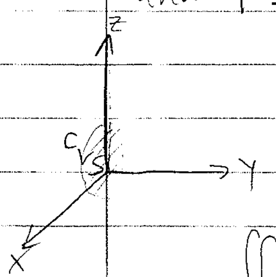
Hutchings Final Exam Spring 2007 — Jeremy Sasson

Problem 8

Calculate $\int_C \vec{F}\cdot d\vec{r}$, where C is the space curve $\vec{r}(t)=\langle \cos t, 0, \sin t\rangle$ for $0\le t\le 2\pi$, and $\vec{F} = \langle \underbrace{\sin(x^3)+z^3}_{P}, \underbrace{\sin(y^3)}_{Q}, \underbrace{\sin(z^3)-x^3}_{R}\rangle$ Hint: Use Stokes' Theorem



$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot d\vec{S}$$

Let S be the unit disk in the xz-plane bound by the curve C, oriented with $\hat{j}$ as its normal vector

$$\iint_S \text{curl}\,\vec{F}\cdot d\vec{S} = \iint_S (\text{curl}\,\vec{F}\cdot\vec{n})\,dA$$

$$\text{Curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

, but only the $\hat{j}$ component is necessary when dotted with $\vec{n}$

$$(\text{curl}\,\vec{F}\cdot\vec{n}) = -\left(\frac{\partial}{\partial x}\left(\sin(z^3)-x^3\right) - \frac{\partial}{\partial z}\left(\sin(x^3)+z^3\right)\right)$$

$$= -\left(-3x^2 - 3z^2\right)$$

$$= 3x^2+3z^2$$

$$\iint_S 3z^2+3x^2\,dA = 3\int_0^1\int_0^{2\pi} r^2\cdot r\,d\theta\,dr = 6\pi\int_0^1 r^3\,dr = \frac{6\pi}{4} = \boxed{\frac{3\pi}{2}}$$