Quiz 11 - Math 53
December 4, 2008

Name Solution

Consider the graph of the function $z = \sin(x)$, $0 \le x \le \pi/2$, in the xz-plane. Let S be the surface obtained by rotating this curve around the z-axis. S looks kind of like a cone that flares out at the wide end. Problems 1a and 1b refer to this surface.

1a)[3pts] Use Stokes' Theorem to compute the flux of $\nabla \times \mathbf{F}$ through S, where $\mathbf{F} = \langle -y, x, z \rangle$ and S is oriented "upward" (in the positive z direction).

The boundary C of S is parameterized by $\mathbf{r}(t) = \langle \frac{\pi}{2}\cos t, \frac{\pi}{2}\sin t, 1 \rangle$, $0 \le t \le 2\pi$, so we have

$$\begin{aligned}
\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} &= \int_C \mathbf{F} \cdot d\mathbf{r} \\
&= \int_0^{2\pi} \left\langle -\frac{\pi}{2}\sin t, \frac{\pi}{2}\cos t, 1 \right\rangle \cdot \left\langle -\frac{\pi}{2}\sin t, \frac{\pi}{2}\cos t, 0 \right\rangle dt \\
&= \int_0^{2\pi} \frac{\pi^2}{4}(\sin^2 t + \cos^2 t)\, dt = \frac{\pi^3}{2}
\end{aligned}$$

1b)[3pts] Parameterize S. Don't forget to include the bounds on the parameters.

Two possible answers:
$\mathbf{r}(x, \theta) = \langle x\cos\theta, x\sin\theta, \sin x \rangle$, with $0 \le x \le \pi/2$ and $0 \le \theta \le 2\pi$
$\mathbf{r}(x, y) = \langle x, y, \sin(\sqrt{x^2 + y^2}) \rangle$, with $x^2 + y^2 \le (\pi/2)^2$. (note: the x in the second parameterization means something different from the x in the first one)

2)[3pts] Compute the flux of $\mathbf{G} = \langle x, y, z \rangle$ through the part of the cylinder $x^2 + y^2 = 1$ where $0 \le z \le 1$, oriented "outward" (away from the z-axis).

A parameterization for the surface is $\mathbf{r}(z, \theta) = \langle \cos\theta, \sin\theta, z \rangle$, $0 \le z \le 1$, $0 \le \theta \le 2\pi$. Then $\mathbf{r}_\theta = \langle -\sin\theta, \cos\theta \rangle$ and $\mathbf{r}_z = \langle 0, 0, 1 \rangle$, so $\mathbf{r}_\theta \times \mathbf{r}_z = \langle \cos\theta, \sin\theta, 0 \rangle$ (this points out, not in, so it is the right normal vector). So

$$\begin{aligned}
\iint_{\text{cylinder}} \mathbf{G} \cdot d\mathbf{S} &= \int_0^1 \int_0^{2\pi} \langle \cos\theta, \sin\theta, z \rangle \cdot \langle \cos\theta, \sin\theta, 0 \rangle \, d\theta\, dz \\
&= \int_0^1 \int_0^{2\pi} 1 \, d\theta\, dz = 2\pi.
\end{aligned}$$