Quiz 11 - Math 54
November 17, 2010

Name Solution

Let $A = \begin{bmatrix} 6 & 1 \\ -4 & 2 \end{bmatrix}$.

1)[6pts] Compute e^{At}. Use this to find the general solution to $\mathbf{x}' = A\mathbf{x}$. (Hint: find an r so that $(A - rI)^2 = 0$.)

We compute $\det(A - rI) = r^2 - 8r + 16 = (r-4)^2$, so $r = 4$ is the only eigenvalue of A. We check that $(A - 4I) = \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}$ squares to zero, so we can compute

$$e^{(A-4I)t} = I + (A - 4I)t = \begin{bmatrix} 1+2t & t \\ -4t & 1-2t \end{bmatrix} \qquad e^{At} = e^{4t}e^{(A-4I)t} = \begin{bmatrix} (1+2t)e^{4t} & te^{4t} \\ -4te^{4t} & (1-2t)e^{4t} \end{bmatrix}$$

Since e^{At} is a fundamental matrix, the general solution to the differential equation is $c_1 e^{4t}\begin{bmatrix} 1+2t \\ -4t \end{bmatrix} + c_2 e^{4t}\begin{bmatrix} t \\ 1-2t \end{bmatrix}$.

2)[4pts] Find a particular solution to $\mathbf{x}' = A\mathbf{x} + \begin{bmatrix} e^{3t} \\ 0 \end{bmatrix}$.

We try a solution of the form $e^{3t}\mathbf{b}$, where $\mathbf{b}$ is a constant vector. Then $\mathbf{b}$ must satisfy the equation $3e^{3t}\mathbf{b} = e^{3t}A\mathbf{b} + e^{3t}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, so we must have $(3I - A)\mathbf{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. We solve for $\mathbf{b}$ by forming the augmented matrix and row reducing:

$$\begin{bmatrix} -3 & -1 & 1 \\ 4 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ -3 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 4 \end{bmatrix}$$

So $\mathbf{b} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$, so $\begin{bmatrix} e^{3t} \\ -4e^{3t} \end{bmatrix}$ is a particular solution.