Definition. Let $\mathcal{C}$ and I be categories. For an object $X \in \mathcal{C}$ we define $k_X : I \to \mathcal{C}$ to be the constant functor, which sends all objects of I to X and all morphisms of I to the identity morphism of X. Let $D : I \to \mathcal{C}$ be a functor (a "diagram" in $\mathcal{C}$). Then we define the *limit (projective limit) of* D to be the object $\varprojlim D$ which represents the functor $X \mapsto \mathrm{Nat}(k_X, F)$, if such an object exists. The *colimit (injective limit or direct limit) of* F is the object $\varinjlim D$ which represents the functor $X \mapsto \mathrm{Nat}(D, k_X)$, if it exists.

Theorem. *Right (resp. left) adjoint functors commute with limits (resp. colimits).*

Proof. Let $\mathcal{C}$ and $\mathcal{D}$ be categories, let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be adjoint functors, with F the right adjoint, and let $D : I \to \mathcal{C}$ be a diagram in $\mathcal{C}$. Then for any object $X \in \mathcal{D}$, we have that

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{D}}\big(X, F(\varprojlim D)\big) &\cong \mathrm{Hom}_{\mathcal{C}}(GX, \varprojlim D) && \text{(adjunction)} \\
&= \mathrm{Nat}(k_{GX}, D) && \text{(definition of } \varprojlim\text{)} \\
&\cong \mathrm{Nat}(k_X, FD) && \text{(adjunction)} \\
&= \mathrm{Hom}\big(X, \varprojlim(FD)\big) && \text{(definition of } \varprojlim\text{)}
\end{aligned}$$

By Yoneda's lemma, it follows that $F(\varprojlim D) \cong \varprojlim(FD)$. The proof that left adjoint functors commute with colimits is basically the same, switching left and right (in particular, using the covariant Yoneda lemma rather than the contravariant one). $\square$