# ANALYSIS STUFF I SHOULD KNOW

ANTON

## Real Analysis

**Definition.** $K$ is a *compact* space (or a compact subset of a metric space) if for any sequence $\{x_n\}$ in $K$, there is a subsequence $\{x_{n_k}\}$ which converges to some $x \underline{\in K}$.

**Heine-Borel.** *$K$ compact iff every open cover of $K$ has a finite subcover.*

Note that compact implies closed and bounded. In $\mathbb{R}^n$, the converse is also true.

**Definition.** A space $X$ is *connected* if there is no representation of $X$ as the disjoint union of two non-empty proper open subsets. i.e. No proper subset of $X$ is both open and closed.

**Definition.** A *path* in $X$ is a continuous function $\gamma : [a, b] \to X$.

**Definition.** A *homotopy* of $\gamma_0$ to $\gamma_1$ is a continuous map $H : [0, 1] \times [a, b] \to X$ such that $H(0, s) = \gamma_0(s), H(1, s) = \gamma_1(s)$.

**Definition.** $X$ is *simply connected* if $\pi_1(X) = 0$ and $X$ is connected.

*Remark.* Continuous functions take compact sets to compact sets and connected sets to connected sets, but do not necessarily take simply connected sets to simply connected sets.

**Banach Contraction Principle/Fixed Point Theorem.** *If $X$ is a complete metric space, and $f : X \to X$ is a map such that $d(f(x), f(y)) \leq k \cdot d(x, y)$ for every $x, y \in X$ and for some constant $k < 1$, then $f$ has a unique fixed point.*

*Proof.* Let $x_0 \in X$, and define $x_{n+1} = f(x_n)$. Then we have that for $n < m$

$$d(x_n, x_m) \leq d(x_n, x_{n+1}) + \cdots + d(x_{m-1}, x_m).$$

Since $d(x_l, x_{l+1}) \leq k^n \cdot d(x_0, x_1)$, we have that $d(x_n, x_m) \leq \frac{k^n}{1-k} d(x_0, x_1)$, which goes to 0. By completeness, $x_i \to x \in X$ gives us a fixed point. $d(f(x), f(y)) < d(x, y)$ implies uniqueness. □

Here is an important variation. If $f^n = f \circ \cdots \circ f$ has a *unique* fixed point, then so does $f$.

*Proof.* If $x$ is fixed by $f$, then we have that $f^n(f(x)) = f(f^n(x)) = f(x)$, so $f(x)$ is also fixed. Uniqueness implies $f(x) = x$. If $f$ had any other fixed point, it would also be a fixed point of $f^n$, contradicting uniqueness. □

**Theorem.** *If $F \subseteq X$ is closed and $K \subseteq X$ is compact with $F \cap K = \varnothing$, then $d(F, K) > 0$.*[1]

*Proof.* Let $f : K \to \mathbb{R}$ be defined by $f(k) = d(k, F)$. This is continuous on a compact set, so it attains a minimum. If that minimum is 0, then let $d(k_0, F) = 0$. Let $\{x_n\} \subseteq F$ be a sequence so that $d(k_0, x_n) \to 0$. Since $F$ is closed, the limit of the $x_n$, which is $k_0$, is in $F$, contradicting $F \cap K = \varnothing$. □

In $\mathbb{R}^n$, the distance between a compact set and a closed set is always attained.

This document is based on Tony's notes from Yonatan's prelim workshop.

[1] $d(F, K) = \inf\{d(f, k) | f \in F, k \in K\}$.

*Proof.* Since $d(k,F)$ is continuous and $K$ is compact, we may choose $k \in K$ so that $d(k,F) = d(K,F)$. Let $\{y_n\}$ be a sequence of points in $F$ so that $d(k,y_n) \to d(K,F)$. This sequence is bounded (in $\mathbb{R}^n$), so there is a convergent subsequence with limit $y \in F$. Then $d(K,F) = d(k,y)$. □

If $f : X \to Y$ is open and closed, and if $Y$ is connected, then $f$ is surjective, for otherwise, $f(X)$ would be a proper subset of $Y$ which is both open and closed.

**Arzela-Ascoli Theorem.** *$K$ a compact space, $A \subseteq C^0(K)$, then $A$ is compact if and only if it is closed, bounded, and (uniformly) equicontinuous*[2].

**Properties of Operator Norm.**

(1) $||T|| = \sup_{||x||=1} ||Tx||$ *(definition).*
(2) $||Tx|| \leq ||T|| \cdot ||x||$ *for all $x$, and $||T||$ is the minimal such number.*
(3) $||T + S|| \leq ||T|| + ||S||$
(4) $||T \circ S|| \leq ||T|| \cdot ||S||$
(5) $||T||^2 \leq \sum_{i,j} a_{ij}^2$ *where $(a_{ij})$ is the matrix for $T$ in some orthonormal basis.*

*Proof of 5.* We want to show that for all $||x|| = 1$, $||Tx||^2 \leq \sum_{i,j} a_{ij}^2$.

$$||Tx||^2 = \left\| \begin{pmatrix} \sum_j a_{1j}x_j \\ \vdots \\ \sum_j a_{nj}x_j \end{pmatrix} \right\|^2 = \sum_i \left( \sum_j a_{ij}x_j \right)^2$$

$$\leq \sum_i \left( \sum_j a_{ij}^2 \right) \underbrace{\left( \sum_j x_j^2 \right)}_{1} = \sum_{i,j} a_{ij}^2 \qquad \text{(Cauchy-Schwartz)}$$

□

**Definition.** Say $A \subseteq \mathbb{R}^n$ with an interior point $a \in A$ and $f : A \to \mathbb{R}^m$. We say $f$ is *differentiable at $a$* with derivative $T : \mathbb{R}^n \to \mathbb{R}^m$ if

$$\lim_{x \to 0} \frac{f(x+a) - f(a) - Tx}{||x||} = 0.$$

Alternatively, $f(a + x) = f(a) + Tx + \alpha(x)$, where $\frac{\alpha(x)}{||x||} \to 0$ as $x \to 0$. We write $T = D_f(a)$ or $D_a f$.

*Remark.* If $T$ exists, it is equal to $\left(\frac{\partial f_i}{\partial x_j}\right)$. All $\frac{\partial f_i}{\partial x_j}$ continuous $\Rightarrow$ $f$ differentiable $\Rightarrow$ All $\frac{\partial f_i}{\partial x_j}$ exist at $a$. However, neither of the converses are true.

The following observations yield a practical way to compute $Ty$ for some $y \in \mathbb{R}^n$:

$$\lim_{h \to 0} \frac{f(a+hy) - f(a) - T(hy)}{||hy||} = 0 \Rightarrow \frac{Ty}{||y||} = \lim_{h \to 0} \frac{f(a+hy) - f(a)}{||hy||}$$

$$\Rightarrow Ty = \lim_{h \to 0} \frac{f(a+hy) - f(a)}{h}$$

**The Chain Rule.** *If $\mathbb{R}^n \xrightarrow{f} \mathbb{R}^m \xrightarrow{g} \mathbb{R}^l$ with $f$ differentiable at $a$ and $g$ differentiable at $f(a)$, then*

$$D_{g \circ f}(a) = D_g(f(a)) \cdot D_f(a)$$

[2] A family of functions, $A$, is *(uniformly) equicontinuous* if $\forall \varepsilon > 0\ \exists \delta > 0$ such that $\forall x, y \in K, \forall f \in A$, $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$

**Theorem.** *If $x, y \in A \subseteq \mathbb{R}^n$, let $I = [x, y]$ be the straight line segment between $x$ and $y$. If $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at every point in $I$, then*

$$\frac{||f(x) - f(y)||}{||x - y||} \leq \sup_{z \in I} ||D_f(z)||.$$

*Proof.* Define $g : [0, 1] \to \mathbb{R}$ by $g(t) = \langle f(y) - f(x), f((1-t)x + ty) - f(x)\rangle$. Then $g(1) = ||f(y) - f(x)||^2$ and $g(0) = 0$. By the mean value theorem, there is some $t \in (0, 1)$ so that $g(1) - g(0) = g'(t)$. So

$$\begin{aligned} ||f(y) - f(x)||^2 &= |g'(t)| \\ &= |\langle f(y) - f(x), D_f((1-t)x + ty) \cdot (y - x)\rangle| \\ &\leq ||f(y) - f(x)|| \cdot ||D_f|| \cdot ||y - x||. \end{aligned}$$

Remember that the derivative of $\langle u, v\rangle$ is $\langle u', v\rangle + \langle u, v'\rangle$. □

**Inverse Function Theorem.** *Say $A \subset \mathbb{R}^n$ open, $f : A \to \mathbb{R}^n$ is $C^1$ on $A$ and $a \in A$ such that the Jacobian $J_f(a) = \det(D_f(a)) \neq 0$, then*

1. *there is a neighborhood $U$ of $a$ such that $f(U) = V$ is open.*
2. *$f : U \to V$ is bijective.*
3. *$f^{-1} : V \to U$ is $C^1$.*
   *Note: $D_{f^{-1}}(f(a)) = (D_f(a))^{-1}$.*

**Open Mapping Theorem.** *Say $A \subseteq \mathbb{R}^n$ open and $f : A \to \mathbb{R}^m$ is $C^1$ on a neighborhood of $A$. If rank$(D_f(a)) = m$, then there is a neighborhood of $f(a)$ which is in the image of $f$(i.e. $f$ is "open at $a$").*

**Implicit Function Theorem.** *Say $A \subseteq \mathbb{R}^n$, $B \subseteq \mathbb{R}^m$. Assume*

1. *$f : A \times B \to \mathbb{R}^n$ is $C^1$ in a neighborhood of $(a, b) \in A \times B$*
2. *$f(a, b) = 0$*
3. *$\left(\frac{\partial f_i}{\partial x_j}\right)_{\substack{1\leq i\leq n\\1\leq j\leq n}}$ is invertible at $(a, b)$. Note that this is a minor of the determinant $\left(\frac{\partial f_i}{\partial x_j}\right)_{\substack{1\leq i\leq n\\1\leq j\leq n+m}}$*

*Then there are open sets $V \subseteq B, U \subseteq A \times B$ and a $C^1$ function $g : V \to A$ such that for all $(x, y) \in U$, $f(x, y) = 0 \Leftrightarrow y = g(x)$.*

**Lemma.** *Say $B \subseteq \mathbb{R}^n$ open, $f, g_1, \ldots, g_k : B \to \mathbb{R}$ are $C^1$, $A = \{x \in B | g_i(x) = 0\ \forall i\}$. If $f|_A$ has a local minimum/maximum at $a \in A$, then the matrix $\begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_k}{\partial x_1} & & \frac{\partial g_k}{\partial x_n} \\ \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_n} \end{pmatrix}$ has rank less than $k + 1$. That is, the vectors $\nabla f$, $\nabla g_i$ (the rows of the matrix) are linearly dependent at $a$.*

**Corollary** (Lagrange Multipliers). *$f, g_1, \ldots, g_k, B, A, a$ as in the lemma. If the $\nabla g_i$ are linearly independent at $a$, then $\nabla f = \sum_{i=1}^k \lambda_i \nabla g_i$ at $a$ for some numbers $\lambda_i$.*

Some stuff about ODEs:

**Existence.** *Let $(x_0, y_0) \in \mathbb{R}^2$, $f : \mathbb{R}^2 \to \mathbb{R}$ continuous in a box $Q$ containing $(x_0, y_0)$ in its interior. Then there is a solution to the differential equation $y' = f(x, y)$ passing through $(x_0, y_0)$. The solution exists as long as it is inside $Q$.*[3]

[3]If $M = \sup_Q |f|$, then the solution lies in the "cone with slope M" through $(x_0, y_0)$.

**Uniqueness.** *If the above $f$ is Lipschitz in $y$ uniformly in $x$, i.e. there is a constant $K$ such that*

$$|f(x,\xi) - f(x,\eta)| \leq K|\xi - \eta|$$

*for all $x$, then the solution to $y' = f(x,y)$ is unique.*

**Smoothness.** *Given a family of differential equations $y' = f_\lambda(x,y)$, where $f_\lambda(x,y)$ is continuous in all three variables and Lipschitz in $y$ uniformly in $x$ and $\lambda$, then the (unique) solution $y_\lambda(x)$ depends continuously on $\lambda$. If $f$ is $C^1$ in all variables, then $y_\lambda(x)$ is also $C^1$.*

*Remark.* All of this works if $\vec{y} \in \mathbb{R}^n$ and $f(x, \vec{(y)})$ is $\mathbb{R}^n$-valued. It follows that these results hold for higher order ODEs, for if we have

$$y^{(n)} = f(x, y, y', \ldots, y^{(n-1)}$$

we may make the substitution $z_i = y^{(i)}$ to reduce to the system

$$\begin{pmatrix} z_0 \\ \vdots \\ z_{n-1} \\ z_n \end{pmatrix}' = \begin{pmatrix} z_1 \\ \vdots \\ z_n \\ f(x, z_0, \ldots, z_{n-1}) \end{pmatrix}$$

Some techniques:

If $y' + P(x)y = Q(x)$, then we multiply both sides by $e^{\int P(x)}$ and observe that the left hand side is the derivative of $y \cdot e^{\int P(x)}$. Integrating and solving for $y$, we have

$$y(x) = e^{-\int P(x)} \int Q(x) e^{\int P(x)}\, dx.^4$$

If $a_n y^{(n)} + \cdots + a_0 y = 0$, and if $r$ is a root of $p(x) = a_n x^n + \cdots + a_0$ with multiplicity $m$, then $e^{rx}, xe^{rx}, \ldots, x^{m-1}e^{rx}$ are solutions to the differential equation. This gives us $n$ linearly independent solutions. Any other solution is a linear combination of these.[5]

If $a_n y^{(n)} + \cdots + a_0 y = Q(x)$, and if $y_p$ is a solution, then any other solution differs from $y_p$ by a solution of the homogeneous version of the differential equation.[6] You can find a $y_p$ using the method of undetermined coefficients or the method of variation of parameters.

## Sample Problems

**Exercise** (4.1.2)**.** Let $K \subseteq \mathbb{R}^k$ be a compact set, and let $\{U_j\}$ be an open cover of $K$. Show that there is some $\lambda > 0$ such that every ball of radius $\lambda$ around some point in $K$ is contained in one of the $B_j$'s.

*Solution.* Suppose not. Then choose $x_n \in K$ such that $B(x_n, \frac{1}{n})$ is not contained in any $B_j$. There is a convergent subsequence (wlog the sequence converges). Let $x \in K$ be the limit point. Since the $B_j$ cover $K$, $x \in B_j$ for some $j$, so $B(x,r) \subseteq B_j$. Choose $n$ large enough so that $|x - x_n| < \frac{r}{2}$ and $\frac{1}{n} < \frac{r}{2}$. Then we have that $B(x_n, \frac{1}{n}) \subseteq B(x,r) \subseteq B_j$, a contradiction.

**Exercise** (4.1.6)**.** Look at me

**Exercise** (4.3.4)**.** Let $K$ be compact, and let $\phi : K \to K$ satisfy $d(\phi(x), \phi(y)) < d(x,y)$ for all $x \neq y$. Show that $\phi$ has a unique fixed point.

*Solution.* Define $f : K \to \mathbb{R}$ by $f(x) = d(x, \phi(x))$. This is a continuous function on a compact set, so it attains its minimum at some $x_0 \in K$. If the minimum is zero, we have a fixed point. Otherwise, note that $f(\phi(x_0)) < f(x_0)$, contradicting minimality. Uniqueness is obvious.

[4]By $\int P(x)$, I really mean $\int_0^x P(\xi)\, d\xi$.

[5]Solutions to a homogeneous ODE form a vector space!

[6]Solutions to a non-homogeneous ODE form an affine space!

Also look at problems 4.*.* from years after 1995.

**Exercise** (Sp00)**.** Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be differentiable with $F(0) = 0$ and $\displaystyle\sum_{j,k} \left|\frac{\partial F_j}{\partial x_k}(0)\right|^2 = c < 1$. Show that there is a ball $B \subseteq \mathbb{R}^n$ around 0 with $F(B) \subseteq B$.

*Solution.* $F(x) = F(0) + Tx + \alpha(x)$ where $\frac{\alpha(x)}{||x||} \to 0$ as $x \to 0$. We also have that $||T||^2 \leq c \leq 1$. Choose $r$ small enough so that $\frac{\alpha(x)}{||x||} < 1 - ||T||$ for $||x|| < r$. Then we have that $||F(x)|| = ||Tx + \alpha(x)|| < ||x|| \leq r$, so $F(B(0, r)) \subseteq B(0, r)$.

**Exercise.** Define $f : \mathcal{M}_{n\times n} \to \mathcal{M}_{n\times n}$ by $f(X) = X^2$. Find the derivative of $f$.

*Solution.* It is enough to show how $D_f(X)$ acts on a matrix $Y$.

$$\begin{aligned} D_f(X)Y &= \lim_{h\to 0} \frac{f(X + hY) - f(X)}{h} \\ &= \lim_{h\to 0} \frac{X^2 + hXY + hYX + h^2Y^2 - X^2}{h} \\ &= XY + YX \end{aligned}$$

**Exercise.** Consider the map $\det : \mathcal{M}_{n\times n} \to \mathbb{R}$ for $n \geq 2$. Show that the derivative at $A$ is 0 if and only if $A$ has rank $\leq n - 2$.

*Solution.* For any $i$, we may write $\det(X) = \sum_j (-1)^{i+j} x_{ij} \det X_{ij}$, so $\dfrac{\partial \det}{\partial x_{i_0 j_0}}(X) = (-1)^{i_0+j_0} \det X_{i_0 j_0}$. Thus, the derivative at $A$ of the determinant is zero exactly when every minor has determinant zero, which happens exactly when $\text{rank}(A) \leq n - 2$.

**Exercise.** Find the maximum value of $\prod_{i=1}^n x_i$ given that $\sum_{i=1}^n x_i = S$ and $x_i \geq 0$.

*Solution.* Use Lagrange multipliers with $f = \prod x_i$ and $g = \sum x_i - S$. Then $\nabla g = (1, \ldots, 1)$, which is a linearly independent set. So a maximum/minimum occurs when

$$\nabla f = (x_2 \cdots x_n, \ldots, x_1 \cdots \hat{x}_i \cdots x_n, \ldots, x_1 \cdots x_{n-1})$$

is a equal to $\lambda\nabla g = (\lambda, \ldots, \lambda)$. It follows that $x_i = x_j$ for all $i, j$. This is a global maximum because $\sum_{i=1}^n x_i = S$ and $x_i \geq 0$ defines a compact set, and the value is larger than any value on the boundary.

**Exercise.** Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be $C^1$ and $\lambda > 0$ such that for all $x, y \in \mathbb{R}^n, ||F(x) - F(y)|| \geq \lambda||x - y||$. Show that $F$ is bijective with continuous inverse.

*Solution.* It is easy to see that $F$ is injective and that its inverse is continuous (the inverse function has Lipschitz constant $\frac{1}{\lambda}$). To see that $F$ is surjective, we will show that it is both open and closed.

$F$ closed: if $y_n = F(x_n) \to y$, then $||y_n - y_m|| \geq \lambda||x_n - x_m||$, so $\{x_n\}$ is Cauchy with limit $x$. Since $F$ is continuous, $F(x) = y$. Thus, $F$ is closed.

$F$ open: $D_F(x)v = \lim_{h\to 0} \frac{F(x+hv)-F(x)}{h}$. Since $||F(x + hv) - F(x)|| \geq \lambda \cdot h||v||$, $||D_F(x)v|| \geq \lambda||v||$, so $\text{rank}(D_F(x)) = n$ for all $x$. By the open mapping theorem, $F$ is open.

**Exercise.** Let $\phi : (a, b) \to [0, \infty]$ be differentiable with $\phi(x_0) = 0$ for some $x_0 \in (a, b)$ and

$$|\phi'(x)| \leq K|\phi(x)|$$

for all $x$ for some fixed $K > 0$. Show that $\phi \equiv 0$. Extend to the case where the range of $\phi$ is $\mathbb{R}$.

*Solution* (1). We may assume $x_0 = 0$. Assume $\phi \not\equiv 0$ on $[0, \frac{1}{2K}]$, and say that $|\phi|$ attains a maximum at $\xi$. Then

$$\begin{aligned} |\phi(\xi)| &= |\phi(\xi) - \phi(0)| = |\xi \cdot \phi'(\eta)| && (\text{MVT} \Rightarrow \exists \eta \in (0,\xi)) \\ &\le K|\xi \cdot \phi(\eta)| \le K|\xi| \cdot |\phi(\xi)|. \end{aligned}$$

So $1 \le K|\xi| \le \frac{1}{2}$. Contradiction. Similarly, $\phi \equiv 0$ on $[\frac{-1}{2K}, 0]$. Thus, $\phi$ is locally constant on a connected set, so it is constant. Note that this proof works what the range of $\phi$ is $\mathbb{R}$.

*Solution* (2). Define $\psi(x) = \phi(x)e^{-Kx}$, so that $\psi(x_0) = 0$. Note that $\psi(x) \ge 0$ for all $x$, and that $\psi'(x) = (\phi'(x) - K\phi(x))e^{-Kx} \le 0$. So $\psi$ is a non-negative function which is zero at a point, and it has non-positive derivative. Thus, $\psi(x) = 0$ for all $x \ge x_0$, so $\phi(x) = 0$ for $x > x_0$. Similarly, we get $\phi(x) = 0$ for $x < x_0$. To get the result when the range is $\mathbb{R}$, note that if $\phi$ satisfies the hypotheses, then so does $|\phi|$. This argument shows that $|\phi| \equiv 0$, from which it follows that $\phi \equiv 0$.

**Exercise.** Say $f : \mathbb{R} \to \mathbb{R}$ is $C^1$. Show that any solution to $y'(x) = f(y(x))$ is monotonic.

*Solution.* Assume $y$ is a non-monotonic solution, then we may assume that $y'(0) = f(y(0)) = 0$. In this case, $y \equiv y(0)$ is also a solution. Since $f$ is Lipschitz on all compact neighborhoods of 0, we have uniqueness of solutions on these neighborhoods, so we have uniqueness globally, so $y \equiv y(0)$, which is monotonic.

## Complex Analysis

Let $u(x,y), v(x,y)$ be $C^1(\mathbb{R})$ functions. $f(x+iy) = u + iv$ is *analytic* or *holomorphic* if and only if $u$ and $v$ satisfy the **Cauchy-Riemann** equations:

$$u_x = v_y \qquad\qquad u_y = -v_x$$

in which case $(u,v)$ are a *harmonic pair.*

**Definition.** If $f : \mathbb{C} \to \mathbb{C}$ and $\gamma : [a,b] \to \mathbb{C}$ is $C^1$, then

$$\int_\gamma f(z)\,dz := \int_a^b f(\gamma(t))\gamma'(t)\,dt.$$

If $f : \mathbb{C} \to \mathbb{C}, \gamma : [a,b] \to \mathbb{C}$ continuous and piecewise $C^1$, then we have the ML-inequality:

$$\left|\int_\gamma f(z)\,dz\right| \le \max|f| \cdot \text{length}(\gamma).$$

**Cauchy's Theorem.** *If $\gamma$ is the boundary of a simply connected domain $D \subset \mathbb{C}$ and $f$ analytic in a neighborhood of $D$ (or analytic in $D^\circ$ and continuous on $\overline{D}$), then*

$$\int_\gamma f(z)\,dz = 0.$$

**Cauchy's Formula.** *If $f$ and $\gamma$ are as above, then $f$ is $C^\infty$ and*

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_\gamma \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

*where the integral is in the counter-clockwise direction.*

**Liouville's Theorem.** *$f$ entire and bounded$\Rightarrow$ $f$ constant.*

**Strong Liouville # 1.** *$f$ entire non-constant $\Rightarrow$ $f$ has dense image.*

*Proof.* If the image of $f$ is not dense, then there is some ball $B(a,r)$ disjoint from the image of $f$. Let $g(z) = \frac{1}{f(z)-a}$, then $g$ is entire and bounded by $\frac{1}{r}$. Thus, $g$ is constant, which implies that $f$ is constant. □

**Strong Liouville # 2.** *$f$ entire and $|f(z)| \leq A|z|^{1-\varepsilon}$ for all $z$ for some $A, \varepsilon > 0 \Rightarrow f$ constant.*

*Proof.* We show that if $f$ is dominated by $|z|^{1-\varepsilon}$, then $f' \equiv 0$. $f'(z_0) = \frac{1}{2\pi i}\int_{|z|=R} \frac{f(z)}{(z-z_0)^2}\,dz$ by Cauchy's formula. Choosing $R > |z_0|$ and applying the ML-inequality, we have that

$$|f'(z)| \leq \frac{1}{2\pi} \cdot 2\pi R \cdot \frac{R^{1-\varepsilon}}{(R-|z_0|)^2}.$$

Letting $R$ blow up, we have that $f'(z_0) = 0$, so $f$ is constant. □

**Morera's Theorem.** *If $f$ is continuous on some domain $D$, and $\int_\gamma f(z)\,dz = 0$ for all closed paths $\gamma$ in $D$, then $f$ is holomorphic.*

**Riemann's Theorem.** *If $f$ is holomorphic and bounded on a punctured neighborhood of $z_0$, then there is a holomorphic extension of $f$ to the unpunctured neighborhood.*

**Taylor's Theorem.** *If $f$ is holomorphic on the open ball $B(a,r)$, then*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \cdots$$

*for all $z \in B(a,r)$.*

**Corollary.** *If $f$ has an accumulation point of zeroes inside its domain, then $f \equiv 0$.*

**Laurent's Theorem.** *If $f$ is holomorphic on the open annulus $A(z_0, r, R)$, then*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$$

*for all $z \in A(z_0, r, R)$*

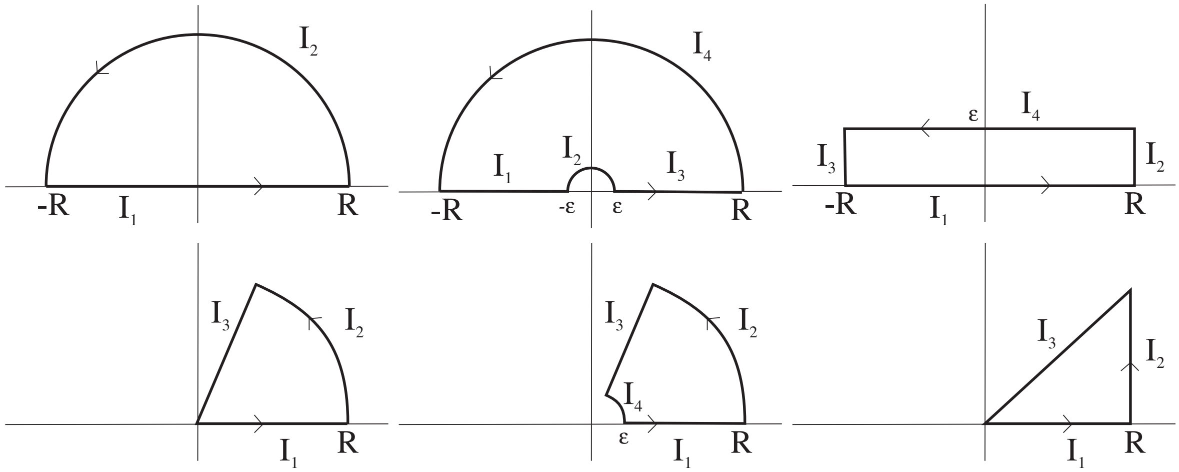
*Remark.*

(1) These expansions are unique.
(2) The series converge uniformly on compact sets, so you can do the stuff you want to do with the series.
(3) Radius of convergence of the Taylor series is the distance to the nearest "bad point".
(4) Taking $r = 0$ for the Laurent series is ok. The *order of the pole* of $f$ at $z_0$ is the smallest (positive) $n$ such that $a_{-n} \neq 0$. If $a_{-n} \neq 0$ for arbitrarily large $n$, then $f$ has an *essential singularity* at $z_0$.

**Cassorati-Weierstrass Theorem.** *If $z_0$ is an isolated essential singularity of $f$, then the image of any neighborhood of $z_0$ is dense in $\mathbb{C}$.*

**The Residue Theorem/Definition.** *Say $f$ is holomorphic in $D \smallsetminus \{z_i\}_{i=1}^n$, and let $f(z) = \sum_{n=-\infty}^{\infty} a_n(z_i)(z-z_i)^n$ be the Laurent expansion around $z_i$. Then $\int_\gamma f(z)\,dz = 2\pi i \sum_{i=1}^{n} \mathrm{Res}(f, z_i)$ where $\mathrm{Res}(f, z_i) = a_{-1}(z_i)$ and $\gamma$ is the boundary of $D$ (counterclockwise).*

Real integrals can often be computed using the Residue Theorem. Some basic shapes of curves that might be used are the semicircle (if you can show $I_2 \to 0$ as $R \to \infty$), the semi-circle with a bite taken out of it (if the function has a pole at 0 ... more bites can be removed if there are more poles), and the rectangle. There are also some useful wedge-shaped variants.




**The Argument Principle.**

$$\frac{1}{2\pi i}\int_{\partial D}\frac{f'(z)}{f(z)}\,dz = Z - P$$

*where $Z$ and $P$ are the number of zeroes and poles of $f$ in $D$, counting multiplicity.*

This is used to prove

**Rouche's Theorem.** *If $f, g$ are holomorphic on $D$ with $|g| < |f|$ on $\partial D$, then $f$ and $f+g$ have the same number of zeroes in $D$.*

**Corollary.** *A non-constant holomorphic function is open.*

*Proof.* We may assume $f(0) = 0$, and we wish to show that the image of $f$ contains a neighborhood of 0. Since $f$ is holomorphic and $f \not\equiv 0$, there is a (punctured) ball of radius $\delta$ around zero on which $f$ has no zero. Let $m = \inf_{|z|=\delta}|f(z)|$; this infimum is attained. For any $\omega$ of modulus less than $m$, $f$ and $f-\omega$ both have one zero in the ball $B(0,\delta)$ by Rouche's theorem. Thus, for any $\omega \in B(0,m)$, $\omega$ is in the image of $f$. So $B(0,m)$ is in the image of $f$. □

**Corollary** (Strong Maximum Principle)**.** *If a non-constant function $f$ is holomorphic on $D^\circ$ and continuous on $\overline{D}$, then it achieves $\sup_{z\in D}|f(z)|$ only on $\partial D$.*

**Schwarz's Lemma.** *If $f$ is holomorphic on $D = \{|z| < 1\}$ with $f(0) = 0$ and $f(D) \subseteq D$, then $|f(z)| \le |z|$ for all $z \in D$. Furthermore, if equality holds for any non-zero $z$, then $f(z) = e^{i\theta}z$ for some $\theta$.*

*Proof.* $f(z) = a_1 z + a_2 z^2 + \cdots$ converges in $D$. Consider the holomorphic function $g(z) = f(z)/z = a_1 + a_2 z + \cdots$. Then for any non-zero $z \in D$, choose $|z| < r < 1$, so that

$$\begin{aligned}
|g(z)| &\le \max_{|\xi|=r}|g(\xi)| && \text{(maximum principle)}\\
&= \max_{|\xi|=r}\left|\frac{f(\xi)}{\xi}\right| = \frac{1}{r}\max_{|\xi|=r}|f(\xi)\\
&\le \frac{1}{r}. && \text{(since } f(D)\subseteq D\text{)}
\end{aligned}$$

Letting $r$ increase to 1, we have that $|f(z)/z| = |g(z)| \le 1$, proving the first statement. If $|f(z_0)| = |z_0|$ for some $z_0 \ne 0$, then $g(z)$ must be constant by the strong maximum principle. □

*Remark.* $f'(0) = g(0)$, so $|f'(0)| \le 1$, and equality implies that $f(z) - e^{i\theta}z$.

If $f_n$ is a sequence of holomorphic functions, and $f_n \rightrightarrows f$, then $f$ is holomorphic[7]. If $f$ is non-constant and $f(z_0) = 0$, then for any $\varepsilon > 0$, there is some $N$ such that $f_n$ has a zero within $\varepsilon$ of $z_0$ for $n > N$.

*Proof.* Use Rouche: $f_n = f + (f_n - f)$ has the same number of zeros as $f$ when $n$ is large. □

This shows that if the $f_n$ are non-vanishing on some domain, then $f$ is either non-vanishing, or identically zero.

**Corollary.** *If $f_n$ are injective and $f_n \rightrightarrows f$, then $f$ is injective or constant.*

**Corollary.** *$f'(z_0) \neq 0$ if and only if $f$ is injective in a neighborhood of $z_0$.*

*Proof.* We may assume that $z_0 = 0$ and $f(0) = 0$.
($\Rightarrow$) Since $f'(0) \neq 0$, $f(z) = a_1 z + a_2 z^2 + \cdots$ with $a_1 \neq 0$. This is the uniform limit of the partial sums $f_n = a_1 z + \cdots a_n z^n$, so it is enough to show that the $f_n$ are locally injective. If $z_1, z_2$ have modulus less than $\delta$, then we have

$$\begin{aligned} f_n(z_1) - f_n(z_2) &= a_1(z_1 - z_2) + a_2(z_1^2 - z_2^2) + \cdots a_n(z_1^n - z_2^n) \\ &= \underbrace{(z_1 - z_2)}_{\neq 0}(a_1 + \underbrace{a_2(z_1 + z_2) + \cdots + a_n(z_1^{n-1} + \cdots + z_2^{n-1})}_{\text{bdd by } \delta(2|a_2| + 3\delta|a_3| + 4\delta^2|a_4| + \cdots) < |a_1| \text{ for small } \delta}) \\ &\neq 0 \end{aligned}$$

($\Leftarrow$) Since $f'(0) = 0$, $f(z) = a_m z^m + \cdots$ for $m > 1$. Say $f$ is injective on $B(\delta, 0)$, and let $m = \inf_{|z|=\delta} |f(z)| > 0$. Then for any $\omega$ of modulus less than $m$, Rouche's Theorem gives us that $f(z) - \omega$ has the same number of zeros as $f$ on $B(\delta, 0)$. Since $f$ is injective, $f(z) - \omega$ must have an $m$-fold zero at some $z = z_\omega$, so $f'(z_\omega) = 0$. Thus, $f'$ is identically zero on some neighborhood of 0, so it is identically zero, contradicting the assumed form of $f$. □

**Riemann Mapping Theorem.** *Any simply connected non-empty domain, $A$, which is not all of $\mathbb{C}$ is holomorphically homeomorphic[8] to $D = \{|z| < 1\}$. For $z_0 \in A$, there is a unique such map $f$ so that $f(z_0) = 0$ and $f'(z_0) > 0$.*

A complex function $f$ is conformal at $z_0$ if and only if it is holomorphic at $z_0$ with $f'(z_0) \neq 0$. If $f$ is holomorphic with $f'(z_0) = 0$, then $f(z) = a_0 + a_n(z - z_0)^n + a_{n+1}(z - z_0)^{n+1} + \cdots$ with $a_n \neq 0$ for some $n > 0$. In this case, $f$ multiplies angles at $z_0$ by $n$.
A *linear fractional transformation* is of the form $Tz = \frac{az+b}{cz+d}$. The inverse of $T$ is $T^{-1}z = \frac{dz-b}{-cz+a}$. LFTs take circles to circles on the Riemann sphere[9].

**Theorem.** *Any holomorphic homeomorphism of $D$ onto itself is of the form $Tz = e^{i\theta}\dfrac{z - z_0}{1 - \overline{z}_0 z}$.*

**Theorem.** *If $z_1, z_2, z_3$ are distinct points if $\mathbb{C}$, then $Tz = \dfrac{z - z_1}{z - z_3} \cdot \dfrac{z_2 - z_3}{z_2 - z_1}$ is the unique LFT taking $\{z_1, z_2, z_3\}$ to $\{0, 1, \infty\}$. Thus, LFTs act uniquely transitively on distinct ordered triples in $\mathbb{C}$.*

## Sample Problems

**Exercise.** Let $f$ be an entire function and let $L \subset \mathbb{C}$ be a line such that $f(\mathbb{C}) \cap L = \varnothing$. Show that $f$ is a constant function.

---

[7] $f_n \rightrightarrows f$ indicates uniform convergence.
[8] *conformally equivalent*
[9] Circles on the Riemann sphere are circles and lines in the complex plane.

*Solution.* $\mathbb{C}$ is connected and $f$ is continuous, so $f(\mathbb{C})$ is connected, so it lies on one side of $L$. Any open ball on the other side of $L$ shows that the image of $f$ is not dense, so Strong Liouville # 1 implies that $f$ is constant.

**Exercise** (Sp01)**.** Let $F(z) = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be a matrix of entire functions. Show that if $F(z) + F(z)^*$ is positive definite, then $F$ is constant.

**Exercise.** Let $f : [0,1] \to \mathbb{C}$ be continuous. Prove that $g(z) = \displaystyle\int_0^1 f(t)e^{tz}\,dt$ is holomorphic.

*Solution.* By Morera's Theorem, it is enough to show that for any closed path $\gamma$, $\int_\gamma g(z)\,dz = 0$.

$$\begin{aligned}
\int_\gamma g(z)\,dz &= \int_\gamma \int_0^1 f(t)e^{tz}\,dt\,dz \\
&= \int_0^1 f(t)\left(\int_\gamma e^{tz}\,dz\right)dt && \text{(by Fubini)} \\
&= \int_0^1 f(t)\cdot 0\,dt = 0. && (e^{tz} \text{ holomorphic})
\end{aligned}$$

**Exercise** (Fa96)**.** Does there exist a function $f$, holomorphic on $\mathbb{C} \setminus \{0\}$ such that $|f(z)| \geq \frac{1}{\sqrt{|z|}}$ for all $z \neq 0$?

*Solution.* Assume yes. Then $g(z) = \frac{1}{f(z)}$ is holomorphic on $\mathbb{C}$ (by Riemann's Theorem, we may add the point at 0) and is dominated by $\sqrt{z}$. By Strong Liouville #2, $g$ is constant, and so $f$ is constant. But $f$ cannot be constant.

**Exercise.** Let $A = \{0\} \cup \{\frac{1}{n} | n \in \mathbb{N}\}$ and $D = \{|z| < 1\}$. Show that any bounded holomorphic function on $D \setminus A$ can be extended to a holomorphic function on $D$.

*Solution.* Each point of the form $\frac{1}{n}$ is isolated, so we may repeatedly apply Riemann's Theorem to extend $f$ to $D \setminus \{0\}$. Then apply Riemann's Theorem again to extend $f$ to $D$.

**Exercise.** If $f, g$ are entire functions, and $|f(z)| \leq |g(z)|$ for all $z$, then show that $f(z) = c\,g(z)$ for some constant $c$.

*Solution.* If $g \equiv 0$, we are done. Otherwise all of the zeroes of $g$ are isolated, so $\frac{f(z)}{g(z)}$ is holomorphic and bounded in a neighborhood of any zero of $g$. By Riemann's Theorem, there is a (bounded) holomorphic extension to all of $\mathbb{C}$. By Liouville, $\frac{f}{g}$ is constant.

**Exercise.** Find the number of roots of $z^7 - 4z^3 - 11$ with $1 < |z| < 2$.

*Solution.* If $|z| \leq 1$, then $|z^7 - 4z^3 - 11| \geq 11 - 4 - 1 = 6$, so there are no zeroes. On the circle $|z| = 2$, we have that $|z^7| > |-4z^3 - 11|$, so $z^7$ and $z^7 - 4z^3 - 11$ have the same number of roots with $|z| < 2$, and that number is 7.

**Exercise.** If $f$ is entire and $|f(z)| = |\sin z|$ for all $z$, then prove that there is some constant $c$ such that $f(z) = c\sin z$.

*Solution.* $g(z) = \frac{f(z)}{\sin z}$ is holomorphic on $\mathbb{C} \setminus \{n\pi | n \in \mathbb{Z}\}$. If $g$ is non-constant, then it is an open mapping. But the image of $g$ is contained in the unit circle, which does not contain any open sets. Thus, $g$ is constant.

**Exercise** (Sp96)**.** Suppose $f = u + iv$ is a holomorphic on some domain $D$. Suppose also that there are real numbers $a, b, c$ with $a^2 + b^2 \neq 0$ such that $au + bv = c$ in $D$. Show that $f$ is constant.

*Solution.* If $f$ is non-constant, then it is an open map. But the image of $f$ lies on the line $ax+by=c$, which does not contain any open sets.

**Exercise.** Determine the group $\mathrm{Aut}(\mathbb{C})$ of all holomorphic bijections $f:\mathbb{C}\to\mathbb{C}$.

*Solution.* Write $f(z)=a_0+a_1z+a_2z^2+\cdots$.
Case 1: If the series is finite, then $f$ is a polynomial. Any polynomial of degree $N$ is an $N$-to-1 map, so $f=a_0+a_1z$.
Case 2: If the series is infinite, then let $g(z)=\sum_{n=0}^{\infty}a_nz^{-n}=f(1/z)$. Since $z\to 1/z$ is bijective on $\mathbb{C}\setminus\{0\}$, $g$ must be bijective (between $\mathbb{C}\setminus\{0\}$ and its image). $g$ is also open, as it is the composition of open maps. Given any $z_0\neq 0$, $g(\{z\big||z-z_0|<\frac{|z_0|}{2}\}$ contains an open neighborhood, $U$ of $g(z_0)$. By Cassorati-Weierstrass, $g(\{z\big||z|<\frac{z_0}{2}\})$ also contains a point in $U$, contradicting bijectiveness of $g$.

**Exercise.** Suppose $f$ is a holomorphic function on the unit disc $D$, $f(-\ln 2)=0$, and $|f(z)|\leq e^z$ for all $|z|=1$. How large can $|f(\ln 2)|$ be?

*Solution.* Let $g(z)=e^{-z}f(z)$. Then $g$ is holomorphic and maps $D$ into $D$. Let $Tz=\frac{z+\ln 2}{z\ln 2+1}$, $T^{-1}z=\frac{z-\ln(2)}{-z\ln 2+1}$, and define $h(z)=g\circ T^{-1}(z)$. We have that $h$ maps $D$ into $D$, and that $h(0)=0$, so by Schwarz's Lemma, $|h(\frac{2\ln 2}{1+(\ln 2)^2})|\leq|\frac{2\ln 2}{1+(\ln 2)^2}|$. But $h(\frac{2\ln 2}{1+(\ln 2)^2})=g(\ln 2)=\frac{1}{2}f(\ln 2)$, so $|f(\ln 2)|\leq\frac{4\ln 2}{1+(\ln 2)^2}$. $f(z)=e^z\frac{z+\ln 2}{z\ln 2-1}$ attains this maximum.

**Exercise.** Let $f$ be holomorphic on the upper half plane with $|f(z)|<1$, $f(i)=0$. How large can $f(2i)$ be?

**Exercise.** Let $f$ be holomorphic on $\{\mathrm{Re}(z)>0\}$ such that $|f(z)|\leq 1$ and $f(1)=0$. How large can $|f'(1)|$ be?

**Exercise.** Say $U\subset\mathbb{C}$ is a simply connected domain with $f:U\to\mathbb{C}$ holomorphic. If the Taylor series for $f$ converges on an open disc $D$ with $D\cap(\mathbb{C}\setminus U)\neq\varnothing$. Does it follow that $f$ can be holomorphically extended to $U\cup D$?

*Solution.* No! Consider $U=\mathbb{C}\setminus\mathbb{R}_{\leq 0}$ and $f(z)=\log(z)$.

**Exercise** (Sp04)**.** Let $f$ :