Quiz 6 - Math 53
October 16, 2008

Name______________________________

1a)[3pts] Consider the function $f(x, y) = x^3 + y^3$ on the domain $x^4 + y^4 \leq 2$. Which points in the *interior* of the domain are candidates to be maxima or minima? For each of these points, what does the second derivative test tell you?

Only critical points can be maxima or minima in the interior. We have that $\nabla f = \langle 3x^2, 3y^2 \rangle = \langle 0, 0 \rangle$ only when $(x, y) = (0, 0)$. $D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 6x & 0 \\ 0 & 6y \end{vmatrix} = 36xy$, which is zero at $(0, 0)$. Thus, the second derivative test doesn't give us any information about what kind of critical point $(0, 0)$ is.

1b)[3pts] Use Lagrange multipliers to determine which points on this boundary $x^4 + y^4 = 2$ can be minima or maxima of the function $f(x, y) = x^3 + y^3$. Evaluate f at these points and any candidate points from part (a) to determine the maximum and minimum values of f.

The only points on the boundary that can be extrema are points (x, y) where $\nabla f = \lambda \nabla g$ for some λ, i.e. where $\langle 3x^2, 3y^2 \rangle = \lambda \langle 4x^3, 4y^3 \rangle$ and $x^4 + y^4 = 2$. Solving these equations, we get $(x, y) = (0, \pm\sqrt[4]{2}), (\pm\sqrt[4]{2}), (1, 1), (-1, -1)$. We have $f(1, 1) = 2$ is a maximum, $f(-1, -1) = -2$ is a minimum, $f(\pm\sqrt[4]{2}, 0) = f(0, \pm\sqrt[4]{2}) = \pm 2^{3/4}$, and $f(0, 0) = 0$.

2)[3pts] Use Lagrange multipliers to determine which point on the surface $z = x^2 + y^2$ is closest to the point $(3, 3, 1)$. (Hint: solve for x, y, and z in terms of λ and try guessing $\lambda = -2$.)

We wish to minimize the function $f(x, y, z) = (x - 3)^2 + (y - 3)^2 + (z - 1)^2$ (the square of the distance to $(3, 3, 1)$) subject to the constraint $g(x, y, z) = x^2 + y^2 - z = 0$. This can only happen when $\nabla f = \lambda \nabla g$ for some λ, i.e. where $\langle 2x - 6, 2y - 6, 2z - 2 \rangle = \lambda \langle 2x, 2y, -1 \rangle$ and $x^2 + y^2 - z = 0$. Solving these equations, we get $x = y = \frac{3}{1-\lambda}$, $z = 1 - \frac{\lambda}{2}$ and $\frac{18}{(1-\lambda)^2} - 1 + \frac{\lambda}{2} = 0$. So $\lambda^3 - 4\lambda^2 + 5\lambda + 34 = (\lambda + 2)(\lambda^2 - 6\lambda + 17) = 0$. The only solution is $\lambda = -2$, so we get $x = y = 1$ and $z = 2$, which we can check satisfies $x^2 + y^2 - z = 0$. Thus, $(1, 1, 2)$ is the closest point to $(3, 3, 1)$.