1 Tom Bridgeland - Stability conditions

The general idea is to start with a triangulated category D (e.g. the bounded derived category of coherent sheaves on a variety). To this, we associate a complex manifold $\mathrm{Stab}(\mathsf{D})$, the *space of stablity conditions*. Each point $\sigma \in \mathrm{Stab}(\mathsf{D})$ defines a subcategory of *semi-stable objects* $\mathsf{P} \subseteq \mathsf{D}$.

Motivations:

1. String theory (this is really where this stuff came from). The goal is to understand the "stringy Kähler moduli space" $\mathcal{M}_K(X)$. By mirror symmetry, this is supposed to be $\mathcal{M}_{\mathcal{C}}(\check{X})$. This came from Mike Douglas' work on Π-stability for D-branes). We won't talk about any string theory here. There are no examples $\mathsf{D}(CY_3)$ (though this may change very soon).

2. $\mathrm{Stab}(\mathsf{D})$ helps us to understand the structure of D (e.g. gives a space on which $\mathrm{Aut}(\mathsf{D})$ acts).

3. To try to define classes of objects in D which have nice moduli spaces. It would be really useful to find moduli spaces parameterizing complexes (not just sheaves). There are some situations where we can do this (e.g. that's how you show equivalence of derived categories under 3-fold flops). There has been some work by Abramovich and Polishchuk.

4. Try to understand wall-crossing for Donaldson-Thomas invariants. The relevant names are Joyce, Toda, and Kontsevich-Soibelman.

These are fine motivations, but they haven't really born fruit yet.

The plan of these talks is roughly as follows:

1. Stability conditions on abelian categories (this may be a bit boring)

2. Hall algebras

3. Triangulated case

4. Examples (in particular, the conifold)

5. Counting invariants (Seandroi's product formula)

Abelian case

Let A be an abelian category (e.g. coherent sheaves, or modules over a ring). $K(\mathsf{A})$ is the Grothendieck group, the free abelian group on isomorphism classes of objects of A, modulo the relation that $[B] = [A] + [C]$ whenever there is a short exact sequence

$$0 \to A \to B \to C \to 0.$$

Definition 1.1. A *stability function* on A is a homomorphism of abelian groups $Z\colon K(\mathsf{A}) \to \mathbb{C}$ such that for $E \neq 0$, $Z(E) \in \mathcal{H} = \{re^{i\pi\theta} | r > 0, 0 < \theta \leq 1\}$ (notice that this is half-closed). ⋄

Example 1.2. Take $\mathsf{A} = \mathsf{Coh}(X)$, where X is some smooth projective curve over $\mathbb{C}$, and take $Z(E) = -\deg(E) + i\,\mathrm{rk}(E) = i(\mathrm{rk}(E) + i\deg(E))$ (notice that this lands in this sends non-zero things into the upper half plane because you either have positive rank, or you are torision, so you have positive degree). ⋄

Example 1.3. Take $\mathsf{A} = R\text{-}\mathsf{mod}$, the category of finitely generated modules over some finitely generated $\mathbb{C}$-algebra R. In this case, $K(\mathsf{A}) = \mathbb{Z}^{\oplus N} = \mathbb{Z}[S_1] \oplus \cdots \oplus \mathbb{Z}[S_n]$ where S_i are the simple modules up to isomophism. As long as we make sure that the S_i are mapped into the upper half plane, then everything else will be. So the set of stability conditions is $\mathcal{H}^N$. We tend to think of stability conditions in algebraic geometry as essentially unique, but in algebra, there are lots of choices. ⋄

Example 1.4. Let $X = \mathbb{P}^2$ and $\mathsf{A} = \mathsf{Coh}(X)$. The obvious thing to do is to set $Z(E) = -\deg(E) + i\,\mathrm{rk}(E)$, but that doesn't work, because a sky-scraper sheaf has rank and degree zero. In fact, there are no stability functions on A. This might make you think that this is a very bad definition. It turns out that the derived category has some interesting stability conditions, but you have to use some other t-structure. This also somehow agrees with what you would expect from physics. So although this looks very bad, I claim it is actually correct. There are generalizations of this definition which would allow from something else, but if you use those, you don't get a complex manifold. There are millions of definitions you could use, but the interesting thing is that you get a complex manifold with this definition. ⋄

Let $Z\colon K(\mathsf{A}) \to \mathbb{C}$ be a stability function. Then every non-zero object has a phase $\phi(E) = \frac{1}{\pi}\arg(Z(E)) \in (0,1]$. A non-zero object E is called *semi-stable* if for every non-zero sub-object $A \subseteq E$, $\phi(A) \leq \phi(E)$. Equivalently, for every non-zero quotient $E \twoheadrightarrow Q$, $\phi(E) \leq \phi(Q)$. This is sometimes called the "see-saw property." For $\phi \in (0,1]$, define $\mathsf{P}(\phi)$ to be the full subcategory of A consisting of semi-stable objects E with $\phi(E) = \phi$ (and the zero object).

Remark 1.5. In Example 1.2, this corresponds to Mumford stability. $\mathsf{P}(1) = \{\text{torsion sheaves}\}$ and $\mathsf{P}(\phi) = \{\mu\text{-stable for } \mu = \cot(\pi\phi)\}$. In Example 1.3, a result of A. King implies that there exists a projective scheme which is a coarse moduli space for semi-stable object of a fixed class $\alpha \in K(\mathsf{A})$. The functor send S to isomorphism classes of bundles $E \to S$ with an action of the algebra R on E. ◇

Lemma 1.6. *If $\phi_1 > \phi_2$ and $E_i \in \mathsf{P}(\phi_i)$, then* $\mathrm{Hom}_{\mathsf{A}}(E_1, E_2) = 0$.

Proof. If $f\colon E_1 \to E_2$ is a non-zero map, then we get two short exact sequences

$$\ker f \to E_1 \to \operatorname{im} f \qquad \operatorname{im} f \to E_2 \to \operatorname{coker} f$$

This implies that $\phi_1 = \phi(E_1) \leq \phi(\operatorname{im} f) \leq \phi(E_2) = \phi_2$, a contradiction. □

Definition 1.7. A *Harder-Narasimhan filtration* for an object $E \in \mathsf{A}$ is a filtration

$$0 = E_1 \subset E_1 \subset \cdots \subset E_n = E$$

such that $F_i = E_i/E_{i+1}$ is semi-stable and $\phi(F_1) > \cdots > \phi(F_n)$. ◇

Lemma 1.8. *If such a filtration exists, it is unique.*

Proof. Suppose you have two filtrations. Look at the last bits

$$\begin{tikzcd}
E_{n-1} \arrow[r, hook] \arrow[d] & E \arrow[r, two heads] \arrow[d, equal] & F_n \arrow[d] \\
E'_{n'-1} \arrow[r, hook] & E \arrow[r, two heads] & F'_{n'}
\end{tikzcd}$$

Assume that $\phi(F'_{n'})$ is the smallest phase of any filtration factor. This implies that $\mathrm{Hom}(E_{n-1}, F'_{n'}) = 0$. This means that you can fill in the vertical maps. Thus, we know that $\phi(F_n) = \phi(F'_{n'})$. Now we can do the argument the other way and get some maps "up". A standard arugment shows that the compositions have to be the identity (because the map $E_{n-1} \to E$ is an inclution and $E = E$ is the identity), so the vertical maps are isomorphisms. □

Definition 1.9. A *stability condition* on A is a stability function $Z\colon K(\mathsf{A}) \to \mathbb{C}$ such that every object has a Harder-Narasimhan filtration. ◇

This is not always easy to check. You often have to do some work to show that the filtrations exist.

Suffient condition: (is it necessary?) for existence of a HN filtration. First of all, there should be no infinite chains of quotients $E_1 \twoheadrightarrow E_2 \twoheadrightarrow \cdots$ with descending phases $\phi(E_1) > \phi(E_2) > \cdots$ (weakly noetherian). Secondly, there must be no infinite chains of subobjects $\cdots \subset E_2 \subset E_1$ with $\cdots > \phi(E_2) > \phi(E_1)$ (weakly artinian). You should try to prove this yourself (it is a bit tricky), and if you get stuck, look at my paper, "Stability conditions on triangulated categories" Prop. 2.4.

Examples 1.2 and 1.3 satisfy this. Example 1.2 (coherent sheaves on a curve over $\mathbb{C}$) is weakly noetherian for free because we get actual noetherian-ness. To show weakly artinian, we note that eventually, $\operatorname{rk} E_i = \operatorname{rk} E_{i-1}$, so the degrees must keep getting bigger (degree means first chern class). Then $\phi(E_i) > \phi(E_{i-1})$, which implies $d(E_i) > d(E_{i-1})$, which implies E_{i-1}/E_i has rank 0 and degree negative [[★★★]].

We've gotten some categorical stuff out of this. We start with a big category, and broken it up into small categories so that each object in the original category can be built up in a unique way from objects in the smaller categories.

The Harder-Narasimhan filtration is extremely important.

Suppose $\mathsf{A} = R\text{-}\mathsf{mod}_{fd}$, where R is a finite-dimensional algebra over a finite field $k = \mathbb{F}_q$ (bear with me, we'll eventually do it in characteristic zero with coherent sheaves). Define $\hat{H}(\mathsf{A}) = \{f\colon (\mathsf{A}/\cong) \to \mathbb{C}\}$, the set of functions on isomorphism classes of A. And define $H(\mathsf{A}) \subseteq \hat{H}(\mathsf{A})$ as the functions with finite support. Define the convolution product

$$(f * g)(M) = \sum_{A \subset M} f(A)g(M/A)$$

(note that M has finitely many sub-objects, so it is a finite sum).

Theorem 1.10. *Under the convolution product, $H(\mathsf{A})$ and $\hat{H}(\mathsf{A})$ become associative algebras with unit $1(0) = 1$ and $1(M) = 0$ for $M \ncong 0$.*

1 Alessio Corti

Most of these lectures will be based on joint work with T(om?) Coates, H(iroshi?) Iritari, and H. H. Tseng. There is a list of problems for this course. References for today: two papers by Abramovich-Graber-Vistoli and the original paper by CR.

Quantum cohomology of stacks

From tomorrow onwards, we will be working with toric stacks, but today we'll be more general. Let $f\colon \Gamma \to \mathcal{X}$ be a stable representable morphism from an orbi-curve to a stack.

An *orbi-curve* is a proper, projective, algebraic curve Γ with some marked points x_i. Each of the marked points has a chart of the form $[\Delta/\mu_{r_i}]$ (where Δ is a disk and μ_r is the group of r-th roots of unity). An orbi-curve has a fundamental group $\pi_1^{orb} = \pi_1(\Gamma \smallsetminus \{x_i\})/\langle \gamma_i^{r_i}\rangle$, where the γ_i is a small loop around x_i.

Orbi-curves have line bundles on them, which are line bundles $\mathcal{L}$, together with an action of μ_{r_i} on $\mathcal{L}_{x_i}$, given by $v \mapsto \zeta^{k_i} v$. Riemann-Roch tells you that

$$\chi(\Gamma, \mathcal{L}) = \deg \mathcal{L} + 1 - g - \sum k_i/r_i$$

I hope you'll accept this stuff without worrying too much about the precise definitions.

Exercise. $\mathbb{P}_{r_1,r_2}$. *Then you can convince yourself that the Picard group is* $\mathbb{Z} \oplus \mathbb{Z}/gcd(r_1, r_2)$.

That's all I'll say about orbi-curves. Now on to stacks.

A stack $\mathcal{X}$ is locally Δ^n/G, where G is a finite group. Stacks have points, and points have stabilizers. A point x has stabilizer G_x. A morphism of stacks $f\colon \mathcal{X} \to \mathcal{Y}$ is *representable* if it induces injections $G_x \to G_{f(x)}$. If G is a finite group, there is a very important stack called BG, which is the quotient $[*/G]$. That is, to give a morphism $\mathcal{X} \to BG$ is the same as to give a principal G-bundle on $\mathcal{X}$. We need this in the case where $\mathcal{X}$ is an orbi-curve.

Example 1.1. If $\mathcal{X} = \Gamma$ is an orbi-curve, then a morphism $\mathcal{X} \to BG$ is a homomorphism $f\colon \pi_1^{orb}\Gamma \to G$ is representable if $f(\gamma_i)$ has order r_i. ⋄

Today I want to discuss orbifold cohomology and orbifold quantum cohomology. For this, we have to introduce the *inertia stack* $I_{\mathcal{X}} = \bigcup_{r\geq 0} \mathrm{Hom}^{rep}(B\mu_r, \mathcal{X})$ (where Hom^{rep} means representable morphisms). Such a morphism is the same as giving a point $x \in \mathcal{X}$ and an injection $\chi\colon \mu_r \hookrightarrow G_x$.

Aside from doing some examples, I'm not really sure how to explain this, so let's do some examples.

Example 1.2. Let $\mathcal{X} = \mathbb{P}^{w_0,\dots,w_n}$ (weighted projective space), which we will think of as $\mathbb{C}(-w_0) \oplus \cdots \oplus \mathbb{C}(-w_n)$ (where $\mathbb{C}(-w)$ is the representation of $\mathbb{C}^\times$ of weight $-w$, so $\lambda\colon x \mapsto \lambda^{-w}x$). In this case, $\mathrm{Box}\mathcal{X} = \{k/w_i | 0 \leq k < w_i\}$ and $I_{\mathcal{X}} = \bigcup_{b\in \mathrm{Box}} \mathbb{P}(V^b)$, where $V^b = \bigoplus_{w_i b\in\mathbb{Z}} \mathbb{C}(-w_i)$.

If we take $\mathcal{X} = \mathbb{P}(1,1,3)$, $I_{\mathcal{X}} = \mathbb{P}(1,1,3) \sqcup \mathbb{P}(3)_{1/3} \sqcup \mathbb{P}(3)_{2/3}$ (subscripts are Box levels). ⋄

Example 1.3. $\mathcal{X} = [M/G]$. In this case, $I_{\mathcal{X}} = \bigsqcup_{g\in C}[M^g/Z(g)]$ (C is conjugacy classes) ⋄

In general, there is a graph of groups $B = \mathrm{Box}\mathcal{X}$, whose elements are injective group homomorphisms χ from μ_r into some stabilizers. The inertia stack is $I\mathcal{X} = \bigsqcup_{\chi\colon \mu_r \to G_\eta} \mathcal{X}_\chi$.

Definition 1.4. $H^\bullet_{orb}\mathcal{X} = H^{\bullet - a(\chi)}(I\mathcal{X})$, where the *age* of χ is $a(\chi)$ (defined below). ⋄

$\chi\colon \mu_r \to G_\eta$, and G_η acts on the tangent space $T_\eta\mathcal{X}$, so we get an induced action given by k_i/r for some $0 \leq k_i < r$. Then we define $a(\chi) = \sum k_i/r$. I still have to tell you what the cup product is.

I have to talk about stable morphisms. For $\beta \in H_2(\mathcal{X})$, let $\mathcal{X}_{0,n,\beta} = \{$stable (no automorphisms) representable morphisms $f\colon (\Gamma, \mu_{r_i}(x_i))_{1\leq i\leq n} \to \mathcal{X}$ of degree β, where Γ is genus zero$\}$. Let me remind you that Γ could be a nodal curve; it doesn't have to be a smooth orbi-curve. The marked points x_i have these little charts $[\Delta/\mu_{r_i}]$. Only the marked points (and sometimes the nodes) have these charts.

Some features:

1. There are evaluation maps $ev_i\colon \mathcal{X}_{0,n,\beta} \to I\mathcal{X}$, given by $ev_i(f) = f(x_i)$. Rather, there aren't, but we can pretend that there are. We have that $\mathcal{X}_{0,n,\beta} = \bigsqcup_{b_1,\dots,b_n\in\mathrm{Box}} X_{0,n,\beta}(b_1,\dots,b_n)$, to be made sense of later.

2. $\mathcal{X}_{0,n,\beta}$ has a virtual dimension

$$\begin{aligned} \mathrm{vdim}_f &= \chi(\Gamma, f^*T_{\mathcal{X}}) + n - 3 \\ &= -K_{\mathcal{X}} \cdot \beta + \dim \mathcal{X} - \sum_{i=1}^{n} \sum_{j=1}^{\dim \mathcal{X}} \frac{w_{i,j}}{r_i} + n - 3 \quad \text{(Riemann-Roch)} \end{aligned}$$

and a virtual class $\mathbb{1}_{vir} \in CH_{\mathrm{vdim}}$.

Product on $H^\bullet_{orb}(\mathcal{X})$. Consider $\mathcal{X}_{0,3,0} \xrightarrow{e_{1,2,3}} I\mathcal{X}$. Define $a \cup b = \overline{e_3} * (e_1^* a \cup e_2^* b) \cap \mathbb{1}_{vir}$. We have $\iota\colon I\mathcal{X} \to I\mathcal{X}$; if $\chi\colon \mu_r \to G_x$, then ι_χ is χ composed with conjugation.

This is a $\mathbb{P}^1$ with three marked points. Two of them are $|a|$ and $|b|$, and the third one is the intersection of $|a|$ and $|b|$.

Example 1.5. $\mathcal{X} = \mathbb{P}(1,1,3)$, so $I\mathcal{X} = \mathbb{P}(1,1,3) \sqcup \mathbb{P}(3)_{1/3} \sqcup \mathbb{P}(3)_{2/3}$. Then $H^\bullet_{orb}$ has a basis $1, \eta, x, \eta', x^2$ (η and η' come from the $\mathbb{P}(3)$'s) of degrees $0, 2/3, 1, 4/3, 2$, respectively (you have to get over the degrees being fractional). These are Chow degrees; if something, then you should double these. We have that $\eta \cup \eta = \eta'$, $\eta' \cup \eta = x^2$, and $\int_{\mathcal{X}} x^2 = 1/3$. Let's explain $\eta \cup \eta' = x^2$. Consider $\mathcal{X}_{0,3,0}(1/3, 2/3, 0)$. In $\mathbb{P}(1,1,3)$, we have the stacky point $\frac{1}{3}(1,1)$, which is a $\mathbb{P}(3)$. There are three marked points; two of them are marked by $1/3$ and $2/3$. $\eta \cup \eta' = e_3 * \mathbb{1} = \frac{1}{3}\{pt\}$. ◇

Quantum orbifold cohomology. The quantum product is defined (using Poincaré duality) by $(a * b, c) = \sum_\beta \langle a, b, c\rangle_\beta Q^\beta$, where $\langle a, b, c\rangle_\beta = \int_{\mathcal{X}_{0,3,\beta}} (e_1^* a \cup e_2^* b \cup e_3^* c) \cap \mathbb{1}_{vir}$.

Example 1.6. Define $\mathcal{X} = X_3^2 \subset \mathbb{P}(1,1,2)$ be a surface of degree 3. Suppose I blow up $\mathbb{P}^2$ at three colinear points. Then the line containing them is a -2 curve. If I contract that curve, I get this $\mathcal{X}$.

Take $A \subset H^\bullet_{orb}(\mathcal{X}, \mathbb{Q})$ with basis $\mathbb{1}, x, u, x^2$, in degrees $0, 1, 1, 2$, respectively. You can figure out that $\int_{\mathcal{X}} x^2 = 3/2$ and $\int_{\mathcal{X}} u^2 = 1/2$.

$$\begin{aligned} \langle x, x, u\rangle_{1/2} &= \frac{1}{4}\langle u\rangle_{1/2} = 3/4 \\ \langle x, x, pt\rangle_1 &= 3 \\ \langle pt, pt, u\rangle_{3/2} &= 1 \end{aligned}$$

Given this, you can write the matrix of quantum multiplication by x (in this basis). It is

$$\begin{pmatrix} 0 & 3Q & 0 & 0 \\ 1 & 0 & \frac{1}{2}Q^{1/2} & 3Q \\ 0 & \frac{3}{2}Q^{1/2} & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Also, $\langle u, u, u\rangle_{1/2} = 3/4$. In this example, $\deg Q = -5/3$. ◇

$D\psi = \psi M$. $\psi\colon \mathbb{C}^\times \to \operatorname{End} H^\bullet_{orb}(\mathcal{X}, \mathbb{C})$, $D = Qd/dQ$, $\psi = (\psi_0, \ldots, \psi_3)$, and ψ_0 satisfies

$$2D^3(2D-1) - 3Q(3D+2)(3D+1)$$

This is the differential equation you expect for something.

1 Valery Alexeev

I will try to give an account of the complete moduli of higher dimensional varieties. Let me begin by giving an overview of what we know about the dimension 1 case. We have a moduli space $\mathcal{M}_g$, introduced 150 years ago by Riemann. There is this wonderful compactification $\overline{\mathcal{M}}_g$ (the Deligne-Mumford compactification, also due to Grothendieck, ...). The two are quite similar. Then there is the space $\overline{\mathcal{M}}_{g,n}$, which again looks bigger, but the differences are quite minor. In particular, there is $\overline{\mathcal{M}}_{0,n}$, which is really very easy. It is a very explicit combinatoria object, some blowup of $\mathbb{P}^{n-3}$. There is also the moduli space $\overline{\mathcal{M}}_{g,n,\beta}$, where you add some weights between zero and 1. Again, we have the special case $\overline{\mathcal{M}}_{0,n,\beta}$. Then we have the Kontsevich maps; the moduli space of stable maps $\overline{\mathcal{M}}_{g,n}(V)$. There are many papers about these first of all because of the importance of applications (e.g. Gromov-Witten theory), and secondly because you can compute things.

I will speak about the dimension $n > 1$ case. The analogue of $\mathcal{M}_g$ is the moduli space of surfaces of general type $\mathcal{M}_{c_1^2,c_2}$. This space is already very hard and very complicated. $\mathcal{M}_g$ is mysterious, but at least it is smooth as a stack. $\mathcal{M}_{c_1^2,c_2}$ is not even equi-dimensional, and even describing its irreducible components is hard. As I said, the difference between $\mathcal{M}_g$ and $\overline{\mathcal{M}}_g$ is very minor, so maybe we can still go somewhere. Also, even if the general case is hopeless, there may be some examples we can work with. In particular, there are analogues of $\overline{\mathcal{M}}_{0,n}$ and $\overline{\mathcal{M}}_{0,n,\beta}$ that can be described in complete detail. Another special case is the case of abelian varieties; stable abelian varieties are quite nice and can be described quite explicitly.

The plan for the course is this course

1. Complete moduli and MMP
2. Stable toric varieties
3. Hyperplane arrangements
4. Abelian varieties
5. Surfaces

The first lecture is introductory. The first four lectures should be quite explicit. The last lecture is the case of the moduli space of surfaces of general type.

The plan for today:
The very first example: degrees of curves
Redo for surfaces (KSB (Kollar? Sheppard? Bard?) 1989).
Redo for n-dimensional pairs; for stable maps.
Sings of MMP: k, klt, dlt, slc (sklt?, sdlt?).
Ex: curves, hyy arrs, toric vars.
Polytopes and toric vars.
$(X, B_1 + \varepsilon B_2)$ lc $\Leftrightarrow B_2 \not\supset T$-orbits
$\overline{\mathcal{M}}_{g,n,\beta}$ after Hassett
Exs: surfaces

The very first example. Suppose you have a 1-dimensional family X of curves of genus g over some base S which is not complete. How do you complete it. First you apply the stable reduction theorem, which says that after some base change on S, th fiber can be made into a curve with simple normal crossings. This may not be stable. How do we make it stable? If there are (-1)-curves, you can contract them, leaving the surface smooth, so you contract them all. If there is a (-2)-curve, you can contract it to a singular point, but the singularities are rational double points of type A_n. After that, X_0 is a stable curve which is nodal with canonical class $K_{X_0} > 0$ ample if and only if $|\operatorname{Aut}(X_0)| < \infty$. For a curve E in the central fiber, $K_{X_0} \cdot E < 0$ if and only if $E \cong \mathbb{P}^1$ and $E^2 = -1$ and $K_{X_0} \cdot E = 0$ if and only if $E \cong \mathbb{P}^1$ and $E^2 = -2$.

Theorem 1.1. *For every $X \to S^0$, there is a finite base change $S' \to S$ and a completion such that $X' \to S'$ is a flat family of stable curves.*

The condition that K was ample means that $K_{X'/S'} + X'_0$ is ample and X'_0 nodal if and only if (X', X'_0) has log canonical (lc) singularities. The nice thing about curves is that you still have a reduction theorem in mixed characteristic.

So what do we do in dimension n? You have to give a label to everything. We started with $X' \to S'$ and you did some stuff, ending up with the *log canonical model* $X'_{can} \to S'$ of the pair (X', X'_0). You know that there is a theory of minimal models in all dimensions, so we can repeat the procedure in higher dimensions.

Suppose we have a family of surfaces. Then after base change, we get a surface with normal crossings. Instead of contracting this and that, you just go straight to the canonical model. $K_{X'/S'} + X'_0$ ample and (X', X'_0) lc, then we say that X'_0 is *semi-log canonical* (slc). $X' = \operatorname{Proj}_{S'} \bigoplus_{d \geq 0} \pi_* \mathcal{O}_{X'}(d(K_{X'} + X'_0))$. Compare to $\operatorname{Proj} \bigoplus_{d \geq 0} H^0(\mathcal{O}_X(d(K_{X'} + X'_0)))$; there is very little difference.

You have to prove existence and uniqueness of the model. uniqueness is easy, and existence is a recent result. The outcome is that for any 1-parameter family of varieties of general type, there is a unique limit with ample canonical class and something slc.

Where do we go from here? You can try to construct this moduli space in general, or you can look at the special cases. You can use the theorem to guess the answer, and then construct your moduli space by various other methods (in the cases of toric, hyperplane arrangements, abelian varieties).

You can redo this for n-dimensional pairs. So we have a family of surfaces with divisors. The stable reduction theorem still works. The log canonical model is not for (X', X'_0) but for $(X', X'_0) + \sum b_i B_i$.

When you do this stuff carefully, you run into hard technical problems for surfaces. Q: does the formation of the log canonical ring commute with base change. That depends on the moduli functor. If you do it carefully, you run into problems (not in the special cases), and I will try to delay them until Saturday.

You can redo this for stable maps. Suppose you have a variety (say a curve) X and a stable map $X \to V$ (parts of X collapse). We say X is stable if $K_X > 0$ and X nodal, and we say the map is stable if $K_{X/V} > 0$ and X nodal. So you just think of families $X \to S^0 \times V$, and in the construction, you do everything over $S' \times V$. All the same general machinery works to give you the unique limit of any family of stable maps. This higher dimensional moduli should exist in this case as well.

Singularities of MMP: lc, klt, dlt, slc. You know the first three (from the pre-reading). slc will be the generalization to the "nodal case."

For a pair (X, B) to be lc, X should be a normal variety over $k = \bar{k}$, and $B = \sum b_i B_i$, where $0 < b_i \le 1$ and B_i are (not necessarily distinct) Weil divisors. There should exist a log resolution $f\colon Y \to X$ (i.e. Y smooth and the exceptional set of f union $f^{-1}_{strict\ trans} \operatorname{Supp}(B)$ has simple normal crossings). We need $K_X + B$ to be a $\mathbb{Q}$-Cartier divisor, so $N(K_X + B)$ is Cartier. In this case, we can write $K_Y = f^*(K_X + B) + \sum_{E_j\ irr\ divs} a_j E_j$. Then lc means that all $a_j \ge -1$ (which implies $b_i \le 1$), klt means that all $a_j > -1$ (which implies $b_i < 1$), and dlt means that there is a Y with $Exc(f)$ is a union of divisors and all $a_j > -1$ for all exceptional divisors. Note that dlt depends on the resolution; if you keep going, you might get some -1's. Some finite generation result for klt which can be pushed to dlt.

Example 1.2 (Curves)**.** Let X be a curve, with some divisors B_i (may not be distinct). What does it mean for the pair (X, B) to be lc? It means that X is smooth and whenever B_i coincide for $i \in I$, then $\sum_{i\in I} b_i \le 1$. It is klt if for every such colletion, $\sum b_i < 1$ (in particular, this implies all $b_i < 1$). In this case, dlt is the same as lc. ◇

Example 1.3 (Hyperplane arrangements)**.** You have hyperplanes B_i intersecting in $\mathbb{P}^{r-1}$. What does it mean for $(\mathbb{P}^{r-1}, B)$ to be lc? It means that for every $I \subset \{1, \ldots, n\}$, $\sum_{i\in I} b_i \le \operatorname{codim} B_i$ (if the intersection is non-empty. klt means that this inequality is strict. ◇

Example 1.4 (Toric varieties)**.** Suppose X is a toric variety with a torus T acting on it. Let $B_1 = X \smallsetminus T$. Then toric geometry tells us two standard facts: (1) $K_X + B_1 = 0$ in a canonical way, and (2) (X, B_1) is lc (this follows from the first fact because a toric variety always has a toric resolution; pull back $K_X + B_1 = 0$ to get $0 = K_Y + f^{-1}B$+exceptional divisors with $a_j = -1$). If you add another divisor B_2, then $(X, B_1 + \varepsilon B_2)$ for $0 < \varepsilon \ll 1$ is lc if and only if $B_2 \not\supset T$-orbits. The reason is that when you resolve, you add exceptional divisors with $a_j = -1$, so you are maxxed out. This already tells you that if you work with coeffiecients 1 and ε, then you are in the toric situation. ◇

I cannot teach you about polytopes and toric varieties in 5 minutes; I hope you already know how to see a variety if I show you a polytope.

1 Martin Olsson

I want to talk about log geometry in the sense of Fontaine, Illusie, and Kato. The history of the subject makes it a little inaccessable sometimes. It came out of trying to prove some conjectures of Fontaine. I want to start by explaining an old example.

Let Δ be a smooth curve over $\mathbb{C}$ and $o \in \Delta$, with $\Delta^\times = \Delta \smallsetminus \{o\}$. Let $t \in \Gamma(\Delta, \mathcal{O}_\Delta)$ a uniformizer at o. Choose $f\colon X \to \Delta$ proper and semi-stable (i.e. étale locally $\mathcal{O}_\Delta[x_1, \ldots, x_n]/(x_1, \ldots, x_n - t)$).[1] We then get a local system $V = (R^i f_*^{an}\mathbb{C})/\Delta^\times$. This is the same thing as a representation of $\pi_1(\Delta^\times)$.

There is an algebraic construction which tells you more. Consider $X^\times := f^{-1}(\Delta^\times) \xrightarrow{j} X \xleftarrow{i} X_0$. We have $\Omega^1_{X/\Delta}(\log)$, the subsheaf of $j_*\Omega^1_{X^\times/\Delta^\times}$ which in local coordinates is generated by $\Omega^1_{X/\Delta}$ and the dx_i/x_i. This is a locally free sheaf. You get a complex

$$\Omega^\bullet_{X/\Delta}(\log) : \mathcal{O}_X \to \Omega^1_{X/\Delta}(\log) \to \Omega^2_{X/\Delta}(\log) \to \cdots$$

You can then define $E = R^i f_* \Omega^\bullet_{X/\Delta}(\log)$. This is a locally free sheaf on Δ. It has more structure, namely the Gauss-Manin connection $\nabla\colon E \to E \otimes \Omega^1_\Delta(\log)$. This is all in the algebraic category, but I can think of it as an analytic vector bundle, so we get the local system $V = \ker\big(\nabla^{an}\colon E_{an} \to E_{an} \otimes \Omega^1_\Delta(\log)\big)|_{\Delta^\times}$. We have $i_\Delta\colon \operatorname{Spec}\mathbb{C} \xhookrightarrow{0} \Delta$, and $\mathbb{C} \cong i_\Delta^*\Omega^1_\Delta(\log)$, given by $1 \mapsto dt/t$. You get $N\colon E(0) \to E(0)$. You have $\nabla(tE) \subseteq tE$.

Let $D \subseteq \Delta_{an}$ be a disk around 0 and $s \in D^\times = D \smallsetminus \{0\}$. Then $\pi_1(D^\times) = \mathbb{Z}$, generated by a loop around zero. Then I have an action of $\mathbb{Z}$ on the vector space V_s, with 1 acting by $T\colon V_s \to V_s$. It is a theorem that this T is a unipotent operator, so I can take it's log to get a nilpotent matrix acting on this vector space.

Theorem 1.1. *The conjugacy class of* $\log T$ *is* N.

That stuff is very old. The question it begs is the following. This E lives in the closed fiber. Do you really need the whole family over the disk to get this N. So the question is, "what extra structure do you need in addition to X_0 to recover $E(0)$ and N?"

[1] I'll assume you know about the étale topology, but if you aren't too familiar with it, think of it as the analytic topology.

Let me pose another question that is closer in spirit to this meeting. This second problem concerns main components/deformation theory. This is an experimental science; you go example by example. It is just a fact of life that when you have a moduli space of higher dimensional things, you get lots of irreducible components.

Example 1.2. Let (E, e) be an elliptic curve over $k = \bar{k}$. It is an exercise to check that you can find an embedding $j\colon E \hookrightarrow P$, where P is a rational surface, with $E \in |-K_P|$ and $j^*I_E \simeq \mathcal{O}_E$. Let $X_0 = P \cup_E P$. This is called a log K3. If you wanted to study a moduli space of K3 surfaces, this is the kind of thing you'd stick at the boundary. An old paper of Friedman showed that the versal deformation space (we're looking at the complete local ring at the point in the moduli space corresponding to X_0) looks like $V_1 \cup V_2$ where

1. V_1 and V_2 are smooth,
2. $\dim V_2 = 20$ and $V_2 \smallsetminus (V_2 \cap V_1)$ correspond to smoothings of X_0, and
3. V_1 classifies locally trivial deformations.

Let me say what this last thing means. You have two components, V_1 (singular deformations) and V_2 (smooth deformations). The question is, "how do you isolate V_2?" This is perhaps the most important question for this series of lectures. You can answer it with log geometry. ◇

There is a third question, which is the connection with stacks (e.g. orbicurves). Q: what is a locally trivial deformation? MO: X_0 locally looks like $\mathbb{C}[x, y, z]/xy$. Locally trivial means that the local ring after deforming still looks like that. The smoothings look like $\mathbb{C}[t][x, y, z]/(xy - t)$.

Now I'll start with foundations. You'll have to bear with me for a lecture and a half or so.

Monoids

You have to be very careful with monoids. You want them to be like groups, but they are very subtle. We will usually write the composition additively.

Definition 1.3. A *monoid* is a commutative semi-group with unit. Morphisms of monoids preserve the unit. ◇

Abelian groups are monoids, so $\mathsf{Ab} \subseteq \mathsf{Mon}$. This inclusion has a left adjoint $M \mapsto M^{gp} = \{(a,b) | (a,b) \sim (c,d)$ if there is an s such that $s+a+d = s+b+c\}$. In particular, any map from M to an abelian group factors uniquely through $M \to M^{gp}$.

Definition 1.4. M is *integral* if for all $m \in M$, $+m\colon M \to M$ is injective (equivalently, if $M \to M^{gp}$ is injective). It is called *saturated* if it is integral and if $M = \{m \in M^{gp} | \exists n > 0$ such that $nm \in M\}$ ◇

If you know about toric varieties, when you dualize a cone, you always get a saturated monoid.

Definition 1.5. M is *fine* if it is integral and finitely generated. It is fs if it is fine and saturated. ◇

Definition 1.6. A *prelog structure* on a scheme X is a sheaf of monoids M and a map of sheaves of monoids $\alpha\colon M \to (\mathcal{O}_X, \cdot)$.[2] A prelog structure is a *log structure* if $\alpha\colon \alpha^{-1}(\mathcal{O}_X^\times) \to \mathcal{O}_X^\times$ is an isomorphism. ◇

Example 1.7. Let k be a field, let X be smooth over k, and let $D \subseteq X$ be a divisor with normal crossings (it could look like components *étale* locally; it could be a nodal cubic, for example). Let $M = \{f \in \mathcal{O}_X | f|_{X \smallsetminus D} \in \mathcal{O}_{X \smallsetminus D}^\times\}$. Here, M is a subsheaf of $\mathcal{O}_X$, but it need not be in general.

If I have an étale morphism $\pi\colon X \to \mathbb{A}^n$ such that $D = \pi^{-1}(V(x_1 \cdots x_r))$ (the first r hyperplanes). Then M is the subsheaf of $\mathcal{O}_X$ generated by $\mathcal{O}_X^\times$ and $x_1, \dots, x_r$. ◇

The next example looks stupid but is very important

Example 1.8. Let $X = \operatorname{Spec} k$, and $M = k^\times \oplus \mathbb{N}$, with $k^\times \oplus \mathbb{N} \to k$ given by $(u,n) \mapsto u(0)^n$ (where $0^0 = 1$ and $O^n = 0$ for $n \neq 0$). This arises from $0 \in \Delta$. Example 1.7 gives M_Δ on Δ. $i_\Delta^* M_\Delta$ on $\operatorname{Spec} \mathbb{C}$ is this example. ◇

Lemma 1.9. *The inclusion (log structures on X) $\hookrightarrow$ (prelog structures on X) has a left adjoint $M \mapsto M^a$.*

[2]This is the only case were we'll use multiplicative notation for a monoid.

Proof. Define M^a as the pushout in the category of sheaves of monoids

$$\begin{tikzcd}
\alpha^{-1}(\mathcal{O}_X^\times) \arrow[r] \arrow[d] & M \arrow[d] \arrow[dr, "\alpha"] & \\
\mathcal{O}_X^\times \arrow[r] \arrow[rr, bend right] \arrow[ur, phantom, "\lrcorner"] & M^a \arrow[r, dashed] & \mathcal{O}_X
\end{tikzcd}$$

□

Example 1.10. Let X be a scheme, P a monoid, and $\beta\colon P \to \Gamma(X, \mathcal{O}_X)$ a morphism of monoids (e.g. Example 1.7 with $\mathbb{N}^r \to k[x_1, \dots, x_n]$ given by $e_i \mapsto x_i$). β corresponds to $P \to \mathcal{O}_X$. This leads to a log structure $P^a \to \mathcal{O}_X$. ◇

Notation: Let P be a monoid, and R a ring. Write $\operatorname{Spec}(P \to R[P])$ for $\operatorname{Spec} R[P]$ with the log structure associated to the natural map $P \to R[P]$.

Definition 1.11. A *log scheme* is a pair (X, M_X), where X is a scheme and M_X is a log structure (the α is omitted from the notation). ◇

Considering these pairs gives you a good category with deformation theory.

Definition 1.12. Let $f\colon Y \to X$ be a morphism of schemes and let M be a log structure on X. Then the composite $f^{-1}M \to f^{-1}\mathcal{O}_X \to \mathcal{O}_Y$ is a prelog structure. We define the pullback f^*M to be the associated log structure. ◇

I have to tell you what morphisms of log schemes are. If (X, M_X) and (Y, M_Y) are log schemes, then a morphism $(Y, M_Y) \to (X, M_X)$ is a pair $(f, f^\flat)$ where $f\colon Y \to X$ is a morphism of schemes and $f^\flat\colon f^*M_X \to M_Y$ is a morphism of log structures (i.e. a morphism over $\mathcal{O}_X$).

Exercise. *Say (X, M_X) is a log scheme and P is a monoid. Then* $\operatorname{Hom}_{\mathsf{logSch}}((X, M_X), \operatorname{Spec}(P \to \mathbb{Z}[P]) \simeq \operatorname{Hom}_{\mathsf{Mon}}(P, \Gamma(X, M_X))$. *This is an exercise in adjoints.*

Exercise. *$f\colon Y \to X$ and P a monoid, and $\beta\colon P \to \Gamma(X, \mathcal{O}_X)$. Then $f^*(P^a) = (P \to \Gamma(X, \mathcal{O}_X) \to \Gamma(Y, \mathcal{O}_Y))^a$.*

Definition 1.13. A log structure M on X is called *fine* if étale locally there is a fine monoid P and a map $\beta\colon P \to \Gamma(X, \mathcal{O}_X)$ such that $M \simeq P^a$. ◇

Remark 1.14. In the Zariski topology, the plane with the nodal cubic divisor is not right. You really want to use the étale topology. ◇

A *chart* for a fine log structure M on X is a fine monoid P and a map $P \to \Gamma(X, M)$ such that $\big(P \to \Gamma(X, M) \to \Gamma(X, \mathcal{O}_X)\big)^a \to M$ is an isomorphism. This is the same thing as a map $(f, f^\flat) \colon (X, M) \to \operatorname{Spec}(P \to \mathbb{Z}[P])$ such that $f^\flat$ is an isomorphism.

2 Tom Bridgeland - Hall Algebras

You'll have to take my word for it that this is an interesting example to think about. It looks funny, but that is to make it very explicit.

Let $\mathsf{A} = R\text{-}\mathsf{mod}$, where R is a finite dimensional algebra (if it is not finite dimensional, you can take nilpotent modules) over $\mathbb{F}_q$ (this is to make only finitely many isomorphism classes of module in each class of K-theory). Last time, we introduced $\hat{H}(\mathsf{A})$, the set of all $\mathbb{C}$-valued functions on isomorphism classes, and $H(\mathsf{A})$, the finitely supported ones.

We introduced the convolution product

$$(f * g)(M) = \sum_{A \subset M} f(A) g(M/A).$$

Lemma 2.1. *Under this multiplication, $\hat{H}(\mathsf{A})$ is associative, with unit $1 = 1_0$ (the zero module).*

Proof. It is clear that $1 * f = f * 1 = f$ because 0 only has itself as a submodule. Next, associativity:

$$\begin{aligned}
[(f * g) * h](M) &= \sum_{B \subset M} (f * g)(B) h(M/B) \\
&= \sum_{A \subset B \subset M} f(A) g(B/A) h(M/B) \\
[f * (g * h)](M) &= \sum_{A \subset M} f(A)(g * h)(M/A) \\
&= \sum_{A \subset M, C \subset M/A} f(A) g(C) h\big((M/A)/C\big)
\end{aligned}$$

These agree because submodule $C \subset M/A$ are in bijection with modules B such that $A \subset B \subset M$, and $(M/A)/C \cong M/B$. □

Hence, we see that

$$(f_1 * \cdots * f_n)(M) = \sum_{0 = M_0 \subset \cdots \subset M_n = M} f_1(M_1/M_0) \cdots f_n(M_n/M_{n-1}). \qquad (\dagger)$$

Now suppose we have a stability condition $Z\colon K(\mathsf{A}) \to \mathbb{C}$. For $0 < \phi \le 1$, we can define elements of the Hall algebra

$$1_{ss}^{\phi}(M) = \begin{cases} 1 & M \in \mathsf{P}(\phi) \\ 0 & \text{else} \end{cases} \qquad 1_{\mathsf{A}}(M) = 1$$

Lemma 2.2 (Reneke)**.** *The Harder-Narasimhan property implies that* $1_{\mathsf{A}} = \prod_{\phi}^{\rightarrow} 1_{ss}^{\phi}$.

Note that this is an infinite product, but it will be finite on any module. The point is that 1_{ss}^{ϕ} will only give something non-zero if the filtration factor is semi-stable with phase ϕ, so there is only one filtration that contributes to the sum (†).

$H(\mathsf{A}) = \oplus_{\alpha\in K_{\ge 0}(\mathsf{A})} H_\alpha(\mathsf{A})$, and $\hat{H}(\mathsf{A})$ is the completion with respect to this filtration. If you're interested, look at a paper of Schiffman "Intro to Hall algebras."

Integration map

Write $(f*g)(M) = \sum_{A,B\subset \mathsf{A}/\cong} n_{AB}^M f(A)g(B)$, where $n_{AB}^M = \big|\{A' \subset M | A' \cong A, M/A' \cong B\}\big|$.

Lemma 2.3. $n_{AB}^M = \dfrac{|\operatorname{Ext}^1(B,A)_M|}{\operatorname{Hom}(B,A)} = \dfrac{|\operatorname{Aut} M|}{|\operatorname{Aut} A|\cdot|\operatorname{Aut} B|}$, *where* $\operatorname{Ext}^1(B,A)_M \subset \operatorname{Ext}^1(B,A)$ *is the set of extensions isomorphic to* M*.*

Proof. Define V_{AB}^M the variety parameterizing

$$0 \to A \xrightarrow{f} M \xrightarrow{g} B \to 0$$

Then $(\alpha,\beta,\gamma) \in \operatorname{Aut}(A)\times\operatorname{Aut}(B)\times\operatorname{Aut}(M)$ acts on V_{AB}^M by $(f,g) \mapsto (\gamma\circ f\circ\alpha, \beta\circ g\circ\gamma^{-1})$. Then $\operatorname{Aut}(A)\times\operatorname{Aut}(B)$ acts freely and $\big|V_{AB}^M/\operatorname{Aut}(A)\times\operatorname{Aut}(B)\big| = n_{AB}^M$. The action of $\operatorname{Aut}(M)$ is not free, and

$$\operatorname{Stab}_{(f,g)} = \{1 + f\eta g | \eta \in \operatorname{Hom}(B,A)\}$$

and $V_{AB}^M/\operatorname{Aut}(M) \cong \operatorname{Ext}^1(B,A)_M$. □

Now I will make a big assumption. Assume that A has global dimension 1 (i.e. $\operatorname{Ext}^p_{\mathsf{A}}(M,N) = 0$ for $p > 1$). By Kontsevich and Soibelman, you get existence of an integration map for Calabi-Yau 3-folds (where you don't have dimension 1). The examples we're left with now are path algebras of quivers with no loops. That will have global dimension 1.

Whenever we have finite global dimension for any category, we can define $\chi(M,N) = \dim\operatorname{Hom}(M,N) - \dim\operatorname{Ext}^1(M,N)$.

Define $\mathbb{C}_q[K_{\ge 0}(\mathsf{A})] = \langle x^\alpha | \alpha \in K_{\ge 0}(\mathsf{A})\rangle/(x^\alpha * x^\beta = q^{-\chi(\beta,\alpha)}x^{\alpha+\beta})$,[1] where q is the size of $\mathbb{F}_q$ (though perhaps secretly you want to think of q as indeterminate).

Lemma 2.4. $I\colon H(\mathsf{A}) \to \mathbb{C}_q[K_{\ge 0}(\mathsf{A})]$*, given by* $I(f) = \sum_{M\in\mathsf{A}/\cong} \frac{f(M)}{|\operatorname{Aut} M|} x^{[M]}$*, is a ring homomorphism.*

I can complete on both sides to get a map $\hat{H}(\mathsf{A}) \to \mathbb{C}_q[\![K_{\ge 0}(\mathsf{A})]\!]$

Proof. Any function is a sum of things of characteristic functions, so let f and g be characteristic functions on A and B, respectively. Then

$$\begin{aligned} I(f*g) &= I\Big(\sum_{M\in\mathsf{A}/\cong} \frac{|\operatorname{Ext}^1(B,A)_M|}{|\operatorname{Hom}(B,A)|}\cdot\frac{x^{[M]}}{|\operatorname{Aut} A||\operatorname{Aut} B|}\Big) \\ &= \frac{|\operatorname{Ext}^1(B,A)|}{|\operatorname{Hom}(B,A)|}\frac{x^{[A\oplus B]}}{|\operatorname{Aut} A||\operatorname{Aut} B|} \\ &= q^{-\chi(B,A)}\frac{x^{[A\oplus B]}}{|\operatorname{Aut} A||\operatorname{Aut} B|} \end{aligned}$$

We also get $I(f) = \frac{x^{[A]}}{|\operatorname{Aut} A|}$ and $I(g) = \frac{x^{[B]}}{|\operatorname{Aut} B|}$. □

Example 2.5. Let $\mathsf{A} = R\text{-}\mathsf{mod}$, where R is the path algebra on the quiver $(\bullet \to \bullet)$. A module is just an assignment of a vector space to each vertex and a linear map for each arrow, so in this case, a module is a map of vector spaces. The indecomposible modules are $S = (\mathbb{C} \to 0)$, $T = (0 \to \mathbb{C})$ (these two are simple), and $E = (\mathbb{C} \xrightarrow{\text{id}} \mathbb{C})$. There is a short exact sequence

$$0 \to T \to E \to S \to 0$$

So $K(\mathsf{A}) = \mathbb{Z}^{\oplus 2}$.

[1] $K_{\ge 0}(\mathsf{A})$ means the cone spanned by actual isomorphism classes in A.

The space $\mathrm{Stab}(\mathsf{A})$ is $\mathcal{H}^2$, and there is one wall. On one side of the wall, $\phi(S) > \phi(T)$ and on the other side $\phi(T) > \phi(S)$, and the wall is where $\phi(S) = \phi(T)$. On the side where $\phi(T) > \phi(S)$, E is unstable. On the other side, E is stable. On the wall, E is semi-stable. Evaluating on the different sides of the wall, we have $1_{ss}^{\phi(T)} * 1_{ss}^{\phi(S)} = 1_{\mathsf{A}} = 1_{ss}^{\phi'(S)} * 1_{ss}^{\phi'(E)} * 1_{ss}^{\phi'(T)}$.

Assuming we are not on the wall (in which case everything would be semi-stable), $\mathsf{P}(\phi(T)) = \{T^{\oplus n} | n \geq 0\}$. Similarly for S. If we let $\alpha = [S]$ and $\beta = [T]$, we get

$$\Phi(x^\beta) + \Phi(x^\alpha) = \Phi(x^\alpha) + \Phi(x^{\alpha+\beta}) + \Phi(x^\beta) \tag{‡}$$

where $\Phi(x) = \sum_{n\geq 0} \frac{x^n}{|GL_n(q)|} = \sum_{n\geq 0} \frac{x^n}{(q^n-1)\cdots(q^n-q^{n-1})}$. This function is sometimes called the q-exponential or the quantum dilog. The identity (‡) is called the 5-term relation for the quantum dilog. ◇

Another identity: Suppose P is a projective module. Then define $1_{\mathsf{A}}^P(M) := |\operatorname{Hom}(P,M)| = q^{\chi(P,M)}$ (again, remember that q is the size of $\mathbb{F}_q$) and $\operatorname{Quot}_{\mathsf{A}}^P(M) = |\operatorname{Hom}^{\twoheadrightarrow}(P,M)|$ (number of surjections $P \twoheadrightarrow M$).

Lemma 2.6. $1_{\mathsf{A}}^P = \operatorname{Quot}_{\mathsf{A}}^P * 1_{\mathcal{A}}$.

Proof. $|\operatorname{Hom}(P,M)| = \sum_{A\subset M} |\operatorname{Hom}^{\twoheadrightarrow}(P,A)|$. This is just the statement that every map has an image. $1_{\mathsf{A}} = \prod_\phi 1_{ss}^\phi$. Both the outer things have product decompositions and you're interested in the guy in the middle. □

Exercise. *Apply this to* $\mathsf{A} = \mathsf{Vect}$ *and* $P = \mathbb{C}^{\oplus N}$ *[[★★★ maybe* Vect *over a finite field?]]. Use the integration map. What do you get?*

1 Kai Behrend - Foundations of Donaldson-Thomas theory

These foundations on Donaldson-Thomas theory are undergoing some change, so some of what I'll say is work in progress with Mike Rose and I. Ciocon-Fontanine, inspired by a paper by Kapranov and Ciocon-Fontanine RQuot.

Everything will happen over $\mathbb{C}$. Y will be a Calabi-Yau 3-fold (i.e. a connected, projective, smooth $\mathbb{C}$-scheme of dimension 3 with a chosen isomorphism $\omega_Y = \wedge^3\Omega_Y \cong \mathcal{O}_Y$). For example, Y could be the generic quintic hypersurface in $\mathbb{P}^4$. X will be a moduli space of coherent sheaves on Y (with trivialized determinant), assumed to be compact (e.g. stable of Hilbert polynomial $p(n)$). To X, we associate a number $\#^{vir}X$, the *virtual count* of points in X. Goal of Donaldson-Thomas theory: (1) define $\#^{vir}X$, and (2) compute $\#^{vir}X$.

Example 1.1. $X = \operatorname{Hilb}^n Y$, the moduli scheme of ideal sheaves $\mathcal{I} \subseteq \mathcal{O}_Y$ such that $\mathcal{O}_Y/\mathcal{I}$ is a skyscraper sheaf (could be supported at many points) of length n. Then

$$\sum_{n=0}^{\infty} \#^{vir}(\operatorname{Hilb}^n Y)t^n = \left(\prod_{n=1}^{\infty} \frac{1}{(1-(-t)^n)^n}\right)^{\chi(Y)}$$

was conjectured by Maulik-Nekrasov-Okounkov-Panthenpole in 2003. ◇

Outline of the lectures. Roughly, I hope to get through two subjects per lecture.

1. moduli space
2. deformation theory
3. virtual fundamental class (which gives rise to the definition of $\#^{vir}X$ from deformation theory, when X is compact)
4. symmetric obstruction theory (this is where the assumption of Calabi-Yau 3-fold comes in
5. obstruction cone is Lagrangian
6. the microlocal function $\nu\colon X \to \mathbb{Z}$. Define $\chi(X,\nu)$, the weighted Euler characteristic (breaking up X into strata where ν is constant; for this, you don't need X compact).
7. The main theorem: $\#^{vir}X = \chi(X,\nu)$ in the compact case
8. equivariant case if there is a $\mathbb{C}^\times$ action on Y
9. Hilbert scheme
10. conics on the quintic.

The moduli space

All sheaves I'm going to be interested in on Y are torsion-free. For a sheaf E, $p(n) = \chi(Y, E(n))$ is the Hilbert polynomial of E (it is a polynomial of degree 3).

Definition 1.2. E is *stable* if for every proper subsheaf $0 \subsetneq E' \subsetneq E$, $p_{E'}(n)/\operatorname{rk} E' < p_E(n)/\operatorname{rk} E$ for $n \gg 0$. ◇

We assume $p(n)$ is chosen so that there exist no semi-stable sheaves which are not stable of the Hilbert polynomial $p(n)$. For example, we can assume rank is 1. If you want to learn more about stability of sheaves, look at Huybrechts-Lehn.

Determinant: If you have a torsion-free sheaf E, you can resolve it by locally free sheaves

$$\cdots \to E^{-1} \to E^0 \to E \to 0$$

and you can make it finite by the syzygy theorem (the third guy is automatically locally free). Then $\det E = \wedge^{\operatorname{rk} E^0}E^0 \otimes (\wedge^{\operatorname{rk} E^{-1}}E^{-1})^{-1} \otimes \cdots$. Trivial determinant means $\det E \cong \mathcal{O}_Y$.

Remark 1.3. If E is rank 1, then $\det E = E^{\vee\vee}$. This is an exercise. If you know a proof or know one, let me know. Somebody: there is a proof in the book by Okenek.

There is always a canonical map $E \to E^{\vee\vee} = \mathcal{O}_Y$ ideal sheaf. ◇

Theorem 1.4 (See a paper by Simpson or Huybrechts-Lehn)**.** *There exists a fine moduli scheme of ideal sheaves (I use "ideal sheaf" to mean torsion-free rank 1 sheaves with trivialized determinant). For higher rank, it is a Deligne-Mumford stack (always assuming the degree and rank are coprime so that I don't have to worry about the strictly semi-stable thing).*

Remark 1.5 (the construction)**.** For a given $p(n)$, choose $q \gg p \gg 0$. $V = V_{[p,q]}$ is a graded vector space with $\dim V_n = p(n)$ for all $p \le n \le q$. Let $A = \bigoplus_{n\ge 0} \Gamma(Y, \mathcal{O}(n))$ and $A' = \bigoplus_{n>0} \Gamma(Y, \mathcal{O}(n))$, $G = GL(V)^{gr} = \prod_{n=p}^{q} GL(V_n)$. Take $L^i = \mathrm{Hom}_{\mathbb{C}}(A'^{\otimes i}, \mathrm{End}_{\mathbb{C}} V)^{gr}$ for $i \ge 0$, and $L = \bigoplus_{i\ge 0} L^i$ (this is finite dimensional). Make L into a differential graded Lie algebra by defining a differential. If $\mu \in L^r$ and $\nu \in L^s$ are elements, then

$$
\begin{aligned}
d\mu(a_1, \dots, a_{r+1}) &:= \sum_{i=1}^{n-1} (-1)^i \mu(a_1, \dots, a_i a_{i+1}, \dots, a_{r+1}) \\
[\mu, \nu](a_1, \dots, a_{r+s} &:= \mu(a_1, \dots, a_r) \circ \nu(a_{r+1}, \dots, a_{r+s}) \\
&\quad - (-1)^{rs} \nu(a_1, \dots, a_s) \circ \mu(a_{s+1}, \dots, a_{r+s})
\end{aligned}
$$

G acts on L by conjugation. Then $\mathfrak{g} = L^0$. The derivative of $G \to GL(L^n)$ is $\mathfrak{g} = L^0 \to \mathfrak{gl}(L^n)$, given by $x \mapsto [x, -]$.

Now define $F\colon L^1 \to L^2$ by $\mu \mapsto d\mu + \frac{1}{2}[\mu, \mu] = d\mu + \mu \circ \mu$. This is a quadratic function. The zero scheme of F (the subscheme of L^1 cut out by F) is $Z(F) = \{\mu \in L^1 | d\mu + \frac{1}{2}[\mu, \mu] = 0\} = MC(L) \xhookrightarrow{\text{closed}} L^1$. The equation $d\mu + \frac{1}{2}[\mu, \mu] = 0$ is called the Mourer-Cartan equation.

Exercise. *Prove that $\mu\colon A \otimes V \to V$ satisfies the MC equation if and only if it is an action (if and only if it makes V into a graded A-module).*

Thus, $\mathcal{X} = [Z(F)/G]$ is the quotient stack, the stack of graded A-modules such that the underlying $\mathbb{C}$ vector space is isomorphic to V, modulo isomorphisms as graded A-modules.

Stability: You want to count the stable points under the G action. You modify slightly. Consider the torus $T = (\mathbb{C}^\times)^{q \cdot p}$ acting by rescaling on each V_n for $p \le n \le q$. Then $\mathbb{P}(L^1) = L^1/T$. The group $\tilde{G}$ is $\prod_{n=p}^{q} PGL(V_n)$ and $[Z(F)/G] \supset \big[\big(Z(F) \subset \mathbb{P}(L^1)\big)/\tilde{G}\big]$ is an open substack. GIT stability of $\tilde{G}$ on $\mathbb{P}(L^1)$ gives you a notion of stable points $Z(F)^{\mathrm{Stab}}$. Then $X = [Z(F)^{\mathrm{Stab}}/G]$ is a projective scheme (if there are semi-stable points, it is quasi-projective). ◇

Exercise. *Note that we have $\{$stable sheavs with Hilbert polynomial $p(n)\}/\cong \xrightarrow{\Gamma_*} \{A$-modules V with HP $p(n)\} \xrightarrow{truncate} X$. Check that Giesecker stability corresponds to GIT stability.*

Deformation Theory

If $\mu \in Z(F)$, then $T_{Z(F)}(\mu) = \{\nu \in L^1, d(\mu + \varepsilon\nu) + \frac{1}{2}[\mu + \varepsilon\nu, \mu + \varepsilon\nu] = 0\}$.

Exercise. $T_{Z(F)}(\mu) = \{\nu \in L^1 | d\nu + [\mu, \nu] = 0\} = \{\mu \in L^1 | d^\mu \nu = 0\}$ *where* $d^\mu = d + [\mu, -]$.

Exercise. *Because μ satisfies the MC equation,* $(d^\mu)^2 = 0$.

Over $Z(F)$, we have the trivial graded vector bundle L, with differential d^μ. Let's denote this by $\mathcal{E}$, a "perfect complex" on $Z(F)$. This $\mathcal{E}$ descends to X. On the quotient X, $T_X(\mu) = \ker(d^\mu\colon L^1 \to L^2)/\operatorname{im}(d^\mu\colon L^0 \to L^1) = H^1(L, d^\mu)$.

Definition 1.6. The *higher tangent* spaces are $T^i_X(\mu) = H^{i+1}(\mathcal{E}|\mu) = H^{i+1}(L, d^\mu)$. ◇

If $\mu \in Z(F)$ (this means that μ makes V into a graded A-module), it makes $\mathrm{End}_{\mathbb{C}} V$ into an A-bimodule by $(a, \mu b)(x) = a\mu(bx)$. Then (L, d^μ) is a well-known thing; it is the Hochschild complex of A' with values in $\mathrm{End}\, V$.

Fact: If E and F are stable sheaves, then $\mathrm{Ext}^i_{\mathcal{O}_Y}(E, F) = \mathrm{Ext}^i_A(\Gamma_* E_{\ge p}, \Gamma_* F_{\ge p})^{gr}$. This is basically Serre's theorem.

Another fact: $\mathrm{Ext}^i_A(\Gamma_* E_{\ge p}, \Gamma_* F_{\ge p})^{gr} = \mathrm{Ext}^i_A(\Gamma_* E_{[p,q]}, \Gamma_* F_{[p,q]})^{gr}$.

Another fact: The Hochschild cohomology $H^i\big(HC^\bullet(A, \mathrm{End}_{\mathbb{C}} V)^{gr}\big) = \mathrm{Ext}^i_A(V, V)^{gr}$.

Putting these facts together, we get

Corollary 1.7. $T^i_X(\mu) = \mathrm{Ext}^{i+1}_{\mathcal{O}_Y}(E, E)$ *if E is the stable sheaf corresponding to μ.*

Remark 1.8. $l_{\ge 1}[1]$ with G-action is a dg scheme, and $\mathcal{E}$ is the tangent complex of the dg scheme. It is not the tangent complex of X. $\mathcal{E}^\vee$ has a canonical map to the cotangent complex L_X by obstruction theory. I will not talk much about this dg scheme structure. ◇

There was a question. I think $F = df$ is not true.

2 Alessio Corti - Toric stacks

Today I want to do an introduction to toric stacks. The references: a paper of BCS and a paper by Fantechi et. al. Unfortunately, if you don't already know something about toric varieties, it will be hard to get much out of this. Toric stacks are a good way to write down examples of stacks, so this is a good way to learn about stacks.

Definition 2.1. A *simplicial stacky fan* is a triple (N, Σ, ρ), where N is a finitely generated abelian group (allowed to have torsion), Σ is a rational simplicial fan in $N_{\mathbb{R}}$, and $\rho\colon \mathbb{Z}^m \to N$ is a homomorphism with finite cokernel such that $\mathbb{R}_+\overline{\rho}_i$ are the the 1-dimensional rays of the fan (where $\overline{\rho}_i$ are the images of the coordinate axes of $\mathbb{R}^m$ in $N_{\mathbb{R}}$). ◇

There is an equivalence of categories between stacky fans and toric stacks. How do you make a stack out of a stacky fan? Let $\mathbb{L} = \ker(\rho\colon \mathbb{Z}^m \to N)$. There is a "Gale dual" sequence

$$0 \to \mathbb{L} \to \mathbb{Z}^m \xrightarrow{\rho} N \qquad \text{(fan sequence)}$$

$$\mathbb{L}^\vee \xleftarrow{D} Z^{\times m} \leftarrow M \leftarrow 0 \qquad \text{(Gale dual)}$$

where $M = \mathrm{Hom}(N, \mathbb{Z})$, and D has finite cokernel. What is $\mathbb{L}^\vee$? It is not too easy; here is the construction. Let $\mathbb{Z}^m \xrightarrow{\rho} N^\bullet \to \mathbb{L}^\bullet \xrightarrow{+1}$ be a mapping cone , so $\mathbb{L} = \mathbb{L}^{-1}$. Dualize and take cohomology gives you

$$0 \to M \to \mathbb{Z}^{m^\bullet} \to H^1(\mathbb{L}^\bullet) =: \mathbb{L}^\vee$$

So $\mathbb{L}^\vee$ is a finitely generated abelian group, and it could have torsion.

Fact: $\mathcal{L}^\vee$ is the Picard group of the corresponding toric stack $\mathcal{X}$.

Think of $\mathbb{L}^\vee$ as the group of characters on an abelian algebraic group $\mathbb{G}$, $\mathrm{Hom}(\mathbb{G}, \mathbb{C}^\times)$. Similarly, $\mathbb{Z}^m$ is the group of characters of $(\mathbb{C}^\times)^m$. Then $\mathcal{X} = [\mathbb{C}^M /\!\!/ \mathbb{T}]$.

That requires a stability condition. If $\sigma \in \Sigma$ is a maximal cone (assume maximal cones are of maximal dimension), then $\bigoplus_{i\in\sigma} \mathbb{Z}e_i \to N$. This leads to $\mathcal{X}_\sigma \subset \mathcal{X}$ and open substack. Q: all these things are simplicial . . . one could contemplate non-simplicial toric stacks, right? AC: sure.

Assumptions: I always assume that the natural map $\mathcal{X} \to \operatorname{Spec} H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is projective (this is made sense of purely in terms of the coarse moduli space). In particular, the support $|\Sigma| \subseteq N_{\mathbb{R}}$ is convex. Q: is that convexity condition equivalent to saying that the map is proper? AC: yes. I also assume that $\mathcal{X}$ is weak Fano, meaning that $-K_{\mathcal{X}}$ is nef. Equivalently, $\Delta(-K_{\mathcal{X}})$ is weakly convex. Q: why do you want to make these assumptions? AC: there are various issues. The projectivity is needed for the equivariant cohomology is sensible. It is also needed to make sense of Gromov-Witten theory. The weak Fano assumption. . . you'll see.

Example 2.2. $\mathcal{X} = \mathbb{P}^{w_1,w_2}$. Then we have

$$0 \to \mathbb{Z} \xrightarrow{\binom{w_1}{w_2}} \mathbb{Z}^2 \xrightarrow{\rho} \mathbb{Z}$$

$$\mathbb{L}^\vee = \mathbb{Z} \xleftarrow{(w_1\ w_2} \mathbb{Z}^2 \leftarrow M \leftarrow 0$$

So we have the quotient of $(\mathbb{C}^\times)^2$ by the action. $\mathbb{C}^\times \to (\mathbb{C}^\times)^2$, $\lambda \mapsto (\lambda^{w_1}, \lambda^{w_2})$. ◇

Example 2.3. $\mathcal{X} = \mathbb{P}_{w_1,w_2}$. Then

$$0 \to \mathbb{Z} \xrightarrow{\binom{w_1'}{w_2'}} \mathbb{Z}^2 \xrightarrow{(-w_2, w_1)} \mathbb{Z}$$

$\mathbb{L}' = \mathbb{Z} \oplus \mathbb{Z}/gcd(w_1, w_2)$, $\mathbb{P}_{2,2} = \mathbb{P}^/\mu_2$ ◇

Example 2.4. $\mathcal{X} = \frac{1}{3}(1,1)$. Then $\rho\colon \mathbb{Z}^2 \to \mathbb{Z}^2 + \frac{1}{3}(1,1)\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z} \leftarrow \mathbb{Z}^2$ given by $a + b \leftarrow (a, b)$. $\mathcal{X} = \mathbb{C}^2/\mu_3$ where $\mu_3 \to (\mathbb{C}^\times)^2$ is given by $\omega \mapsto (\omega, \omega)$ ◇

Example 2.5. $\mathcal{X} = \mathbb{P}(1,1,3)$, $N = \mathbb{Z}^2 + \frac{1}{3}(1,1)\mathbb{Z}$. $\rho\colon \mathbb{Z}^3 \to N$. Here $\rho_1 = (1,0)$, $\rho_2 = (0,1)$, and $\rho_3 = -\frac{1}{3}(1,1)$. Ther are two lattice points of N in the convex hull of these three. They will play an important role later. ◇

Some facts:

– N_{tors} is the generic stabilizer

– the rays determine some divisors; write $N_i = \{v \in N | \overline{v} \in \mathbb{Q}_+\rho_i\}$ [[★★★ what is $\overline{v}$]]. Then $N_i/\langle\rho_i\rangle$ is the stabilizer of D_i.

Enhanced Picard group. Let $\mathcal{X}$ be a stack. Define the enhanced Picard group $\widehat{Pic}(\mathcal{X})$ of $\mathcal{X}$ by the following exact sequence.

$$0 \to \widehat{Pic}(\mathcal{X}) \to Pic(\mathcal{X}) \oplus \mathbb{Z}^{\text{Box}} \to \bigoplus_{b\in\text{Box}} \mathbb{Z}/r_b\mathbb{Z}$$

So $\widehat{Pic}(\mathcal{X}) = \{(L,m)|L \in Pic(\mathcal{X}), m\colon \text{Box} \to \mathbb{Z}$ such that for $\chi\colon B\mu \to \mathcal{X}$, $\chi^*L = m(\chi)\}$.

Remark 2.6. If $f\colon \mathcal{X} \to \mathcal{Y}$ is a representable morphism of stacks, then you get $f^*\colon \widehat{Pic}(\mathcal{Y}) \to \widehat{Pic}(\mathcal{X})$. ◇

If I have a representable morphism from an orbi-curve $f\colon (\Gamma, x_i(r_i)) \to \mathcal{X}$, then f has an enhance degree $\widehat{\deg} f\colon \widehat{Pic}(X) \to \mathbb{Z}$, given by taking (L,m) to $\deg(f^*L) - \sum \frac{m_i}{r_i}$. Q: have you said what the degree of f^*L is? AC: the degree is the thing that makes the Riemann-Roch formula work.

Next I have to tell you how to calculate this for toric stacks. If $\mathcal{X}$ is a toric stack, then $\text{Box} = \bigcup_{\sigma\in\Sigma} \text{Box}(\sigma)$ where $\text{Box}(\sigma) = \{v \in N|\overline{v} = \sum_{i\in\sigma} v_i\overline{\rho}_i, 0 \le v_i < 1\}$. This is the justification for the name "Box." We have $\rho\colon \mathbb{Z}^m \to N$; we augment this go get

$$0 \to \widehat{\mathbb{L}} \to \mathbb{Z}^m \oplus \bigoplus_{v\in\text{Box}} \mathbb{Z} \to N$$

where the second map takes elements of the box to themselves (in N). It turns out that $\widehat{Pic}(\mathcal{X}) = \widehat{\mathbb{L}}^*$ (dual). So you don't have to do the complicated homological algebra from before.

If I have $\mathbb{P}_{r_1,r_2}$, I have a μ_{r_1} at zero and a μ_{r_2} at infinity, so I have sheaves like $\mathcal{O}(k_1/r_1)$ and $\mathcal{O}(k_2/r_2)$. I can pull back a line bundle from the coarse moduli space ($\mathbb{P}^1$), you get [[★★★ something something]] with just integers. Q: what about $\mathbb{P}_{1,1}$? AC: there is no stackiness; there is no box, so I can't play the game. Q: what is the enhanced Picard group of this $\mathbb{P}_{r_1,r_2}$? AC: we can do it. We had $\mathbb{Z}^2 \xrightarrow{\binom{r_1}{r_2}} \mathbb{Z}$. We have to augment this by the box. In this case, $\text{Box} = \{-k/r_1|0 \le k < r_1\} \cup \{k/r_2|0 \le k < r_2\}$ (I guess for $\mathbb{P}_{1,1}$ you can choose a random integer, so the augmented Picard group is a line bundle plus an integer). So the enhanced picard group is the kernel of $\mathbb{Z}^2 \oplus \mathbb{Z}^{r_1} \oplus \mathbb{Z}^{r_2} \to \mathbb{Z}$.

MO: is there always a map $\widehat{Pic}(\mathcal{X}) \to Pic(\mathcal{X})$ with kernel $\mathbb{Z}^{\text{Box}}$? AC: yes.

Stanley-Reisner rings

Given a stacky fan (Σ, N, ρ), we define $SR^\bullet_{\mathbb{R}}(\mathcal{X}, \mathbb{Q}) = \mathbb{Q}[\Sigma]$. Explicitly, there are generators u^e, where $e \in N$ and $\overline{e} \in \Sigma$. The product rule is that $u^{e_1}u^{e_2} = u^{e_1+e_2}$ whenever e_1 and e_2 belong to the same cone, and $u^{e_1}u^{e_2} = 0$ otherwise. This is a graded ring, with grading given by age. So $\deg u^e = a(e)$. Recall that for $\overline{e} = \sum_{i\in\sigma} a_i\overline{\rho}_i$, $a(e) = \sum a_i$.

Example 2.7. If $\mathcal{X} = \mathbb{P}^2$, then $SR_{\mathbb{T}}(\mathbb{P}^2) = R[u_1, u_2, u_3]/(u_0u_1u_2)$. You can think of the ring SR as the ring of polynomial functions on the polytope. ◇

Take $\mathbb{R} = \text{Sym}^\bullet M$ as the "base ring". Then SR is an $\mathbb{R}$-algebra. An element $m \in M$ maps to $\sum\langle\rho_i, m\rangle\rho_i \in SR^1$.

Facts:

1. $SR^\bullet_{\mathbb{T}} = H^\bullet_{orb,\mathbb{T}}\mathcal{X}$ is the $\mathbb{T}$-equivariant orbifold cohomology of $\mathcal{X}$.
2. $SR_{\mathbb{T}} \otimes_{\text{Sym}^\bullet M} \mathbb{Q} = H^\bullet_{orb}\mathcal{X}$.

2 Valery Alexeev

Fix a weight $\beta = (b_1, \dots, b_n)$ (rational numbers $0 < b_i \leq 1$).

Definition 2.1. A stable pair is $(X, B = \sum b_i B)$ where X is projective connected reduced, and B_i are Weil divisors such that

1. (on singularities) slc
2. (numberical) $K_X + B$ ample

A stable map is $f\colon (X, B) \to Z$ satisfying two conditions; the first are the same, and the second is changed to saying that $K_X + B$ is ample over Z (e.g. if the map is finite, this is a non-condition). $\diamond$

Ideal theorem: Fix a dimension, β, and some other invariants, then there exists a projective moduli space $\overline{\mathcal{M}}$ of stable maps.

Example 2.2. The weighted moduli spaces $\overline{\mathcal{M}}_{g,\beta}$ due to Hassett. This is indeed a projective smooth stack. If $g = 0$, it is a fine moduli space (i.e. it is a smooth projective variety). $\diamond$

The goal is to generalize to higher dimensions. We'll sometimes use this dream theorem for inspiration.

When you talk about moduli spaces, you need a functor. The minimal condition is that you look at flat families. Here you have to be more careful because the B_i are only Weil divisors, not Cartier divisiors. We'll be more careful about this later.

Let's review the dimension 1 case again. What does the mysterious condition slc mean for curves? (X, B) is a curve with points, then

1. (slc) when $\{B_i | i \in I\}$ coincide, $\sum_{i \in I} b_i \leq 1$.
2. (numerical) for all irreducible components $E \subseteq X$, $\deg(K_X + B)|_E > 0$. This degree is $2p_0(E) - 2\sum_{B_i \subseteq E} 1 + E(X - E)$.

What is the definition of slc in higher dimensions? () You should require that in codimenison 1, it is at worst nodal. This already implies that it is Gorenstein in codimension 1, so you have the notion of ω_X. () You also require that the B_i do not contain the components of the double locus. () We ask that $[\omega_X^{\otimes N}(N \sum b_i B_i)]^{\vee\vee}$ be invertible. This allows us to talk about $K_X + B$; it will be a $\mathbb{Q}$-Cartier divisor. Next, you can take a normalization, in which you will have the divisors B_i and the double locus. () We would like to require that $(X^\nu, B^\nu + \text{(double locus)})$ is lc. We're almost done. (4) We ask that X is (S2) (Serre condition 2, which is normal minus R1).

The other condition is dlt, which is better than lc because it implies Cohen-Macaulay, and lc only implies normal. Similarly we may want sdlt, which would imply Cohen-Macaulay, whereas slc only implies S2. I will not give a definition of sdlt, but there is a reasonable candidate.

Stable toric varieties

I use the word stable by analogy with stable curves; some people use the word "broken" toric varieties, which kind of gives you an idea of what they are.

(TV) the segment is the polytope for $\mathbb{P}^1$. (STV) two intervals glued at ends should be two $\mathbb{P}^1$'s meeting at a point. (TV) square is $\mathbb{P}^1 \times \mathbb{P}^1$, triangle is $\mathbb{P}^2$. Triangle with corner cut is $Bl_{pt}\mathbb{P}^2$. With two corners cut is is the blow-up at two points; one curve can be blown down to get $\mathbb{P}^1 \times \mathbb{P}^1$. (STV) [[★★★ picture]] If we glue two triangles to two adjacent edges of a square, that is two $\mathbb{P}^2$'s glued to a $\mathbb{P}^1 \times \mathbb{P}^1$ along a couple of $\mathbb{P}^1$'s, and all three of these intersect at a point. There is a 1-parameter family where this guy is a limit of $\mathbb{P}^2$'s ...you "break" two corners of the triangle and leave them hinged.

(TV) In toric geometry, there is a correspondence. Fix a lattice $\Lambda \cong \mathbb{Z}^r$ and a torus $T = (\mathbb{C}^\times)^r$ (you don't have to work over $\mathbb{C}$, but I will for simplicity). Then there is a correspondence between {integral polytopes with vertices in Λ} and $\{(X, L)$ polarized linearized toric variety$\}$ (X normal projective toric variety and L is an ample line bundle with T action). Q: does toric variety mean normal. VA: yes, I do require normal.

(STV) To $\Delta = \bigcup P^\alpha \in$ {coplex of integral polytopes} we associate an element of $\{$family of (X, L) polarized STV$\}$. Stanley-Riesner varieties are the ones that come from breaking a polytope, but you can have two triangles joined at a vertex or multiple edges between two vertices. $H^0(\Delta, \underline{\mathrm{Aut}}) = \mathrm{Aut}(X)$. Something is parameterized by $H^1(\Delta, \underline{\mathrm{Aut}})$. We have $C^0 \to C^1 \xrightarrow{\partial} C^2$. In our example, $C^1 = \mathbb{C}^\times \oplus \mathbb{C}^\times$, $C^0 = (\mathbb{C}^\times)^2 \oplus (\mathbb{C}^\times)^2 \oplus (\mathbb{C}^\times)^2$, $C^2 = 0$, and the first homology is zero.

Example 2.3. [[★★★ picture: triangle in a triangle; corresponding vertices joined]] Here, we will get $H^1 = \mathbb{C}^\times$. $\diamond$

In a huge class of examples, the varieties I get are slc.

Our first example [[★★★ two triangles on a square]] the topological space $|\Delta|$ is a manifold with boundary. This implies that X is Cohen-Macaulay.

Example 2.4. [[★★★ picture: two triangles glued at a vertex]] is not S2 and not CM. ◇

Consider the variety [[★★★ first example; two triangles on a square]], with the three boundary edges are B_1, B_2, and B_3.

Lemma 2.5. *$(X, \sum B_i)$ is slc and $K_X + \sum B_i = 0$.*

Proof. It is S2 because it is CM. The next condition is to look at the normalization which is [[★★★ picture: break off the hinged parts]] and check that you get lc. □

Now condier adding an additional divisor [[★★★ picture]]. This is a tropical picture

Lemma 2.6. *$(X, \sum B_i + \varepsilon B_{n+1})$ is slc if and only if B_{n+1} does not contain any T-orbits.*

The proof is the same. When you break off the hinged parts, the extra divisor is a line that intersects each of the other lines (on the boundary) at one point each.

We either work with all weights 1, or with a bunch of 1's and an ε. The 1's correspond to the boundary and the ε corresponds to an ample Cartier divisor.

Definition 2.7. A stable toric variety *over Z* is as follows. We have a torus $T = (\mathbb{C}^\times)^r$ acting on $\mathbb{P}^N$ $(r < N)$, and in $\mathbb{P}^N$, we have a closed T-invariant subvariety Z. We define a stable toric variety over Z to be a finite morphism $f\colon X \to Z$ from a stable toric variety X. ◇

Theorem 2.8 (main theorem)**.** *Fix Δ. Then there exists a projective moduli space (which is a scheme) of stable toric varieties over Z, $M^T(Z, \Delta)$.*

The theorem is much more general; this corresponds to the multiplicity-free case. Doesn't have to be a torus; you can do it for spherical varieties. The proof is in one of my papers with .... I won't try to give the proof.

How is this different from the toric Hilbert scheme? In that case, you'd look at subschemes of Z. These would normally be non-reduced and non-normal. I insist that we work with nice normal S2 toric varieties. Why can't I just restrict to the reduced case?

Example 2.9. $\{tx_0x_2 - x_1^2 = 0\}$ is a family (the parameter is t). As $t \to 0$, in Hilb, we get $x_1^2 = 0$, a double line. In M^T, you break the $\mathbb{P}^1$ segment by removing a point. This somehow says that the map to $\mathbb{P}^2$ is a 2-to-1 map, not an embedding. ◇

1-parameter degenerations

I have the curves X_t from the previous example, with maps $X_t \to \mathbb{P}^2$, and I have $f^*\mathcal{O}(1) = L_t$, and a morphism $\bigoplus H^0(\mathbb{P}^2, \mathcal{O}(d)) \to \bigoplus H^0(X_t, L_t^d)$. $H^0(\mathcal{O}_{\mathbb{P}^2}(1))$ has a basis x_0, x_1, x_2 (corresponding to the two endpoints of the segement and the one in the middle). Fix isomorphisms $(X_t, L_t) \cong (\mathbb{P}^1, \mathcal{O}(2))$. Then the map $H^0(\mathcal{O}_{\mathbb{P}^2}(1)) \to H^0(X_t, L_t)$ is given by $x_i \mapsto t^{\delta_{i,1}} e_i$. To compute the limit (how to break the picture), you take the lower convex hull of the height function and project the linear parts down. This is how you prove that every family has a unique limit point.

You have this height function h. You can take the discrete Laplace dual. Every face will correspond to a point in the dual space. Dimension 1 is too small, so let's do a 2-dimensional picture.

[[★★★ picture big triangle breaking into our first example]] This should be the projection of some height function. The height function is defined on the six lattice points. If you take the discrete Laplace dual, you get [[★★★ tropical picture: trivalent tree to depth 2]]. This is $\operatorname{trop}(f_t\colon (X, L)_t \to Z)$. If you take a different family, you may still get the same limit, but the tropical thing will change.

So far, I have stable toric varieties over Z. The MMP interpretation is that $(X, \sum B_i) \to Z$ is stable map. I proved that this is slc and the other condition. Here, we have all weights are 1. Stable toric pairs (X, D) will have weights all 1's and an ε. This is a special case of the previous on. If we have $L = \mathcal{O}_X(D)$, we get $\phi_{|H^0(X,L)|^*}\colon X \to \mathbb{P}^N = Z \supset H = \{x_0 + \cdots + x_n = 0\}$, then $D = f^*H$.

Conclusion for today: there is a moduli space of stable toric varieties. I used MMP here for motivation, but then independently on constructs this moduli space. I will use this in the later lectures in two ways. Tomorrow I'll describe higher dimensional generalizations of one of the guys. On Friday, I'll consider the compactification of the moduli space of abelian varieties, and this will correspond to stable toric pairs.

When you look at 1-parameter degenerations, they are described by height functions. How many height functions do you have (if you allow real heights)? It looks like $\mathbb{R}^m$. You can say that two heights are equivalent if they give you the same subdivision. This gives a fan on $\mathbb{R}^m$. This defines a secondary toric variety. the main component of $M^T(\mathbb{P}^n, D)$ is a possibly non-normal toric variety and its normalization is this guy.

2 Martin Olsson

Recall that we have the category of log schemes, logSch, whose objects are pairs (X, M_X), where X is a scheme and M_X is a sheaf of monoids with a map of sheaves of monoids $\alpha\colon M_X \to \mathcal{O}_X$ such that $\alpha^{-1}(\mathcal{O}_X^\times) \to \mathcal{O}_X^\times$ is an isomorphism. A morphism $(X, M_X) \to (Y, M_Y)$ is a morphism of schemes $f\colon X \to Y$ and a map $f^\flat\colon f^*M_Y \to M_X$.

The goal for today is to say something about differentials. I want to explain how to do algebraic differential geometry in this category.

Definition 2.1. A morphism $(f, f^\flat)\colon (X, M_X) \to (Y, M_Y)$ is *strict* if $f^\flat\colon f^*M_Y \to M_X$ is an isomorphism. ◇

You should think of strictness as the anology of being a closed immersion.

Consider the diagram (with all log structures integral, to be careful)

$$\begin{tikzcd} (T_0, M_{T_0}) \arrow[r, "a"] \arrow[d, hook, "J", "j"'] & (X, M_X) \arrow[d] \\ (T, M_T) \arrow[r, "b"'] \arrow[ur, dashed, "g"] & (Y, M_Y) \end{tikzcd} \tag{$*$}$$

If $T_0 \hookrightarrow T$ is a closed immersion defined by ideal J, with $J^2 = 0$ and j is strict (sometimes called a log closed immersion with $J^2 = 0$), we're interested in filler arrows.

First let's fix two filler arrows g_1 and g_2. The difference should correspond to a derivation. Let me remind you how that goes. T_0 and T have the same "étale topological space" (by which I mean that they have equivalent categories of étale sheaves). I have the diagram of sheaves of algebras

$$\begin{tikzcd} a^{-1}\mathcal{O}_X \arrow[r] \arrow[dr, dashed, shift left, "g_2"] \arrow[dr, dashed, shift right, "g_1"'] & \mathcal{O}_{T_0} \\ b^{-1}\mathcal{O}_Y \arrow[r] \arrow[u] & \mathcal{O}_T \arrow[u] \end{tikzcd} \quad \supset \quad J$$

Then $g_1 - g_2\colon a^{-1}\mathcal{O}_X \to J$ is a derivation $\partial_{g_1-g_2}\colon a^{-1}\mathcal{O}_X \to J$. In our situation, we also have the log structures

$$\begin{tikzcd} a^{-1}M_X \arrow[r] \arrow[dr, dashed, shift left, "g_2"] \arrow[dr, dashed, shift right, "g_1"'] & M_{T_0} \\ b^{-1}M_Y \arrow[r] \arrow[u] & M_T \arrow[u] \end{tikzcd} \quad \supset \quad 1+J$$

with $(1+a)(1-a) = 1$.

Lemma 2.2. $M_T|_{1+J} \xrightarrow{\sim} M_{T_0}$

Proof. □

$$1+J \quad \subset \quad \begin{tikzcd} \mathcal{O}_T^\times \arrow[r, hook] \arrow[d] & M_T \arrow[d] \arrow[dr] & \\ \mathcal{O}_{T_0}^\times \arrow[r] \arrow[ur, phantom, "\lrcorner"] & M_{T_0} \arrow[r] & \mathcal{O}_{T_0} \end{tikzcd}$$

I get a map $D_{g_1-g_2}\colon a^{-1}M_X \to J$ such that for every section $m \in a^{-1}M_X$, we have $g_1(m) = (1 + D_{g_1-g_2}(m)) + g_2(m)$. You have to check that this is actually additive. Passing to the associated groups, we get $D_{g_1-g_2}: a^{-1}M_X^{gp} \to J$. There is a map of diagrams going from the log structures to the sheaves of rings. (1) That means that for every local section $m \in a^{-1}M_X$,

$$a^\#(m)D_{g_1-g_2}(m) = \partial_{g_1-g_2}(\alpha(m))$$

where $a^\#\colon a^{-1}M_X \xrightarrow{\alpha} a^{-1}\mathcal{O}_X \to \mathcal{O}_{T_0}$. That is, we want to say $D_{g_1-g_2}(m) =$"$d\log(\alpha m)$". (2) $D_{g_1-g_2}|_{b^{-1}M_Y} = 0$.

Remark 2.3. $D_{g_1-g_2}$ determines $\partial_{g_1-g_2}$. ◇

Define $\Omega^1_{(X,M_X)/(Y,M_Y)} := \frac{\Omega^1_{X/Y} \oplus (\mathcal{O}_X \otimes_{\mathbb{Z}} M_X^{gp})}{I}$ where I is the $\mathcal{O}_X$-submodule generated by

(i) $(d\alpha(m), 0) - (0, \alpha(m) \otimes m)$

(ii) $(0, 1 \otimes m)$ where $m \in \operatorname{im}(f^{-1}M_Y \to M_X)$.

Summary: if a dotted arrow filling in $(*)$ exists, then the set of such g is a torsor under $\mathrm{Hom}(a^*\Omega^1_{(X,M_X)/(Y,M_Y)}, J)$. I explained one direction (that any two maps differ by an element of this Hom). You have to check that if you add an element of the Hom to a given filler, then you get another filler.

There are two approaches you could take. I wanted to be very explicit and write down the formula for this Ω^1, but it is completely characterized by this property (you do the construction to prove existence, but then you never care about the formula again). The point is that you have a nice sheaf with this nice property.

Example 2.4. Say P is a fine monoid, k is a field. Let's compute $\Omega^1_{\mathrm{Spec}(P\to k[P])/k}$, where k means $\mathrm{Spec}\, k$ with the trivial log structure (meaning the monoid $\mathcal{O}^\times$ with the inclusion; this is the initial object in the category of log structures). we have

$$\begin{tikzcd} (T_0, M_{T_0}) \arrow[r, "a"] \arrow[d, "j"'] & \mathrm{Spec}(P\to k[P]) \arrow[d] \\ (T, M_T) \arrow[r, "b"'] \arrow[ur, dashed, "g"] & (\mathrm{Spec}\, k, k^\times) \end{tikzcd}$$

From last time, a corresponds to a map $\gamma_0\colon P \to \Gamma(T_0, M_{T_0})$. A g would correspond to $\gamma\colon P\to\Gamma(T, M_T)$. If we fix one such γ, another map g' would correspond to a map γ' of the form $\gamma+\rho$, where $\rho\colon P\to\Gamma(T, 1+J)$, which is exactly the same as a map $\rho\colon P^{gp}\to\Gamma(T_0, J)$. The universal property implies that $\Omega^1_{\mathrm{Spec}(P\to k[P])/k}\cong \mathcal{O}_{\mathrm{Spec}\, k[P]}\otimes_{\mathbb{Z}} P^{gp}$ (using the universal property of tensor product). ◇

In general, there is a derivation $d\colon \mathcal{O}_X\to\Omega^1_{(X,M_X)/(Y,M_Y)}$. One way to see that is that there is a map $\mathcal{O}_X\to\Omega^1_{X/Y}\to\Omega^1_{(X,M_X)/(Y,M_Y)}$. [[★★★ I missed the other explanation]]

Example 2.5. $P=\mathbb{N}^r$, so $\mathrm{Spec}\, k[P]$ is $\mathbb{A}^r_k$. We have that $\Omega^1_{\mathrm{Spec}(P\to k[P])/k} = \mathcal{O}_{\mathbb{A}^r}\otimes_{\mathbb{Z}}\mathbb{Z}^r$. Maybe that's not so interesting, but let's think about what is d. You have the standard generators for $\mathbb{N}^r$; call the corresponding variables $x_1,\dots,x_r$. Then

$$d\colon k[x_1,\dots,x_r]\to k[\underline{x}](1\otimes e_1)\oplus\cdots\oplus k[\underline{x}](1\otimes e_r)$$

◇

Exercise. *$d(x_i)=x_i(1\otimes e_i)$. That is, we can think of $1\otimes e_i$ as dx_i/x_i.*

Fact 1: if $(X,M_X)\to(Y,M_Y)$ is strict, then $\Omega^1_{(X,M_X)/(Y,M_Y)}=\Omega^1_{X/Y}$. Fact 2: if (X,M_X) and (Y,M_Y) are fine (which means that étale locally, they are the associated log structure to the prelog structure coming from some fine monoid), then $\Omega^1_{(X,M_X)/(Y,M_Y)}$ is quasi-coherent and is coherent if locally noetherian and f is of finite type.

Depending on how you learned algebraic geometry, the whole theory of differentials either goes through, or it seems very mysterious. Hopefully, you learned by following SGA1 or EGA. The point is that this lifting property is what you need to develop most of the theory of differentials.

Definition 2.6. A morphism $f\colon (X,M_X)\to(Y,M_Y)$ is *log smooth* (or *smooth*, if there is no confusion about what category we're in) if $X\to Y$ is locally of finite presentation and for every diagram $(*)$ (reproduced below) of solid arrows

$$\begin{tikzcd} (T_0, M_{T_0}) \arrow[r, "a"] \arrow[d, hook, "j", "J"'] & (X, M_X) \arrow[d] \\ (T, M_T) \arrow[r, "b"'] \arrow[ur, dashed, "g"] & (Y, M_Y) \end{tikzcd} \qquad (*)$$

there exists étale locally on T a dashed arrow. ◇

Remark 2.7. The definition of log étale can be obtained by requiring that the dashed arrow is unique. This is equivalent to log smooth plus $\Omega^1_{(X,M_X)/(Y,M_Y)}=0$. ◇

Example 2.8. $(Y,M_Y)=(\mathrm{Spec}\, k, k^\times)$ and $(X,M_X)=\mathrm{Spec}(P\to k[P])$, where P is fine.

$$\begin{tikzcd} M_{T_0} \arrow[r] & P \arrow[dl, dashed, "g"] \\ M_T \arrow[u] & \end{tikzcd}$$

I have

$$\begin{tikzcd} 0 \arrow[r] & 1+J \arrow[r] & M_T^{gp} \arrow[r] & M_{T_0}^{gp} \arrow[r] & 0 \\ & & & P \arrow[u] \arrow[ul, dashed, "g"] & \end{tikzcd}$$

pull back to get an exact sequence

$$0 \to 1 + J \to E \to P^{gp} \to 0$$

If P^{gp} is a free group, then I can split this. If the torsion of P is invertible in k, then something. The upshot is that $\mathrm{Spec}(P \to k[P]) \to (\mathrm{Spec}\, k, k^\times)$ is log smooth if and only if the order of $(P^{gp})_{tors}$ is invertible in k. ◇

Example 2.9. Take $X = \mathrm{Spec}\, k[x_1, \ldots, x_n]/(x_1 \ldots x_r) = \mathrm{Spec}\, k[\mathbb{N}^r][x_{r+1}, \ldots, x_n] \otimes_{\Delta, k[\mathbb{N}], \beta} k$, where $\Delta\colon \mathbb{N} \to \mathbb{N}^r$ is the diagonal map and $\beta\colon \mathbb{N} \to k$ is given by $n \mapsto 0^n$. M_X is the log structure associated to $\mathbb{N}^r \to \mathcal{O}_X$ and $M_k = k^\times \oplus N \to k$ given by $(u, n) \mapsto u \cdot \beta(n)$.

The claim is that $(X, M_X) \to (\mathrm{Spec}\, k, M_k)$ is log smooth. This is good news from the point of view of moduli because it means that we will get log smooth things on the boundary.

Why is the claim true? Consider the case $n = r$ to not be too confusing

$$\begin{tikzcd} (T_0, M_{T_0}) \arrow[r, "a"] \arrow[d, hook, "j"'] & (X, M_X) \arrow[d] \\ (T, M_T) \arrow[r, "b"'] \arrow[ur, dashed, "g"] & (Y, M_Y) \end{tikzcd} \quad \leftrightarrow \quad \begin{tikzcd} M_{T_0} & \mathbb{N}^r \arrow[l] \arrow[dl, dashed] \\ M_T \arrow[u, two heads] & \mathbb{N} \arrow[l] \arrow[u, "\Delta"'] \end{tikzcd}$$

you pick any lift on the right, then the diagram doesn't commute, but it commutes up to a unit, so you change something a little bit.

The same argument show that $\Omega^1_{(X,M_X)/(Y,M_Y)}$ is a free module on generators dx_i/x_i modulo the relations $\sum_{i=1}^r dx_i/x_i = 0$. ◇

3 Tom Bridgland

I want to start by correcting a mistake in a calculation from yesterday. Remember I wanted to compute $I(1_{ss}^{\phi}) = \sum_{n\geq 0} \frac{x^{[S^{\oplus n}]}}{|GL_n(q)|}$, but it is not true that $x^{[S^{\oplus n}]} = q^{\frac{1}{2}n(n-1)}(x^{[S]})^n$ because $x^\alpha * x^\beta = q^{-\chi(\beta,\alpha)}x^{\alpha+\beta}$. So we actually get $I(1_{ss}^{\phi}) = \Phi(x^{[S]})$, where $\Phi(x) = \sum_{n\geq 0} \frac{q^{n(n-1)/2}}{|GL_n(q)|}x^n = \sum_{n\geq 0} \frac{x^n}{(q-1)\cdots(q^n-1)}$

Triangulated categories

Let D be a triangulated category (e.g. $\mathsf{D} = \mathsf{D}^b\mathsf{Coh}(X)$).

Definition 3.1. A *heart* (of a bounded t-structure) is a full subcategory $\mathsf{A} \subset \mathsf{D}$ such that

(a) $\operatorname{Hom}_{\mathsf{D}}(A_1, A_2[k]) = 0$ for all $A_i \in \mathsf{A}$ and $k > 0$, and

(b) for all $E \in \mathsf{D}$, there exists integers $k_1 > \cdots > k_n$ and triangles $E_i \to E_{i+1} \to F_i \to E_i[1]$ with $E_0 = 0$ and $E_n = E$ with $F_i \in \mathsf{A}[k_i]$.[[★★★ add diagram?]] ◇

Remark 3.2. Condition (a) implies that the "filtration" in (b) is unique. The argument is the same as the argument for uniqueness of the HN filtration. ◇

Remark 3.3. If we define $\mathsf{D}^{\leq 0} = \{E \in \mathsf{D} | k_i \geq 0 \text{ for all } i\}$ (this is an unavoidable clash in notation) and $\mathsf{D}^{\geq 1} = \{E \in \mathsf{D} | \text{all } k_i \leq -1\}$. Then $(\mathsf{D}^{\leq 0}, \mathsf{D}^{\geq 1})$ a *bounded t-structure*. Conversely, given a bounded (non-degenerate) t-structure $(\mathsf{D}^{\leq 0}, \mathsf{D}^{\geq 1})$, we can define $\mathsf{A} = \mathsf{D}^{\leq 0} \cap \mathsf{D}^{\geq 1}[1]$. ◇

Example 3.4. The inclusion of an abelian category in it's derived category, $\mathsf{A} \subseteq \mathsf{D}^b(\mathsf{A})$, is a heart. Part (a) comes from the fact that there are no negative Ext's, and part (b) comes from truncations. Define $\tau_{\leq i}(\cdots E_i \xrightarrow{d^i} E_{i+1} \cdots) = (\cdots E_i \to \ker d^i \to 0 \cdots)$. Then you get triangles $\tau_{\leq i-1}(E) \to \tau_{\leq i}(E) \to H^i(E)[-i] \to$, and because we have a bounded derived category, we finish. ◇

So a triangulated category is a floppy thing (it could be the derived category of many different things, for example), and a t-structure rigidifies it.

Remark 3.5. If A is an abelian category which is a heart, then $\mathsf{D} \ncong \mathsf{D}^b(\mathsf{A})$ in general. In fact, you don't even have a map in general. This has to do with the fact that Ext^n is generated by Ext^1 for an abelian category. ◇

What does $\mathsf{D}^b(\mathbb{P}^1)$ look like? The sheave on $\mathbb{P}^1$ look like

$$\begin{tikzcd}
\cdots & \mathcal{O}(3) & \mathcal{O}(1) \arrow[dl, two heads] & \cdots \arrow[dl, two heads] \\
\cdots & \mathcal{O}(2) \arrow[u, two heads] & \mathcal{O} \arrow[u, two heads] &
\end{tikzcd}$$

But then you get all the shifts of this picture (think of this as a frame in a film strip). But you can take all the $\mathcal{O}(-n)$ and $\mathcal{O}(n)[1]$ as another heart. This category A is equivalent to the category of representations of the quiver $(\bullet \rightrightarrows \bullet)$. In general, your derived category doesn't natually sit in a filmstrip like this and you have to pick a direction to slice it.

Definition 3.6. A *stability condition* on D is a heart $\mathsf{A} \subseteq \mathsf{D}$ together with a stability condition $Z \colon K(\mathsf{A}) \to \mathbb{C}$ on A. ◇

I've just combined two things that don't look like they have anything to do with eachother. They both involve filtrations. I'll give you an alternative definition which is more symmetric.

Remark 3.7. A triangulated category also has a Grothendieck group $K(\mathsf{D})$, the free abelian group on isomorphism classes, modulo the relation $[B] = [A] + [C]$ when there is a triangle $A \to B \to C \to$.

Note that rotating the triangle, we can compute that $[E] = -[E[1]]$. Also note that if A is a heart, then $K(\mathsf{A}) \xrightarrow{\sim} K(\mathsf{D})$. ◇

Definition 3.8 (Alternative)**.** A *stability condition* on D consists of a group homomorphism $Z \colon K(\mathsf{D}) \to \mathbb{C}$ (usually called *central charge*) and full subcategories $\mathsf{P}(\phi) \subseteq \mathsf{D}$ for all $\phi \in \mathbb{R}$ satisfying

(a) $Z(\mathsf{P}(\phi)) \subseteq \mathbb{R}_{>0}e^{i\pi\phi}$

(b) $\mathsf{P}(\phi)[1] = \mathsf{P}(\phi + 1)$

(c) $\phi_1 > \phi_2$ and $A_i \in \mathsf{P}(\phi_i)$ implies $\operatorname{Hom}_{\mathsf{D}}(A_1, A_2) = 0$, and

(d) for all $E \in \mathsf{D}$, there exist $\phi_1 > \cdots > \phi_n$ and triangles $E_{i-1} \to E_i \to F_i \to$ such that $E_0 = 0$ and $E_n = E$ with $F_i \in \mathsf{P}(\phi_i)$. ◇

If you're familiar with t-structures, a t-structure is where you have filtrations like this with ϕ_i integers. This is some kind of more refined thing.

Proof (equivalence of definitions). Given the heart $\mathsf{A} \in \mathsf{D}$ and $Z\colon K(\mathsf{A}) \to \mathbb{C}$, we get $\mathsf{P}(\phi) \subset \mathsf{A}$ for $0 < \phi \leq 1$. Axiom (b) gives $\mathsf{P}(\phi)$ for all $\phi \in \mathbb{R}$. (c) is easy because we know it when $\phi_i \in (0, 1]$ and because of the axioms of a heart. Finally, the filtrations in (d) will come from combining the filtrations from the t-structure with Harder-Narasimhan filtrations. If you want to understand this, you should check it carefully.

To go the other way, how do we define a heart from these data. As usual, (c) tells us that the filtrations in (d) are unique up to isomorphism. For any interval $I \subset \mathbb{R}$, define $\mathsf{P}(I) = \{E \in \mathsf{D} | \phi_i \in I \text{ for all } i\}$. Now set $\mathsf{A} = \mathcal{P}((0, 1])$. (c) and (d) imply that this is a heart. If you think about it, you see that Z is a stability condition on A by (a). □

The later definition is actually much more symmetric. Choosing a heart is a choice of $\mathsf{P}((0, 1])$. But there is no reason to choose 0. Every $\mathsf{P}((\alpha, \alpha + 1])$ defines a heart. Q: in this particular case, is the triangulated category always the derived category of any heart? TB: I don't think so; let's discuss this after lecture.

Technical point: If $I \subseteq \mathbb{R}$ is an interval of length less than 1, then $\mathsf{P}(I)$ is not abelian, but there is still a notion of short exact sequences. A short exact sequence is just a triangle $A \to B \to C \to$ with $A, B, C \in \mathsf{P}(I)$. If you're categorically inclined, you might not like that this is not intrinsic. There is an intrisic version. $\mathsf{P}(I)$ is a quasi-abelian category.

Definition 3.9. A stability condition is *locally finite* if there is an $\varepsilon > 0$ such that for all $\phi \in \mathbb{R}$, $\mathsf{P}(\phi + \varepsilon, \phi - \varepsilon)$ is finite length (i.e. noetherian and artinian). Here a subobject is something that fits into an exact sequence, not just a categorical subobject. We write $\operatorname{Stab} \mathsf{D}$ for the set of locally finite stability conditions on D. ◇

Theorem 3.10. *There is a natural topology on* $\operatorname{Stab} \mathsf{D}$ *such that every connected component* $\operatorname{Stab}^* \mathsf{D} \subseteq \operatorname{Stab} \mathsf{D}$*, there exists a linear subspace* $V \subseteq \operatorname{Hom}_{\mathbb{Z}}(K(\mathsf{D}), \mathbb{C})$ *with a linear topology such that* $\operatorname{Stab}^* \mathsf{D} \to \operatorname{Hom}_{\mathbb{Z}}(K(\mathsf{D}), \mathbb{C})$*, given by* $(Z, \mathsf{P}) \mapsto Z$*, is a local homeomorphism onto an open subset of* V*. In particular,* $\operatorname{Stab} \mathsf{D}$ *is a (possibly infinite dimensional) complex manifold.*

This tells you that deformations of Z lift uniquely to deformations of the whole stability condition. Note that as Z changes, the t structure changes.

Remark 3.11. In practice, we insist that $Z\colon K(\mathsf{D}) \to \mathbb{C}$ factors via a finite dimensional quotient. For example, if $\mathsf{D} = \mathsf{D}^b\mathsf{Coh}(X)$ for X smooth and projective over $\mathbb{C}$, we insist that Z factors through the chern character $ch\colon K(\mathsf{D}) \to H^*(X, \mathbb{Q})$. If we make this constraint, then $\operatorname{Stab} \mathsf{D}$ is finite dimensional. ◇

2 Kai Behrend

Remember last time I constructed the moduli space X of stable sheaves on a Calabi-Yau 3-fold. It was $MC(L)^{st}/G$. I want to give some justfications for going through that construction:

1. It is a direct construction, avoiding the Quot scheme.

2. It gives X as a differential graded scheme. Let $W = L_{\geq 1}[1]$, and let $\mathcal{A} = \operatorname{Sym} W^*$ (this is the *graded* symmetric algebra). Define a derivation $Q\colon \mathcal{A} \to \mathcal{A}$ by defining it on generators: $Q\colon W^* \to \operatorname{Sym} W^*$. Make it by summing two parts, $Q_1\colon W^* \to W^*$, the dual of $d\colon L \to L$, and $Q_2\colon W^* \to \operatorname{Sym}^2 W^*$, the dual of the Lie bracket.

   Exercise. *$d^2 = 0$, d is a derivation on the Lie bracket, and Jacobi identity are equivalent to the single condition that $Q^2 = 0$.*

   $(\mathcal{A}, Q)$ is a differential graded algebra, and G acts on this. $\mathcal{A}^0 = \operatorname{Sym} L^{1*}$ and $\mathcal{A}$ is an $\mathcal{A}^0$-module, with Q $\mathcal{A}^0$-linear. Then $\mathcal{A}$ is a sheaf of differential graded algebras on L^1. G acts on it, so it descends to a sheaf on $M = L^{1\,\mathrm{Stab}}/G$. You have $(\cdots \xrightarrow{Q} \mathcal{A}^{-1} \xrightarrow{Q} \mathcal{A}^0)$, and you get that $h^0(\mathcal{A}, Q) = \mathcal{A}^0/Q\mathcal{A}^{-1}$. You can check that $\operatorname{Spec}(\mathcal{A}^0/Q\mathcal{A}^{-1}) = MC^{\mathrm{Stab}}(L) = Z(F)$. Then $X = \operatorname{Spec} h^0(\mathcal{A})/G$.

   We will need this differential graded stuff for "categorification", which has not yet been worked out.

   It is not easy to write down a category of dg schemes, for which you need some heavy duty homotopy theory. The underlying classical scheme has a universal property and it has a tangent complex that gives the right deformation theory.

3. Indoctrination. These days, differential graded Lie algebras show up a lot in deformation theory in characteristic zero (Manetti). I'm convinced that eventually, differential graded Lie algebras will take over moduli theory. If you find a finite dimensional dg algebra, you get a global moduli space. You really should always construct moduli spaces as $MC(L)/G$ for some dg Lie algebra L. One very popular differential graded Lie algebra is the Dolbeout Lie algebra. Take $\Omega^{0,*}(Y, \operatorname{End}_{\mathcal{O}_Y}(E))$, so L^1 are $\bar\partial$ operators, MC is integrability, and G is the guage group. To construct stuff this way, you have to do infinite dimensional stuff which is not very algebraic.

4. Replace V by a graded vector space (different from the grading it already has) $V = V^{-1} \oplus V^0$. $L^n = \bigoplus_{i+j=n} \operatorname{Hom}(A^{\otimes i}, \operatorname{End}^j_{\mathbb{C}}(V))$. You get a doubly graded dg Lie algebra.

   Maybe you get a moduli of complexes over derived category objects. I have no idea if this works, but if you're looking for a research problem, I think this might be promising. If you do, let me know about it so that we avoid duplication of research.

3. The virtual fundamental class

The basic setup that is called the toy model is as follows. M is smooth variety over $\mathbb{C}$. E/M is a vector bundle, $s \in \Gamma(M, E)$, and $X = Z(s)$ (we hope that this is our moduli space). We can draw this as a cartesian square

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & M \arrow[d, "s"] \\ M \arrow[r, "0"] & E \end{tikzcd}$$

In this case, $[X]^{vir}$ is what Fulton calls the localized top chern class of E, $0^!_{E|X}[C_{X/M}] \in A_{\dim M - \operatorname{rk} E}(X)$, defined as follows. If $\mathcal{I}$ is the ideal sheaf of X in M, $c_{X/M} = \operatorname{Spec} \bigoplus I^n/I^{n+1}$. $C_{X/M} \hookrightarrow E|_X \hookrightarrow E$ is a scheme of cones over X. Something $\mathbb{C}^\times$-action flow to ∞. You can multiply a section by an element of $\mathbb{C}^\times$ and you can let it go to ∞ to get a cone on X [[★★★ picture getting vertical lines on the zeros of the section s]]. All of this is explained in Fulton's book on intersection theory. $C_{X/M}$ is pure dimension of the same dimension as M.

$E^\vee \xrightarrow{s^\vee} \mathcal{O}_M$, and by definition, the image is the ideal sheaf $\mathcal{I}$ of X in M. Restricting to X, we have the diagram of sheaves on X

$$\begin{array}{ccccc} E^\vee|_X & \longrightarrow & \Omega_M|_X & = & \mathcal{F} \in \mathsf{D}(\mathcal{O}_X) \\ {\scriptstyle s^\vee}\downarrow & & \| & & \\ I/I^2 & \xrightarrow{\text{“}d\text{”}} & \Omega_M|_X & = & \tau_{\geq 1} L_X \in \mathsf{D}(\mathcal{O}_X) \end{array}$$

$\Omega_M|_X$ in the derived category is perfect of amplitude in $[-1,0]$. The bottom row in the derived category is completely intrinsic to X, $\tau_{\geq 1}L_X$. Let $\mathcal{F}$ be the top row., $\phi\colon \mathcal{F} \to \tau_{\geq 1}L_X$ in $\mathsf{D}(\mathcal{O}_X)$ has properties: (1) $h^0(\phi)$ is an isomorphism (2) $h^{-1}(\phi)$ is an epimorphism. This is called a "perfect obstruction theory" on X.

Theorem 2.1. *(1) $[X]^{vir}$ depends only on the perfect obstruction theory $\mathcal{F} \to \tau_{\geq 1}L_X$. (2) $\mathcal{F} \to \tau_{\geq 1}L_X$ defines $[X]^{vir}$.*

Now I'll try to explain how this fits into our example. Remember we have $MC^{st}(L) \hookrightarrow (L^1)^{st}$ and G acts. I'm going to define $G' = G/\mathbb{C}^\times$. The $\mathbb{C}^\times$ acts trvially, so the quotient will be a stack even if I pass to the stable locus, so I get rid of it to get a scheme by taking the quotient by G'. So we have

$$\begin{tikzcd} MC^{st}(L)/G' \arrow[r, hook] \arrow[d, equal] & (L^1)^{st}/G' \arrow[d, equal] \\ X \arrow[r, hook] & M \end{tikzcd}$$

Graded trivial vector bundle on L^1: $L^0 \to L^1 \xrightarrow{d^\mu} L^2 \to L^3$, where $d^\mu = d + [\mu, -]$ for $\mu \in L^1$. It is easy to check that $(d^\mu)^2 = 0$ if and only if $\mu \in MC$. This descends to M to give a vector bundle on M and d^μ. At $\mu \in X$, we get a complex, the Hochschild chain complex $HC^\bullet(A, \mathrm{End}^\mu_{\mathbb{C}}(V))$, which computes $\mathrm{Ext}^i_{\mathcal{O}_Y}(E_\mu, E_\mu)$.

Now use Serre duality and CY3 condition (for the first time) to get a perfect pairing

$$\mathrm{Ext}^i_Y(E,E) \otimes \mathrm{Ext}^{3-i}(E,E) \to \mathbb{C}.$$

You also use stability to conclude that $\mathrm{Ext}^0_Y(E,E) = \mathbb{C}$. This implies that $\mathrm{Ext}^3_Y(E,E) = \mathbb{C}$. It also says that $\mathrm{Ext}^2_Y(E,E) = \mathrm{Ext}^1_Y(E,E)^\vee$ and everything else vanishes.

Now assume also that $H^1(Y,\mathcal{O}) = H^2(Y,\mathcal{O}) = 0$. Many people put this in the definition of a Calabi-Yau manifold. Under this assumption, $H^1(Y,\mathcal{O}) = 0$ means that there are no deformations of line bundles, so if you have the correct Hilbert polynomial, your determinant is automatically trivial. Of course, there is still a choice of isomorphism, given by $\mathbb{C}^\times$, which we've already made up for. So in this case, our X is the moduli of sheaves with trivial determinant. Basically, by changing the group slightly, we killed Ext^0. To preserve symmetry, we will kill Ext^3. Replace $(L^\bullet, d^\mu)$ on X by $\tau_{[1,2]}(L^\bullet, d^\mu)$. Before truncation, $(L^\bullet, d^\mu) = R\,\mathrm{Hom}(E,E)$. Once I truncate, I get $\tau_{[1,2]}(L^\bullet, d^\mu) = R\,\mathrm{Hom}(E,E)_0$, the so-called "traceless ext". We've replaced (L, d^μ) by $\big(L^1/dL^0 \to \ker(L^2 \to L^3)\big)$.

Lemma 2.2. *$W^0 = L^1/dL^0$ and $W^1 = \ker(L^2 \to L^3)$ are vector bundles on X.*

So my perfect object is $\mathcal{F} = (W^0 \xrightarrow{d^\mu} W^1)$. It is actually not difficult to see that $W^0 \cong T_M|_X$.

$$\begin{tikzcd} W^0 \arrow[r, "d^\mu"] & W^1 \\ T_M|_X \arrow[u, "\wr"] \arrow[r] & \mathcal{N}_{X/M} \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} W^{1*} \arrow[r] \arrow[d] & W^{0*} \arrow[d, "="] \\ I/I^2 \arrow[r] & \Omega_{A/X} \end{tikzcd}$$

The map $\mathcal{N}_{X/M} \to W^1$ is induced by $\mathcal{F}$. f section of L^2 over L^1 is a section of W^1. The diagram is dual to the one on the right, which is the map $\mathcal{F}$ down to $\tau_{\geq 1}L_X$. $C_{X/M} \hookrightarrow \mathcal{N}_{X/M} \hookrightarrow W^1$. Check that $[X]^{vir} = 0^!_{W^1}[C_{X/M}] \in A_0(X)$. So $\#^{vir}X$ is the proper pushforward to $*$.

3 Martin Olsson

Last time, I introduced the notion of log smoothness. A morphism $(X, M_X) \to (Y, M_Y)$ is log smooth if it is locally of finite presentation and for every $T_0 \subseteq T$ defined by a square zero ideal with j strict, there is a filler arrow g.

$$\begin{tikzcd} (T_0, M_{T_0}) \arrow[r, "a"] \arrow[d, hook, "J"', "j"] & (X, M_X) \arrow[d] \\ (T, M_T) \arrow[r, "b"'] \arrow[ur, dashed, "g"] & (Y, M_Y) \end{tikzcd}$$

Classically, to be smooth, it is the same as saying that the morphism is étale locally affine space over the base.

Theorem 3.1 (Kato's structure theorem)**.** *Let $f\colon (X, M_X) \to (Y, M_Y)$ is a morphism of fine log structures and assume $\beta\colon Q \to \Gamma(Y, M_Y)$ is a chart. Then the following are equivalent:*

1. *f is log smooth.*
2. *étale locally on X, there is a chart $P \to \Gamma(X, M_X)$ and a map of monoids $\theta\colon Q \to P$ so that*

$$\begin{tikzcd} P \arrow[r] & M_X \\ Q \arrow[r] \arrow[u, "\theta"] & M_Y \arrow[u] \end{tikzcd}$$

such that (i) $\ker \theta^{gp}$ and the torsion part of $\operatorname{coker}(\theta^{gp})$ have order invertible on X, and (ii) the natural map $X \to Y \times_{\operatorname{Spec} \mathbb{Z}[Q]} \operatorname{Spec} \mathbb{Z}[P]$ is étale.

Exercise. *(2)⇒(1). The other direction is harder.*

Loosly speaking, being log smooth means you're toric.

Corollary 3.2. *Suppose $(B_0, M_{B_0}) \hookrightarrow (B, M_B)$ is a strict closed immersion (of fine log schemes) defined by a nilpotent ideal, and $(X_0, M_{X_0}) \to (B_0, M_{B_0})$ is log smooth. Then étale locally on X_0, there exists a log smooth lifting $(X, M_X) \to (B, M_B)$ (i.e. this morphism is log smooth) such that*

$$\begin{tikzcd} (X_0, M_{X_0}) \arrow[r, hook, "strict"] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & (X, M_X) \arrow[d] \\ (B_0, M_{B_0}) \arrow[r, hook] & (B, M_B) \end{tikzcd}$$

If you know about deformation theory of smooth schemes, this is very promising.

Remark 3.3. In general, when you do deformation theory, you really want the underlying morphisms of schemes to be flat. The underlying morphisms of log smooth morphisms need not be flat. This can be problematic. ◇

Definition 3.4. A map of fine monoids $\theta\colon Q \to P$ is called *integral* if the map on monoid algebras $\mathbb{Z}[Q] \to \mathbb{Z}[P]$ is flat. ◇

If you write out the equational condition for flatness of a map of rings, this is the following condition. For every $p_1, p_2 \in P$ and $q_1, q_2 \in Q$ such that $p_1 + \theta(q_1) = p_2 + \theta(q_2)$, there exists $p \in P$ and $q_3, q_4 \in Q$ such that $p_1 = p + \theta(q_3)$, $p_2 = p + \theta(q_4)$, and $q_1 + q_3 = q_2 + q_4$.

A map of fine log schemes $f\colon (X, M_X) \to (Y, M_Y)$ is integral if for every geometric point $\overline{x} \to X$, the map on monoids $M_{Y,f(\overline{x})}/\mathcal{O}^\times =: \overline{M}_{Y,f(\overline{x})} \to \overline{M}_{X,\overline{x}} := M_{X,\overline{x}}/\mathcal{O}^\times$.

Being integral has nothing to do with being an integral scheme. It means it is universally integral in the category of monoids (i.e. any pushout remains an integral monoid).

Fact: If $(X, M_X) \to (Y, M_Y)$ is log smooth and integral, then $X \to Y$ is flat, and in (2), you can take $Q \to P$ to be an integral morphism.

Remark 3.5. In the corollary, if $(X_0, M_{X_0}) \to (B_0, M_{B_0})$ is integral, then any lifing $(X, M_X) \to (B, M_B)$ is also integral. $M_{X_0} = M_X/1 + J$, so when you quotient out by $\mathcal{O}^\times$, they are the same. So if you have a log smooth integral morphism, then its log smooth deformations are automatically integral. ◇

Q:The property of being integral is stbale under pullback? MO: yes.

Setup: Start with a strict closed immersion defined by square zero ideal J and $(X_0, M_{X_0}) \to (B_0, M_{B_0})$ log smooth integral.

$$\begin{tikzcd} (X_0, M_{X_0}) \arrow[r, hook, dashed] \arrow[d, "f_0"'] \arrow[dr, phantom, "\ulcorner", very near start] & (X, M_X) \arrow[d, dashed] \\ (B_0, M_{B_0}) \arrow[r, hook] & (B, M_B) \end{tikzcd}$$

We will call an extension a log smooth deformation.

1. étale locally on X_0, there is a log smooth deforamtion,
2. Given an (X, M_X), the automorphism group is given by $\mathrm{Hom}(\Omega^1_{(X_0,M_{X_0})/(B_0,M_{B_0})}, J \otimes \mathcal{O}_X) = T_{X_0/B_0}(log) \otimes J$. The point is that any dashed arrow

$$\begin{tikzcd}
(X_0, M_{X_0}) \arrow[r] \arrow[d] & (X, M_X) \arrow[d] \\
(X, M_X) \arrow[r, hook] \arrow[ur, dashed] & (B, M_B)
\end{tikzcd}$$

   Then $0 \to J \otimes \mathcal{O}_{X_0} \to \mathcal{O}_X \to \mathcal{O}_{X_0} \to 0$. Any morphism is an isomorphism and ...

3. Any two log smooth liftings are étale locally isomorphic

$$\begin{tikzcd}
(X_0, M_{X_0}) \arrow[r] \arrow[d] & (X', M_{X'}) \arrow[d] \\
(X, M_X) \arrow[r] \arrow[ur, dashed] & (B, M_B)
\end{tikzcd}$$

   So the stack of log smooth deformations is a gerbe.

Theorem 3.6. *(1) there is a canonical obstruction $\eta \in H^2(X_0, T_{X_0/B_0}(\log) \otimes J)$ such that $\eta = 0$ if and only if there exists a log smooth deformation. (2) if $\eta = 0$, then the set of log smooth deformations form a torsor under $H^1(X_0, T_{X_0/B_0}(\log) \otimes J)$. (3) the automorphism group of any deformation is isomorphic to $H^0(X_0, T_{X_0/B_0}(\log) \otimes J)$.*

Part (3) is the universal property of differentials. Q: is this false if the map is not flat? MO: I think you run into trouble; the kernel won't be $J \otimes \mathcal{O}_X$. You can make some statement if you're over the dual numbers.

Let me tell you what the obstruction is. It's exactly how it is if you read about ordinary smooth deformations in SGA1. For simplicity, let's assume that X_0 is separated. First, choose an (étale) covering $\mathcal{U} = \{U_i\}$ of X_0 and choose liftings $(\tilde{U}_i, M_{\tilde{U}_i}) \to (B, M_B)$ of (U_i, M_{U_i}). Now we try to patch them together. We have $(U_{ij}, M_{U_{ij}}) \hookrightarrow (\tilde{U}_i|_{U_{ij}}, M_{\tilde{U}_i}|_{U_{ij}})$ and $(U_{ij}, M_{U_{ij}}) \hookrightarrow (\tilde{U}_j|_{U_{ij}}, M_{\tilde{U}_j}|_{U_{ij}})$. By the comment and cohomology of a quasi-coherent sheaf vanishes, we know that there is an isomorphism $\theta_{ij} \colon (\tilde{U}_i|_{U_{ij}}, M_{\tilde{U}_i}|_{U_{ij}}) \to (\tilde{U}_j|_{U_{ij}}, M_{\tilde{U}_j}|_{U_{ij}})$. But now we we need to satisfy a cocycle condition $\partial_{ijk} = \theta_{ij} + \theta_{jk} - \theta_{ik} \in T_{X_0/B_0}(\log) \otimes J$ (everything restricted to U_{ijk}).

Exercise. *$\{\partial_{ijk}\}$ is a Cech 2-cocycle.*

η is the corresponding cohomology class. You can check that η is a boundary if and only if we could have chosen our θ's better so that $\partial_{ijk} = 0$.

Now let's apply this to some examples. Probably, you really just care about schemes, so let's just start with a scheme. Suppose k is a field, and $X_0 \to \operatorname{Spec} k$ is some scheme we're interested in. Suppose that étale locally, $X_0 = \operatorname{Spec} k[x_1, \ldots, x_n]/x_1 \cdots x_r$. Let M_k be the log structure on k given by $k^\times \oplus \mathbb{N} \to k$ given by $(u, n) \mapsto u \cdot 0^n$. Question: When does there exist a log structure M_{X_0} and a morphism $(X, M_{X_0}) \to (\operatorname{Spec} k, M_k)$ which locally is "the standard one" (one of the examples from before)? You view your X_0 as $(k \otimes_{k[\mathbb{N}]} k[\mathbb{N}^r])[x_{r+1}, \ldots, x_n]$ and you get a natural log structure which we call the standard one. Answer: d-semistability: when the line bundle $\mathcal{E}xt^1(\Omega^1_{X_0/k}, \mathcal{O}_{X_0})$ on $D = X_0^{sing}$ is trivial. In fact, there is more you can say; the log structure is unique up to something.

Example 3.7. (1) nodal curve. (2) Exercise where you blow up the 3-torsion points on $E \hookrightarrow \mathbb{P}^2$ and take X_0 to be the gluing of two copies. $H^2(X_0, T_{X_0} \otimes J) = 0$. The Hodge diamond $H^i(X_0, \Omega^j_{X_0/k}(\log))$ is as on the exercises:

$$\begin{matrix} 1 & 0 & 1 \\ 0 & 20 & 0 \\ 1 & 0 & 1 \end{matrix}$$

$\Omega^2(\log) = \mathcal{O}_{X_0}$ is the dualizing sheaf implies X_0 is smoothable. $\diamond$

4 Tom Bridgeland

Last time we were talking about the space of stability conditions. When you combine t-structures and stability conditions on an abelian category, you get this quite interesting thing. Basically, we get that $\operatorname{Stab}\mathsf{D} \to \operatorname{Hom}_{\mathbb{Z}}(K(\mathsf{D}), \mathbb{C})$, given by $(Z, \mathsf{P}) \mapsto Z$, is a local homeomorphism.

Today I want to talk about the example of the conifold. If you really want to know about it, it is in the paper: Tom Bridgeland, "Stability conditions on triangulated categories". Consider $X = \{x_1x_2 - x_3x_4\} \subseteq \mathbb{C}^4$. We have two resolutions $Y^{\pm}$, which contain curves $C^{\pm}$ as fibers over the singularity. The common resolution is $f^{\pm}\colon Z \to Y^{\pm}$. You can look at Toda's paper "stability conditions and crepand small resolutions".

Theorem 4.1 (Bondal, Orlav, ...). *There exist equivalences* $\Psi = Rf^{+}_{*} \circ L(f^{-})^{*}\colon \mathsf{D}^b\mathsf{Coh}(Y^-) \to \mathsf{D}^b\mathsf{Coh}(Y^+)$ *(meaning that there is an equivalence commuting with the pushdown to X), and* $\Phi^{\pm}\colon \mathsf{D}^b\mathsf{Coh}(Y^{\pm}) \to \mathsf{D}^b R\text{-}\mathsf{mod}$ *with* $\Phi^- = \Phi^+ \circ \Psi$*, where R is the algebra of the quiver* ($\bullet \underset{b_1,b_2}{\overset{a_1,a_2}{\rightleftarrows}} \bullet$) *with relations of superpotential* $W = a_1b_1a_2b_2 - a_1b_2a_2b_2$ *(Klebanov-Witten).*

You can do this for any flop; things become a bit more difficult. The black magic rule to interpret the word superpotential is that you think $dW = 0$, which is four relations, one of them being $\partial_{a_2}W = b_2a_1b_1 - b_1a_1b_2 = 0$.

Consider representations of $\mathbb{C}Q$ (the quiver without the relations) of given dimension (d_1, d_2). It is $\big(\operatorname{Hom}(\mathbb{C}^{d_1}, \mathbb{C}^{d_2})^{\otimes 2} \otimes \operatorname{Hom}(\mathbb{C}^{d_2}, \mathbb{C}^{d_1})^{\otimes 2}\big)/GL(d_1) \times GL(d_2)$. Let A_i and B_i be the images of a_i and b_i. Then we get $\Phi = tr(W(A_1, A_2, B_1, B_2))\colon Q\text{-}\mathsf{mod} \to \mathbb{C}$. The moduli stack of representations of $R = \mathbb{C}[Q]/I = \{d\Phi = 0\}$ sits inside the representations of Q. [[★★★ somehow]] the black magic rule I gave before comes from this.

If I write a quiver with some relations, and the algebra you get is CY symmetric, it has to be given by a superpotential (Segal, Bocklandt).

How do you get R in this situation? You get it geometrically by considering $Y^+ = Tot(\mathcal{O}(-1,-1)) \xrightarrow{\pi} C^+ = \mathbb{P}^1$. The tilting object is $E^+ = \pi^*(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$. Then $R = \operatorname{End}_{Y^+}(E^+)$ and $\Phi^+ = R\operatorname{Hom}(E^+, -)$. Similarly, for Φ^-, you take $E^- = \pi^*(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1))$ and $\Phi^- = R\operatorname{Hom}_{Y^-}(E^-, -)$.

Under these equivalences, $\mathcal{O}_{C^-}(-1)$ and $\mathcal{O}_{C^-}(-2)[1]$ go to the simple modules S and T, respectively, where $\dim S = (1,0)$ and $\dim T = (0,1)$. Similarly, S and T correspond to $\mathcal{O}_{C^+}(-1)[1]$ and $\mathcal{O}_{C^+}$, respectively. This should be pretty clear from applying these functors. You get the impression that people spend their time shuffling around adjunctions, but you eventually have to think about specific objects and where they go.

The above equivalences restrict to {objects (topologically) supported on C^-} $\leftrightarrow$ {objects supported on C^-} and {objects supported on $C^{\pm}$} $\leftrightarrow$ {objects with nilpotent cohomology modules}. $R = \bigoplus_{n \geq 0} R_n$ is a graded algebra. $R_nM = 0$ for all $n \gg 0$ if and only if M is finite length and all simple factors are S and T (the vertex simples); we say such an M is nilpotent. I think you can get a lot by just considering these subcategories. They are all equivalent, so call them D. Then $K(\mathsf{D}) = \mathbb{Z}^{\oplus 2} = \mathbb{Z}[S] \oplus \mathbb{Z}[T]$. Note that the class of a point is $[\mathcal{O}_x] = [S] + [T]$.

Consider "normalized" stability stability conditions, where we assume $Z([S]+[T]) = -1$. I didn't tell you this, but there is always an action of $\mathbb{C}$ on the stability condition, where $\mathbb{C}^{\times}$ rotates Z. Consider the map $\pi\colon \operatorname{Stab}_n(\mathsf{D}) \to \mathbb{C}$, given by $(Z, \mathsf{P}) \mapsto Z([T])$. This will be a local homeomorphism.

Theorem 4.2 (Toda). *There is a connected combonent* $\operatorname{Stab}^*(\mathsf{D})$ *such that* $\pi\colon \operatorname{Stab}^*(\mathsf{D}) \to \mathbb{C} \smallsetminus \mathbb{Z}$ *is the universal covering.*

Physicists would say the $\mathcal{O}_C[-n]$ are the branes and their mass is not allowed to vanish.

I'll define a point in Stab^* that lies in the upper half plane, and we'll go on a little journey down (between -1 and 0) and see what happens.

We have the heart $\mathsf{Coh}_{C^+}(Y^+)$, the category of coherent sheaves on Y^+ supported on C^+. Note that I'm not claiming that the bounded derived category of this is equal to D; I suspect it's not. The stability function is $Z(E) = ch_2(E) \cdot (\beta + i\omega) - ch_3(E)$, where $\beta, \omega \in H^2(Y^+, \mathbb{R}) \cong \mathbb{R}$, and $\omega > 0$ is ample, and $\beta + i\omega$ is in the complexified Kähler cone, which in this case is just the upper half plane $\mathcal{H}$. You have to check the HN property.

Now let's suppose $\omega \to 0$ with $\beta \cdot C \in (-1, 0)$. We have that $Z(\mathcal{O}_C)$ is in the upper half plane and $Z(\mathcal{O}_x)$ is on the real line. So as $Z(\mathcal{O}_C)$ goes to the real line, it just decides which things end up on which side of the real line. The stuff that ends up on $\mathbb{R}_+$ is no longer in the heart. $\mathsf{P}((0,1])$ changed so that we lose $\mathcal{O}_{C^+}(-k)$ for $k \geq 1$, but we gain $\mathcal{O}_{C^+}(-k)[1]$ for $k \geq 1$. This is like one of those filmstrip pictures from before (in some sense, this is the $\mathbb{P}^1$ diagram I drew before). Call our new heart A. $\mathsf{A} = R\text{-}\mathsf{mod}_{nil} \subset \mathsf{D}$. So you can continuously get across the flop if you've complexified the Kähler class, but in between, you're naturally talking about this non-commutative guy. As an abelian category, it is equivalent, but it is sitting inside the derived category differently.

If we continue our journey a little bit more, we need to tilt again. We lose $\mathcal{O}_{C^+}(k)$ for $k \geq 0$ and we gain $\mathcal{O}_{C^+}(k)[-1]$. I'll leave it to you to check that the new heart is naturally $\mathsf{Coh}_{C^-}(Y^-) \subset \mathsf{D}$. Now you can think about going back along another path (looping around the integer point $k \in \mathbb{Z}$). You get the action of the Seidel-Thomas twist functor $\Phi_{\mathcal{O}_C(k)}$.

Next time I'll come back to Hall algebras and tell you how to do it in characteristic zero. I also want to explain [[★★★ some other stuff]].

3 Kai Behrend

Microlocal geometry

If you want to learn more about microlocal geometry, there are some notes by MacPherson from Park City. There is also a book that contains all the results and all the details: Kashiwara-Shapira, Sheaves over manifolds.

Everything will be over $\mathbb{C}$. Let X be a singular scheme, embedded in a smooth scheme M. The content of microlocal geometry is that you can study X by means of the symplectic geometry of the cotangent bundle Ω_M of M.

Remark 3.1. If $V \subset M$ is a closed subvariety and V^0 is the smooth locus of V. Then define $\ell(V) = \overline{\mathcal{N}^\vee}_{V^0/M} \subseteq \Omega_M$ to be the closure conormal bundle in the cotangent bundle of M. This is a Lagrangian cone. ⋄

A conic Lagrangian subvariety of Ω_M is a closed subvariety of dimension of $\dim M$ such that the restriction of the symplectic form $\sigma = d\alpha = \sum dp_i \wedge dx_i$ (where $\alpha = \sum p_i dx_i$) vanishes and invariant with respect to the $\mathbb{C}^\times$ action on Ω_M (given by scaling the fibers). The property of being Lagrangian is a generic property, so we can just require that α vanishes at the generic point. We have coordinates (x_i, p_i). We may as well assume $V^0 = \{x_i = 0 | i \le k\}$. Then $\mathcal{N}^\vee_{V^0/M} = \{x_i = 0, p_j = 0 | i \le k < j\}$.

Exercise. *All conic Lagrangian prime cycles (i.e. irreducible) on Ω_M are obtained in this way. That is, every such cycle is the closure of the conormal bundle of a variety on the base.*

So $L\colon Z_*(M) \to \mathcal{L}(\Omega_M)$, where $Z_*(M)$ are the algebraic cycles on M and $\mathcal{L}(\Omega_M)$ are the conic Lagrangian cycles, given by $V \mapsto (-1)^{\dim V}\ell(V)$, is an isomorphism of groups. We can restrict this to an isomorphism $L\colon Z_*(X) \xrightarrow{\sim} \mathcal{L}_X(\Omega_M)$ (everything supported on X).

Example 3.2 ("distinguished cycle")**.** $c_X = \sum_{c'} (-1)^{\dim \pi(c')} mult(c')\pi(c')$, where the sum is over all c' irreducible components of $c_{X/M}$ and $\pi\colon c_{X/M} \to X$. It turns out that this cycle is independent of the choice of embedding of X in M. $L(c_X)$ could be called the "distinguished Lagrangian cycle of X in Ω_M". Of course, this does depend on the embedding of X in M. ⋄

Q: are there any assumptions on X; can it be reducible of various dimensions? KB: X can be any scheme which is embeddable into something of finite type; it can be any closed subscheme of M.

We get the commutative diagram

$$\begin{tikzcd} Z_*(X) \arrow[r, "L"] \arrow[dr, "c_0^M"'] & \mathcal{L}_X(\Omega_M) \arrow[d, "0^!_{\Omega_M|X}"] \\ & A_0(X) \end{tikzcd}$$

where c_0^M is the degree zero part of the Chern-Mather class.

The isomorphism L factors as $Z_*(X) \xrightarrow{Eu} \mathrm{Con}(X) \xrightarrow{Char} \mathcal{L}_X(\Omega_M)$, where $\mathrm{Con}(X)$ is the constructible functions $X \to \mathbb{Z}$. Eu is MacPherson's local Euler obstruction (you can find this in Fulton's book on intersection theory), and $Char$ is the characteristic cycle. both of these are isomorphisms. The easiest thing to define is $Char^{-1}$, and this was first done by Ginzburg. $Char^{-1}\colon \mathcal{L}_X(\Omega_M) \to \mathrm{Con}(X)$ is given by $[c] \mapsto \big(P \mapsto I_P([c], [\Delta])\big)$, where Δ is the graph $d\rho$ of the square of a Euclidean distance function $\rho\colon M \to \mathbb{R}$ from P (basically, $\rho(x) = \sum x_i^2$, $d\rho = \sum 2x_i dx_i$). Ginzburg proves that P is an isolated point of the intersection, so the intersection multiplicity $I_P([c], [\Delta])$ makes sense. [[★★★ picture: symplectic manifold Ω_M with zero section M. Lagrangian cone looks like a collection of vertical lines.]] At every point in M, you compute the intersection number of the graph with the cone.

Really, we're intersecting the cone $[c]$ (vertical lines) with $[0]$. The dimensions are complementary, so I should be getting a cycle of degree zero. Every point of M has a well-defined contribution, which is that intersection number. This kind of vague statement is justified by this theorem.

Theorem 3.3 (MacPherson/Kashiwara, 1970s)**.** *If X is compact, then for every cycle $c \in Z_*(X)$, $\chi(X, Eu(c))$ (this means chop X into pieces where the function is constant; compute their Euler characteristics, and add them up with weight given by the function) is equal to $\int_X c_0^M$. Kashiwara's formulation is $\chi(X, Char^{-1}(c)) = \int_X 0^! c$.*

I think of this theorem as a generalization of the Gauß-Bonnet theorem to singular schemes. If X is smooth, $(-1)^{\dim X}\chi(X) = \int_X e(\Omega_X)$.

Example 3.4 (distinguished cycle)**.** Let $\nu_X := Eu(c_X)$ be the "distinguished constructible function on X". This is intrinsic to X. It is the microlocal function in the story. If you read some recent paper, some joker started calling it χ^B, but I won't use that notation.

In this case, the theorem says $\boxed{\int_X 0^! L(c_X) = \chi(X, \nu_X).}$ ◇

Symmetric obstruction theories

An obstruction theory for the scheme X is an arrow in $\mathsf{D}(\mathcal{O}_X)$, $\mathcal{F} \to \tau_{\geq -1} L_X$, with certain properties which I won't repeat. The obstruction theory is *symmetric* if it is endowed with $\beta\colon \mathcal{F} \otimes \mathcal{F} \to \mathcal{O}[1]$ (this is of course derived tensor product and this is a morphism in $\mathsf{D}(\mathcal{O}_X)$) such that the induced map $\alpha\colon \mathcal{F} \to \mathcal{F}^\vee[1]$ is (1) an isomorphism in $\mathsf{D}(\mathcal{O}_X)$, and (2) symmetric in the sense that $\alpha^\vee[1] = \alpha$ in $\mathsf{D}(\mathcal{O}_X)$. Of course, there are signs involved, but I thought about it for months and determined that that was the right sign ... but that was years ago.

In our case, we're interested in X, the moduli space of sheaves on the Calabi-Yau Y. We have $\pi\colon X \times Y \to X$, with $\mathcal{F}^\vee = R\pi_* R\mathcal{H}om(E, E)_0$, and β comes from Serre duality.

Remark 3.5. You can write $\mathcal{F}$ much more explicitly in my construction of the moduli space: $\mathcal{F} = [W^0 \to W^1]$ was the trunction of the obstruction complex I constructed on X. I explained that X came with an embedding into some M, and $W^0 = T_M|_X$. What would have been nice, but I cannot prove, is for $W^1 = \Omega_M|_X$. It is a vector bundle of the right dimension, but it is not straightforward to prove. So I really wanted that $\mathcal{F} = [T_M|_X \to \Omega_M|_X]$, and the map is self-dual. If you manage to prove this, let me know. In the absence of that, I have to throw all this homological algebra at you. ◇

On X itself (not on M), I do have the exact sequence

$$0 \to T_X \to W^0 \to W^1 \to ob \to 0.$$

The kernel is the tangent bundle and the cokernel is defined to be the obstruction sheaf. If you take the dual of this obstruction theory, you're supposed to get the same thing back. There is an isomorphism (in the derived category) to $[W^{1*} \to W^{0*} \to ob \to 0]$. So $T_X = ob^\vee$.

Last time I did construct $c_{X/M} \hookrightarrow W^1$, and $[X]^{vir} = 0^![c_{X/M}]$.

$$\begin{tikzcd} W^1 \arrow[r] & ob & \Omega_M|_X \arrow[l, two heads] \\ c_{X/M} \arrow[u, hook] \arrow[r] & cv \arrow[u, hook] & C \arrow[l] \arrow[u] \arrow[ul, phantom, "\lrcorner"] \end{tikzcd}$$

Where cv is the subsheaf of cones (big sheaves on some big site; you can also do it with stacks, but you don't have to), it stands for "curvilinear obstructions". The big thing is that $\boxed{ob = \Omega_X \text{ and } T_X = ob^\vee.}$ Forming the pullback C gives a scheme of cones in $\Omega_M|_X$. Q: what is the property of X that allows you to understand the dual of T_X? KB: it is the other way around $T_X = ob^\vee$, but $ob \neq T_X^\vee$.

Proposition 3.6. $[X]^{vir} = 0^!_{\Omega_M|_X}[C]$.

Now I'm in the situation where I can apply microlocal geometry.

Toy models for symmetric obstruction theories

ω is a 1-form on M and $X = Z(\omega) \subset M$. Then (repeating what I said last time) how do you get the obstruction theory?

$$\mathcal{F} \quad = \quad \begin{tikzcd} {[T_M|_X} \arrow[r] \arrow[d] & {\Omega_M|_X]} \arrow[d, equal] \\ {[\mathcal{I}/\mathcal{I}^2} \arrow[r] & {\Omega_M|_X]} \end{tikzcd} \qquad \begin{tikzcd} {[T_M|_X} \arrow[r] \arrow[d, equal] & {\Omega_M|_X]} \arrow[d] \\ {[T_M|_X} \arrow[r] & {\mathcal{N}_{X/M}]} \end{tikzcd}$$

(the diagram on the right is the dual) If ω is closed, then the obstruction theory is symmetric. $\Gamma_\omega \subset \Omega_M$ is Lagrangian, and $\omega\colon M \hookrightarrow \Omega_M$ with $\omega^*(\sum dp_i \wedge dx_i) = \sum df_i \wedge dx_i = d\omega = 0$. Here Γ_ω is the graph of ω. In the limit (rescaling), $c_{X/M} \hookrightarrow \Omega_M$ is the Lagrangian cone.

Theorem 3.7. *You need only* $\frac{\partial f_i}{\partial x_j} \equiv \frac{\partial f_j}{\partial x_i}(f_1, \ldots, f_n)$ *to get the result that* $c_{X/M} \hookrightarrow \Omega_M$ *is Lagrangian and that the obstruction theory is symmetric. In this case we say that* ω *is* almost closed.

Proposition 3.8. *If* X *has a symmetric obstruction theory and* $X \hookrightarrow M$ *embedded, then étale locally in* M *there exists a closed 1-form cutting out* X *and giving rise to the given symmetric obstruction theory.*

Q: what if X is smooth of the wrong dimension? KB: If X is smooth and $M = X$, with $\omega = 0$. Then the obstruction theory is

$$\begin{array}{ccc} [T_M|_X & \xrightarrow{0} & \Omega_M|_X] \\ \downarrow & & \| \\ [0 & \longrightarrow & \Omega_X] \end{array}$$

You check locally that $C \hookrightarrow \Omega_M$ is Lagrangian. Because C is locally isomorphic to the normal cone, we get the corollary.

Corollary 3.9. *The underlying cycle $[C]$ is the distinguished Lagrangian cycle, so $\#^{vir} X = \int_X 0^! L[c_X] = \chi(X, \nu_X)$.*

3 Alessio Corti

References for today: CCIT, in preparation; CCLT, weighted projective spaces.

Plan: the J-function, S-extended stuff, I-function, mirror theorem, and a simple example ($\mathbb{P}^{2,2}$). Tomorrow, I'll try to write down some presentations for quantum cohomology for toric stacks (with some assumptions)

The J-function

$J(\tau, z) = z + \tau + \sum_{\ell,n} \frac{Q^\ell}{n!} ev_{x+i*}(ev_1^*\tau \cdots ev_n^*\tau \cdot \frac{\mathbb{1}}{z-\psi_{n+1}}) \in H^\bullet_{\pi,orb}(\mathcal{X}, \mathbb{C})$, where the things in the sum are happening on $\mathcal{X}_{0,n+1,\ell}$ and $ev_i\colon \mathcal{X}_{0,n+1,\ell} \to I\mathcal{X}$. L_i are line bundles, and $L_{i,f} = T^\vee_{C,x_i}$, and $\psi_i = c_1(L_i)$.

Goal: if $\mathcal{X}$ is a toric stack, write down J explicitly. Why, Alessio? For one thing, J contains all information about quantum cohomology. How does one come to consider this power series? J is the fundamental class in some cohomology theory, $H^{\infty/2}_{S^1}(\mathcal{L}_0\mathcal{X}, \mathbb{C})$. In some sense, when interpreted correctly, J has degree 1.

S-extended stuff

Let $\mathcal{X}$ be a toric stack with stacky fan (N, Σ, ρ). To this, we attached the fan sequence and the divisor sequence

$$0 \to \mathbb{L} \to \mathbb{Z}^m \xrightarrow{\rho} N$$

$$0 \to M \to \mathbb{Z}^{*m} \xrightarrow{D} \mathbb{L}^\vee$$

where $\mathbb{L}^\vee = Pic(\mathcal{X})$ and $\mathbb{L} = N_1(\mathcal{X}, \mathbb{Z})$. Let $S \subset N$ be a (finite, for today at least) subset which containes the rays ρ_i. The key examples of S will be just the set of rays, or $S = B = \mathrm{Box}(\mathcal{X})$ (which will lead to the enhanced stuff). Recall that $\mathrm{Box}(\mathcal{X}) = \bigcup_{\sigma\in\Sigma}\{v \in N | \overline{v} \in \sum_{i\in\sigma} a_i\overline{\rho}_i, 0 \le a_i < 1\}$ (remember that the bar just means image in N modulo torsion). Perhaps the most important example is where $S = \overline{B}^{\le 1} = \{v \in \overline{B} | \sum a_i \le 1\}$, the closed box. I think of $\overline{B}^{\le 1}$ as a basis for $H^{\le 2}_{\mathbb{T},orb}(\mathcal{X}, \mathbb{C})$. This will be the space of parameters for small quantum cohomology.

We have $\rho^S : \mathbb{Z}^S \mapsto N$, given by $e_s \mapsto s$, and let $\mathbb{L}^S$ be the kernel of this map. The get the Gale dual

$$\mathbb{L}^{S\vee} \xleftarrow{D^s} \mathbb{Z}^{*S} \leftarrow M \leftarrow 0.$$

I think of these as being some kind of glorified Picard group and topological classes of stable morphisms.

If $\sigma \in \Sigma$, we write $C^S_\sigma = \{\sum_{i\in S\smallsetminus\sigma} r_i D^S_i | r_i \ge 0\} \subseteq \mathbb{L}^{S\vee}_\mathbb{R}$ is a cone. We define $NE^S_\sigma = C^{S\vee}_\sigma \subseteq \mathbb{L}^S_\mathbb{R}$ and $NE^S = \sum_{\sigma\in\Sigma} NE^S_\sigma$. Define $\Lambda^S_\sigma = \{\lambda = \sum_{i\in S} f_i e_i \in \mathbb{L}^S_\mathbb{R} | j \notin \sigma \Rightarrow f_j \in \mathbb{Z}\}$, and $\Lambda^S = \bigcup_{\sigma\in\Sigma}\Lambda^S_\sigma$. The goal is to tell you exactly, inside $\mathcal{X}$, what are all the possible degrees of stable maps from an orbi-curve. There is a map $v\colon \Lambda^S \to B$, given by $v(\lambda) = \sum_{i\in S}\lceil f_i \rceil_{\rho_i} \in B$. $\Lambda E^S = \Lambda^S \cap NE^S$. These S's allow you to keep track of which torus invariant loci the various marked points are in.

The I-function

$$I^S(\tilde{Q}, z) = z \sum_{v\in B} \sum_{\substack{\lambda\in\Lambda E^S \\ v(\lambda)=v}} \tilde{Q}^\lambda \mathbb{1}_v \Box_\lambda(z)$$

where

$$\Box_\lambda(z) = \frac{\prod_{i\in S}\prod_{\langle b\rangle=\langle\lambda_i\rangle, b\le 0}(ui + bz)}{\prod_{i\in S}\prod_{\langle b\rangle=\langle\lambda_i\rangle, b\le\lambda_i}(ui + bz)} \in SR^\bullet_\mathbb{T}[z, z^{-1}]\!]$$

The box corresponds to irreducible components of $I\mathcal{X}$, $\mathbb{1}_v \in H^\bullet_{orb}(\mathcal{X})$ corresponding fundamental class, $u_i = u^{\rho_i} \in SR^\bullet_\mathbb{T}(\mathcal{X})$ if i is one of the rays (and $u^i = 0$ for $i \in S \smallsetminus \{\rho_i\}$), and $\lambda_i = \lambda \cdot D^S_i$.

One place where you can find this in a slightly less general context is in a paper of Bousov and Horja, Mellin-Baues, etc. Givental wrote it down in an article on dark manifolds. Q: was this inspired by mirror symmetry? AC: yes.

Mirror theorem

Theorem 3.1 (CCIT)**.** *Assume $\mathcal{X}$ is weak Fano and $S \subset \overline{B}^{\le 1}$. Then (t is more or less Q) $I^S(t; z) = F(t)z + \mathbb{G}(t) + O(z^{-1})$. $J^S(\tau(t), z) = \frac{I^S(t,z)}{F(t)}$, where $\tau(t) = \mathbb{G}(t)/F(t)$.*

$J\colon H^\bullet_{\mathbb{T},orb} \to ?$. By Stanley-Riesner, $S \subset H^\bullet_{\mathbb{T},orb}(\mathcal{X}, \mathbb{C})$, with $\langle S\rangle$ the subspace generated by S, then $J^S = J|_{\langle S\rangle}$.

Special cases:

1. If $\mathcal{X}$ is Fano ($-K_{\mathcal{X}}$ is ample, not just nef) and has canonical singularities and $S = \{\rho_i\}$, then $I^S = J^S$. In the case where $\mathcal{X}$ is a manifold, this recovers Givental's theorem.

2. If $\mathcal{X} = \mathbb{P}^{w_0,\ldots,w_n}$ is a weighted projective space and $S = \{\rho_i\}$, then $I^S = J^S$. This was proven in CCLT.

Example 3.2 ($\mathbb{P}^{2,2}$)**.** The fan diagram for $\mathbb{P}^{2,2}$ is

$$0 \to \mathbb{Z} \xrightarrow{\binom{2}{2}} \mathbb{Z}^2 \xrightarrow[\left(\begin{smallmatrix} -1 & 1 \\ 0 & 1 \end{smallmatrix}\right)]{\rho} N = \mathbb{Z} \oplus \mathbb{Z}/2$$

Box $= \{0, 1\}$. The fan looks like [[★★★ ρ_1, ρ_3, ρ_6 at "height" 0 and ρ_5, ρ_4, ρ_2 at height ε]] Take $S = \overline{B}^{\leq 1}$.

$$0 \to \mathbb{Z}^5 \cong 1^S \to \mathbb{Z}^6 \xrightarrow[\left(\begin{smallmatrix} -1 & 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{smallmatrix}\right)]{\rho^S} N$$

$$I^S(Q; , s, t, z) = z e^{\frac{s_1 u_1 - s_2 u_2}{z}} \sum_{\ell, k_0, \ldots, k_3 \in \mathbb{N}} \frac{Q^\ell e^{\ell(s_1+s_2)} t_0^{k_0} \cdot t_3^{k_3}}{z^{\sum k_i} k_0! k_1! k_2! k_3!} \mathbb{1}\Big\langle \frac{\ell + k_0 + k_1 + k_2 + k_3}{2} \Big\rangle \\ \cdot \frac{\prod_{b \leq 0}(u_1 + bz) \prod_{b \leq 0}(u_2 + bz)}{\prod_{b \leq \ell - k_2}(u_1 + bz) \prod_{b \leq \ell - k_3}(u_2 - bz)}$$

and the mirror map is

$$\tau(t) = u_1 s_1 + u_2 s_2 + t_0 \mathbb{1} + t_1 \mathbb{1}_{1/2} - \frac{u_1}{2} \log(1 - t_2^2) + \frac{u_1 \mathbb{1} v_2}{2} \log(??) \cdots$$

[[★★★ somebody fill in the rest of the mirror map]] ◇

3 Valery Alexeev

Quiz for today: Suppose you have a family of [[★★★ triangle]] $X_t \cong \mathbb{P}^2 \to Z$ degenerating to [[★★★ that breaking of the triangle you always draw for proper non-projective]] $X_0 = \lim_{t\to 0} \to Z$. Can this happen? If you recall, you have to have some height function that the broken picture is a projection of. This should remind you of that famous picture of Escher. If you didn't know, that picture is not really possible; it is an optical trick. However, there is a point in the moduli space for this stable toric variety. The result is that the moduli space of stable toric varieties has multiple components; not everything can be written as a limit of things in the main component.

I will state three main theorems and try to give as many examples as possible. Fix positive integers $r, n \in \mathbb{N}$ (n divisors in $\mathbb{P}^{r-1}$), and a weight $\beta = (b_1, \dots, b_n)$, with b_i rational $0 < b_i \le 1$.

Theorem 3.1. *There exists a family $(\mathcal{X}, B_1, \dots, B_n) \to \overline{M}_\beta(r,n)$ such that every fiber $(X, \sum b_i B_i)$ is a stable curve. Moreover, there is an open subset $M_\beta(r,n) \subseteq \overline{M}_\beta(r,n)$ such that the restriction of the family is a family of lc pairs $(\mathbb{P}^{r-1}, B_i)$. Furthermore, all fibers are non-isomorphic.*

The *weight domain* (the possible values of β) is $D = \{\beta = (b_i) | 0 < b_i \le 1, \sum b_i > r\}$. It looks like a cube, with a corner cut off by the inequality. We take a chamber decomposition, where the walls are $\sum_{i\in I} b_i = k$ for all $I \subseteq \{1, \dots, n\}$ and for all $1 \le k \le n-1$. For example, something on the boundary lies on a different chamber from something in the interior. [[★★★ picture]]

Theorem 3.2. *(1) If $Ch(\beta) = Ch(\beta')$, then $\overline{M}_\beta = \overline{M}_{\beta'}$ and $(\mathcal{X}, B_i)_\beta = (\mathcal{X}, B_i)_{\beta'}$. (2) if $\beta' \in \overline{Ch(\beta)}$, then we get a commutative (not cartesian) diagram*

$$\begin{tikzcd} \mathcal{X}_\beta \arrow[r] \arrow[d] & \mathcal{X}_{\beta'} \arrow[d] \\ \overline{M}_\beta \arrow[r] & \overline{M}_{\beta'} \end{tikzcd}$$

Moreover, if $\beta' > \beta$ (in every coordinate), then $\overline{M}_\beta \xrightarrow{\sim} \overline{M}_{\beta'}$ and $\mathcal{X}_\beta \to \mathcal{X}_{\beta'}$ is birational (when you go down, it doesn't have to be birational). (3) For all $\beta > \beta'$, we have morphisms (dashed is rational map)

$$\begin{tikzcd} \mathcal{X}_\beta \arrow[r, dashed] \arrow[d] & \mathcal{X}_{\beta'} \arrow[d] \\ \overline{M}_\beta \arrow[r] & \overline{M}_{\beta'} \end{tikzcd}$$

and on fibers, X' is the log canonical model for $(X, \sum b_i' B_i)$ (in particular, the model exists).[1]

Theorem 3.3. *(1) Every X is Cohen-MaCaulay, and $X \setminus \bigcup B_i$ is Gorenstein. (2) for β in the maximal chambers, then X is Gorenstien and B_i are Cartier.*

Example 3.4. Suppose we have $(\mathbb{P}^2 + 5 \text{ lines})_t$ in general position, and as $t \to 0$, the lines converge to two triple points [[★★★ picture, with numbered lines]]. Let's take $\beta = (1,1,1,1,1)$. If the sum of the weights is less than one, the lines can all coincide. If the sum of weights is less than 2, then three lines can go through the same point. Here the sum is 5, so we get general position. What happens in the limit? In the central fiber, you'd blow up the two points. You get the blowup of $\mathbb{P}^2$ at two points. You've blown up a 3-fold, so you get two extra $\mathbb{P}^2$'s. The lines will break up [[★★★ picture with ears + picture with triangle ears (tropical?)]]. You would think that this is the limit, but it's not. If you try this with weights $(1,1,1,1,1-\varepsilon)$, then this is indeed a stable pair ($K+B$ is ample). If you compute $(K+B)\cdot C$, where C is a piece of line number 5, you get ε. So so long as $\varepsilon > 0$, you're stable, but for $\varepsilon = 0$, that curve has to be contracted, so the actual picture is [[★★★ picture + picture with triangles]]. Let's call the weight $\beta' = (1,1,\dots,1)$; this is in the closure. There is another way to go to the closure; consider the weights $(\frac{1+\varepsilon}{2}, \frac{1+\varepsilon}{2}, 1, 1, 1-\varepsilon)$. Now there was no reason to blow up the first point; you only had to blow up the second point, so the picture is [[★★★ picture]]. We have morphisms [[★★★ triangle picture]]; one is birational, but the other is not; we lost a whole $\mathbb{P}^2$. ⋄

In addition to producing progressively cooler pictures, I'd like to tell you about how to construct these things.

I will start with the Grassmanian of r-dimensional subspaces of $\mathbb{C}^n$ with the Plüker embedding $G(r,n) \hookrightarrow \mathbb{P}(\bigwedge^r \mathbb{C}^n)$. We have the torus $\widetilde{T} = (\mathbb{C}^\times)^n$ acts

[1] The map to the log canonical model is only rational. For surfaces, it is usually an acutal map if you work with normal surfaces.

on the Grassmanian, $T = \widetilde{I}$ the diagonal copy of $\mathbb{C}^\times$. there are $\binom{n}{r}$ Plüker coordinates $p_{i_1,\dots,i_r}$ with $i_1 < \cdots < i_r$, which have characters in $\mathbb{Z}^n$ (each entry is zero or 1). There is a hypersimplex $\Delta(r,n)$, which is (1) the convex hull of these (2) $\{(x_i) \in \mathbb{R}^n | 0 \le x_i \le 1, \sum x_i = r\}$. $[V \subset \mathbb{C}^n] \in G(r,n)$ has an embedded toric variety $\overline{T[V]}$ (over $G(r,n)$). There is a moment polytope P_V (called the metroid polytope). P_V is the convex hull of the $V(p_I)$ such that $p_I(V) \neq 0$. It is also $\{(x_i) \in \mathbb{R}^n | K_{\mathbb{P}^{r-1}} + \sum x_i B_i = 0, (\mathbb{P}^{r-1}, \sum x_i B_i \text{ lc}\}$. $V^r \hookrightarrow \mathbb{C}^n$ fixed, so we get $\mathbb{P}^{r-1} \cong \mathbb{P}V \hookrightarrow \mathbb{P}^{n-1}$, with $B_i = \mathbb{P}V \cap H_i$ where $H_i = \{z_i = 0\}$. Note that the definition still works if something is contained in something.

Example 3.5. Begin with a hyperplane arrangement $(\mathbb{P}^{r-1}, \sum b_i B_i) = (\mathbb{P}V, \sum b_i B_i)$ which is lc. Over $G(r,n)$, I have the universal family $U \subseteq \mathbb{P}^{n-1} \times G(r,n)$, $\pi\colon U \to G(r,n)$. I have the point $[\mathbb{P}V \subseteq \mathbb{P}^{r-1}] \in G(r,n)$. I take the orbit $T \cdot [\mathbb{P}V \subseteq \mathbb{P}^{r-1}]$. I claim that the stabilizer is trivial, so the orbit is isomorphic to T. Take the preimage of the orbit $\pi^{-1}(U)$. I can take the quotient $\pi^{-1}(U)/T$, which will recover the pair I started with. ◇

Example 3.6. Take $r = 2$ and $n = 4$, four points on $\mathbb{P}^1$. The easiest degeneration is where you break the $\mathbb{P}^1$ to get points 1 and 2 on one piece and 3 and 4 on the other piece. Start with $\Delta(2,4)$, a hypersimplex (looks like an octahedron) with vertices labelled by distinct pairs of numbers between 1 and 4. What is the configuration where the first two points coincide. What is the metroid polytope of this arrangement? You see that the Plüker coordinate $p_{12} = 0$ and $p_{ij} \neq 0$ for $(i,j) \neq (i,j)$. So we get the lower pyramid (the top vertex is 12). What is the condition for this to be log canonical? it is that $P_V = \{x_1 + x_2 \le 1\}$, which is the lower pyramid. I am working with $\beta = (1,1,1,1)$. What is the locus where the pairs are log canonical? They are the places where 1 and 2 do not coincide. I'm looking at a certain open subset (given by GIT) $\pi^{-1}(U)^{ss}_\beta$ where the pair is log canonical. When I divide by the torus action, I get the line with points 3 and 4 and a point missing. I can now redo this for the configuration with 3 and 4 coincide. Then I redo it where 1 and 2 coincide and 3 and 4 coincide. Then the torus action downstairs is not free, but the action upstairs is free. You end up with a line with two points missing, which you have to divide by $\mathbb{C}^\times$, which gives you a point. When you stick these together, you get the two lines with 1 and 2 on one side and 3 and 4 on the other.

Somehow, the base is a stable toric variety, and I throw away its boundary to get $Y \to G(r,n)$. The GIT quotient is $X = \pi^{-1}(Y)/\!\!/T$.

If you study GIT, you know there is a choice of line bundle and linearization of it. In this case, we need an ample line bundle on $U \subset \mathbb{P}^{r-1} \times G(r,n) \hookrightarrow \mathbb{P}^{r-1} \times$Plüker and a linearization. It turns out that this information is equivalent to the weight β. If I have a weight, then the line bundle is $p_1^*\mathcal{O}(\sum b_i - r) \otimes p_2^*\mathcal{O}(1)$. If $\sum b_i - r \to 0$, then the first factor will disappear. This shows that $\overline{M}_\beta$ will be the GIT quotient $G(r,n)/\!\!/_\beta T$ for generic β. It is well-known that this is also the GIT quotient $\mathbb{P}^{r-1}/\!\!/_\beta PGL(r)$. ◇

When the weights are $\beta = (1,\dots,1,\varepsilon,\dots,\varepsilon)$, with $K + B > 0$ by $K + B \approx 0$, then this is the toric case.

Example 3.7. $(\mathbb{P}^2, B_1, \dots, B_n)$ a configuration of lines, so I have $(1,1,1,\varepsilon,\dots,\varepsilon)$ ($n-3$ ε's). Then all X's are stable toric varieties. If $n = 5$ you get pictures of triangles where the three sides have coefficient 1 and there are two more divisors. [[★★★ picture]] These are described by puzzles like this, where the pieces are either triangles or rhombuses. Here are some examples: [[★★★ pictures]] Your homework is to count these puzzles. I think you can get the staircase with 6 ε's, showing that that moduli space is not irreducible. ◇

4 Martin Olsson

Today I want to discuss Alexeev's moduli stack of broken toric varieties from the point of view of log geometry.

X will be a free abelian group of finite rank. $Q \subseteq X_{\mathbb{R}}$ will be an integral polytope. $T = \mathrm{Hom}_{\mathsf{Gp}}(X, \mathbb{G}_m)$ will be the torus.

Example 4.1. $X = \mathbb{Z}$, $Q = [-1, 1] \subseteq \mathbb{R}$. The quiz is: what is the moduli space of polarized toric varieties for this polytope: [[★★★ picture interval broken at 0]] ⋄

We have the associated toric variet. Take M to be the integral points of the $Cone(1, Q) \subseteq \mathbb{R} \times X_{\mathbb{R}}$, which form a graded monoid, and take $\mathrm{Proj}\, \mathbb{Z}[M]$. I can write this as $\big(\mathrm{Spec}\, \mathbb{Z}[M] \smallsetminus \{0\}\big)/\mathbb{G}_m$. Consider the log scheme $\big(\mathrm{Spec}(M \to \mathbb{Z}[M]) \smallsetminus \{0\}\big)/\mathbb{G}_m$, which is the same scheme, but with a log structure.

Now we want to degenerate this guy, so we should be thinking of functions on integral points. Say $Z \subseteq Q$ are the integral points. Let $\psi\colon Z \to \mathbb{R}$ be a function. Then we are supposed to consider the set $G_\psi = \{(h, x) \in \mathbb{R} \times X_{\mathbb{R}} | x \in Z, h \geq \psi(x)\}$. The lower boundary of G_ψ is a piece-wise linear function on Q, which gives me a *paving* of Q (which is what you think it is; you break your polytope into sub-polytopes with some expected properties). Associated to this we're supposed to get a degeneration. I want to actually degenerate it as a log scheme.

Example 4.2. On the polytope $[-1, 1]$ I could consider the function $(7, 9, 2)$, which gives me the interval, or $(3, 2, 4)$, which gives the broken interval. ⋄

Definition 4.3. A *paving* S of Q is a collection of integral sub-polytopes of Q such that (1) if $\omega, \eta \in S$, $\omega \cap \eta \in S$, (2) any face of $\omega \in S$ is in S, and (3) $Q = \bigcup_{\omega \in S} \omega$ and the ω have disjoint interior. ⋄

Define an associated monoid H_S in as follows. For all ω, define N_ω to be the integral points of $Cone(1, \omega) \subseteq \mathbb{R} \times X_{\mathbb{R}}$. Let $N_S^{gp} := \mathrm{colim}_{\omega \in S} N_\omega^{gp}$.

Example 4.4. Consider the broken interval (with points x, z, y), then I get the diagram of groups

$$\begin{tikzcd} \mathbb{Z}x \arrow[r, hook] & \mathbb{Z}x \oplus \mathbb{Z}z \\ \mathbb{Z}z \arrow[ur, hook] \arrow[dr, hook] & \\ \mathbb{Z}y \arrow[r, hook] & \mathbb{Z}z \oplus \mathbb{Z}y \end{tikzcd}$$

Thus, $N_S^{gp} = \mathbb{Z}x \oplus \mathbb{Z}y \oplus \mathbb{Z}z$. ⋄

There is a set map $\rho\colon M \to N_S^{gp}$. Because of how we did the limit, ρ is well-defined. This is not a monoid map. Define H_S to be the submonoid of N_S^{gp} generated by things of the form $p * q = \rho(p) + \rho(q) - \rho(p + q)$. In our example, $H_S \cong \mathbb{N}$ is generated by $x + y - 2z$.

Facts: (1) H_S is finitely generated; (2) $H_S^\times = \{0\}$ if and only if the paving comes from a height function as Valery explained.

Define $M \rtimes H_S$ to have elements pairs (m, h), where $m \in M$ and $h \in H_S$, where $(m, h) + (m', h') := (m + m', h + h' + m * m')$. In our example, $M = \langle x, y, z\rangle/(x + y = 2z)$. Then $M \rtimes H_S = \langle x, y, z, t\rangle/(x + y = t + 2z)$. There is a map of monoid algebras $\mathbb{Z}[H_S] \to \mathbb{Z}[M \rtimes H_S]$.

In our example, this map is $\mathbb{Z}[t] \to \mathbb{Z}[t][x, y, z]/(xy = tz^2)$. If I take Proj, I get a family over the affine line: $\mathrm{Proj}\, \mathbb{Z}[t][x, y, z]/(xy = tz^2) \to \mathbb{A}^1_t$. This is a degeneration of $\mathbb{P}^1$ corresponding to $\mathcal{O}(2)$ (the interval had length 2) with the paving we had. This has a natural log structure because it arose as a monoid algebra.

In general, $\mathrm{Proj}(M \rtimes H_S \to \mathbb{Z}[M \rtimes H_S]) \to \mathrm{Spec}(H_S \to \mathbb{Z}[H_S])$. This is a degeneration of the toric variety $\mathrm{Proj}(M \to \mathbb{Z}[M])$ as a log scheme. Note that the log structure on the base could be quite complicated.

Definition 4.5 ("sort of a cop out")**.** A *standard object* over $k = \bar{k}$ is the data $\big(M_k, f\colon (X, M_X) \to (\mathrm{Spec}\, k, M_k)$, T-action, line bundle L with T-action$\big)$, where M_k is a log structure on k, f is a log smooth proper map, and (X, M_X) isomorphic to the closed fiber of a family coming from a convex paving S (of Q) as above. ⋄

$\mathcal{K}_Q$ is a stack over Z which to any scheme B associates the groupoid data $\big(M_B, f\colon (X, M_X) \to (B, M_B), L, \theta, \rho\big)$, where M_B is a log structure on B, f is log smooth with $X \to B$ proper, L is a relatively ample line bundle on X, ρ is an action of T on (X, M_X, L) over (B, M_B), and $\theta \in f_* L$ such that

- for every geometric point $\bar{s} \to B$, the zero locus of $\theta_{\bar{s}}$ in $X_{\bar{s}}$ does not contain any T-orbit, and
- $\big(M_{\bar{s}}, (X_{\bar{s}}, M_{X_{\bar{s}}}) \to (\bar{s}, M_{\bar{s}}), L_{\bar{s}}\big)$ is a standard object.

Theorem 4.6. *$\mathcal{K}_Q$ is an algebraic stack with finite diagonal and toric singularities (i.e. is log smooth), and is equal to the main component in Alexeev's moduli space.*

We have a natural log structure $(\mathcal{K}_Q, M_{\mathcal{K}_Q})$ and toric singularities means log smooth over $\mathbb{Z}$ with trivial log structure (this was Kato's theorem).

Exercise. *$X = \mathbb{Z}$ and $Q = [-1, 1]$. I think $\mathcal{K}_Q \cong \mathbb{P}^{2,1}$. The coordinate is t; there is a μ_2 at 0 and the log structure $M_{\mathcal{K}_Q}$ is defined by the divisor at ∞.*

Let's try to verify that this guy is log smooth. What does it mean to say that $(\mathcal{K}_Q, M_{\mathcal{K}_Q})$ is log smooth (I can verify this even if I don't yet know it is algebraic).

$$\begin{tikzcd}
(T_0, M_{T_0}) \arrow[r, "a"] \arrow[d, hook, "J"'] & (\mathcal{K}_Q, M_{\mathcal{K}_Q}) \\
(T, M_T) \arrow[ur, dashed, "?"'] &
\end{tikzcd}$$

Lemma 4.7. *To check that something is smooth, it is enough to consider the case where a is strict.*

So I need to fine a lifting

$$\begin{tikzcd}
(X_0, M_{X_0}, L_0, \theta_0, \rho_0) \arrow[r, dashed] \arrow[d, "\mathcal{K}_Q \ni"'] & (X, M_X, L, \theta, \rho) \arrow[d, dashed] \\
(T_0, M_{T_0}) \arrow[r, hook] & (T, M_T)
\end{tikzcd}$$

Exercise. $\Omega^1_{(X_0, M_{X_0})/(T_0, M_{T_0})} \cong$ *Lie(torus)* $\otimes \mathcal{O}_{X_0}$*, so to prove log smoothness, it is enough to show that $H^2(X_0, \mathcal{O}_{X_0}) = 0$ because the obstruction to lifting the log scheme is .... In fact, by standard reduction, it is enough to consider the case where T_0 is a field k.*

how do you compute it? You go back to your picture of the paving. You get an exact sequence (where $m = \dim Q$)

$$\mathcal{O}_{X_0} \to \prod_{\dim \omega = m} \mathcal{O}_{X_0, \omega} \to \prod_{\dim \eta = m-1} \mathcal{O}_{X_0, \eta} \to \cdots$$

This implies that the cohomology of $\mathcal{O}_{X_0}$ is computed by

$$\prod_{\dim \omega = m} k \to \prod_{\eta} k \to \cdots$$

which just computes $H^*(|Q|, k) = k$. This proves that you can always lift the scheme. The line bundle and the section are not so bad. Lifting the torus action is a little more complicated, so I won't talk about it.

Example 4.8. Recall the embedding $E \hookrightarrow P$ obtained from blowing up the 3-torsion, and X_0 the gluing of two of them along E. We have $(X_0, M_{X_0}) \to (\operatorname{Spec} k, M_k)$, where M_k is given by $k^\times \oplus \mathbb{N} \to k$. Consider the function $F\colon$ (artinian local k-algebras) $\to \mathsf{Set}$ given by $A \mapsto \{$log smooth deformations of (X_0, M_{X_0}) to $\operatorname{Spec} A$ with log structure associated to $\mathbb{N} \to A$ given by $1 \mapsto$ image of $t\}$.

We know that F is unobstructed (because $H^2(X_0, T_{X_0}(\log)) = 0$). We also know that the tangent space is 20-dimensional because $h^1(X_0, T_{X_0}(\log)) = 20$. That means that F is (pro)represented by $k[\![t]\!][\![s_1, \ldots, s_{20}]\!]$, which is 21-dimensional. We were expecting 20-dimensional, so what is the extra dimension?

The extra dimension comes from $\operatorname{Aut}(M_k) = k^\times$. Part of the data of $(X_0, M_{X_0}) \to (k, M_k)$ is $f^\flat\colon M_k \to M_{X_0}$. ◇

Q: could you say something about how you compute that 20? MO: first you show that the dualizing sheaf is trivial. Then by Serre duality, it is easy to fill in all the other parts of the Hogde diamond. Then there is some argument that the Euler characteristic should be 24.

If I fix the log structure on the base, I get the wrong tangent space. You really have to allow different log structures on the base, and allow isomorphisms of those as part of the structure.

5 Tom Bridgeland

It's the last lecture, so I'm allowed to talk about things I don't entirely understand.

Recall the picture from before (it was kind of a baby case with finite fields). You have an abelian category A, to which you associate a Hall algebra $H(\mathsf{A})$. You associate a stability condition to A so that $1_{\mathsf{A}} = \prod_{\phi}^{\rightarrow} 1_{ss}^{\phi}$ (this is basically the Harder-Narasimhan property). In the case of global dimension 1, we had an integration map $I\colon H(\mathsf{A}) \to \mathbb{C}_q[K(\mathsf{A})]$, where the formal identity $1_{\mathsf{A}} = \prod_{\phi}^{\rightarrow} 1_{ss}^{\phi}$ becomes something interesting. A reference is a paper of Kontsevich and Soibelman, which hopefully is coming soon. They say something highly non-trivial, and it works in incredibly general context (they deal with the triangulated case, but I won't). The referecne for this stacky Hall algebra is Joyce's paper "Configurations in abelian categories I, II,…".

Stacky Hall algebras

Let A be an abelian category, equal to $R\text{-}\mathsf{mod}_{fg}$, where R is a finitely generated algebra over $\mathbb{C}$ (for concreteness). You could take $\mathsf{Coh}(X)$, for X a projective variety over $\mathbb{C}$. There is an Artin stack of objects $\mathcal{M} = \coprod_{d\geq 0} \mathcal{M}_d = \coprod_{d\geq 0}[V_d/GL(d)]$, where V_d is some space of matrices (it's an affine variety), and I get rid of the framing by modding out by $GL(d)$. $\mathcal{M}(S)$ is the groupoid of vector bundles $\mathcal{E}$ over S with $R \to \mathcal{E}nd_{\mathcal{O}_S}(\mathcal{E})$.

For $n \geq 1$, $\mathcal{M}^{(n)}$ is the stack of n-flags in A, so $\mathcal{M}^{(n)}(S)$ is the groupoid of flags of vector bundles $0 = \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_n$, with $R \to \mathcal{E}nd_{\mathcal{O}_S}(\mathcal{E}_n)$ preserving the flag, with $\mathcal{F}_i = \mathcal{E}_i/\mathcal{E}_{i-1}$ a vector bundle for all i. Note that $\mathcal{M}^{(1)} = \mathcal{M}$.

We have morphisms $a_i\colon \mathcal{M}^{(n)} \to \mathcal{M}$, given by $(\mathcal{E}_1 \subset \cdots \subset \mathcal{E}_n) \mapsto \mathcal{F}_i = \mathcal{E}_i/\mathcal{E}_{i-1}$, and $b\colon \mathcal{M}^{(n)} \to \mathcal{M}$, given by $(\mathcal{E}_1 \subset \cdots \subset \mathcal{E}_n) \mapsto \mathcal{E}_n$.

Lemma 5.1. *There is a cartesean square*

$$\begin{tikzcd} \mathcal{M}^{(n+1)} \arrow[r, "g"] \arrow[d, "f"'] & \mathcal{M}^{(2)} \arrow[d, "a_1"] \\ \mathcal{M}^{(n)} \arrow[r, "b"'] & \mathcal{M} \end{tikzcd}$$

Proof. Take $f(\mathcal{E}_1 \subset \cdots \subset \mathcal{E}_{n+1}) = (\mathcal{E}_1 \subset \cdots \subset \mathcal{E}_n)$ and $g(\mathcal{E}_1 \subset \cdots \subset \mathcal{E}_{n+1}) = (\mathcal{E}_n \subset \mathcal{E}_{n+1})$. □

Suppose I have a "cohomology theory" for stacks. That is, I have a vector space for each stack $\mathcal{X} \mapsto H(\mathcal{X})$ such that for every (representable, proper) $f\colon \mathcal{X} \to \mathcal{Y}$, I get $f_*\colon H(\mathcal{X}) \to H(\mathcal{Y})$, and for every (finite type) $f\colon \mathcal{X} \to \mathcal{Y}$, I get $f^*\colon H(\mathcal{Y}) \to H(\mathcal{X})$, and these should be functorial (in the correct 2-categorical way). These should have properties:

1. Künneth formula. I want $H(\mathcal{X} \times \mathcal{Y}) \cong H(\mathcal{X}) \otimes H(\mathcal{Y})$. MO: so here H is cohomology with compact support. TB: yeah. I'll give an example of something satisfying these axioms. There may be more interesting examples.

2. Base change. For a cartesean square

$$\begin{tikzcd} \mathcal{X} \arrow[r, "f"] \arrow[d, "g"'] & \mathcal{Y} \arrow[d, "h"] \\ \mathcal{Z} \arrow[r, "j"] & \mathcal{W} \end{tikzcd}$$

   We have $f_* \circ g^* \cong h^* \circ j_*$.

Example 5.2. $H(\mathcal{X})$ is the vector space with basis given by representable maps of finite type $T \to \mathcal{X}$, moduli isomorphism over $\mathcal{X}$. If I have $f\colon \mathcal{X} \to \mathcal{Y}$, and $g\colon T \to X$, I have $f_*(g) = f \circ g$ and if $h\colon T \to \mathcal{Y}$, I have $f_*(h) = (\mathcal{X} \times_{\mathcal{Y}} T \to \mathcal{X})$. It is easy to verify the two axioms. ◇

Example 5.3. You could also quotient by relations $[T \to \mathcal{X}] = [U \to \mathcal{X}] + [(T \setminus U) \to \mathcal{X}]$ when $U \subset T$ is open. Then I think of $H(\mathcal{X})$ as $K_0(St/\mathcal{X})$. This is something more motivic. ◇

So if you have a "cohomology theory" with these properties, then you'll get an associative algebra. Consider

$$\begin{tikzcd} \mathcal{M}^{(2)} \arrow[r, "b"] \arrow[d, "{(a_1,a_2)}"'] & \mathcal{M} \\ \mathcal{M} \times \mathcal{M} & \end{tikzcd}$$

Then define $m = b_* \circ (a_1, a_2)^*\colon H(\mathcal{M}) \otimes H(\mathcal{M}) \to H(\mathcal{M})$. b is representable because this is the Quot scheme. Q: for this, it doesn't look like you need an isomorphism for Künneth, you just need a map. TB: good point, I just need a map.

Lemma 5.4. *m is associative and unital.*

The unit is $i\colon \operatorname{Spec}\mathbb{C} \to \mathcal{M}$ given by $pt \mapsto 0$, then the unit is $i_*(1) \in H(\mathcal{M})$. Q: the Künneth isomorphisms need to have some properties; fro example, they should be associative. [[★★★ some other stuff]] TB: oh, so maybe you want H to always be a ring. Q: I think the Künneth formula is more or less equivalent. Take something with the diagonal map. TB: I'll have to think about that some more.

Proof. (associativity) fill in the cartesean square

$$\begin{tikzcd}
\mathcal{M}^{(3)} \arrow[r, "g"] \arrow[d, "{(f,a_2\circ g)}"] \arrow[dd, bend right, "{(a_1,a_2,a_3)}"'] \arrow[rr, bend left, "b"] & \mathcal{M}^{(2)} \arrow[r, "b"] \arrow[d, "{(a_1,a_2)}"] & \mathcal{M} \\
\mathcal{M}^{(2)} \times \mathcal{M} \arrow[r, "{(b,\mathrm{id})}"] \arrow[d, "{(a_1,a_2,\mathrm{id})}"] & \mathcal{M} \times \mathcal{M} & \\
(\mathcal{M} \times \mathcal{M}) \times \mathcal{M} & &
\end{tikzcd}$$

□

Integration map

These ideas are from Kontsevich-Soibelman (if I haven't messed anything up). Assume R has finite global dimension (so the Ext's don't go on forever). Hall algebra: $[f\colon \mathcal{X} \to \mathcal{M}]$ where $\mathcal{X}$ is finite type and f is representable; take $\mu\colon K_0(\text{varieties})\big[[GL(n)]^{-1} | n \geq 1\big] \to \Lambda = \mathbb{Q}(s)$ to be the ring homomorphism given by the Poincaré polynomial. Define $\mathbb{C}_s[K_{\geq 0}(\mathsf{A})] = \Lambda \otimes_{\mathbb{C}} \mathbb{C}[K_{\geq 0}(\mathsf{A})]$ with multiplication $x^\alpha * x^\beta = s^{\chi(\alpha,\beta)} x^{\alpha+\beta}$.[1]

Define an integration map. Given a constructible function $\omega\colon \mathcal{M} \to \Lambda$, define $[f\colon \mathcal{X} \to \mathcal{M}_\alpha] \mapsto \big[\int_{\mathcal{X}} f^*(\omega) d\mu\big] x^\alpha$. So $f^*(\omega)$ is a constructable function; I break up $\mathcal{X}$ according to the values of the function and and add up the pieces with weights. This gives an integration map $I\colon H(\mathsf{A}) \to \mathbb{C}_s[K_{\geq 0}(\mathsf{A})]$.

[1] Next we'll assume A is CY_3. then χ is skew symmetric, so we get the same answer as somewhere else.

Lemma 5.5. *I is a ring homomorphism if and only if for $A, B \in \mathsf{A}$, $\int_{\operatorname{Ext}^1(B,A)/\operatorname{Hom}(B,A)} W(E) d\mu = s^{\chi(A,B)} W(A) W(B)$.*

$$\int_{\operatorname{Ext}^1(B,A)/\operatorname{Hom}(B,A)} W(E) d\mu = s^{-2\dim \operatorname{Hom}(B,A)} \int_{\operatorname{Ext}^1(B,A)} W(E) d\mu.$$

Claim (conjectural)**.** *Suppose A is CY_3 (I'll in fact assume $R = \mathbb{C}[Q]/I$ for some quiver Q and I given by cyclic derivatives of a polynomial superpotential. This implies that framed modules, V_α, sits inside $\mathbb{C}^N$, framed representations of Q with no relations). Then $\omega(E) = s^{[E,E]} MF_W(E)$, where MF_W is the reduced Poincaré polynomial of the Milnor fiber of W at $E \in \mathbb{C}^N$.*

Here $[E, E]$ is the Euler form on the quiver without relations; it is $\dim \operatorname{Hom}(E, E) - N$. In Kontsevich-Soibelman, they have $\dim \operatorname{Hom}(E, E) - \dim \operatorname{Ext}^1(E, E)$, which is something on the quiver with relations. This is not an Euler form because of something.

The claim boils down to the following. If I have a polynomial map of vector spaces $W\colon \mathbb{C}^{n_1} \oplus \mathbb{C}^{n_2} \oplus \mathbb{C}^{n_3} \to \mathbb{C}$, invariant under the $\mathbb{C}^\times$ action with weight $(1, -1, 0)$, then $\frac{1}{s^{2n_1}} \int_{X \in \mathbb{C}^n} MF_W(x)\, d\mu = MF_{W|_{\mathbb{C}^{n_3}}}(0)$. I don't know how to prove this or if it is true.

4 Kai Behrend

Let me start today by going over the argument from the end of last lecture.

(1) Recall that $X \hookrightarrow M$ is an arbitrary scheme embedded into a smooth scheme. There is no canonicalway that the normal cone $C_{X/M}$ is embedded into the cotangent bundle Ω_M. If X were smooth, the conormal bundle would be embedded in Ω_M, not the normal bundle. But $C_{X/M}$ defines $c_X \in Z_*X$. For every cycle, I pass to it's smooth locus and then I get an embedding into Ω_M. And hence I get $L(c_X) \in \mathcal{L}_X(\Omega_M)$. The mirolocal index theorem (reviewd last time) says that if X is compact, then $\int_X 0^! L(c_X) = \chi(X, \nu_X)$, where $\nu_X = Char^{-1}(L(c_X)) = Eu(c_X)$. In the case where X is smooth, we get $\int_X 0^![N^\vee_{X/M}]$, but because I have a short exact sequence of vector bundles on X

$$0 \to N^\vee_{X/M} \to \Omega_M|_X \to \Omega_X \to 0$$

this is $\int_X e(\Omega_X) = \chi(X, Eu[X]) = \chi(X, (-1)^{\dim X}) = (-1)^{\dim X}\chi(X)$. This is the Gauss-Bonnet theorem.

(2) If $\omega \in \Gamma(M, \Omega_M)$ such that $X = Z(\omega)$, then we get $\omega^\vee : T_M \to I$, where I is the ideal sheaf of X. I can restrict this to X, $T_M|_X \to I/I^2$, and take duals to get

$$C_{X/M} \hookrightarrow \mathcal{N}_{X/M} \hookrightarrow \Omega_M|_X \hookrightarrow \Omega_M$$

so I get the cone $C_{X/M}$ as a closed subscheme of Ω_M. If ω is almost closed, then $\omega\colon C_{X/M} \to \Omega_M$ is Lagrangian. Then it follows that $\omega_*[C_{X/M}] = L(c_X)$.

(3) If X has an obstruction theory $\mathcal{F} \to \tau_{\geq -1}L_X$, then taking duals we get $(\tau_{\geq -1}L_X)^\vee \to \mathcal{F}^\vee$, which induces a subsheaf of cones $cv \hookrightarrow ob = h^1(\mathcal{F}^\vee)$ as we saw last time. If $F \twoheadrightarrow ob$ is an epimorphism from a vector bundle F, then the pullback $C = cv \times_{ob} F$ is a cone scheme $C \hookrightarrow F$ such that $[X]^{vir} = 0^![C]$. If the obstruction theory is symmetric, then $ob = \Omega_X$ canonically and the embedding $X \hookrightarrow M$ defines $\Omega_M|_X \twoheadrightarrow \Omega_X$. So we get $C \hookrightarrow \Omega_M|_X \hookrightarrow \Omega_M$, and $[X]^{vir} = 0^![C]$. Then $[C] = L(c_X)$. This can be checked locally, where you can put yourself in the situation explained in (2) where you have an almost closed 1-form, so locally $(C \hookrightarrow \Omega_M) = (\omega\colon C_{X/M} \hookrightarrow \Omega_M)$. Q: when did you prove that last equality? KB: I didn't go into details about why that is true. If the obstruction theory is given by an almost closed 1-form, then I can take [[★★★ something something]]. Then $[X]^{vir} = 0^! L(c_X)$, so $\#^{vir}X = \int_X 0^! L(c_X) = \chi(X, \nu_X)$.

Remark 4.1 (A few remarks on derived geometry)**.** We have $W = L_{\geq 1}[1]$, where L is a dg Lie algebra. We have the guage group G acting (this is the Lie group with Lie algebra given by L_0, which we threw away in $L_{\geq 1}$). W is a graded linear manifold. $\mathcal{A} = \operatorname{Sym} W^*$, the graded commutative algebra of functions on W, had this derivation of degree 1 $Q\colon \mathcal{A}^i \to \mathcal{A}^{i+1}$. This is a vector field of degree 1 on W. You should think of the moduli space X as the zero locus $Z(Q)$. ◇

Remark 4.2 (Speculation)**.** Using CY3, we get Serre duality, which gives rise to an inner product (analogue of the Killing form) $\kappa\colon L \otimes L \to \mathbb{C}[-3]$. I have $\operatorname{Ext}^1(E,E) \otimes \operatorname{Ext}^2(E,E) \to \mathbb{C}$ because we are on a Calabi-Yau 3-fold. This κ is a product of degree -3, by we pass to W and then we have $\kappa\colon L[1] \otimes L[1] \to \mathbb{C}[-1]$, so κ is really of degree -1 on W: we have $\kappa\colon W \otimes W \to \mathbb{C}[-1]$, or $\kappa\colon W \to W^*[-1]$. Think of κ as a differential 1-form on W. Then $\sigma := d\kappa$ is a symplectic form of degree -1 on W. "Cyclicity"[1] translates into the fact that Q is a Hamiltonian vector field for σ. I get a function $f = \langle Q, \kappa\rangle$ of degree 0 (since $\deg Q = 1$ and $\deg \kappa = -1$), so it is in $\mathcal{A}^0 = \operatorname{Sym} L^{1*} = \mathcal{O}_{L^1}$. Contracting, I get $Q \lrcorner \sigma = df$, where f is the Hamiltonian. To make this precise, you have to allow that κ is non-constant of Q has higher degree terms.

The main point I want to make is that the extra geometry that comes from CY 3-fold is that your moduli space is a symplectic manifold with symplectic form of degree -1. $X = Z(Q)$, so it looks like $X = Z(df)$, but this is only true if σ is really non-degenerate. $X = Z(Q) \subsetneq Z(df)$ because σ is only non-degenerate on cohomology, not on W. Anyway, eventually I think some good will come out of all this. The (symmetric) obstruction theory $(L, d + [\mu, -], \beta)$ on X is just the shadow of the degree -1 symplectic structure on the classical scheme X. ◇

By the way, $M = \operatorname{Hom}(A \otimes V, V)^{\mathrm{Stab}}/G$ is space of the tensor algebra TA-module structure (TA is non-commutative) on V. So you can think of M as a moduli space of sheaves on a non-commutative scheme Y. I think this is an important principle.

Take infinite dimensional model such as $\Omega^{0*}(Y, \operatorname{End}_{\mathcal{O}} E)_0 = L$ (Delbeout something). Here cyclicity holds $\kappa([x,y],z) = \kappa(x,[y,z])$. This more or less

[1] The two main properties of cyclicity are (1) $\kappa(dx, y) \pm \kappa(x, dy) = 0$ and (2) $\kappa([x,y],z) = \kappa(x,[y,z])$.

directly follows from the usual linear algebra that the trace satisfies that equation.

There is the Transfer Theorem for cyclic L_∞-algebras, which says that as complexes of vector spaces with inner product, $L \cong \mathrm{Ext}^1(E,E)_0 \oplus \mathrm{Ext}^2(E,E)_0 = W^0 \oplus W^1$. Of course, this is very non-canonical. So if I have a dg Lie algebra structure on L, it transfers to an L_∞ structure on $W = W^0 \oplus W^1$, which is a derivation Q (which can have terms of arbitrarly high order, not just linear and quadratic) on $\mathcal{A} = \mathrm{Sym}\, W^*$ such that $Q^2 = 0$. If you really think about what this means, you get infinitely many operations $\mu_n \colon \mathrm{Ext}^1_Y(E,E)_0^{\otimes n} \to \mathrm{Ext}^2_Y(E,E)$. If the L_∞-algebra only has components in degree 1 and 2, then this only amounts to operations in degrees 1 and 2. Then you write down $f(X) = \sum_{n=-2}^{\infty} \frac{(-1)^{n(n-1)/2}}{(n+1)!} \kappa(\mu_n(x,\dots,x),x) \colon \mathrm{Ext}^1(E,E) \to \mathbb{C}$, which is a formal function such that $Z(df) \subset \mathrm{Ext}^1(E,E)$ is isomorphic to the completion of X at E. One of the big questions is to understand the radius of convergence (it is bigger than 0?).

$X \subseteq M$ is locally $Z(\omega)$, where ω is almost closed. We'd like to have ω closed, or even exact, so $X = Z(df)$. I can't prove that, but this works in a formal neighborhood at every point.

In the end, I also want to explain how to do some computations. I'll start by explaining some properties of the microlocal function ν_X. Some properties:

1. If X is smooth of dimension n, then $\nu_X = (-1)^n$. This follows directly from the definition.

2. $\nu_{X \times Y} = \nu \boxdot \nu_Y$.

3. If $f \colon X \to Y$ smooth of relative dimension n, then $\nu_X = (-1)^n f^* \nu_Y$.

4. In particular, if $f \colon X \to Y$ is étale, then $\nu_X = f^* \nu_Y$.

5. If $X = Z(df)$ for $f \colon M \to \mathbb{C}$, and $P \in X$, then $\nu_X(P) = (-1)^{\dim M}(1 - \chi(F_P))$ = "Milnor number", where F_P is the Milnor fibre of f at P.More generally, ν_X is the fiberwise Euler characteristic (Φ_f), where $\Phi_f \in \mathrm{Perv}(X)$ is the perverse sheaf of vanishing cycles of f. A fact from microlocal geometry is that "the characteristic variety of Φ_f" is equal to $C_{X/M} \overset{df}{\hookrightarrow} \Omega_M$ (due to Lê and Mebkhout)

$$\begin{tikzcd}
\mathrm{Perv}(X) \arrow[r, leftrightarrow, "\text{Riemann-Hilbert}", "\text{correspondence}"'] \arrow[d, "\text{fiberwise } \chi"] & \text{reg. holonomic } \mathcal{D}\text{-mods} \arrow[d, "\text{Char}"] \\
Con(X) \arrow[r] & \mathcal{L}_X(\Omega_X)
\end{tikzcd}$$

going down is decategorification and going up is categorification.

In the case $X = Z(df)$, $\chi(X, \nu_X) = \sum_i (-1)^i \dim_{\mathbb{C}} \mathbb{H}^i(X, \Phi_f)$. "categorified Donaldson-Thomas invariants".

6. $\nu_X(P)$ depends only on analytic neighborhood of X at P.

7. Conjecture: $\nu_X(P)$ depends only on formal neighborhood of X at P.

4 Alessio Corti

Quantum cohomology

Today I want to discuss quantum cohomology and wall crossings. Let $\mathcal{X}$ be a toric stack with stacky fan (Σ, N, ρ). We have the ring $SR^\bullet_{\mathbb{T}}(\mathcal{X}, \mathbb{Q}) = \mathbb{Q}[\Sigma]$, where the elements are of the form u^e where $e \in N$ so that $\bar{e} \in |\Sigma|$, and $u^{e_1}u^{e_2} = u^{e_1+e_2}$ if e_1 and e_2 are in the same cone and $u^{e_1}u^{e_2} = 0$ otherwise. Let $\mathbb{R} = S^\bullet M$, an algebra, and $M \ni \chi \mapsto div(\chi) = \sum_{i=1}^m \langle \chi, \rho_i \rangle u^{\rho_i}$, so we have

$$0 \to \mathbb{L} \to \mathbb{Z}^m \xrightarrow{\rho} N$$

$\Lambda E \subset \mathbb{L} \otimes \mathbb{L}$.

For $QSR^\bullet(\mathcal{X}, \mathbb{Q})$, we modify the relations. If $e_1 \in \sigma_1$, $\bar{e}_1 = \sum_{i\in\sigma} a_i \bar{\rho}_i$ and $e_2 \in \sigma_2$, $\bar{e}_2 = \sum_{i\in\sigma} b_i\bar{\rho}_i$ with $\sigma_1 \neq \sigma_2$, let $e = e_1 + e_2 \in \sigma$, with $\bar{e} = \sum_{i\in\sigma} c_i\bar{\rho}_i$. Then we impose the relation $\ell(e_1, e_2) = \sum_{i=1}^m (a_i + b_i - c_i)e_i \in \mathbb{L}_{\mathbb{Q}}$. If $i \notin \sigma$, then the coefficient $(a_i + b_i - c_i) \geq 0, \in NE_\sigma \subseteq NE\mathcal{X}$.

Theorem 4.1. *If $\mathcal{X}$ is weak Fano and $I^{\{}\rho_i\} = J$ (i.e. $I(t, z) = \mathbb{1} + t + O(z^{-1})$), then $QH^\bullet_{\mathbb{T},orb}(\mathcal{X}) = \mathbb{Q}[\Lambda E][N]/(u^{e_1}u^{e_2} = Q^{\ell(e_1,e_2)}u^e)$.*

Remark 4.2. (1) Baryrev was the first to say what the quantum cohomology of a toric Fano manifold was. BCS told us what the orbifold cohomology of a stack was. The natural pushout of these two statements is the theorem above, so it was not difficult to guess the right answer.

(2) We should be able to do this for any $S \subseteq \overline{B}^{\leq 1}$. ◇

Example 4.3 ($\mathbb{P}^{1,2}$)**.** The fan sequence is

$$0 \to \mathbb{L} = \mathbb{Z} \xrightarrow{\binom{2}{1}} \mathbb{Z}^2 \xrightarrow[\rho]{(-1\ 2)} \mathbb{Z} = N \to 0$$

There are two cones σ_1 and σ_2 (and zero), with generators $u_1 = \rho_1$, $u_2 = \rho_2$, and w, with $w^2 = u_2$. Then $\rho_1 + \frac{1}{2}\rho_2 = 0$, so $\ell = 0$ (taking $e_1 = \rho_1$ and $e_2 = 1$). $uw = Q^{1/2}\mathbb{1}$.

$QH^\bullet_{\mathbb{T},orb}(\mathbb{P}^{1,2}) = \mathbb{Q}[q, u, w]/(uw = Q^{1/2}\mathbb{1})$, which contains (?) the $\mathbb{R}$-algebra $\mathbb{R} = \mathbb{Q}[\chi]$, $\chi = -u_1 + 2w^2$. There is the non-equivariant limit, where you take $\chi = 0$, so you obtain $QH^\bullet_{orb}(\mathbb{P}^{1,2}) = \mathbb{Q}[q, u, w]/(uw - Q^{1/2}\mathbb{1}, -u_1 + 2w^2)$. There is the classical limit, where $Q = 0$, in which you get the Stanley-Riesner ring $H^\bullet_{\mathbb{T},orb} = \mathbb{Q}[u, w]/uw$. Then there is the case where you do both, to get $H^\bullet_{orb}(\mathcal{X})$ where you take $\chi = Q = 0$ ◇

Proof. All of this comes from the GKZ differential system. If $\ell \in \mathbb{Z}(?)$, □

Wall crossing

[C,I,T]

Example 4.4. $\mathbb{P}(1, 1, 2)$ over $\mathbb{F}_2$. then I have [[★★★ picture]]. The fan sequence is

$$0 \to \mathbb{L} + \mathbb{Z}^2 \to \mathbb{Z}^4 \xrightarrow{\left(\begin{smallmatrix}1 & -1 & 0 & 0\\ 0 & 2 & -1 & 1\end{smallmatrix}\right)} \mathbb{Z}^2 = N \to 0$$

and the Gale dual is

$$0 \leftarrow Pic(\mathcal{X}) = \mathbb{L}^\vee = \mathbb{Z}^2 \xleftarrow{D=\left(\begin{smallmatrix}1 & 1 & 0 & -2\\ 0 & 0 & 1 & 1\end{smallmatrix}\right)} \mathbb{Z}^4 \leftarrow \mathbb{Z}^2 = M \leftarrow 0.$$

$(\mathbb{C}^\times)^2 \to (\mathbb{C}^\times)^4$ acts on $\mathbb{C}^4$ with weights $(1, 1, 0, -2)$ and $(0, 0, 1, 1)$. In $\mathbb{L}^\vee$, I have the picture [[★★★ picture: $D_1 = D_2 = P_1$, $D_3 = P_2$, and $D_4 = -2P_1 + P_2$, K_1 first quadrant, K_2 the part of the second quadrant above D_4]]

I'll think of an element $\psi \in \mathbb{L}^\vee = \mathrm{Hom}_{\mathsf{Gp}}((\mathbb{C}^\times)^2, \mathbb{C}^\times)$ as a $(\mathbb{C}^\times)^2$-linearized line bundle on $\mathbb{C}^4$. The stable points of that linearization will be $U^s = \{s \in \mathbb{C}^4 | \exists P(\vec{x}) \in \mathbb{C}[x_1, \ldots, x_4], P(g\vec{x}) = \psi(g)P(\vec{x})$ such that $P(\vec{a}) \neq 0\}$. You can check that if $\psi \in K_1$, then $U^s = \mathbb{C}^2 \smallsetminus \{0\} \times \mathbb{C}^2 \smallsetminus \{0\}$ and $U^s/(\mathbb{C}^\times)^2 = \mathbb{F}_2$. I hope you're familiar with this as the standard way to construct the surface $\mathbb{F}_2$. On the other hand, if $\psi \in K_2$, then $U^s = \mathbb{C}^3 \smallsetminus \{0\} \times \mathbb{C}^\times$ and $U^s/(\mathbb{C}^\times)^2 = \mathbb{P}(1, 1, 2)$. ◇

I didn't fully explain to you how to get a toric stack from a stacky fan. I explained how to get an open cover. This wall crossing, when you cross from K_1 to K_2, is somehow responsible for the birational transformation between $\mathbb{F}_2$ and $\mathbb{P}(1, 1, 2)$.

If you look at the picture D with K_1 and K_2, it looks like the fan of a toric stack, so let's consider the toric stack with that fan, $\mathcal{M}$, which has two charts. $\mathbb{F}_2$ corresponds to the chart $\mathbb{C}^2$ with coordinates q_1 and q_2, dual to P_1 and P_2. And $\mathbb{P}(1, 1, 2)$ corresponds to a stacky chart $\mathbb{C}^2/\mu_2$ with coordinates $\tilde{q}_1 = q_1^{-1/2}$ and $\tilde{q}_2 = q_1^{1/2}q_2$, dual to $-2P_1 + P_2$ and P_2.

The I-function of $\mathbb{F}_2$ is a function of q_1 and q_2, for q_1 and q_2 small. I won't write it down; you can write it down $I_{\mathbb{F}_2}(q_1, q_2) \in H^\bullet(\mathbb{F}_2, \mathbb{C})[z, z^{-1}]\!] = \mathbb{C}[P_1, P_2][z, z^{-1}]\!]/(P_1^2, P_2^2 - 2P_1P_2)$. Imagine now that you use yesterday's procedure to write down the I-function with basis 1, P_1, P_2, and P_1P_2.

We have that $I_{\mathbb{P}(1,1,2)}(q) \in H^\bullet_{orb}(\mathbb{P}(1,1,2), \mathbb{C})[z, z^{-1}]\!] = \mathbb{C}[P, \mathbb{1}_{1/2}]/(P^3 = P \cdot \mathbb{1}_{1/2} = 0, \dots)$. The right basis for the I-function, for some reason, is $\mathbb{1}$, $P - i\mathbb{1}$, $2P$, $2P^2$. People know why this is the right basis, but I can't say why.

I analytically continue to the other chart to get $I_{\mathbb{F}_2}(\tilde{q}_1, \tilde{q}_2)|_{\tilde{q}_1=0, \tilde{q}_2=\sqrt{q}} = U(z) I_{\mathbb{P}(1,1,2)}(q)$, where

$$U(z) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -i\pi/2 & 0 & 0 & i \\ i\pi/2z & 1/2 & 0 & -i/2 \\ \pi^2/4z^2 & 0 & 1/2 & 0 \end{pmatrix}$$

This has no positive powers of z, which gives you some crepant resolution. If you do this with the next hardest case [[★★★ $\mathbb{P}(1,1,1,3)$ or something]], you get a positive power of z, which screws things up.

4 Valery Alexeev

Nobody turned in the homework. The Quiz for today is for you to stare at these two pictures [[★★★ pictures]] and see that they are basically the same. They are both stairways to heaven, going up and up and up. The second one is seven lines in $\mathbb{P}^2$, so $r = 3$ and $n = 7$, with $\beta = (1, 1, 1, \varepsilon, \varepsilon, \varepsilon, \varepsilon)$. You complete the puzzle by adding the divisors like this [[★★★ picture]]. The implies that $\overline{\mathcal{M}}_\beta(3, 7)$ is not irreducible, which implies that $\overline{\mathcal{M}}_{\mathbb{1}}(3, 7)$ is not irreducible either. Last time I told you you need 9 lines, but you can do it with 7.

Abelian varieties

Abelian varieties are of the form $A = \mathbb{C}^g/\mathbb{Z}^{2g} = (\mathbb{C}^\times)^g/\mathbb{Z}^g$ (the later form is the more general version).

Ideal Theorem: If you fix $\beta = (b_1, \ldots, b_n)$, a dimension, and some other stuff, then there exists a projective $\overline{\mathcal{M}}$, the moduli space of stable pairs $(X, B = \sum b_i B_i)$ satisfying (1) (X, B) slc, and (2) $K_X + B > 0$.

I will attempt to give a more complete picture for surfaces tomorrow, but for now we look at special cases, taking inspiration from this thing we wish were a theorem.

What is a polarization on an abelian variety A? A polarization λ is an ample divisor Θ, modulo algebraic equivalence. If a polarization is principal, then Θ is unique up to translation. Of course, $K_A = 0$, so $K_A + \varepsilon\Theta$ will be ample. If we pick ε very small, then singularities of the pair $(A, \varepsilon\Theta)$ will be essentially the same as those of A.

When the weights are 1 or ε, and $K + B \approx 0$ (but positive), then we are in the toric situation (i.e. X has to be something like a stable toric variety). The ideal theorem says that there has to be a compact moduli space here, of toroidal nature. If the polarization is principal, then the divisor is essentially unique, so it is the same as the moduli space of polarized varieties: $(A, \varepsilon\Theta) \leftrightarrow (A, \lambda)$. You have to be careful about the divisors matching up; in the ideal theorem, B has to be an actual divisor, not a divisor up to linear equivalence.

So one has to switch somehow to another variety $(A, \lambda) \leftrightarrow (X, \Theta)$. In doing so, we give up the notion of $0 \in A$. So X is a torsor under A, but Θ is an actual divisor. One instance of this is very familiar. $(Pic^0 C, \lambda) \leftrightarrow (Pic^{g-1}, \Theta_{g-1})$. If you work with one thing, you hardly see the difference, but in families, these things behave differently. Theorem: there is an equivalence of categories between principally polarized abelian varieties and torsors with divisor. The bad news is that this only works for principal polarizations. For the non-principal case, Martin Olsson suggested a solution with log structures. It is not in the spirit of the ideal theorem. I believe it can be done without log structures as well.

Consider [[★★★ broken interval, labelled $(0, 1, 1)$]]. I have a family X_t, where for $t \neq 0$, $(X_t, \varepsilon B_t) = (\mathbb{P}^1, \varepsilon(2\text{pts})$. In the limit, where $t = 0$, I have (X_0, B_0) a stable toric pair, a couple of $\mathbb{P}^1$'s joined at a point, where B_0 is one point on each $\mathbb{P}^1$.

If I have a toric variety X and an ample divisor B, let $L = \mathcal{O}_X(B)$. Then $H^0(X, L) = \bigoplus \mathbb{C}e_i$, where the e_i are the lattice points in the polytope. We have $\theta \in H^0(X, L)$, with $(\theta) = B$, $\theta = \sum c_i e_i$. In a family, $\mathbb{C}[\![t]\!][1/t]$ or meromorphic functions on $\{0 < |z| < \varepsilon\}$. Then $c_i(t) = c_i' t^{h_i}$, where c_i' are invertible and h_i are the heights of the lattice points. The projection of the lower convex hull of the height function gives the paving of the polytope that gives you the limit.

In tropical geometry, you look at some tropical polynomials like "$h_0 x^0 + h_1 x^1 + h_2 x^2$"$= \max(0 \cdot x + h_0, 1 \cdot x + h_1, 2 \cdot x + h_2)$. This gives a piecewise linear thing. Looking at the points where it breaks, you get the associated tropical variety.[[★★★ picture]] This is related to the toric picture by the Laplace transform. The tropical picture is like N space and the toric picture is like M space. At every point on the piecewise linear function gives you a slope in the dual space. You take some difference to get the values. The transform of the picture is the function with heights (h_1, h_2, h_3).

Let's do this with a slightly more complicated 2-dimensional picture [[★★★ picture triangle with two ears]], then the tropical picture is [[★★★ same, with dualish lines on it]].

Now I'm going to do something like this for families of abelian varieties. I'll start with the simplest picture. Start with $\lambda = \mathbb{Z}^g$ (the pictures are for $g = 1$). In the dual space $\Lambda^* \otimes \mathbb{R} = \mathbb{R}^g$, you'll get something tropical. For the height function, I'll take a non-homogeneous quadratic form, $h = q +$ linear, and I'll require that the quadratic form q is positive definite. If I take the lower convex envelope and project down, I'll get some sub-division. From this, I can construct some graded algebra R, and $\operatorname{Proj} R \to \operatorname{Spec} \mathbb{C}[\![t]\!][1/t]$ is a family (or use the meromorphic functions on $\{0 < |z| < \varepsilon\}$ as a base). When you do the construction, you get that for $t \neq 0$, $X_t = \mathbb{C}^{*g}/\mathbb{Z}^g$, and for $t = 0$, X_0 is the stable toric variety for the periodic decomposition, quotiented by $\mathbb{Z}^g$. Each of the intervals is a $\mathbb{P}^1$, and the periodic decomposition is an infinite chain of

$\mathbb{P}^1$'s. When you quotient by $\mathbb{Z}^g$, you get (X_0, D_0) [[★★★ picture like nodal cubic, with D_0 a point on it]]. What is different from the previous case is that $\mathbb{C}^{*g}/\mathbb{Z}^g$ is half way to algebraic, but it is not algebraic; you can't make sense of this quotient algebraically. You have to do something; there are three ways to solve the problem. One way is to work in the complex analytic topology, so you have a family of complex analytic varieties. The nice thing is that once you quotient by the action, you get an algebraic variety. Then there is the approach of Tate and Mumford. Mumford's approach is purely algebraic. You look at the central fiber first, where you get this infinte chain of $\mathbb{P}^1$'s. Though this is not a variety, it is a scheme, and it is locally of finite type. On such a thing, you can still define an ample line bundle and an ample divisor. Then this action by $\mathbb{Z}$ is properly discontinuous in the Zariski topology (it makes perfect sense in the algebraic category). Then after you quotient, you can descend the line bundle. That's only for the central fiber. You can replace the central point by some artinian ring to thicken it up; you can then get a thickening of the central fiber. After you do it for all such artinian rings, you can use Grothendieck's algebraization theorem to extend to a family. Mumford got his Fields medal for this stuff. There is a third solution, which is to use rigid algebraic geometry.

Now let's understand the tropical side of things. The Laplace transform of this picture is again a piecewise linear quadratic function, which you can project down. The corner locus will be some tropical variety, which will be periodic (you'll still have an action of $\mathbb{Z}^g$). If you vary the heights, the sub-division will change abruptly. One picture lives in Λ and the other lives in Λ^*, but we can identify them using the quadratic function q. There is an associated bilinear form $q\colon \Lambda \times \Lambda \to \mathbb{R}$, which gives us an isomorphism $\Lambda \xrightarrow{\sim} \Lambda^*_{\mathbb{R}}$.

A higher-dimensional picture is [[★★★ picture of two interlaced square lattices]]. [[★★★ In the tropical side?]] the 4-gons will become hexagons and the other 4-gons will become triangles, giving [[★★★ picture with hexagons dual to triangles]]. These decompositions have names. The square one (white) is called the Delanay decomposision (1920s), and the other one is called the [[★★★]] decomposition (1908). In 2007, something called the tropical theta divisor ("Tropical Jacobians"). It tells you that a tropical variety is something which describes a 1-parameter degeneration. Let's see what the result of the degeneration is in this case (the triangle tiling). Each triangle is a $\mathbb{P}^2$, and modulo the period, there are only two of them. When you divide, you'll get the two $\mathbb{P}^2$'s glued to eachother along three $\mathbb{P}^1$'s, and this is a degeneration of abelian surfaces. There is a divisor on it; algebraically, you have a line in each plane, and they intersect at three points (one on each of the three shared lines). Some people draw the divisor like this \$ and call it a dollar sign.

The picture with the squares. Each square is a $\mathbb{P}^1 \times \mathbb{P}^1$, and modulo the period, there is only one copy. So when you quotient, you get a $\mathbb{P}^1 \times \mathbb{P}^1$ glued to itself along two lines (you can introduce a twist (shift) in the gluing). The degenerations are described by $H^1(\underline{P}, \mathrm{Aut})$. There is a $\mathbb{C}^\times$ of abelian varieties here.

Start with an abelian variety $\mathbb{C}^\times/\mathbb{Z}$. On this, there is a divisor Θ. Then we go to a $\mathbb{Z}$-cover, which is $\mathbb{C}^\times$, on which we have a periodic divisor, given by function $\theta = \sum_{i\in\mathbb{Z}} c_i z^i$, where the c_i are quadratic non-homogeneous (in i). Now we repeat it in a family. Then $\theta = \sum_{i\in\mathbb{Z}} c_i(t) z^i$, where $c_i = c_i' t^{h_i}$, where $h_i\colon \mathbb{Z} \to \mathbb{Z}$ is quadratic non-homogeneous: $h_i = q +$ linear, with $q \geq 0$.

Theorem 4.1. *There exists a space $\overline{AP}_g$, the moduli space of stabil semiabelic ($\leftrightarrow$ toric) pairs, with an open (but possibly not dense) subspace $AP_g = A_g$, the moduli space of principally polarized abelian varieties. The normalization of the main irreducible component of this space is $\overline{A}_g^{vor}$, a toroidal compactification of A_g for the second Voronai fan.*

Back to the first picture, where you have a polytope (broken interval) and a height function. If you look at all possible height function, it is a vector space, and it is broken into cones depending on the decomposition of the polytope that they give you. This gives you a fan, called the secondary fan, which gives the secondary toric variety. Now consider the height functions $q\colon \mathbb{Z}^g \to \mathbb{R}$ which are quadratic. Projecting the lower convex hull, you get a periodic decomposition. Breaking up the vector space of such q by the decomposition they give, you get the second Voronai fan.

5 Martin Olsson

This is the last lecture about log geometry. I want to switch gears a bit and talk about the connection between log geometry and algebraic stacks, and do a bunch of examples.

Warm-up. Let X be a scheme and let $r \geq 1$ be a integer. Consider the category C whose objects are pairs (M, β), where M is a fine log structure on X and $\beta\colon \mathbb{N}^r \to \overline{M} = M/\mathcal{O}^\times$ such that β locally on X lifts to a chart for M.[1] The morphisms $(M, \beta) \to (M', \beta')$ are isomorphisms of log structures $\sigma\colon M \to M'$ such that $\beta' = \overline{\sigma} \circ \beta$.

Claim. *C is equivalent to the category D, defined as follows. The objects are collections $(\gamma_1\colon L_1 \to \mathcal{O}_X, \dots, \gamma_r\colon L_r \to \mathcal{O}_X)$, where the L_i are line bundles and the γ_i are $\mathcal{O}_X$-module morphisms (need not be isomorphisms; could all be zero). The morphisms are isomosphisms of such data (i.e. isomorphisms of the L_i over $\mathcal{O}_X$).*

Proof. I'll sketch one direction. Given (M, β), construct the (L_i, γ_i) as follows. We have

$$\begin{tikzcd} & & \mathbb{N}^r \arrow[d, "\beta"'] & e_i \arrow[d, mapsto] \\ M \arrow[r, "\pi"] & M/\mathcal{O}_X^\times \arrow[r, equal] & \overline{M} & \beta(e_i) \end{tikzcd}$$

Take L_i to be the line bundle associated to the $\mathcal{O}_X^\times$-torsor $\pi^{-1}(\beta(e_i))$. There is a map of sheaves $\pi^{-1}(\beta(e_i)) \to \mathcal{O}_X$ induced by the given map $M \to \mathcal{O}_X$.

I'll leave it to you to go the other way. □

On the other hand, the groupoid D is equivalent to the groupoid of maps $X \to [\mathbb{A}^r/\mathbb{G}_m^r] = [\mathbb{A}^1/\mathbb{G}_m]^r$, maps from X to the stack quotient. The stack $[\mathbb{A}^1/\mathbb{G}_m]$ parameterizes line bundles with maps to $\mathcal{O}_X$.

More generally, if P is a fine monoid, let $\mathcal{S}_P = \big[\operatorname{Spec} \mathbb{Z}[P]/D(P^{gp}) = \operatorname{Hom}(P^{gp}, \mathbb{G}_m) = \operatorname{Spec} \mathbb{Z}[P^{gp}]\big]$. This classifies pairs (M, β) where M is a fine log structure and $\beta\colon P \to \overline{M}$ which locally lifts to a chart.

[1] $$\begin{tikzcd} & M \arrow[d] \\ \mathbb{N}^r \arrow[r] \arrow[ur, dotted, "\text{locally}"] & \overline{M} \end{tikzcd}$$

Example 5.1. $\mathbb{A}^1_t \to [\mathbb{A}^1/\mathbb{G}_m]$, in this dictionary, is a line bundle with a map $(L \to \mathcal{O}_{\mathbb{A}^1})$. It is given by $L = (t)$ with the map $(t) \hookrightarrow \mathcal{O}_{\mathbb{A}^1}$. $\Omega^1_{\mathbb{A}^1/[\mathbb{A}^1/\mathbb{G}_m]}$ is computed as the ideal of the diagonal mod its square.

$$\begin{tikzcd} & (x,1) & x \arrow[l, mapsto] \\ \mathbb{A}^1 \arrow[d] & \mathbb{A}^1 \times \mathbb{G}_m \arrow[l, "pr_1"'] \arrow[d, "\rho"'] & \mathbb{A}^1 \arrow[l, "\Delta"'] \\ {[\mathbb{A}^1/\mathbb{G}_m]} & \mathbb{A}^1 \arrow[l] & \end{tikzcd}$$

So $\Omega^1_{\mathbb{A}^1/[\mathbb{A}^1/\mathbb{G}_m]} = k[t]\Delta^*(u-1)$. You should think of $d\colon k[t] \to k[t]\Delta^*(u-1)$ given by $t \mapsto ut - t$. That is, "dt/t" $= \Delta^*(u-1)$. ◇

Example 5.2. Let X be a toric variety over a field k (so it is normal, with an action of the torus T, which is dense in X). Then you can consider the stack quotient $\Omega^1_{X/[X/T]}$, which will be a subsheaf of $j_*\Omega^1_T$. It is exactly the subsheaf $\Omega^1_X(\text{log along } \partial X)$. If you write $X = \operatorname{Spec} k[P]$, then this is our old friend $\Omega^1_X(\log)$.

Q: if you take $[\mathbb{P}^1/\mathbb{G}_m]$, what is it? MO: it is two guys glued together, so two log structures marked by a color. Whenever you have an algebraic stack, you can make a space out of if (for $\mathbb{A}^1/\mathbb{G}_m$, there is a closed point and a generic point), which will be the fan of the toric variety.

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \operatorname{Spec} k \arrow[d] & T \arrow[l] \arrow[d] & \operatorname{Spec} k \arrow[l, hook'] \\ {[X/T]} \arrow[r] & {[\operatorname{Spec} k/T]} & \operatorname{Spec} k \arrow[l] & \end{tikzcd}$$

So $\Omega^1_{X/[X/T]} = g^*\Omega^1_{\operatorname{Spec} k/[\operatorname{Spec} k/T]} = Lie(T)^\vee \otimes_k \mathcal{O}_X$. ◇

Example 5.3. Say X is a scheme, $D \subset X$ is a Cartier divisor, and $r \geq 1$ is an integer. Let's construct the universal r^{th} root of D. Let L be the ideal of D, which has a map $\gamma\colon L \to \mathcal{O}_X$. We want to classify line bundles $\mathcal{K}$ with maps

$\delta\colon \mathcal{K} \to \mathcal{O}$ and $\iota\colon \mathcal{K}^{\otimes r} \xrightarrow{\sim} L$ such that the diagram below commutes

$$\begin{tikzcd} \mathcal{K}^{\otimes r} \arrow[rr, "\sim"] \arrow[dr, "\delta^{\otimes r}"'] & & L \arrow[dl, "\gamma"] \\ & \mathcal{O} & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{X}_{D,r} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & {[\mathbb{A}^1/\mathbb{G}_m]} \arrow[d, "t\mapsto t^r"] \\ X \arrow[r, "{(L,\gamma)}"] & {[\mathbb{A}^1/\mathbb{G}_m]} \end{tikzcd}$$

This $\mathcal{X}_{D,r}$ defines $(\mathcal{K},\delta,\iota)$. If $D=(f)\subset\mathcal{O}_X$, then $\mathcal{X}_{D,r}=\big[(\operatorname{Spec}\mathcal{O}_X[z]/z^r=f)/\boldsymbol{\mu}_r\big]$. So this gives you a global construction, which you understand how it looks like locally. ◇

Example 5.4. Let X be a nodal curve over a field k. In the other talks, we've seen something about putting stacky structure at various points, but how do you actually do that? How would I put a $\boldsymbol{\mu}_3$ at the node for example? Here is one way to do it. First, consider the case of a single node. You can't write a Zariski local neighborhood where the node looks like two axes. You have to do it étale locally, and then you have to descend a stack in the étale topology, which it's not so clear how to do.

Give it the canonical log structure, $(X,M_X)\to(\operatorname{Spec}k, k^\times\oplus\mathbb{N}\to k)$, where the $\mathbb{N}$ is a local parameter at the node. You want $k[t][x,y]/(xy-t^3)$. You have $\left[\frac{k[t,z,w]}{(zw-t)}/\boldsymbol{\mu}_3\right]\to k[t][x,y]/(xy-t^3)$, where $\zeta\in\boldsymbol{\mu}_3$ acts by $z\mapsto\zeta z$ and $w\mapsto\zeta^{-1}w$, $x\mapsto z^3$ and $y\mapsto w^3$. We have

$$\begin{tikzcd} M_X & \\ k^\times\oplus\mathbb{N} \arrow[u] \arrow[r] & k^\times\oplus\mathbb{N} \\ (u,n) \arrow[r, mapsto] & (u,3n) \end{tikzcd} \qquad \begin{tikzcd} M_X \arrow[r] & \mathcal{N} \\ k^\times\oplus\mathbb{N} \arrow[u] \arrow[r] & k^\times\oplus\mathbb{N} \arrow[u] \\ (u,n) \arrow[r, mapsto] & (u,3n) \end{tikzcd}$$

Let $\mathcal{X}\to X$ be the stack over X classifying diagrams of fine log structures on the right, such that for every geometric point $\overline{x}\to X$, the diagram on the left is isomorphic to the middle one if $\overline{x}$ hits the node and the one on the right otherwise

$$\begin{tikzcd} \overline{M}_{X,x} \arrow[r] & \overline{\mathcal{N}}_{\overline{x}} \\ \mathbb{N} \arrow[u] \arrow[r, "\cdot 3"] & \mathbb{N} \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} \mathbb{N} \arrow[r] & \mathbb{N} \\ \mathbb{N} \arrow[u, "\wr"] \arrow[r, "\cdot 3"] & \mathbb{N} \arrow[u, "\wr"'] \end{tikzcd} \qquad \begin{tikzcd} \mathbb{N}^2 \arrow[r, "\cdot 3"] & \mathbb{N}^2 \\ \mathbb{N} \arrow[u, "\Delta"] \arrow[r, "\cdot 3"] & \mathbb{N} \arrow[u, "\Delta"'] \end{tikzcd}$$

◇

Proposition 5.5. *If $X=\operatorname{Spec}(k[x,y]/xy)$, then $\mathcal{X}=[\operatorname{Spec}(k[z,w]/zw)/\boldsymbol{\mu}_3]$.*

Remark 5.6. This construction between "twisted curves" and nodal curves with extra log data. ◇

The general story is this. Consider a finite category (i.e. a directed graph) D, for example $(\bullet\to\bullet\to\bullet)$. For any scheme X, define $\mathcal{L}og^D(X)$ to be the category of functors from D to log structures on X. For example, $\mathcal{L}og^{(\bullet\to\bullet\to\bullet)}(X)$ is the set of diagrams of log structures $M_1\to M_2\to M_3$ on X. Morphisms in $\mathcal{L}og^D(X)$ are isomorphisms of functors.

Theorem 5.7. *$\mathcal{L}og^D$ is an algebraic stack.*

<u>Back to deformation theory.</u> We started with some diagram on the left

$$\begin{tikzcd} (X_0,M_{X_0}) \arrow[r, dashed] \arrow[d, "\text{log smooth}"'] & (X,M_X) \arrow[d, dashed] \\ (T_0,M_{T_0}) \arrow[r, hook, "J"] & (T,M_T) \end{tikzcd} \qquad \begin{tikzcd} & \mathcal{L}og^{(\bullet\to\bullet)} \arrow[d] & (M_1\to M_2) \arrow[d, mapsto] \\ Y \arrow[r, "M_Y"] & \mathcal{L}og^{\bullet} & M_1 \end{tikzcd}$$

For any log scheme (Y,M_Y), define $\mathcal{L}og_{(Y,M_Y)}$ to be the fiber product of the diagram on the right. So $\mathcal{L}og_{(Y,M_Y)}(f\colon X\to Y)$ is the set of pairs $(M,f^\flat)$ M a log structure on X and $f^\flat\colon f^*M_Y\to M$, so we're upgrading a morphism to a morphism of log structures.

This is equivalent to

$$\begin{tikzcd} X_0 \arrow[r, dashed] \arrow[d, "\mathcal{L}og(f_0)"'] & X \arrow[d, dashed, "\text{smooth}"] \\ \mathcal{L}og_{(T_0,M_{T_0})} \arrow[r, hook, "\tilde{J}"] \arrow[d] & \mathcal{L}og_{(T,M_T)} \arrow[d] \\ T_0 \arrow[r, hook, "J"] & T \end{tikzcd}$$

log smoothness is equivalent to the map $X_0\to\mathcal{L}og_{(T_0,M_{T_0})}$ being smooth. Deformation theory should be governed by $H^i(X_0, T_{X_0/\mathcal{L}og_{(T_0,M_{T_0})}}\otimes\tilde{J})$.

$(X_0, M_{X_0}) \to (T_0, M_{T_0})$ is integral if and only if $\mathcal{L}og(f_0)$ has image in flat locus of $\mathcal{L}og_{(T_0, M_{T_0})} \to T_0$. This implies the theorem from before.

Example 5.8. $\operatorname{Proj}\big(M \rtimes H_S \to \mathbb{Z}[M \rtimes H_S]\big) \to \operatorname{Spec}(H_S \to \mathbb{Z}[H_S])$, and we have an action of the torus on $\mathbb{Z}[M \rtimes H_S]$. This is equivalent to

$$\begin{tikzcd} \operatorname{Proj}(\mathbb{Z}[M \rtimes H_S]) \arrow[d] & \subsetneq & {[\operatorname{Spec}\mathbb{Z}[M]/\mathbb{G}_m]} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & B\mathbb{G}_m \arrow[d] \\ \mathcal{L}og_{\operatorname{Spec}(H_S \to \mathbb{Z}[H_S])} \arrow[rr, "\text{étale}"] & & {[\operatorname{Spec}\mathbb{Z}[M \rtimes H_S]/D(M^{gp})]} \arrow[r] & BD(M^{gp}) \end{tikzcd}$$

it can be shown that the bottom map is étale. [[★★★ some stuff]] ◇

5 Valery Alexeev - Moduli of surfaces

Today I'm going to talk about surfaces, and I'll try not to skip technical details. The reason is that for the previous lectures, there are papers with proofs. For surfaces, there is no definite source. There is supposed to be a book, but it has four authors, so it is delayed.

Let $\pi\colon X \to S$ be a flat family of slc surfaces, and let $N \in \mathbb{N}$ such that $Nb_i \in \mathbb{Z}$. So each fiber X_s is slc, in particular is S_2. Let $Z \subseteq X$ be a subset so that for all $s \in S$, $\operatorname{codim}(Z_s, X_s) \geq 2$. On $X \smallsetminus Z$, $\omega_{X/S}$ and $\mathcal{O}_X(N\sum b_iB_i)$ are invertible. So the bad set, where these sheaves are possibly not invertible, is contained in Z. Let $j\colon X \smallsetminus Z \hookrightarrow X$.

Definition 5.1. $L^N_{X\to S} := j_*\big(\omega^{\otimes N}_{X/S} \otimes \mathcal{O}_X(N\sum b_iB_i)|_{X\smallsetminus Z}\big) =$"$N(K+B)$". ◇

Remark 5.2. Note that formation of $L^N_{X\to S}$ does *not* commute with base change. In particular, for the (key) base change $s \to S$, the construction does not commute. In particular, the value of K^2 jumps. ◇

Definition 5.3. Fix $\beta = (b_1, \dots, b_n)$, $V \subseteq \mathbb{P}$ (some projective scheme; for stable pairs, $V = pt$), and coefficients c_1, c_2, and c_3. Define $\mathcal{M}_N(S) = \Big\{$flat projective families $f\colon (X, \sum b_iB_i) \to S\times V$ such that (1) X, B_i are flat over S, (2) $(X_s, \sum b_iB_i)_s \to V$ is a stable map, (3) $L^N_{X\to S}$ is invertible, ample over $S\times V$, and $(L^N_{X\to S})_s = L^N_{X_s\to s}$, and (4) $(K_{X_s}+B_s)^2 = c_1, (K_{X_s}+B_s)H_s = c_2, H_s^2 = c_3$ for $H_s = f_s^*\mathcal{O}_V(1)\Big\}$. ◇

$\mathcal{M}_N$ definitely depends on N. In characteristic p, the moduli space definitly depends on N. In charactaristic zero, I think it may not.

Definition 5.4. $\mathcal{M}^K(S)$ is the same as $\mathcal{M}^N$, but the K stands for "Kollár" and condition (3) is replaced by (3′) for all m such that $m\beta \in \mathbb{Z}^n$, $L^m_{X\to S}$ is flat over S, and $(L^m_{X\to S})_s = L^m_{X_s\to s}$, and some $L^N_{X\to S}$ is invertible. ◇

Remark 5.5. Under assumptions (3,3′), formation of the sheaf $L^N_{X\to S}$ *does* commute with base change $S' \to S$. ◇

Note that $L^N_{X\to S}$ does not depend on the "nice" $Z \subset X$. It should be called "the caturation in codimension 2 relative over S"; it can be defined as $\lim j_{Z,*}(\dots)$. Under conditions (3) or (3'), $L^N_{X\to S}$ is called the *hull* of $\omega^{\otimes N}_{X/S} \otimes \mathcal{O}_X(N\sum b_iB_i)$. The reference is Kollár, Hulls and husks.

Problem 1: embedded components of B.

Example 5.6 (Hacking, Hassett)**.** Consider the surface $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ as the fiber. Take the base $S = \mathbb{A}^1$. In the central fiber, take the divisor $s_0 + 2f$ in $|FF_2$, and blow it up. We get s_4 in $\mathbb{F}_4$. So you have two glued surfaces in the central fiber. Take $2s_0$ on the $\mathbb{F}_0$ and $4f + 4(s_4 + 4f)$ on the $\mathbb{F}_4$, which intersect the curve along which the two surfaces are glued four times. [[★★★ picture]] The key thing is that on the $\mathbb{F}_4$, you have a nodal curve, which we will smooth in the generic fiber (explicitly, it is something with genus 35; you can find this in my paper on stable limits of surfaces). Now, contract the $\mathbb{F}_0$ component to a point. So in the central fiber, we'll have $\mathbb{F}_4$, with the exceptional fiber contracted. It will be a cone on a quartic. The genus of the curve in the central fiber is one higher than the (arithmetic) genus of the curve in the generic fiber. You can see this; when you contract all four intersection points to a point, the genus jumps up by one. $p_a(C_{0,red}) = g + 1$. But in flat families, arithmetic genus is constant. We can conclude that $B_0 \subseteq X_0$ is not reduced. But you can compute that $K_S + \frac{1}{2}B$ is $\mathbb{Q}$-Cartier ($2K_X + B$ is Cartier) and ample over $S = \mathbb{A}^1$, so this is the log canonical model for .... So I have a divisor on the 3-fold, but when I restrict to the central fiber, it is not a divisor, it is only a closed subscheme.

The problem here is that B is not $\mathbb{Q}$-Cartier, but $(B_0)_{red}$ is $\mathbb{Q}$-Cartier. In this situation, you necessarily aquire an embedded component. If B_0 were Cartier, we would just lift that divisor in the family. You should expect this. $K + B$ is Cartier in the log canonical model, but K and B need not be Cartier. ◇

What are we supposed to do now? Work with a subscheme instead of a divisor? What are the definitions of lc and slc in that case? This seems like a very serious problem, but there are several solutions:

1A. This problem does not happen if $b_i = 1$. This is proven in my paper on limits of stable pairs. This is not so good; $\frac{1}{2}$ is a perfectly good coefficient.

1B. Work with subschemes $B_i \subseteq X$ which are closed and flat over S. That is, require $(X, \sum b_i(B_i)^{div})$ to be a stable pair. I don't like this solution. Once you allow these embedded points to be there, then you have a surface, with a divisor, and these embedded points can crawl everywhere, and that is just unnatural. Maybe you could allow them in the bad fiber, but it seems like you have to allow them everywhere.

1C. Replace divisors by finite maps $B_i \to X$, where the B_i are reduced and codimension 1. I call these *branch divisors*. So when you form the divisor, you just take the divisor given by the image. How does this cure the example? It will look like this [[★★★ picture with two nodes]]; this is really a branch divisor, because at one point it is 2-to-1.

1D. (Kollár) "$B = (K+B)-K = L-K$" where L is an ample $\mathbb{Q}$-Cartier divisor. On a smooth surface, you can interpret NB as a morphism $\phi\colon \omega_{X/S}^{\otimes N} \to L_N^{X\to S}$.

1E. Only work with coefficients $(b_i + \varepsilon_i)$ and 1. This means that in the definition, you insist that these divisors are $\mathbb{Q}$-Cartier. This is a cheap way out, because $\frac{1}{2}$ is a perfectly good coefficient.

I like solutions 1C and 1D.

Problem 2: The properness criterion of MMP for a non-normal generic fiber. We started with a normal 3-fold in the picture I described before. What if you start with some family of surfaces where the generic fiber is not normal? Do we have a MMP for such things?

Example 5.7 (Kollár)**.** Start with the surface $\mathbb{F}_n$ and you attach to it an $\mathbb{F}_m$. You attach a divisor, which is the simplest thing you could have: s_n with s_m and $s_n + nf$. [[★★★ picture]] $K + B$ is slc and big, but $\bigoplus_{d\geq 0} H^0(d(K+B)$ is not finitely generated. Kollár has a more sophisticated example where the surface is irreducible. ◇

This example is really not a problem. You take the normalization and run MMP for all the pieces. [[★★★ picture]] Then you want to say that this glues uniquely together. The solution is to require $(K_X + B)|_E$ matches on the "left" and "right". This should be treated in the étale topology (the normalization could be connected, so you take a cover where it breaks into pieces). With this condition, everything glues nicely and the triple points are not a problem. In higher dimensions, there would be more trouble because you'd have things of higher codimension. So the surface in the example does not appear as a limit if you impose this condition.

Construction of moduli

The construction is standard once you have good properties of the stack $\mathcal{M}$. The properties are

1. properness, which is ok by MMP and above
2. boundedness, which is ok (V.A. 1994)
3. local closedness, which is Problem 3.

I could have $(X, \sum b_i B_i, L)$, where L is relatively ample invertible. If you make a base change, to get (X_T, B_T, L_T), would this be in $\mathcal{M}(T)$ (i.e. would it be an admissible family)? Local closedness means that for $S^u = \bigsqcup S_i$ with the S_i closed, then $T \to S^u$.

Fix some N. By boundedness, you can fix it so that L^N is very ample. Then L gives you an embedding into some projective space of fixed dimension. So you are in some Hilbert scheme. You cut out .... The problem begins with the fact that $L_N^{X\to S}$ does not commute with base change (if I don't start with a good family). If you know local closedness, then you know that you can form an admissible family. The only thing that is different from what we want is the embedding. You quotient out by the embedding and you get a quotient stack $\mathcal{M} = U/PGL$. The properness implies that this is algebraic with finite stabilizers.

Problem 3 has been solved. The statement is true, but one has to prove it. HK did it in dimension 2 with $B = \varnothing$. Kollár Husks and hulls gives a comprehensive treatment. According to me, this moduli space exists in complete generality (at least with contastant coefficients).

5 Kai Behrend

Remark 5.1 (Heuristics: why vanishing cycles)**.** Suppose $f\colon X \to \mathbb{C}$ is a 1-parameter family of projective varieties. Suppose that the generic fibers are non-singular, and the singularities are all in the special fiber: $Z(df) = Z(f)$. X_η the generic fiber is smooth. I explained that

$$\#^{vir} X_\eta = \int_X e(\Omega_X) = (-1)^{\dim X_\eta}\chi(X_\eta) = \sum (-1)^{\dim X_\eta - i} \dim H^i(X_\eta).$$

$\#^{vir}$ should be invariant. There is a spectral sequence $H^p(X_0, \Psi_f^q) \Rightarrow H^{p+q}(X_\eta, \mathbb{C})$. This is a Leray spectral sequence of the embedding $X_\eta \subset X$ [[★★★ should be special fiber $X_0 \subset X$?]]. μ_f': fiberwise Euler characteristic of Ψ_f: constructible function on X_0. $\chi(X_0, \mu_f) = \chi(X_\eta)$. This function is supported at singularities. Q: if X projective, it can have a lot of cohomology, but the vanishing cycle around a singular point is a tiny little thing. A: it's all the cycles including the vanishing cycles. KB: this is copied from SGA7 Exp. I. I should probably call this "nearby cycles"; there is always this confusion between vanishing cycles and nearby cycles. $\mu_f' = 1 \pm \mu_f$. I keep saying that the moduli space should be the singular locus in the special fiber, but if I deform .... I don't remember what the relavence was supposed to be. ◇

Now X is back to what it was the whole time (a scheme with a symmetric obstruction theory). One more property of the microlocal function ν_X. Suppose $\mathbb{C}^\times$ acts on X with an isolated fixed point P. This corresponds to a sheaf E on the Calabi-Yau Y. Suppose the $\mathbb{C}^\times$ action preserves the obstruction theory:

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\sim"] & g^*\mathcal{F} \arrow[r] \arrow[d] & \mathcal{F} \arrow[d] \\ & X \arrow[r, "g"] & X \end{tikzcd}$$

and the isomorphism should satisfy a cocycle condition. Then $\mathbb{C}^\times$ acts on the Zariski tangent space $T_X(P) = \mathcal{E}xt^1(E,E)_0$. Suppose all the weights of this action are non-zero (this is what I mean by "P is an isolated fixed point").

Theorem 5.2 (Behrend-Fantechi)**.** *$\nu_X(P) = (-1)^{\dim T_X(P)} = \frac{n_1 \cdots n_d}{(-n_1)\cdots(-n_d)}$ if $n_1, \ldots, n_d$ re the weights of $\mathbb{C}^\times$ on $\mathcal{E}xt^1$ and $-n_i$ are the weights of $\mathbb{C}^\times$ on* $\mathrm{Ext}^\vee = \mathrm{Ext}^2$.

"Proof". Let me do this in the case $X = Z(df)$, where $f\colon M \to \mathbb{C}$ is holomorphic and M is smooth. Suppose $\mathbb{C}^\times$ acts and f is homogeneous of degree 0. Say $M = \mathbb{C}^n$. Then $\mathbb{C}^\times$ acts on the induced obstruction theory in the required way. Then the nodal fiber $F_P = \{x \in M | f(x) = \delta, \|x\| < \varepsilon\}$. Then $S^1 \subset \mathbb{C}^\times$. acts on the Milnor Fiber F_p. Since all the weights are non-zero, you can prove that there are fixed points so $\chi(F_P) = 0$. So $\nu_X(P) = (-1)^{\dim M}(1-0)$. The proof in the general case is similar, but you have to change [[★★★ something]] and find a replacement for the Milnor fiber. □

Mac Mahon function: $M(q) = \prod_{n \geq 1} \frac{1}{(1-q^n)^n} = \sum_{n\geq 0} p(n) q^n$, where $p(n)$ is the number of 3-dimensional partitions of length n. $\mathrm{Hilb}^n Y$, where Y is Calabi-Yau 3-fol (compact or not), then

1. (Y compact) $\sum_{n\geq 0} \#^{vir} \mathrm{Hilb}^n Y \, q^n = M(-q)^{\int_Y e(T_Y \otimes \omega_Y)}$. (Li Levine-Pandhaupale)
2. (Y arbitrary) $\sum_{n \geq 0} \chi(\mathrm{Hilb}^n Y, \nu) q^n = M(-q)^{\chi(Y)}$. (Behrend-Fantechi)

These formulas are identical if Y is compact and Calabi-Yau.

Let $C \subset Y$ be a super rigid rational curve (i.e. $C \cong \mathbb{P}^1$ and $N_{C/Y} = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$) with Y a compact Calabi-Yau 3-fold. It is known that in degrees 1 and 2 these are the only kind of curves on the generic .... Let $X_n(C,Y)$ be the moduli of ideal sheaves $\mathcal{I} \subset \mathcal{O}_Y$ such that $\mathcal{O}_Y/\mathcal{I}$ defines a subscheme of Y whose associated 1-cycle is C and $\chi(\mathcal{O}_Y/\mathcal{I}) = n$. So it is really Y with C and $n-1$ points floating around (which can be on or off of C). This an open and closed subscheme of the moduli with the same Hilbert polynomial (because of the super rigidity). [[Q: It's not of finite type, is it? KB: it is; n is fixed and I'm fixing the Hilbert polynomial, and the associated 1-cycle is one copy of C (no multiplicities).]] Let $N_n(C,Y)$ be the virtual count, contribution of C to the Donaldson-Thomas invariant with this Hilbert polynomial. The generating function for this thing turns out to be

$$\sum_{n\geq 0} N_n(C,Y) q^n = M(-q)^{\chi(Y)} \frac{q}{(1+q)^2}.$$

Now we can stratify the moduli space $X_n(C,Y)$. $Z_{n,0}(C,Y) \subset X_n(C,Y)$ closed. $Z_{n-i,i}(C,Y)$ is where exactly i points (with multiplicity) are off C. $Z_{1,n-1}(C,Y) = \mathrm{Hilb}^{n-1}(Y \smallsetminus C)$ is the open stratum. $X_n(C,Y) = \bigsqcup_{i=0}^{n-1} Z_{n-i,i}(C,Y)$.

$\mathrm{Hilb}^i(Y\smallsetminus C)\times Z_{n-i,0}(C,Y) = Z_{n-i,i}(C,Y)$ is contained as a closed subscheme of U, the open subset of $\mathrm{Hilb}^i(Y\smallsetminus C)\times X_{n-i}(C,Y)$ defined as the open subset where the two subschemes have disjoint support.

$$\begin{tikzcd}
\mathrm{Hilb}^i(Y\smallsetminus C)\times Z_{n-i,0}(C,Y) \arrow[d, "\text{closed}"'] & = & Z_{n-i,i} \arrow[d, "\text{incl. of stratum}"] \\
U \arrow[rr, "\text{étale}"] \arrow[d, "open"'] & & X_n(C,Y) \\
\mathrm{Hilb}^i(Y\smallsetminus C)\times X_{n-i}(C,Y) & &
\end{tikzcd}$$

Notation: if $f\colon X\to Y$ is a morphism of schemes, then $\tilde\chi(X,Y)=\chi(X,f^*\nu_Y)$, and $\tilde\chi(X)=\tilde\chi(X,X)=\chi(X,\nu_X)$.

I want to compute $\tilde\chi\big(X_n(C,Y)\big)=\sum_{i=0}^{n-1}\tilde\chi\big(Z_{n-i,i}(C,Y),X_n(C,Y)\big)$.

$$\begin{aligned}
&\tilde\chi\big(Z_{n-i,i}(C,Y),X_n(C,Y)\big)=\\
&\qquad=\tilde\chi\big(\mathrm{Hilb}^i(Y\smallsetminus C)\times Z_{n-1,0}(C,Y),\mathrm{Hilb}^i(Y\smallsetminus C)\times X_{n-i}(C,Y)\big)\\
&\qquad=\tilde\chi(\mathrm{Hilb}^i(Y\smallsetminus C))\cdot\tilde\chi\big(Z_{n-i,0}(C,Y),X_{n-i}(C,Y)\big)\\
&\qquad=\text{known}
\end{aligned}$$

$\nu_{X_n(C,Y)}=\nu_U=\nu_{\mathrm{Hilb}^i(Y\smallsetminus C)\times X_{n-i}(C,Y)}$. Closed stratum

$$\tilde\chi\big(Z_{n-i,0}(C,Y),X_{n-i}(C,Y)\big)=\tilde\chi\big(Z_{n-i,0}(\mathbb{P}^1,N),X_{n-i}(\mathbb{P}^1,N)\big)$$

$N=\mathcal{O}(-1)\oplus\mathcal{O}(-1)$. [[Q: [[★★★ something]] is very lucky. KB: I could stratify further; I don't think the full power of this method has been exploited.]] Now we are on $N=\mathcal{O}(-1)\oplus\mathcal{O}(-1)$. We have $\mathbb{C}^\times$ action. We get a $\mathbb{C}^\times$ action preserving the CY, with isolated fixed points. We know $\nu_X(P)$ if P is fixed; it is $(-1)^{n-1}$ (MNOP1), and we can count the fixed points, which means you're piling boxes in two corners of the room, not just one corner. In this case, the two corners are connected by an (infinite) row of boxes, so the two piles never meet.

I don't have to worry about the value of the function at any other point because the $\mathbb{C}^\times$ action is something so their stuff cancels out. So I get the formula from earlier.

5 Alessio Corti

Today I want to do two (or perhaps three) things. The main theorem I stated was the mirror theorem with the I-function and J-function. I want to zoom in on one part of the proof.

Let $\mathcal{X}$ be a proper 1-dimensional toric stack. Question: classify all representable toric morphisms $f\colon \mathbb{P}_{r_1,r_2} \to \mathcal{X}$. You have to calculate some Gromov-Witten numbers.

Remark 5.1. If $\mathcal{X}$ is a manifold (not a stack), then $\mathcal{X} = \mathbb{P}^1$. In this case, the only representable morphisms are from $\mathbb{P}^1 = \mathbb{P}_{1,1}$. Toric morphisms are then classified by degree; every such morphism is given by $(x_0, x_1) = (z_0^d, z_1^d)$. ◇

The slogan: all such morphisms are classified by the enhanced degree $\widehat{\deg} \in \operatorname{Hom}(\widehat{Pic}(\mathcal{X}), \mathbb{Z})$. This is the main motivation for introducing $\widehat{Pic}$.

Notation: $\mathcal{X}$ has a fan diagram

$$0 \to \mathbb{Z} \xrightarrow{\binom{w_2}{w_1}} \mathbb{Z}^2 \xrightarrow{\rho} N$$

where N is a rank 1 abelian group (so $\mathbb{Z}$ plus a torsion bit). [[★★★ picture fan for $\mathbb{P}^1$; $\overline{\rho}_1$ negative, $\overline{\rho}_2$ positive, σ_1 negative cone, σ_2 positive gone]]. Let $B = \operatorname{Box}(\mathcal{X})$. Then $B(\sigma_1) = \{v \in N | \overline{v} = a\overline{\rho}_1, 0 \leq a < 1\} = N/\langle \rho_1 \rangle$. So $N_{tors} \subset B(\sigma_i) =: B_i$ and $\operatorname{Box} = B(\sigma_1) \cup B(\sigma_2)$.

Recall that $\mathbb{P}_{r_1,r_2}$ is a $\mathbb{P}^1$ with a μ_{r_1} at zero and μ_{r_2} at infinity. $f\colon \mathbb{P}_{r_1,r_2} \to \mathcal{X}$, $f(0$ and $f(\infty)$ give me $B\mu_{r_i} \to \mathcal{X}$, which give me $v_i \in B_i$. $\rho = \{\rho_1, \rho_2, v_1, v_2\}$. $\overline{v}_i = f_i(\overline{\rho}_i)$ where $0 \leq f_1, f_2 < 1$ are rational. Then $\widehat{\deg} f \in \mathbb{L}^S \subset \widehat{\mathbb{L}} = \operatorname{Hom}(\widehat{Pic}, \mathbb{Z})$.

More explicitly, we enhance the fan map to get

$$0 \to \mathbb{L}^S \to \mathbb{Z}^S = \mathbb{Z}^4 \xrightarrow[(\rho_1,\rho_2,v_1,v_2)]{\rho^S} N$$

So $\widehat{\deg} f = (q_1, q_2, 1, 1) \in \ker \rho^S$ (column vector), where q_1 and q_2 are positive integers. A general enhanced degree would have integers k_1 and k_2 in place of the two 1's.

Remark 5.2 (Exercise)**.** There is a positive rational number $\ell \in \mathbb{Q}_+$ such that $w_i\ell - f_i = q_i$, where w_1 and w_2 are from the fan sequence. This ℓ is the "good old" $\deg f$.

If N has torsion, then there is more information in the enhanced degree. From ℓ, I would not be able to recover the box elements (only modulo torsion). ◇

Proposition 5.3. *The following sets of data are equivalent:*

1. *non-constant representable morphisms $f\colon \mathbb{P}_{r_1,r_2} \to \mathcal{X}$ for some r_1, r_2 (unspecified),*
2. *Box elements $v_1 \in B_1$, $v_2 \in B_2$ and integers $q_1, q_2 > 0$ such that $q_1\rho_1 + q_2\rho_2 + v_1 + v_2 = 0$ in N.*

Proof. We've done one direction; you can convince yourself that it works. Let's do the other direction. We have to construct a morphism of fans.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\binom{r_2'}{r_1'}} & \mathbb{Z}^2 & \xrightarrow{(-r_1, r_2)} & \mathbb{Z} & & (\mathbb{P}_{r_1,r_2}) \\
 & & \downarrow m & & \downarrow \left(\begin{smallmatrix} m_1 & 0 \\ 0 & m_2 \end{smallmatrix}\right) & & \downarrow \eta & & \\
0 & \longrightarrow & \mathbb{Z} & \xrightarrow[\binom{w_2}{w_1}]{} & \mathbb{Z}^2 & \xrightarrow[\rho]{} & N & & (\mathcal{X})
\end{array}$$

First we construct η by $\eta(1) = -v_1 - q_1\rho_1 = v_2 + q_2\rho_2$. Let r_i be the order of v_i as a group element of $B_i = N/\langle \rho_i \rangle$. Then $r_i v_i = k_i \rho_i$ for some non-negative integers $k_i \geq 0$. It is easy to check that $k_i/r_i = f_i$. This tells us what r_1 and r_2 are.

Next set $m_i = r_i q_i + k_i$. We'll define the middle map to be given by $\left(\begin{smallmatrix} m_1 & 0 \\ 0 & m_2 \end{smallmatrix}\right)$. Let's check that the square commutes. We calculate that $m_1\rho_1 = (r_1q_1 + k_1)\rho_1 = r_1(-v_1 + \eta(-1)) + r_1 v_1 = -r_1\eta(1)$.

Finally, we want to construct the last map, but that is easy because they are kernels; the image of the top $\mathbb{Z}$ in the bottom $\mathbb{Z}^2$ is sent to zero because of commutativity of the square we checked. □

This is not difficult, but it took some time to sort it out because we had to find the correct way to package the combinatorics. In the end, this $\widehat{Pic}$ is what did it.

General words on the proof of the mirror theorem. You want to calculate some Gromov-Witten invariants at the end of the day. We want to calculate the J-function

$$J = \sum_\beta Q^\beta \int_{\mathcal{X}_{0,1,\beta}} \frac{ev^* \mathbb{1}}{z - \phi}$$

$\mathbb{T}$ acts on $\mathcal{X}$ and on $\mathcal{X}_{0,1,\beta}$. Inside of $\mathcal{X}$, there is the 1-dimensional stratum. For a map to be $\mathbb{T}$-invariant, your orbi-curve is likely to be extremely reducible. Basically, the J-function breaks up into contributions. A fixed point will be in an affine chart $\mathcal{X}_\sigma$ (σ maximal cone), then there is a point mapping there with some box element $\nu \in B_\sigma$. It is more or less a combinatorial problem. You break it up into pieces where you somehow rip off that component of the source curve. You know the contribution from that component, and the rest has lower degree, so you set up some induction procedure.

Oil Update

Not all of you have heard me lecture about this, so let me give you some basics. Oil is measured in barrels. One barrel is about 140 liters, or 40 gallons (unrefines). Today, a barrel costs about 135 USD, which is more than twice what it cost exactly a year ago. You might think 135 USD is a lot of money, so let me tell you what you buy when you buy a barrel of oil. One barrel equal 25,000 man hours of mechanical energy (never mind how that calculation is done)! You may be asking yourself, why is it that in the last twelve months, oil doubled it's price. Maybe it is speculators in Wall street, who want to make sure you pay because they messed up some prime mortgage thing. So they keep making money why you pay. Actually, the answer is rather different. There are very few studies of the world supply of oil. Given such a serious problem, why is it that nobody trys to study the future supply of oil. Today, world production is estimated to be approximately 85 MB/day (actually 75 MB/day, when it comes to crued oil; they get 85 by tricks called refinery gains and some other stuff, natural gas tricks), which is about the same that was produced in 2005! It is amazing how economists get Nobel prizes by saying that when there is enough demand, the shit will turn up. There are studies by the EUIA (or something), which still tell us that in 2030, the world will produce about 120 MB/day. Why do they say that? They plot the curve and extrapolate! There are very few studies on the supply side, and let me put my favorites

– F Rebelius, March 2007. This is a graduate student doing this! It's on the web; you can look it up.

– C. Skrebowski, Megaproject Update, on Petrolium review; this one is not on the web. Instead of guessing how much oil is in the ground, he knows about the big projects. Journalists say "the high prices are not stimulating investment in oil production". What are they talking about? It takes 8 years after you find an oil field to production. He sees a lot of oil coming in until 2012, and then we're walking into empty space; there is nothing there to fill that hole. Q: is that what the other one predicts? AC: yes, but the other one estimates the oil in the ground, so it is a completely different methodology.

– Energy Watch group, October 2007. These are scientists who were given serious money to buy data from the oil industry (by bribe). Oil data from Saudi Arabia and Russia are classified, so you have to bribe to get the data. Google gave me less than one page on this study. None of the press quoted this study! What these guys say is that world oil production reached its peak in 2006 and that it will be down 50% by 2020 (?).

Open Problem Session

For the next hour, we'll be doing something semi-experimental. The purpose of this hour is to ask ourselves collectively what we feel are important questions. They can be speculative and long-term ("show something like this conjecture"), as well as very specific ("understand this particular special case"). Very briefly, I (Ravi) will be all-powerful. Only I can hold the chalk. If I don't understand you, it's your fault. Anton will take notes. Somebody should start by asking some question you want to know the answer to. Then we'll look for related questions before moving to a new topic.

These notes are very rough, but if you have more to add, including better attributions and references, please e-mail Anton.

Question: what is a good example of a zero Gromov-Witten invariant on $\mathcal{M}_(X, \beta)$ that is zero for a non-trivial reason? Arend Bayer: There are examples on blow-ups. Blow up a point on a surface. β multiple of E. This is in a paper of Gathmann. Tom Graber: these examples are not good enough from the point of view of the motivation. **Motivating question:** find an equivalence between rational connectedness and the non-vanishing of certain global invariants. **The big conjecture:** rational connectedness is equivalent to symplectic rational connectedness. This is believed to be very hard. Dan Abramovich: one direction is known. There is a paper of Voison that deals with dimension 3. Her argument for 3-folds except when those that have projections to surfaces that involve blow-ups. **Possible hope:** If we knew this conjecture, does this help with the problem of finding a variety that is rationally connected but not unirational. Further motivation: if this were true, then rational connectedness would be preserved by symplectic equivalence. So here is a **related question**: Are all rationally connected varieties unirational? Refences include: Kollár, Ruan, Li et. al. for above. A student at Brandeis (who?). Does this question have a name? It is implied by some conjecture of Mumford (the $-\infty$ conjecture).

Question related to Jim Bryan's talk: the Donald-Thomas/Crepant-Resolution-Conjecture story involved a derived equivalence. Is there some way of geometrically interpreting the image of a Fourier-Mukai as a moduli problem? **Question:** Find a single Bridgeland stability condition on a compact Calabi-Yau 3-fold. Back to DT/CRC: **Question:** Find an analogue of the Givental formalism in Donaldson-Thomas theory.

Question. Given a family over a base $X \to Y$, find a relation between the moduli map $Y \to \mathcal{M}$ and the Campana core map. Another formulation: if Y is special, must the family be isotrivial. What hypotheses are implicit here? For example, you must need some assumption like a smooth family of canonically polarized varieties. Response: this is part of the question: find any set of hypotheses that work. make that part of the question. Possibilities: Family = canonically polarized varieties, semi-ample K, existence of a good minimal model? Do we want smooth? Do we need a polarization in the semi-ample case?

Question: If Y is smooth and has a nice compactification $\overline{Y}$ with $S = \partial Y$ and the moduli map is generically finite, does this imply that $\Omega^1_{\overline{Y}}(\log S)$ is in some sense weakly positive (boundary of big). The answer is "yes" if there exists local Torelli. **Question:** There is a sheaf of differentials that Viehweg and Zuo construct. What (if any) are its universal properties and where does it live, really? By construction it lives on Y, but maybe from universal properties it should follow that it lives in some weak way to the moduli space. Is this thing known to be unique? Response: To be unique, of course one should ask for some universal property, so we're led back to this question. **Related question:** characterize rigid (with respect to the morphism i) subvarieties $i\colon Y \hookrightarrow \mathcal{M}$ moduli space (under one of these 3 assumptions listed earlier).

Question: Try to axiomitize (relative) "curve counting" theories and try to prove that such a theory is characterized by its values on some class of varieties. For exaple, perhaps they should just cut and paste well, and then be determined by its values on toric varieties. Cf. J. Li, Levine-Pandharipande, Maulik-Pandharipande.

Question: Find a good definition of the virtual fundamental class when amplitude of tangent obstruction theory is finite but bigger than two. This makes sense in the context of derived or dg algebraic geometry. **Questions** prior to this: Where should the virtual class live? Are there Chow groups in derived geometry? Should they just be the usual Chow groups on the underlying scheme/stack? Intersection theory in on higher stacks would be helpful. **Question:** Develop it. **Question:** Can someone write readable finite length foundations for higher stacks? But back to the virtual fundamental class: someone (H.-L. Chang?) said that on the arXiv, there is some paper that approaches this question in the Donaldson-Thomas case using some modified hermitian Yang-Mills equation. **Question:** For Artin stacks in general, is there some theory of virtual fundamental class? Cf. thesis of Noseda at SISSA.

Questions: What stacks are quotient stacks? (Quotient stack means quotient by affine group schemes.) Are there good conditions on a stack that ensures that it is a quotient stack? If you have a coherent sheaf on a stack, is it a quotient of a vector bundle? This is called the *resolution property*. The resolution property implies that the stack is a quotient stack (cf. Totaro). It is not even known if all smooth proper algebraic spaces over $\mathbb{C}$ have the resolution property. **Question:** Find conditions for the resolution property to hold in this algebraic space case? There are results in this direction. Totaro proved that the resolution property is equivalent to being a quotient of an affine scheme by an affine group under some hypotheses. **Question:** What about conditions for when a stack is a quotient stack? More precisely: for a closed point of an Artin stack with a linearly reductive stabilizer, when is the stack étale locally a quotient by that stabilizer (the expected group). **Motivating problem**: construct good moduli spaces (in the sense of Alper) without GIT. Often constructions using GIT involves tough proofs that your notion of stability agrees with GIT stability.

Question. Remove global resolution and global embedding into a smooth stack hypotheses from Graber-Pand virtual localization and from Ciocan-Fontanine-Kapranov dg schemes. Jim Bryan doesn't care about this question.

Question related to Lucia Caporaso's talk, where she compactified the space of line bundles on a curve. What about a higher dimensional base? **Question:** Extend Caporoso's work on line bundles on nodal curves to surfaces. **Follow-up:** Try to get moduli spaces of sheaves where you have an understandable deformation-obstruction theory At the very least, we would want it to be finite amplitude/length. Perhaps the objects, instead of being sheaves, might best be objects in the derived category, as in Lieblich's work. Vector bundles on curves reference: Schmidt.

Kai Behrend says: take a holomorphic fuction $f\colon \mathbb{C}^n \to \mathbb{C}$. Consider the Milnor fiber at the origin. I'm interested in the Euler characteristic of that. Now consider the Kähler differentials $\Omega^{\bullet}_{\mathbb{C}^n}$ on $\mathbb{C}^n$. You have two things you can do: you can wedge with df and you have d. I take homology first with respect to $\wedge df$ and then d. The conjecture is that that is equal to that Euler characteristic. This is some kind of generalization of the holomorphic Poincaré lemma. There is a reference of this in a paper of Kapranov. (Kapranov, M. M., *On DG-modules over the de Rham complex and the vanishing cycles functor,* Algebraic geometry (Chicago, IL, 1989), 57–86, Lecture Notes in Math., 1479, Springer, Berlin, 1991. The relevant statements are very well hidden in this article, so I should point out that Kapranov proves the finite-dimensionality just before the end of Section 2 (at the bottom of Page 72), and he describes the Conjecture in Remark 2.12(b).) **Question:** Find a proof. Kapranov proves that you get something finite dimensional and well defined. This is also a generalization of the fact that the length of Jacobian ideal is equal to the Milnor number in the isolated singularity case. There is motivation from Donaldson-Thomas theory, in which there is some derived scheme. There is a De Rham model with the two differentials and the perverse sheaf of vanishing cycles and something about them being the same. The De Rham thing natually comes out of the derived stack of DT theory.

Question: Is the moduli space of genus $g \geq 3$ curves maximal in the sense that it does not admit a non-trivial finite étale map to a smooth orbifold? This is minimality for the moduli space and maximality for the Teichmuller space. **Question:** Is there some kind of extension of the Teichmuller group?

Let's move to higher-dimensional moduli. **Question:** Is there a tautological ring for moduli spaces of higher dimensional objects? Possible examples: polarized K3s, mated $\mathbb{P}^2$'s. **Question:** Perhaps Noether-Lefschetz loci in the first case are good? Perhaps no "natural" classes lying outside this ring?

Question: Work out complete nontrivial examples of compactifications of moduli of surfaces of general type. There are some examples where you have a product of curves so you can use the moduli space of curves (van Opstall); a single additional example would be interesting. How about cubic surfaces in $\mathbb{P}^3$, are those understood? Those aren't general type, so never mind. Valery Alexeev proved that there is a compactification of the moduli space of surfaces of general type. It would be good to have an example we understand, cf. Hacking for example in the marked case.