Math 130B Set 2

due Jan. 13

Read Chapter 3 of Vakil (Dec. 20, 2011 version), do problems 3.2.F, 3.3.C, 3.3.I, 3.4.B, 3.5.G, and do the following problem.

1. ("the property $\mathcal{P}$ argument") Suppose $\mathcal{P}$ is a property of morphisms in a category so that

   - $\mathcal{P}$ is *stable under composition*. That is, given morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$, if f and g have $\mathcal{P}$, then so does $g \circ f$.
   - $\mathcal{P}$ is *stable under base change*. That is, if the diagram below is a cartesian square and f has $\mathcal{P}$, then f' has $\mathcal{P}$.

   $$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

   (a) Suppose $f\colon X \to Y$ and $h\colon Z \to W$ have $\mathcal{P}$. Show that the product $f \times h\colon X \times Z \to Y \times W$ has $\mathcal{P}$.

   (b) Suppose we have morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ so that $g \circ f$ has $\mathcal{P}$ and the diagonal $\Delta_g\colon Y \to Y \times_Z Y$ has $\mathcal{P}$ (Δ_g is given by $(\mathrm{id}_Y, \mathrm{id}_Y)$). Show that f has $\mathcal{P}$. (Hint: use the "magic diagram" with $X_2 = Y$.)