Midterm 1 solutions.

1: Take $x = 34 + (83 - 34) \cdot 36 \cdot 255$ and $y = 3 \cdot 255$.

2a: By hypothesis, every exponent in the prime factorization of n is at least 2. To show that n is a product of a square and a cube, it is enough to show that every exponent is a positive integer combination of 2 and 3. If a given exponent is even, then it is $2k + 3 \cdot 0$ for some positive k. If it is odd, then it is at least three, so when you subtract three, you get a non-negative even number, so the exponent is $2k + 3$ for some non-negative k.

2b: We want an answer of the form $n = 2^a \cdot 3^b \cdot 7^c$. Since $n/2$ should be a square, we have that a should be $1 \mod 2$ and b and c should be $0 \mod 2$. Since $n/3$ is a cube, $b \equiv 1 \mod 3$ and $a \equiv c \equiv 0 \mod 3$. Since $n/7$ is a seventh power, $c \equiv 1 \mod 7$ and $a \equiv b \equiv 0 \mod 7$. Either using the CRT algorithm or by just eyeballing, we find that $a = 21$, $b = 28$, and $c = 36$ work. So take $n = 2^{21} \cdot 3^{28} \cdot 7^{36}$.

3a: If $x^2 \equiv 127 \mod 127^2$, then $x^2 \equiv 127 \equiv 0 \mod 127$, so x must be divisible by 127, say $x = 127k$. But then $x^2 \equiv 127^2 k^2 \equiv 0 \mod 127^2$. Thus, there are no solutions to $x^2 \equiv 127 \mod 127^2$.

3b: We have that $n^4 + n^2 + 1 = (n^2 + 1)^2 - n^2 = (n^2 + n + 1)(n^2 - n + 1)$. Since $n > 1$, we have that $n^2 + n + 1 > n^2 - n + 1 = n(n - 1) + 1 > 1$, so this is a proper factorization of $n^4 + n^2 + 1$.

4a: Note that $\gcd(x, 255) = 1$ if and only if $\gcd(x, 3) = \gcd(x, 5) = \gcd(x, 17) = 1$. By CRT, $x^{16} \equiv 1 \mod 255$ if and only if x^{16} is 1 modulo 3, 5, and 17. Applying Euler's theorem three times, we have that $x^{16} \equiv (x^2)^8 \equiv 1^8 \equiv 1 \mod 3$, $x^{16} \equiv (x^4)^4 \equiv 1^4 \equiv 1 \mod 5$, and $x^{16} \equiv 1 \mod 17$.

4b: By Fermat's theorem, every residue modulo 11 is a solution to $f(x) = x^{11} - x \equiv 0 \mod 11$. We have that $f'(x) = 11x^{10} - 1 \equiv -1 \mod 11$ is always non-zero modulo 11, so these 11 root are "non-singular." By Hensel's lemma, each non-singular root lifts to a unique root modulo 11^{1234}, so there are 11 solutions total.