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Sec. 111

Spring 2003 Final - L. Evans

1. Find the length of the curve
$r(t)=\langle \cos t + t\sin t,\ \sin t - t\cos t\rangle$ for $0\le t\le 10$.

Solution: $x=\cos t + t\sin t$, $y=\sin t - t\cos t$

$\frac{dx}{dt} = -\sin t+\sin t + t\cos t$, $\frac{dy}{dt} = \cos t - \cos t + t\sin t$

$\therefore$ Length $= \int_0^{10}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$

$= \int_0^{10}\sqrt{t^2(\cos^2 t+\sin^2 t)}\,dt$

$=\int_0^{10} t\,dt = \frac{t^2}{2}\Big|_0^{10} = \frac{100}{2} = \boxed{50}$

2. Compute $|r_u\times r_v|$ for
$r(u,v)=\langle v, uv, u+v\rangle$.

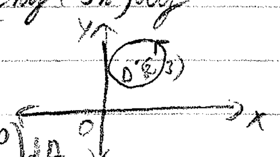
Solution: $r_u=\langle 0,v,1\rangle$, $r_v=\langle 1,u,1\rangle$

$\therefore (r_u\times r_v) = \begin{vmatrix} i & j & k\\ 0 & v & 1\\ 1 & u & 1\end{vmatrix} = (v-u)i+j-vk$

$\therefore |r_u\times r_v| = \sqrt{(v-u)^2+1^2+(-v)^2}$

$\boxed{=\sqrt{2v^2+2uv+u^2+1}}$

3. Let C denote the circle $(x-2)^2+(y-3)^2=4$ in the xy-plane, oriented counterclockwise. Compute $\int_C (2x+y^2)\,dx+(2xy+3x)\,dy$



Solution: Let $P(x,y)=2x+y^2$ and $Q(x,y)=2xy+3x$

By Green's Theorem, $\int_C (2x+y^2)dx+(2xy+3x)dx = \iint_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dA$

$=\iint_D(2y+3-2y)\,dA = 3\iint_D dA = 3\int_0^2\int_0^{2\pi} r\,d\theta\,dr = 3\times 2\pi\times \frac{r^2}{2}\Big|_0^2 = 3\times2\pi\times 2$

$=\boxed{12\pi}$

4. Find the critical points of $f(x,y) = x^4 - 2x^2 + y^2 - 2$ and classify each as a local maximum, a local minimum, or a saddle point.

Solution: $f_x = 4x^3 - 4x$, $f_y = 2y$
$\Rightarrow f_{xx} = 12x^2 - 4$, $f_{yy} = 2$, $f_{xy} = 0$

$\therefore D = f_{xx} f_{yy} - (f_{xy})^2 = \underline{24x^2 - 8}$

$f_x = 0 \Rightarrow x^3 - x = 0 \Rightarrow x(x^2-1) = 0 \Rightarrow \underline{x = 0, 1, -1}$
$f_y = 0 \Rightarrow 2y = 0 \Rightarrow \underline{y = 0}$

$\therefore$ Critical Points : $(0,0)$, $(1,0)$, $(-1,0)$
For $(0,0)$, $D = -8 < 0$. So it is a saddle point.
For $(1,0)$, $D = 16 > 0$, $f_{xx} = 12 - 4 = 8 > 0$. $\therefore$ It is a point of local minimum.
For $(-1,0)$, $D = 16 > 0$, $f_{xx} = 12 - 4 = 8 > 0$. $\therefore$ It is a point of local minimum.

6. Use Lagrange multipliers to find the point on the plane $x + 2y + 3z = 14$ that is closest to the origin.

Solution: Distance from a point (x,y,z) to $(0,0,0)$ is $d = \sqrt{x^2+y^2+z^2}$
$\therefore$ Let $f(x,y,z) = d^2 = x^2 + y^2 + z^2$.
$g(x,y,z) = x + 2y + 3z = 14$

Using Lagrange multipliers, $\nabla f = \lambda \nabla g$.

$\therefore 2x = \lambda$
$2y = 2\lambda$
$2z = 3\lambda$
$\therefore 2y = 4x \Rightarrow y = 2x$
and $2z = 6x \Rightarrow z = 3x$
$g = 14 \Rightarrow x + 4x + 9x = 14 \Rightarrow x = 1 \Rightarrow y = 2, z = 3$.

$\therefore$ The point closest to the origin on the plane is $\boxed{(1,2,3)}$