MIDTERM 1 REVIEW - MATH 53

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Studying tips:

(1) Try doing the quizzes again (without looking at the solutions). Try doing the quizzes that other GSIs have posted. I think three other GSIs have posted quizzes.
(2) Look at the summaries in the appropriate sections of the worksheets. Try doing the *questions* in the worksheets; they take much less time than the problems, but still test your understanding.
(3) Look at the chapter reviews in the book. It doesn't take much time to do the concept checks and the True-False quizzes.

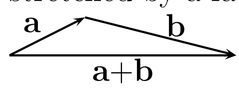
Most of the things I state for three variables can be specialized to two variables simply by removing the third variable, and many of the things stated for two variables can be generalized to more variables in the obvious way. A notable exception is cross product, which is defined only in the case of three-dimensional vectors.

Vectors

When working in more than one dimension or with more than one variable, it is usually much easier to understand concepts, derive formulas, and do calculations using vectors.

- *Vector addition, scalar multiplication*: You can add vectors and multiply them by scalars:

$$\langle a_1, a_2, a_3\rangle + \langle b_1, b_2, b_3\rangle = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3\rangle, \qquad c \cdot \langle a_1, a_2, a_3\rangle = \langle ca_1, ca_2, ca_3\rangle$$

  Since vector addition and scalar multiplication amounts to addition and scalar multiplication of the components, your intuition for how addition and scalar multiplication should behave will work well. Geometrically, $c\mathbf{a}$ is "$\mathbf{a}$ stretched by a factor of k" and $\mathbf{a} + \mathbf{b}$ is the vector that fills in the following picture: 

- *Dot product*: The dot product of $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ with $\mathbf{b} = \langle b_1, b_2, b_3\rangle$ is the number

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3 = |\mathbf{a}|\,|\mathbf{b}| \cos\theta$$

  where θ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$. Geometrically, $\mathbf{a} \cdot \mathbf{b}$ is a number that measures how much $\mathbf{a}$ and $\mathbf{b}$ point in the same direction. In particular, $\mathbf{a}$ and $\mathbf{b}$ are perpendicular if and only if $\mathbf{a} \cdot \mathbf{b} = 0$. Here are some properties of dot product you should know.

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a} \qquad \mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} \qquad \mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2.$$

- *Cross product*: The cross product of $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ with $\mathbf{b} = \langle b_1, b_2, b_3\rangle$ is the vector

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1\rangle$$

Geometrically, the magnitude of $\mathbf{a} \times \mathbf{b}$ is the area of the parallelogram determined by $\mathbf{a}$ and $\mathbf{b}$, and the direction is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ (and in the direction given by the right hand rule). Here are some properties of cross product you should know.

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a} \qquad \mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$$
$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) \qquad \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$
$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}| \sin\theta$$

It is useful to know that $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ is (up to sign) the volume of the parallelopiped determined by $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$. The sign is positive if $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ satisfy the right hand rule.

Warning: cross product is non-commutative and non-associative!

Lines & Planes

- *Lines*: To find a parameterization of a line, you need to find a point $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$ on the line and a *direction vector* $\mathbf{v} = \langle a, b, c \rangle$ that points in the same direction as the line. In that case, the line is parameterized by

$$\langle x, y, z \rangle = \mathbf{r} = \mathbf{r}_0 + t\mathbf{v} \qquad \text{or} \qquad x = x_0 + ta, \quad y = y_0 + tb, \quad z = z_0 + tc$$

  This means that any point on the line can be obtained by starting at $\mathbf{r}_0$ and moving in the direction of $\mathbf{v}$ by some amount.

  If a, b and c are all non-zero, the line can be given by the equations

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

- *Planes*: To find the equation for a plane, it is enough to find a point $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$ on the plane and a normal vector $\mathbf{n} = \langle a, b, c \rangle$ which is perpendicular to the plane. In that case, the plane is given by the equation

$$(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n} = 0 \qquad \text{or} \qquad \mathbf{r} \cdot \mathbf{n} = \mathbf{r}_0 \cdot \mathbf{n}$$

  This means that the vector from $\mathbf{r}_0$ to $\mathbf{r}$, which is $\mathbf{r} - \mathbf{r}_0$ must be perpendicular to $\mathbf{n}$ if $\mathbf{r}$ is to be a point in the plane.
- *Common Tricks*: You can use the properties of cross and dot product to compute angles, direction vectors, and normal vectors.
  (1) If you have two vectors $\mathbf{a}$ and $\mathbf{b}$ *in* a plane, then their cross product must be perpendicular to the plane, so you can use $\mathbf{n} = \mathbf{a} \times \mathbf{b}$ as a normal vector.
  (2) The angle between two intersecting lines is the angle between their direction vectors; the angle between two planes is the angle between their normal vectors. The angle between two vectors can be computed using the dot product: $\cos\theta = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|}$.
  (3) Given two planes with normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$, the direction vector of the line of intersection must be perpendicular to both $\mathbf{n}_1$ and $\mathbf{n}_2$, so you can use $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2$.

Parametric, Polar, and Space Curves

Parametric curves in two dimensions are a special case of space curves, and polar curves are a special case of parametric curves in two dimensions. This makes it easy to understand how different formulas for arc length, tangent direction, and area are related.

- *Space curves*: A space curve is a curve in three dimensions, given by some parameterization $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$. You can think of the curve as the path of a particle flying around in space. The vector $\mathbf{r}(t)$ is the position vector of the particle, so $\mathbf{r}'(t) = \langle x'(t), y'(t), z'(t)\rangle$ is the velocity vector and $\mathbf{r}''(t) = \langle x''(t), y''(t), z''(t)\rangle$ is the acceleration vector. Since $\mathbf{r}'(t_0)$ is tangent to the curve at the point $\mathbf{r}(t_0)$, it is often desirable to compute this derivative. Since $\mathbf{r}(t)$ may be presented to you as built up from other space curves using vector operations, it is useful to know the following rules for differentiation. Suppose $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are vector-valued functions (space curves), and $f(t)$ is an ordinary function.

$$\left.\begin{aligned} \frac{d}{dt}\big(f(t)\mathbf{u}(t)\big) &= f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t) \\ \frac{d}{dt}\big(\mathbf{u}(t)\cdot\mathbf{v}(t)\big) &= \mathbf{u}'(t)\cdot\mathbf{v}(t) + \mathbf{u}(t)\cdot\mathbf{v}'(t) \\ \frac{d}{dt}\big(\mathbf{u}(t)\times\mathbf{v}(t)\big) &= \mathbf{u}'(t)\times\mathbf{v}(t) + \mathbf{u}(t)\times\mathbf{v}'(t) \end{aligned}\right\} \text{Product rules}$$

$$\frac{d}{dt}\mathbf{u}(f(t)) = f'(t)\mathbf{u}'(f(t)) \qquad \text{Chain rule}$$

There are a number of interesting areas you can compute related to a space curve, but I don't think you're responsible for knowing those formulas. You can easily compute the arc length of a space curve by integrating speed (speed is given by $|\mathbf{r}'(t)| = \sqrt{\mathbf{r}'(t)\cdot\mathbf{r}'(t)}$), but I don't think you need to know that either.

To find the tangent line to the curve $\mathbf{r}(t)$ at time t_0, we need a point and a direction vector. The easiest choice of a point is $\mathbf{r}_0 = \mathbf{r}(t_0)$, and the easiest choice of direction vector is $\mathbf{v} = \mathbf{r}'(t_0)$.

- *Parametric curves*: A parametric curve is a two-dimensional space curve $\mathbf{r}(t) = \langle x(t), y(t)\rangle$. At the point $\mathbf{r}(t_0)$, $\mathbf{r}'(t_0) = \langle x'(t_0), y'(t_0)\rangle$ is tangent to the curve, so the slope of the tangent line must be equal to the slope of this vector, which is $y'(t_0)/x'(t_0)$ (provided $x'(t_0) \neq 0$). Thus, we have

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \quad \text{or} \quad \text{“}\frac{d}{dx} = \frac{d/dt}{dx/dt}\text{”} \qquad \frac{d^2y}{dx^2} = \frac{d}{dx}\frac{dy}{dx} = \frac{\frac{d}{dt}\frac{dy}{dx}}{dx/dt}$$

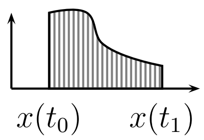
We can use this to find the higher order derivatives $\dfrac{d^ny}{dx^n}$ as demonstrated on the right above.

We can compute the arc length of a parametric curve from time t_0 to times t_1 as follows.

$$\text{Length} = \int_{t_0}^{t_1} \text{speed}\, dt = \int_{t_0}^{t_1} \sqrt{\mathbf{r}'(t)\cdot\mathbf{r}'(t)}\, dt = \int_{t_0}^{t_1} \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

We can also compute area under the curve from times t_0 to t_1 as follows.

$$\text{Area} = \int_{t_0}^{t_1} y(t)x'(t)\, dt$$



You can see that this is the right formula if you think of the area as being divided into skinny rectangles whose height is $y(t)$ at time t and whose width is $x'(t)dt$.

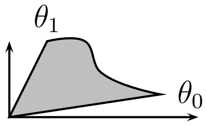
- *Polar curves*: A polar curve is when the distance from the origin $r(\theta)$ is given as a function of the angle with the x-axis θ. You can think of a polar curve as a special case of a parametric curve by taking θ to be the parameter (taking the place of t), with $x(\theta) = r(\theta)\cos\theta$ and $y(\theta) = r(\theta)\sin\theta$. Using this, we can compute the derivative dy/dx and the arc length of the curve between θ_0 and θ_1.

$$\frac{dy}{dx} = \frac{\frac{d}{dt}\big(r(\theta)\sin\theta\big)}{\frac{d}{dt}\big(r(\theta)\cos\theta\big)} = \frac{r'(\theta)\sin\theta + r(\theta)\cos\theta}{r'(\theta)\cos\theta - r(\theta)\sin\theta}$$

$$\text{Length} = \int_{\theta_0}^{\theta_1} \sqrt{x'(\theta)^2 + y'(\theta)^2}\, d\theta = \ldots\text{simplify}\ldots = \int_{\theta_0}^{\theta_1} \sqrt{r(\theta)^2 + r'(\theta)^2}\, d\theta.$$

The formula for area is not obtained by thinking of the curve as a parametric curve. The area you usually compute for a polar curve is the area swept out by the segment connecting to the origin rather than the area under the curve. The formula is given by

$$\text{Area} = \frac{1}{2}\int_{\theta_0}^{\theta_1} r(\theta)^2\, d\theta$$



You can see that this is the right formula if you think of the area as being divided into skinny triangles whose height is $r(\theta)$ and whose base is $r(\theta)d\theta$.

Quadric surfaces & Multivariable functions

- *Quadric surfaces*: It turns out that every surface in three dimensions given by a quadratic equation is either a cylinder or one of the six quadric surfaces. The standard quadric surfaces are

Ellipsoid $(\frac{x}{a})^2 + (\frac{y}{b})^2 + (\frac{z}{c})^2 = 1$	Elliptic paraboloid $z = (\frac{x}{a})^2 + (\frac{y}{b})^2$	Hyperbolic paraboloid $z = (\frac{x}{a})^2 - (\frac{y}{b})^2$
1-sheet hyperboloid $(\frac{x}{a})^2 + (\frac{y}{b})^2 - (\frac{z}{c})^2 = 1$	Cone $(\frac{x}{a})^2 + (\frac{y}{b})^2 - (\frac{z}{c})^2 = 0$	2-sheet hyperboloid $(\frac{x}{a})^2 + (\frac{y}{b})^2 - (\frac{z}{c})^2 = -1$

You should be able to complete the square to reduce any quadric to one of the standard quadrics (possibly with the variables mixed around). You should also be able to sketch these surfaces. For example, the surface $(x-1)^2 - (y+2)^2 + z^2 = 0$ is a cone centered at the point $(1, -2, 0)$ and oriented along the y-axis direction rather thand the standard z-axis direction.

The names come from the shapes of the traces (or slices) of the surfaces (i.e. when you set one variable equal to a constant and see what curve you get). For example, the traces of the hyperbolic paraboloid are hyperbolas and parabolas.

- *Limits and Continuity*: A function $f(x,y)$ is continuous at (a,b) if $\lim_{(x,y)\to(a,b)} f(x,y)$ exists and is equal to $f(a,b)$. You should try to get an understanding of what it means for a function to be continuous. The intuition I use is that a continuous function "takes points that are close to points that are close."

Given a function $f(x,y)$, you may be asked to determine if a limit exists at some point (a,b). As far as I can tell, there are three approaches.

(1) Show the limit does not exist. You can show this by approaching the point (a,b) along different curves and obtaining different limit values. For example, you might set $y = kx$ or $y = kx^2$ and take the limit of $f(x,y)$ as x approaches a (I'm assuming $b = ka$ or $b = ka^2$ here).

(2) Use the conjugation trick. If the denominator is a difference of square roots that becomes zero as you approach (a,b), try multiplying the numberator and denominator by the sum of those square roots. You might get a function which is clearly continuous, in which case the limit exists and is equal to the value of the function at (a,b). For example,

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{\sqrt{x^2-y^2+4}-2} &= \lim_{(x,y)\to(0,0)} \frac{(x^2-y^2)(\sqrt{x^2-y^2+4}+2)}{(\sqrt{x^2-y^2+4}-2)(\sqrt{x^2-y^2+4}+2)} \\ &= \lim_{(x,y)\to(0,0)} \frac{(x^2-y^2)(\sqrt{x^2-y^2+4}+2)}{x^2-y^2} \\ &= \lim_{(x,y)\to(0,0)} (\sqrt{x^2-y^2+4}+2) = 4\end{aligned}$$

(3) Change to polar coordinates by setting $x = r\cos\theta$ and $y = r\sin\theta$. This trick is usually only useful when the limit point is $(0,0)$. For example,

$$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2} = \lim_{r\to 0} \frac{r^3(\cos^3\theta+\sin^3\theta)}{r^2} = \lim_{r\to 0} r(\cos^3\theta+\sin^3\theta) = 0.$$

Approaching $(0,0)$ along a line $y = kx$ is the same as letting r approach zero with a fixed value of θ. So if after changing to polar coordinates you get a function that is independent of r but has *some* dependence on θ, the limit does not exist.

- *Differentials*: If $z = f(x,y)$ is a differentiable function, then if (x,y) is a point close to (a,b),

$$f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

This is a *linear approximation* of f at (a,b) (note that the function on the right hand side is a linear function; its graph is a plane). Another useful way to say this is

$$\Delta f \approx f_x(a,b)\Delta x + f_y(a,b)\Delta y.$$

This form allows you to compute error in z given error in x and y. If we take limits, then we get the *differential* of f:

$$df = f_x(a,b)dx + f_y(a,b)dy = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy.$$

As far as I can tell, you never really used these.

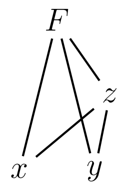
- *Chain Rule & Implicit differentiation*: Multivariable chain rule is just like regular chain rule, but you add up the results of applying regular chain rule to each of the variables. If f is a function of x, y and z, which themselves somehow depend on a variable t, then

$$\frac{\partial}{\partial t}f = f_x\frac{\partial x}{\partial t} + f_y\frac{\partial y}{\partial t} + f_z\frac{\partial z}{\partial t}.$$

Here, by f_x, I mean the rate of change of f as I vary x and hold y and z fixed. Unfortunately, y and z are left out of the notation, which sometimes confuses things. To keep things straight, I recommend drawing a dependency graph. Then whenever you differentiate a function with respect to a variable it directly depends on, you're implicitly assuming all the other variables it depends on are held constant.

Suppose $z = f(x, y)$ is implicitly defined by $F(x, y, z) = 0$, then differentiating with respect to x, we get

$$\frac{\partial}{\partial x} F = F_x \underbrace{\frac{\partial x}{\partial x}}_{1} + F_y \underbrace{\frac{\partial y}{\partial x}}_{0} + F_z \underbrace{\frac{\partial z}{\partial x}}_{f_x} = \frac{\partial}{\partial x}(0) = 0,$$



so

$$f_x(x, y) = \frac{-F_x(x, y, z)}{F_z(x, y, z)}.$$

Note that F_x means that you're moving x and holding y and z constant, whereas $\frac{\partial}{\partial x}F$ means you're moving x and holding y constant (but $z = f(x, y)$ is not constant). This isn't a standard convention, but I like it.

- *The Gradient*: For a function of three variables, $f(x, y, z)$, the gradient,

$$\nabla f = \langle f_x, f_y, f_z \rangle$$

is a vector whose direction is the direction of maximal increase of f and whose magnitude is the largest possible directional derivative. The gradient is always perpendicular to the level curves (or surfaces) of f. Given any unit vector $\mathbf{u}$, the *directional derivative* of f in the direction of $\mathbf{u}$ is

$$D_{\mathbf{u}} f = u \cdot \nabla f.$$

This gives you the rate of change of f in the direction of $\mathbf{u}$.