Quiz 7 - Math 54
October 13, 2010

Name________________________

Let $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 3 \\ 2 & 4 & 6 \end{bmatrix}$.

1)[5pts] Find an orthonormal basis for Col(A). (Hint 1: the last column is the sum of the first two, so how can you get *some* basis for Col(A)? How many vectors does it contain? Hint 2: Don't forget to normalize your vectors so that they have length 1. The numbers should work out so that you don't have any square roots.)

Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be the first two columns. These form a basis for Col(A). We get an orthogonal basis with the Gram-Schmit process:

$$\mathbf{v}_1 = \mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \qquad \mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{v}_1 \cdot \mathbf{x}_2}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix} - \frac{9}{9}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$$

Both of these have length 3, so an orthonormal basis for Col(A) is $\left\{ \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}, \begin{bmatrix} -2/3 \\ -1/3 \\ 2/3 \end{bmatrix} \right\}$.

2)[5pts] Find a 3×3 matrix B so that $B\mathbf{x} = \text{proj}_{\text{Col}(A)}\,\mathbf{x}$ for any vector $\mathbf{x} \in \mathbb{R}^3$. (Hint: start by making a matrix U whose columns are the orthonormal basis you found in problem 1.)

Let $U = \begin{bmatrix} 1/3 & -2/3 \\ 2/3 & -1/3 \\ 2/3 & 2/3 \end{bmatrix}$. Since the columns of U are an orthonormal basis for Col(A), the orthogonal projection onto Col(A) is given by the matrix

$$UU^T = \begin{bmatrix} 1/3 & -2/3 \\ 2/3 & -1/3 \\ 2/3 & 2/3 \end{bmatrix} \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ -2/3 & -1/3 & 2/3 \end{bmatrix} = \begin{bmatrix} 5/9 & 4/9 & -2/9 \\ 4/9 & 5/9 & 2/9 \\ -2/9 & 2/9 & 8/9 \end{bmatrix}.$$