Quiz 3 - Math 53
September 18, 2008

Name______________________________

Let **a**, **b**, **c**, and **d** be vectors in three dimensions, no two of them parallel. Let $V(\mathbf{a}, \mathbf{b})$ denote the plane through the origin consisting of all vectors of the form $\lambda\mathbf{a} + \mu\mathbf{b}$ (λ and μ constants). Let $\mathbf{r} = \langle x, y, z\rangle$ be a variable vector.

a)[3pts] Find vector equations for the planes $V(\mathbf{a}, \mathbf{b})$ and $V(\mathbf{c}, \mathbf{d})$. Find a formula for the cosine of the angle between these two planes. *Do not use components.*

The vectors **a** and **b** are in the plane $V(\mathbf{a}, \mathbf{b})$, so $a \times b$ is normal to $V(\mathbf{a}, \mathbf{b})$. The zero vector is in $V(\mathbf{a}, \mathbf{b})$, so and equation for $V(\mathbf{a}, \mathbf{b})$ is $\mathbf{r} \cdot (\mathbf{a} \times \mathbf{b}) = 0$. Similarly, an equation for $V(\mathbf{c}, \mathbf{d})$ is $\mathbf{r} \cdot (\mathbf{c} \times \mathbf{d}) = 0$. The angle between two planes is the same as the angle between their normal vectors so $\cos\theta = \dfrac{(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d})}{|\mathbf{a} \times \mathbf{b}|\,|\mathbf{c} \times \mathbf{d}|}$.

b)[3pts] Find a parameterization of the line that is the intersection of $V(\mathbf{a}, \mathbf{b})$ and $V(\mathbf{c}, \mathbf{d})$ (Hint: the line is in both planes, so it must be orthogonal to both normal vectors). Demonstrate that the direction vector of this line is in $V(\mathbf{c}, \mathbf{d})$ by writing it in the form $\lambda\mathbf{c} + \mu\mathbf{d}$ for some constants λ and μ. *Do not use components.*

The zero vector is on both planes, so I can use it as my $\mathbf{r}_0$. The direction vector must be perpendicular to both $\mathbf{a} \times \mathbf{b}$ and $\mathbf{c} \times \mathbf{d}$, so I'll take $\mathbf{v} = (\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d})$, so the line is given by $\mathbf{r} = t\big[(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d})\big]$. To see that $\mathbf{v}$ is of the form $\lambda\mathbf{c} + \mu\mathbf{d}$, note that $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = \big((\mathbf{a} \times \mathbf{b}) \cdot \mathbf{d}\big)\mathbf{c} - \big((\mathbf{a} \times \mathbf{b}) \times \mathbf{c}\big)\mathbf{d}$.

c)[3pts] If $\mathbf{a} = \langle 3, 0, 4\rangle$, $\mathbf{b} = \langle 0, 1, 0\rangle$, $\mathbf{c} = \langle 2, 2, 1\rangle$, and $\mathbf{d} = \langle 1, 0, 1\rangle$, Find equations for the planes $V(\mathbf{a}, \mathbf{b})$ and $V(\mathbf{c}, \mathbf{d})$. Compute the *cosine* of the angle between these two planes. Find a parameterization of the intersection of these two planes.

Compute that $\mathbf{a} \times \mathbf{b} = \langle -4, 0, 3\rangle$ has length 5 and $\mathbf{c} \times \mathbf{d} = \langle 2, -1, -2\rangle$ has length 3. $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = \langle 3, -2, 4\rangle$. So $V(\mathbf{a}, \mathbf{b})$ is given by $-4x + 3y = 0$, $V(\mathbf{c}, \mathbf{d})$ is given by $2x - y - 2z = 0$, the cosine of the angle between them is $\dfrac{-8 + 0 - 6}{5 \cdot 3} = \dfrac{-14}{15}$, and the line of intersection is $\mathbf{r} = \langle 3t, -2t, 4t\rangle$.