Quiz 10 - Math 53
November 20, 2008

Name ______Solution______

Let C be the curve parameterized by $\mathbf{r}(t) = \langle t, t^2 \rangle$, $-1 \le t \le 1$, and let B be the curve parameterized by $\mathbf{r}(t) = \langle t, 1 \rangle$, $-1 \le t \le 1$. Let $\mathbf{F} = \langle y, \sin^8(y^2) - 2x \rangle$.

1a)[3pts] Set up an integral to compute $\int_C \mathbf{F} \cdot d\mathbf{r}$. *You don't have to evaluate the integral*, but it should be in a form so that you could plug it into a computer integration program or try to look it up on a table of integrals.

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{-1}^{1} \langle t^2, \sin^8(t^4) - 2t \rangle \cdot \langle 1, 2t \rangle \, dt = \int_{-1}^{1} \left(-3t^2 + 2t \sin^8(t^4)\right) dt.$$

1b)[3pts] Use Green's theorem to find a relationship between $\int_C \mathbf{F} \cdot d\mathbf{r}$ and $\int_B \mathbf{F} \cdot d\mathbf{r}$ (be careful to get the signs right). Use this to compute $\int_C \mathbf{F} \cdot d\mathbf{r}$.

Green's theorem tells us that $\int_C \mathbf{F} \cdot d\mathbf{r} - \int_B \mathbf{F} \cdot d\mathbf{r} = \iint_R (-3)\, dA$, where R is the region enclosed by $y = x^2$ and $y = 1$. We can compute

$$\iint_R (-3)\, dA = \int_{-1}^{1} \int_{x^2}^{1} (-3)\, dy\, dx = \int_{-1}^{1} (3x^2 - 3)\, dx = x^3 - 3x \Big|_{x=-1}^{1} = -4$$

$$\int_B \mathbf{F} \cdot d\mathbf{r} = \int_{-1}^{1} \langle 1, \sin^8(1) - 2t \rangle \cdot \langle 1, 0 \rangle \, dt = \int_{-1}^{1} 1 \, dt = 2$$

So $\int_C \mathbf{F} \cdot d\mathbf{r} = -4 + 2 = -2$.

2)[3pts] Consider the vector field $\mathbf{F} = \langle x + y, y^2 + \sin z, z^3 \rangle$. Compute $\nabla \times \mathbf{F}$ and $\nabla \cdot \mathbf{F}$. Could $\mathbf{F}$ be of the form ∇f for some function f? Could $\mathbf{F}$ be of the form $\nabla \times \mathbf{G}$ for some vector field $\mathbf{G}$? *Be sure you explain your reasoning.*

$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ x + y & y^2 + \sin z & z^3 \end{vmatrix} = \langle -\cos z, 0, -1 \rangle$. Since the curl of a gradient is always zero, $\mathbf{F}$ cannot be of the form ∇f.

$\nabla \cdot \mathbf{F} = \langle \partial_x, \partial_y, \partial_z \rangle \cdot \mathbf{F} = 1 + 2y + 3z^2$. Since the divergence of a curl is always zero, $\mathbf{F}$ cannot be of the form $\nabla \times \mathbf{G}$.