Multivariable Calculus (Math 53) MT2 Review: Max/min, Lagrange multipliers $\iint$, $\iiint$, change of variables, V-fields and line integrals.

Extreme Value thm – If region D is closed and bounded and f is continuous, f has a minimum and maximum on D.

To find min/max,
1. Find critical points of f on interior of D (set $f_x = f_y = 0$)
2. Find min/max of the restriction of f to the boundary of D, $(g=k)$ (parametrize x and y in terms of t, plug into equation of boundary, find where $f_B'(t) = 0$
3. Evaluate function at points in interior (1) and on boundary (2) to see which is the max and which the min

2nd Derivative Test states that given a twice-differentiable function $f(x,y)$, let

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$$

evaluated at point $P(x_0, y_0)$.

IF $\left(D>0 \begin{cases} f_{xx}>0 & \text{minimum} \\ f_{xx}<0 & \text{maximum}\end{cases}\right)$ ELSE IF $(D<0 \Rightarrow \text{saddle point})$ ELSE $\Rightarrow$ inconclusive.

Lagrange Multipliers To find max/min of f subject to constraint $g=k$, where $g(x,y,z)=k$ is the equation describing the boundary, it is sufficient to find the solution to the system of equations

$$\nabla f = \lambda \nabla g \quad \text{and} \quad g=k$$

, assuming $\nabla g \neq 0$ where $g=k$, where λ can be any scalar. In words, $\nabla f \parallel \nabla g$ must hold true because if not, there is a component of ∇f that is $\perp$ to ∇g and therefore f can increase if traveled along g, until $\nabla f \parallel \nabla g$. In other words, the level sets must be $\parallel$ (tangent) to g at solution point. Given 2 constraints $g=k$, $h=c$, set $\nabla f = \lambda \nabla g + \mu \nabla h$

Multiple Integrals

Fubini's Thm: For a bounded continuous function $f(x,y)$ on region E, $\iint_E f(x,y)\,dx\,dy = \iint_E f(x,y)\,dy\,dx$ whenever both integrals exist.

mass of ~~region~~ solid $= m = \iiint_E \rho(x,y,z)\,dV$

average of f on E $= \langle f \rangle = \frac{1}{m}\iiint_E f(x,y,z)\rho(x,y,z)\,dV$

For example, average value of $f(x,y,z) = x$ gives x-coordinate of center of mass: $\bar{x} = \frac{1}{m}\iiint_E x\rho\,dV$

Finding bounds of integration can either be done geometrically or analytically using inequalities

Change of Variables When changing variables from one coordinate system $(x,y,z) \to (u,v,w)$, the integrand must be multiplied by a magnification factor: the absolute value of the Jacobian:

$$\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v & \partial x/\partial w \\ \partial y/\partial u & \partial y/\partial v & \partial y/\partial w \\ \partial z/\partial u & \partial z/\partial v & \partial z/\partial w \end{vmatrix} = \left|\frac{\partial(u,v,w)}{\partial(x,y,z)}\right|^{-1}$$

Ex. for polar coordinates $dA = r\,dr\,d\theta$; cylindrical, $dV = r\,dr\,d\theta\,dz$; spherical, $\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

Vector Fields and Line Integrals To integrate a function f in $\mathbb{R}$ over a curve C, to produce the "area under the fence" or "mass" of C if f is interpreted as mass density, parametrize C into $\vec{r}(t) = \langle x(t), y(t)\rangle$, where $a \le t \le b$, and compute $\boxed{\int_C f(x,y)\,ds = \int_a^b f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt}$ where $|\vec{r}\,'(t)| = \sqrt{(dx/dt)^2 + (dy/dt)^2}$. $\int_C$ wrt x $= \boxed{\int_C f(x,y)\,dx = \int_a^b f(\vec{r}(t))\,x'(t)\,dt}$

A line integral of a vector field $\vec{F}$ along curve C $= \boxed{\int_C \vec{F}\cdot d\vec{r}}$ If $\vec{F}$ is a force, this gives work.

$\boxed{\int_C \vec{F}\cdot d\vec{r} = \int_C \vec{F}\cdot\vec{r}\,'(t)\,dt}$

Fundamental Thm of Line Integrals gives $\boxed{\int_C \vec{F}\cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))}$ given $\vec{F}$ is conservative.

$\{\vec{F}$ is conservative$\} = \{$there exists a potential f such that $\vec{F} = \nabla f\} = \{\int_C$ is path-independent$\} = \{$if $\vec{F} = \langle P(x,y), Q(x,y)\rangle$, $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}\} = \{\oint_C \vec{F}\cdot d\vec{r} = 0$ for every closed path C in $D\}$