline{v} = \sum_{i\in\sigma} v_i \overline{\rho}_i, 0 \le v_i < 1\}$. This is the justification for the name "Box." We have $\rho\colon \mathbb{Z}^m \to N$; we augment this go get

$$0 \to \widehat{\mathbb{L}} \to \mathbb{Z}^m \oplus \bigoplus_{v\in\mathrm{Box}} \mathbb{Z} \to N$$

where the second map takes elements of the box to themselves (in N). It turns out that $\widehat{Pic}(\mathcal{X}) = \widehat{\mathbb{L}}^*$ (dual). So you don't have to do the complicated homological algebra from before.

If I have $\mathbb{P}_{r_1,r_2}$, I have a μ_{r_1} at zero and a μ_{r_2} at infinity, so I have sheaves like $\mathcal{O}(k_1/r_1)$ and $\mathcal{O}(k_2/r_2)$. I can pull back a line bundle from the coarse moduli space ($\mathbb{P}^1$), you get [[★★★ something something]] with just integers. Q: what about $\mathbb{P}_{1,1}$? AC: there is no stackiness; there is no box, so I can't play the game. Q: what is the enhanced Picard group of this $\mathbb{P}_{r_1,r_2}$? AC: we can do it. We had $\mathbb{Z}^2 \xrightarrow{\binom{r_1}{r_2}} \mathbb{Z}$. We have to augment this by the box. In this case, $\mathrm{Box} = \{-k/r_1 | 0 \le k < r_1\} \cup \{k/r_2 | 0 \le k < r_2\}$ (I guess for $\mathbb{P}_{1,1}$ you can choose a random integer, so the augmented Picard group is a line bundle plus an integer). So the enhanced picard group is the kernel of $\mathbb{Z}^2 \oplus \mathbb{Z}^{r_1} \oplus \mathbb{Z}^{r_2} \to \mathbb{Z}$.

MO: is there always a map $\widehat{Pic}(\mathcal{X}) \to Pic(\mathcal{X})$ with kernel $\mathbb{Z}^{\mathrm{Box}}$? AC: yes.

Stanley-Reisner rings

Given a stacky fan (Σ, N, ρ), we define $SR^\bullet_{\mathbb{R}}(\mathcal{X}, \mathbb{Q}) = \mathbb{Q}[\Sigma]$. Explicitly, there are generators u^e, where $e \in N$ and $\overline{e} \in \Sigma$. The product rule is that $u^{e_1}u^{e_2} = u^{e_1+e_2}$ whenever e_1 and e_2 belong to the same cone, and $u^{e_1}u^{e_2} = 0$ otherwise. This is a graded ring, with grading given by age. So $\deg u^e = a(e)$. Recall that for $\overline{e} = \sum_{i\in\sigma} a_i \overline{\rho}_i$, $a(e) = \sum a_i$.

Example 2.7. If $\mathcal{X} = \mathbb{P}^2$, then $SR_{\mathbb{T}}(\mathbb{P}^2) = R[u_1,u_2,u_3]/(u_0u_1u_2)$. You can think of the ring SR as the ring of polynomial functions on the polytope. ◇

Take $\mathbb{R} = \mathrm{Sym}^\bullet M$ as the "base ring". Then SR is an $\mathbb{R}$-algebra. An element $m \in M$ maps to $\sum \langle \rho_i, m\rangle \rho_i \in SR^1$.

Facts:

1. $SR^\bullet_{\mathbb{T}} = H^\bullet_{orb,\mathbb{T}}\mathcal{X}$ is the $\mathbb{T}$-equivariant orbifold cohomology of $\mathcal{X}$.
2. $SR_{\mathbb{T}} \otimes_{\mathrm{Sym}^\bullet M} \mathbb{Q} = H^\bullet_{orb}\mathcal{X}$.

3 Alessio Corti

References for today: CCIT, in preparation; CCLT, weighted projective spaces.

Plan: the J-function, S-extended stuff, I-function, mirror theorem, and a simple example ($\mathbb{P}^{2,2}$). Tomorrow, I'll try to write down some presentations for quantum cohomology for toric stacks (with some assumptions)

The J-function

$J(\tau, z) = z + \tau + \sum_{\ell,n} \frac{Q^\ell}{n!} ev_{x+i*}\left(ev_1^*\tau \cdots ev_n^*\tau \cdot \frac{\mathbb{1}}{z-\psi_{n+1}}\right) \in H^\bullet_{\pi,orb}(\mathcal{X}, \mathbb{C})$, where the things in the sum are happening on $\mathcal{X}_{0,n+1,\ell}$ and $ev_i \colon \mathcal{X}_{0,n+1,\ell} \to I\mathcal{X}$. L_i are line bundles, and $L_{i,f} = T^\vee_{C,x_i}$, and $\psi_i = c_1(L_i)$.

Goal: if $\mathcal{X}$ is a toric stack, write down J explicitly. Why, Alessio? For one thing, J contains all information about quantum cohomology. How does one come to consider this power series? J is the fundamental class in some cohomology theory, $H^{\infty/2}_{S^1}(\mathcal{L}_0\mathcal{X}, \mathbb{C})$. In some sense, when interpreted correctly, J has degree 1.

S-extended stuff

Let $\mathcal{X}$ be a toric stack with stacky fan (N, Σ, ρ). To this, we attached the fan sequence and the divisor sequence

$$0 \to \mathbb{L} \to \mathbb{Z}^m \xrightarrow{\rho} N$$

$$0 \to M \to \mathbb{Z}^{*m} \xrightarrow{D} \mathbb{L}^\vee$$

where $\mathbb{L}^\vee = Pic(\mathcal{X})$ and $\mathbb{L} = N_1(\mathcal{X}, \mathbb{Z})$. Let $S \subset N$ be a (finite, for today at least) subset which containes the rays ρ_i. The key examples of S will be just the set of rays, or $S = B = \mathrm{Box}(\mathcal{X})$ (which will lead to the enhanced stuff). Recall that $\mathrm{Box}(\mathcal{X}) = \bigcup_{\sigma\in\Sigma}\{v \in N | \overline{v} \in \sum_{i\in\sigma} a_i\overline{\rho}_i, 0 \le a_i < 1\}$ (remember that the bar just means image in N modulo torsion). Perhaps the most important example is where $S = \overline{B}^{\le 1} = \{v \in \overline{B} | \sum a_i \le 1\}$, the closed box. I think of $\overline{B}^{\le 1}$ as a basis for $H^{\le 2}_{\mathbb{T},orb}(\mathcal{X}, \mathbb{C})$. This will be the space of parameters for small quantum cohomology.

We have $\rho^S : \mathbb{Z}^S \mapsto N$, given by $e_s \mapsto s$, and let $\mathbb{L}^S$ be the kernel of this map. The get the Gale dual

$$\mathbb{L}^{S\vee} \xleftarrow{D^s} \mathbb{Z}^{*S} \leftarrow M \leftarrow 0.$$

I think of these as being some kind of glorified Picard group and topological classes of stable morphisms.

If $\sigma \in \Sigma$, we write $C^S_\sigma = \{\sum_{i\in S\smallsetminus\sigma} r_i D^S_i | r_i \ge 0\} \subseteq \mathbb{L}^{S\vee}_\mathbb{R}$ is a cone. We define $NE^S_\sigma = C^{S\vee}_\sigma \subseteq \mathbb{L}^S_\mathbb{R}$ and $NE^S = \sum_{\sigma\in\Sigma} NE^S_\sigma$. Define $\Lambda^S_\sigma = \{\lambda = \sum_{i\in S} f_i e_i \in \mathbb{L}^S_\mathbb{R} | j \notin \sigma \Rightarrow f_j \in \mathbb{Z}\}$, and $\Lambda^S = \bigcup_{\sigma\in\Sigma} \Lambda^S_\sigma$. The goal is to tell you exactly, inside $\mathcal{X}$, what are all the possible degrees of stable maps from an orbi-curve. There is a map $v \colon \Lambda^S \to B$, given by $v(\lambda) = \sum_{i\in S} \lceil f_i \rceil_{\rho_i} \in B$. $\Lambda E^S = \Lambda^S \cap NE^S$. These S's allow you to keep track of which torus invariant loci the various marked points are in.

The I-function

$$I^S(\tilde{Q}, z) = z \sum_{v\in B} \sum_{\substack{\lambda\in\Lambda E^S \\ v(\lambda)=v}} \tilde{Q}^\lambda \mathbb{1}_v \Box_\lambda(z)$$

where

$$\Box_\lambda(z) = \frac{\prod_{i\in S}\prod_{\langle b\rangle=\langle\lambda_i\rangle, b\le 0}(ui + bz)}{\prod_{i\in S}\prod_{\langle b\rangle=\langle\lambda_i\rangle, b\le\lambda_i}(ui + bz)} \in SR^\bullet_\mathbb{T}[z, z^{-1}]\!]$$

The box corresponds to irreducible components of $I\mathcal{X}$, $\mathbb{1}_v \in H^\bullet_{orb}(\mathcal{X})$ corresponding fundamental class, $u_i = u^{\rho_i} \in SR^\bullet_\mathbb{T}(\mathcal{X})$ if i is one of the rays (and $u^i = 0$ for $i \in S \smallsetminus \{\rho_i\}$), and $\lambda_i = \lambda \cdot D^S_i$.

One place where you can find this in a slightly less general context is in a paper of Bousov and Horja, Mellin-Baues, etc. Givental wrote it down in an article on dark manifolds. Q: was this inspired by mirror symmetry? AC: yes.

Mirror theorem

Theorem 3.1 (CCIT). *Assume $\mathcal{X}$ is weak Fano and $S \subset \overline{B}^{\le 1}$. Then (t is more or less Q) $I^S(t; z) = F(t)z + \mathbb{G}(t) + O(z^{-1})$. $J^S(\tau(t), z) = \frac{I^S(t,z)}{F(t)}$, where $\tau(t) = \mathbb{G}(t)/F(t)$.*

$J \colon H^\bullet_{\mathbb{T},orb} \to ?$. By Stanley-Riesner, $S \subset H^\bullet_{\mathbb{T},orb}(\mathcal{X}, \mathbb{C})$, with $\langle S\rangle$ the subspace generated by S, then $J^S = J|_{\langle S\rangle}$.

Special cases:

1. If $\mathcal{X}$ is Fano ($-K_{\mathcal{X}}$ is ample, not just nef) and has canonical singularities and $S = \{\rho_i\}$, then $I^S = J^S$. In the case where $\mathcal{X}$ is a manifold, this recovers Givental's theorem.

2. If $\mathcal{X} = \mathbb{P}^{w_0,\dots,w_n}$ is a weighted projective space and $S = \{\rho_i\}$, then $I^S = J^S$. This was proven in CCLT.

Example 3.2 ($\mathbb{P}^{2,2}$)**.** The fan diagram for $\mathbb{P}^{2,2}$ is

$$0 \to \mathbb{Z} \xrightarrow{\binom{2}{2}} \mathbb{Z}^2 \xrightarrow[\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}]{\rho} N = \mathbb{Z} \oplus \mathbb{Z}/2$$

Box $= \{0, 1\}$. The fan looks like [[★★★ ρ_1, ρ_3, ρ_6 at "height" 0 and ρ_5, ρ_4, ρ_2 at height ε]] Take $S = \overline{B}^{\leq 1}$.

$$0 \to \mathbb{Z}^5 \cong 1^S \to \mathbb{Z}^6 \xrightarrow[\begin{pmatrix} -1 & 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}]{\rho^S} N$$

$$I^S(Q; , s, t, z) = ze^{\frac{s_1 u_1 - s_2 u_2}{z}} \sum_{\ell, k_0, \dots, k_3 \in \mathbb{N}} \frac{Q^\ell e^{\ell(s_1+s_2)} t_0^{k_0} \cdot t_3^{k_3}}{z^{\sum k_i} k_0! k_1! k_2! k_3!} \mathbb{1}\left\langle \frac{\ell + k_0 + k_1 + k_2 + k_3}{2} \right\rangle \cdot \frac{\prod_{b \leq 0}(u_1 + bz) \prod_{b \leq 0}(u_2 + bz)}{\prod_{b \leq \ell - k_2}(u_1 + bz) \prod_{b \leq \ell - k_3}(u_2 - bz)}$$

and the mirror map is

$$\tau(t) = u_1 s_1 + u_2 s_2 + t_0 \mathbb{1} + t_1 \mathbb{1}_{1/2} - \frac{u_1}{2} \log(1 - t_2^2) + \frac{u_1 \mathbb{1} v_2}{2} \log(??) \cdots$$

[[★★★ somebody fill in the rest of the mirror map]] ⋄

4 Alessio Corti

Quantum cohomology

Today I want to discuss quantum cohomology and wall crossings. Let $\mathcal{X}$ be a toric stack with stacky fan (Σ, N, ρ). We have the ring $SR^\bullet_{\mathbb{T}}(\mathcal{X}, \mathbb{Q}) = \mathbb{Q}[\Sigma]$, where the elements are of the form u^e where $e \in N$ so that $\bar{e} \in |\Sigma|$, and $u^{e_1}u^{e_2} = u^{e_1+e_2}$ if e_1 and e_2 are in the same cone and $u^{e_1}u^{e_2} = 0$ otherwise. Let $\mathbb{R} = S^\bullet M$, an algebra, and $M \ni \chi \mapsto div(\chi) = \sum_{i=1}^m \langle \chi, \rho_i \rangle u^{\rho_i}$, so we have

$$0 \to \mathbb{L} \to \mathbb{Z}^m \xrightarrow{\rho} N$$

$\Lambda E \subset \mathbb{L} \otimes \mathbb{L}$.

For $QSR^\bullet(\mathcal{X}, \mathbb{Q})$, we modify the relations. If $e_1 \in \sigma_1$, $\bar{e}_1 = \sum_{i\in\sigma} a_i \bar{\rho}_i$ and $e_2 \in \sigma_2$, $\bar{e}_2 = \sum_{i\in\sigma} b_i \bar{\rho}_i$ with $\sigma_1 \neq \sigma_2$, let $e = e_1 + e_2 \in \sigma$, with $\bar{e} = \sum_{i\in\sigma} c_i \bar{\rho}_i$. Then we impose the relation $\ell(e_1, e_2) = \sum_{i=1}^m (a_i + b_i - c_i)e_i \in \mathbb{L}_{\mathbb{Q}}$. If $i \notin \sigma$, then the coefficient $(a_i + b_i - c_i) \geq 0, \in NE_\sigma \subseteq NE\mathcal{X}$.

Theorem 4.1. *If $\mathcal{X}$ is weak Fano and $I^{\{}\rho_i\} = J$ (i.e. $I(t,z) = \mathbb{1} + t + O(z^{-1})$), then $QH^\bullet_{\mathbb{T},orb}(\mathcal{X}) = \mathbb{Q}[\Lambda E][N]/(u^{e_1}u^{e_2} = Q^{\ell(e_1,e_2)}u^e)$.*

Remark 4.2. (1) Baryrev was the first to say what the quantum cohomology of a toric Fano manifold was. BCS told us what the orbifold cohomology of a stack was. The natural pushout of these two statements is the theorem above, so it was not difficult to guess the right answer.

(2) We should be able to do this for any $S \subseteq \overline{B}^{\leq 1}$. ◇

Example 4.3 ($\mathbb{P}^{1,2}$)**.** The fan sequence is

$$0 \to \mathbb{L} = \mathbb{Z} \xrightarrow{\binom{2}{1}} \mathbb{Z}^2 \xrightarrow[\rho]{(-1\ 2)} \mathbb{Z} = N \to 0$$

There are two cones σ_1 and σ_2 (and zero), with generators $u_1 = \rho_1$, $u_2 = \rho_2$, and w, with $w^2 = u_2$. Then $\rho_1 + \frac{1}{2}\rho_2 = 0$, so $\ell = 0$ (taking $e_1 = \rho_1$ and $e_2 = 1$). $uw = Q^{1/2}\mathbb{1}$.

$QH^\bullet_{\mathbb{T},orb}(\mathbb{P}^{1,2}) = \mathbb{Q}[q,u,w]/(uw = Q^{1/2}\mathbb{1})$, which contains (?) the $\mathbb{R}$-algebra $\mathbb{R} = \mathbb{Q}[\chi]$, $\chi = -u_1 + 2w^2$. There is the non-equivariant limit, where you take $\chi = 0$, so you obtain $QH^\bullet_{orb}(\mathbb{P}^{1,2}) = \mathbb{Q}[q,u,w]/(uw - Q^{1/2}\mathbb{1}, -u_1 + 2w^2)$. There is the classical limit, where $Q = 0$, in which you get the Stanley-Riesner ring $H^\bullet_{\mathbb{T},orb} = \mathbb{Q}[u,w]/uw$. Then there is the case where you do both, to get $H^\bullet_{orb}(\mathcal{X})$ where you take $\chi = Q = 0$ ◇

Proof. All of this comes from the GKZ differential system. If $\ell \in \mathbb{Z}(?)$, □

Wall crossing

[C,I,T]

Example 4.4. $\mathbb{P}(1,1,2)$ over $\mathbb{F}_2$. then I have [[★★★ picture]]. The fan sequence is

$$0 \to \mathbb{L} + \mathbb{Z}^2 \to \mathbb{Z}^4 \xrightarrow{\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 2 & -1 & 1 \end{pmatrix}} \mathbb{Z}^2 = N \to 0$$

and the Gale dual is

$$0 \leftarrow Pic(\mathcal{X}) = \mathbb{L}^\vee = \mathbb{Z}^2 \xleftarrow{D=\begin{pmatrix} 1 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{pmatrix}} \mathbb{Z}^4 \leftarrow \mathbb{Z}^2 = M \leftarrow 0.$$

$(\mathbb{C}^\times)^2 \to (\mathbb{C}^\times)^4$ acts on $\mathbb{C}^4$ with weights $(1,1,0,-2)$ and $(0,0,1,1)$. In $\mathbb{L}^\vee$, I have the picture [[★★★ picture: $D_1 = D_2 = P_1$, $D_3 = P_2$, and $D_4 = -2P_1 + P_2$, K_1 first quadrant, K_2 the part of the second quadrant above D_4]]

I'll think of an element $\psi \in \mathbb{L}^\vee = \mathrm{Hom}_{\mathsf{Gp}}((\mathbb{C}^\times)^2, \mathbb{C}^\times)$ as a $(\mathbb{C}^\times)^2$-linearized line bundle on $\mathbb{C}^4$. The stable points of that linearization will be $U^s = \{s \in \mathbb{C}^4 | \exists P(\vec{x}) \in \mathbb{C}[x_1, \ldots, x_4], P(g\vec{x}) = \psi(g)P(\vec{x})$ such that $P(\vec{a}) \neq 0\}$. You can check that if $\psi \in K_1$, then $U^s = \mathbb{C}^2 \smallsetminus \{0\} \times \mathbb{C}^2 \smallsetminus \{0\}$ and $U^s/(\mathbb{C}^\times)^2 = \mathbb{F}_2$. I hope you're familiar with this as the standard way to construct the surface $\mathbb{F}_2$. On the other hand, if $\psi \in K_2$, then $U^s = \mathbb{C}^3 \smallsetminus \{0\} \times \mathbb{C}^\times$ and $U^s/(\mathbb{C}^\times)^2 = \mathbb{P}(1,1,2)$. ◇

I didn't fully explain to you how to get a toric stack from a stacky fan. I explained how to get an open cover. This wall crossing, when you cross from K_1 to K_2, is somehow responsible for the birational transformation between $\mathbb{F}_2$ and $\mathbb{P}(1,1,2)$.

If you look at the picture D with K_1 and K_2, it looks like the fan of a toric stack, so let's consider the toric stack with that fan, $\mathcal{M}$, which has two charts. $\mathbb{F}_2$ corresponds to the chart $\mathbb{C}^2$ with coordinates q_1 and q_2, dual to P_1 and P_2. And $\mathbb{P}(1,1,2)$ corresponds to a stacky chart $\mathbb{C}^2/\mu_2$ with coordinates $\tilde{q}_1 = q_1^{-1/2}$ and $\tilde{q}_2 = q_1^{1/2} q_2$, dual to $-2P_1 + P_2$ and P_2.

The I-function of $\mathbb{F}_2$ is a function of q_1 and q_2, for q_1 and q_2 small. I won't write it down; you can write it down $I_{\mathbb{F}_2}(q_1, q_2) \in H^\bullet(\mathbb{F}_2, \mathbb{C})[z, z^{-1}]\!] = \mathbb{C}[P_1, P_2][z, z^{-1}]\!]/(P_1^2, P_2^2 - 2P_1P_2)$. Imagine now that you use yesterday's procedure to write down the I-function with basis 1, P_1, P_2, and P_1P_2.

We have that $I_{\mathbb{P}(1,1,2)}(q) \in H^\bullet_{orb}(\mathbb{P}(1,1,2), \mathbb{C})[z, z^{-1}]\!] = \mathbb{C}[P, \mathbb{1}_{1/2}]/(P^3 = P \cdot \mathbb{1}_{1/2} = 0, \dots)$. The right basis for the I-function, for some reason, is $\mathbb{1}$, $P - i\mathbb{1}$, $2P$, $2P^2$. People know why this is the right basis, but I can't say why.

I analytically continue to the other chart to get $I_{\mathbb{F}_2}(\tilde{q}_1, \tilde{q}_2)|_{\tilde{q}_1=0, \tilde{q}_2=\sqrt{q}} = U(z)I_{\mathbb{P}(1,1,2)}(q)$, where

$$U(z) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -i\pi/2 & 0 & 0 & i \\ i\pi/2z & 1/2 & 0 & -i/2 \\ \pi^2/4z^2 & 0 & 1/2 & 0 \end{pmatrix}$$

This has no positive powers of z, which gives you some crepant resolution. If you do this with the next hardest case [[★★★ $\mathbb{P}(1,1,1,3)$ or something]], you get a positive power of z, which screws things up.

5 Alessio Corti

Today I want to do two (or perhaps three) things. The main theorem I stated was the mirror theorem with the I-function and J-function. I want to zoom in on one part of the proof.

Let $\mathcal{X}$ be a proper 1-dimensional toric stack. Question: classify all representable toric morphisms $f\colon \mathbb{P}_{r_1,r_2} \to \mathcal{X}$. You have to calculate some Gromov-Witten numbers.

Remark 5.1. If $\mathcal{X}$ is a manifold (not a stack), then $\mathcal{X} = \mathbb{P}^1$. In this case, the only representable morphisms are from $\mathbb{P}^1 = \mathbb{P}_{1,1}$. Toric morphisms are then classified by degree; every such morphism is given by $(x_0, x_1) = (z_0^d, z_1^d)$. ◇

The slogan: all such morphisms are classified by the enhanced degree $\widehat{\deg} \in \mathrm{Hom}(\widehat{Pic}(\mathcal{X}), \mathbb{Z})$. This is the main motivation for introducing $\widehat{Pic}$.

Notation: $\mathcal{X}$ has a fan diagram

$$0 \to \mathbb{Z} \xrightarrow{\binom{w_2}{w_1}} \mathbb{Z}^2 \xrightarrow{\rho} N$$

where N is a rank 1 abelian group (so $\mathbb{Z}$ plus a torsion bit). [[★★★ picture fan for $\mathbb{P}^1$; $\overline{\rho}_1$ negative, $\overline{\rho}_2$ positive, σ_1 negative cone, σ_2 positive gone]]. Let $B = \mathrm{Box}(\mathcal{X})$. Then $B(\sigma_1) = \{v \in N | \overline{v} = a\overline{\rho}_1, 0 \le a < 1\} = N/\langle \rho_1 \rangle$. So $N_{tors} \subset B(\sigma_i) =: B_i$ and $\mathrm{Box} = B(\sigma_1) \cup B(\sigma_2)$.

Recall that $\mathbb{P}_{r_1,r_2}$ is a $\mathbb{P}^1$ with a μ_{r_1} at zero and μ_{r_2} at infinity. $f\colon \mathbb{P}_{r_1,r_2} \to \mathcal{X}$, $f(0$ and $f(\infty)$ give me $B\mu_{r_i} \to \mathcal{X}$, which give me $v_i \in B_i$. $\underline{\rho} = \{\rho_1, \rho_2, v_1, v_2\}$. $\overline{v}_i = f_i(\overline{\rho}_i)$ where $0 \le f_1, f_2 < 1$ are rational. Then $\widehat{\deg} f \in \mathbb{L}^S \subset \widehat{\mathbb{L}} = \mathrm{Hom}(\widehat{Pic}, \mathbb{Z})$.

More explicitly, we enhance the fan map to get

$$0 \to \mathbb{L}^S \to \mathbb{Z}^S = \mathbb{Z}^4 \xrightarrow[(\rho_1,\rho_2,v_1,v_2)]{\rho^S} N$$

So $\widehat{\deg} f = (q_1, q_2, 1, 1) \in \ker \rho^S$ (column vector), where q_1 and q_2 are positive integers. A general enhanced degree would have integers k_1 and k_2 in place of the two 1's.

Remark 5.2 (Exercise)**.** There is a positive rational number $\ell \in \mathbb{Q}_+$ such that $w_i \ell - f_i = q_i$, where w_1 and w_2 are from the fan sequence. This ℓ is the "good old" $\deg f$.

If N has torsion, then there is more information in the enhanced degree. From ℓ, I would not be able to recover the box elements (only modulo torsion). ◇

Proposition 5.3. *The following sets of data are equivalent:*

1. *non-constant representable morphisms $f\colon \mathbb{P}_{r_1,r_2} \to \mathcal{X}$ for some r_1, r_2 (unspecified),*
2. *Box elements $v_1 \in B_1$, $v_2 \in B_2$ and integers $q_1, q_2 > 0$ such that $q_1\rho_1 + q_2\rho_2 + v_1 + v_2 = 0$ in N.*

Proof. We've done one direction; you can convince yourself that it works. Let's do the other direction. We have to construct a morphism of fans.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\binom{r_2'}{r_1'}} & \mathbb{Z}^2 & \xrightarrow{(-r_1, r_2)} & \mathbb{Z} & & (\mathbb{P}_{r_1,r_2}) \\
 & & {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}} & & \downarrow{\scriptstyle \eta} & & \\
0 & \longrightarrow & \mathbb{Z} & \xrightarrow[\binom{w_2}{w_1}]{} & \mathbb{Z}^2 & \xrightarrow[\rho]{} & N & & (\mathcal{X})
\end{array}$$

First we construct η by $\eta(1) = -v_1 - q_1\rho_1 = v_2 + q_2\rho_2$. Let r_i be the order of v_i as a group element of $B_i = N/\langle \rho_i \rangle$. Then $r_i v_i = k_i \rho_i$ for some non-negative integers $k_i \ge 0$. It is easy to check that $k_i/r_i = f_i$. This tells us what r_1 and r_2 are.

Next set $m_i = r_i q_i + k_i$. We'll define the middle map to be given by $\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}$. Let's check that the square commutes. We calculate that $m_1\rho_1 = (r_1 q_1 + k_1)\rho_1 = r_1(-v_1 + \eta(-1)) + r_1 v_1 = -r_1\eta(1)$.

Finally, we want to construct the last map, but that is easy because they are kernels; the image of the top $\mathbb{Z}$ in the bottom $\mathbb{Z}^2$ is sent to zero because of commutativity of the square we checked. □

This is not difficult, but it took some time to sort it out because we had to find the correct way to package the combinatorics. In the end, this $\widehat{Pic}$ is what did it.

General words on the proof of the mirror theorem. You want to calculate some Gromov-Witten invariants at the end of the day. We want to calculate the J-function

$$J = \sum_{\beta} Q^\beta \int_{\mathcal{X}_{0,1,\beta}} \frac{ev^* \mathbb{1}}{z - \phi}$$

$\mathbb{T}$ acts on $\mathcal{X}$ and on $\mathcal{X}_{0,1,\beta}$. Inside of $\mathcal{X}$, there is the 1-dimensional stratum. For a map to be $\mathbb{T}$-invariant, your orbi-curve is likely to be extremely reducible. Basically, the J-function breaks up into contributions. A fixed point will be in an affine chart $\mathcal{X}_\sigma$ (σ maximal cone), then there is a point mapping there with some box element $\nu \in B_\sigma$. It is more or less a combinatorial problem. You break it up into pieces where you somehow rip off that component of the source curve. You know the contribution from that component, and the rest has lower degree, so you set up some induction procedure.

Oil Update

Not all of you have heard me lecture about this, so let me give you some basics. Oil is measured in barrels. One barrel is about 140 liters, or 40 gallons (unrefines). Today, a barrel costs about 135 USD, which is more than twice what it cost exactly a year ago. You might think 135 USD is a lot of money, so let me tell you what you buy when you buy a barrel of oil. One barrel equal 25,000 man hours of mechanical energy (never mind how that calculation is done)! You may be asking yourself, why is it that in the last twelve months, oil doubled it's price. Maybe it is speculators in Wall street, who want to make sure you pay because they messed up some prime mortgage thing. So they keep making money why you pay. Actually, the answer is rather different. There are very few studies of the world supply of oil. Given such a serious problem, why is it that nobody trys to study the future supply of oil. Today, world production is estimated to be approximately 85 MB/day (actually 75 MB/day, when it comes to crued oil; they get 85 by tricks called refinery gains and some other stuff, natural gas tricks), which is about the same that was produced in 2005! It is amazing how economists get Nobel prizes by saying that when there is enough demand, the shit will turn up. There are studies by the EUIA (or something), which still tell us that in 2030, the world will produce about 120 MB/day. Why do they say that? They plot the curve and extrapolate! There are very few studies on the supply side, and let me put my favorites

- F Rebelius, March 2007. This is a graduate student doing this! It's on the web; you can look it up.
- C. Skrebowski, Megaproject Update, on Petrolium review; this one is not on the web. Instead of guessing how much oil is in the ground, he knows about the big projects. Journalists say "the high prices are not stimulating investment in oil production". What are they talking about? It takes 8 years after you find an oil field to production. He sees a lot of oil coming in until 2012, and then we're walking into empty space; there is nothing there to fill that hole. Q: is that what the other one predicts? AC: yes, but the other one estimates the oil in the ground, so it is a completely different methodology.
- Energy Watch group, October 2007. These are scientists who were given serious money to buy data from the oil industry (by bribe). Oil data from Saudi Arabia and Russia are classified, so you have to bribe to get the data. Google gave me less than one page on this study. None of the press quoted this study! What these guys say is that world oil production reached its peak in 2006 and that it will be down 50% by 2020 (?).