Quiz 12 - Math 54
December 1, 2010

Name Solution

The differential equation $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$ for $x \in [0,2]$ and $t > 0$ with fixed end values $u(0,t) = U_1$, $u(1,t) = U_2$ has the general formal solution

$$u(x,t) = U_1 + \frac{U_2 - U_1}{2}x + \sum_{n=1}^{\infty} c_n e^{-(n\pi/2)^2 t} \sin(n\pi x/2).$$

Find the formal solution given the initial values $u(0,t) = 0$, $u(1,t) = 2$, and $u(x,0) = \sin(\pi x)$. *Write out the first five non-zero terms in the series explicitly.*

From the general solution, I know that the formal solution must be of the form

$$u(x,t) = x + \sum_{n=1}^{\infty} c_n e^{-(n\pi/2)^2 t} \sin(n\pi x/2).$$

to find the c_n, I plug in $t = 0$ and rearrange, giving

$$\sin(\pi x) - x = \sum_{n=1}^{\infty} c_n e^{-(n\pi/2)^2 t} \sin(n\pi x/2).$$

Now I find the c_n as usual:

$$\begin{aligned} c_n &= \frac{2}{2}\int_0^2 (\sin(\pi x) - x)\sin(n\pi x/2)\,dx \\ &= \int_0^2 \sin(\pi x)\sin(n\pi x/2)\,dx - \int_0^2 x\sin(n\pi x/2)\,dx \end{aligned}$$

The first integral is 0 unless $n = 2$, and 1 for $n = 2$. Using integration by parts, I compute the second integral to be $\frac{4\cdot(-1)^n}{n\pi}$, so I have that

$$\begin{aligned} u(x,t) = x &+ \frac{-4}{\pi}e^{-(\pi/2)^2 t}\sin(\pi x/2) + \left(1 + \frac{4}{2\pi}\right)e^{-(2\pi/2)^2 t}\sin(2\pi x/2) \\ &+ \frac{-4}{3\pi}e^{-(3\pi/2)^2 t}\sin(3\pi x/2) + \frac{4}{4\pi}e^{-(4\pi/2)^2 t}\sin(4\pi x/2) \\ &+ \frac{-4}{5\pi}e^{-(5\pi/2)^2 t}\sin(5\pi x/2) + \cdots \end{aligned}$$