Spring 2007 Problem 7 — Jonathan Wang

$$\vec{F} = \langle x\sqrt{x^2+y^2+z^2},\ y\sqrt{x^2+y^2+z^2},\ z\sqrt{x^2+y^2+z^2} \rangle$$

(a) To find c, we take...

$$\operatorname{div}\vec{F} = \sqrt{x^2+y^2+z^2} + \frac{x^2}{\sqrt{x^2+y^2+z^2}} + \sqrt{x^2+y^2+z^2} + \frac{y^2}{\sqrt{x^2+y^2+z^2}} + \sqrt{x^2+y^2+z^2} + \frac{z^2}{\sqrt{x^2+y^2+z^2}}$$

• we simplify to get...

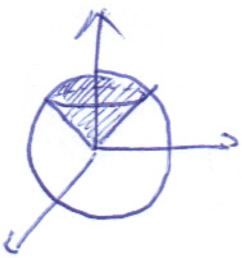
$$= 3\sqrt{x^2+y^2+z^2} + \frac{x^2+y^2+z^2}{\sqrt{x^2+y^2+z^2}}$$

• we combine the two to get...

$$= \frac{3(x^2+y^2+z^2) + (x^2+y^2+z^2)}{\sqrt{x^2+y^2+z^2}}$$

$$= \frac{4(x^2+y^2+z^2)}{\sqrt{x^2+y^2+z^2}} = 4\sqrt{x^2+y^2+z^2} \quad \text{so } \boxed{c=4}$$

$\uparrow c$

(b) 

$$\iint_S F\cdot dS = \iiint (\operatorname{div} F)\,dV = \iiint_E 4\sqrt{x^2+y^2+z^2}$$

$$= 4\int_0^{\pi/4}\int_0^{2\pi}\int_0^1 \rho^3 \sin\phi \, d\rho \, d\theta \, d\phi$$

$$= 4(-\cos\phi)\Big|_0^{\pi/4} 2\pi\left(\tfrac{1}{4}\right)$$

$$\boxed{= 2\pi\left(1-\tfrac{1}{\sqrt{2}}\right)}$$

④ SPRING 2007 PROBLEM 4

Jonathan Wong

$x^2+y^2+2z^2=4$ ①

$z=xy$ ②

Our parametrization is:

$$\left.\begin{aligned} x&=x(t)\\ y&=y(t)\\ z&=z(t)\end{aligned}\right\}\text{ we plug into Eqn 1 \& 2}$$

$$x(t)^2+y(t)^2+2z(t)^2=4$$

$$z(t)=x(t)\,y(t)$$

We then take derivatives w/ respect to t. using chain rule:

$$(2)x(t)\,x'(t)+2y(t)\,y'(t)+4z(t)\,z'(t)=0 \quad ③$$

$$z'(t)=x'(t)\,y(t)+y'(t)\,x(t) \quad ④$$

Since we are looking for $y'(0)$ & $z'(0)$ we rewrite eqns 3 & 4 ~~with~~ when $t=0$

Given: $\langle x(0),y(0),z(0)\rangle=\langle 1,1,1\rangle$ and $x'(0)=1$

Eqn 3:

$$2x(t)\,x'(t)+2y(t)\,y'(t)+4z(t)\,z'(t)=0$$

$$=2(1)(1)+2(1)\,y'(t)+4(1)\,z'(t)=0$$

$$=2+2y'(t)+4z'(t)=0 \quad ⑤$$

Eqn 4:

$$z'(t)=x'(t)\,y(t)+y'(t)\,x(t)$$

$$z'(t)=(1)(1)+y'(t)(1)$$

$$z'(t)=1+y'(t) \quad ⑥$$

Solve for $y'(0)$ and $z'(0)$ using ⑤ & ⑥

⑤ $z'(0)=1+y'(0)$

⑥ $2+2y'(0)+4z'(0)=0$

$$2+2y'(0)+4(1+y'(0))=0$$

$$2+2y'(0)+4+4y'(0)=0$$

$$\boxed{y'(0)=-1\;,\;z'(0)=0}$$

Problem 5 Spring 2007

Jonathan Wang

$$D_{\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\rangle} f(x,y) = \sqrt{2}\,x = \langle \tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\rangle \cdot \langle f_x, f_y\rangle \quad (1)$$

$$D_{\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\rangle} f(x,y) = \sqrt{2}\,y = \langle \tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}}\rangle \cdot \langle f_x, f_y\rangle \quad (2)$$

From (1) we get...

$$\sqrt{2}\,x = \frac{1}{\sqrt{2}}f_x + \frac{1}{\sqrt{2}}f_y$$

From (2) we get...

$$\sqrt{2}\,y = \frac{1}{\sqrt{2}}f_x - \frac{1}{\sqrt{2}}f_y$$

solve for f_x & f_y

$$\sqrt{2}\,x + \sqrt{2}\,y = \frac{1}{\sqrt{2}}f_x + \frac{1}{\sqrt{2}}f_x + \cancel{\frac{1}{\sqrt{2}}f_y - \frac{1}{\sqrt{2}}f_y}^{\,0}$$

$$\sqrt{2}(x+y) = \frac{2}{\sqrt{2}}f_x$$

$$2(x+y) = 2f_x$$

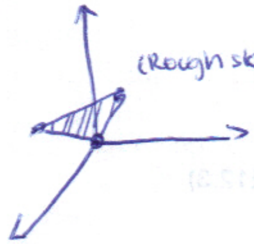
$$\boxed{x+y = f_x}$$

$$\boxed{x-y = f_y}$$

SPRING 2007 Problem 6

Jonathan Wong

⑥



(Rough sketch)

Point A = $(0,0,0)$

Point B = $(1,0,1)$

Point C = $(1,1,2)$

First, we find 2 vectors so that we can find an eqn for the plane of the triangle.

$$\overrightarrow{BA} = \langle 1-0, 0-0, 1-0 \rangle = \langle 1,0,1 \rangle$$

$$\overrightarrow{CA} = \langle 1-0, 1-0, 2-0 \rangle = \langle 1,1,2 \rangle$$

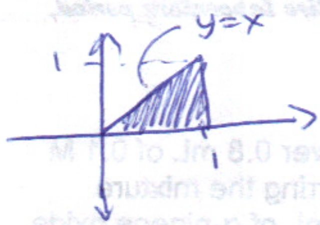
$$\langle 1,0,1 \rangle \times \langle 1,1,2 \rangle = \begin{vmatrix} i & j & k \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{vmatrix} = (-1)i - 1j + (1)k$$

Eqn to plane:

$$(-1)(x-0) + (-1)(y-0) + (1)(z-0) = 0$$

$$-x - y + z = 0 \Rightarrow \underline{z = x+y}$$

The shadow of the triangle in the x-y plane is:



$$\iint_S \langle 3,4,5 \rangle \cdot dS = \int_0^1 \int_0^x (-3-4+5)\, dA$$

$$= -2\int_0^1 \int_0^x dy\, dx$$

$$= -2\int_0^1 x\, dx$$

$$= (-2)\tfrac{1}{2} x^2 \Big|_0^1 = \boxed{-1}$$