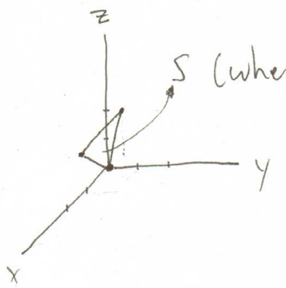
Hutchings Math 53 Final Spring (5/18/07) Solutions by: Ye-Qing Lin

#6 Let S be the triangle with vertices $(0,0,0)$, $(1,0,1)$, $(1,1,2)$ oriented upwards. Calculate the surface integral $\iint_S \langle 3,4,5\rangle \cdot d\vec{S}$

$F = \langle 3,4,5\rangle$ where F is the vector field we integrate.
But, before we do that, we must find an equation for surface S.



S (where S is a plane bounded by 3 lines)

To find the equation of the plane:

Label the 3 points given
$$\underbrace{P=(0,0,0) \quad Q=(1,0,1) \quad R=(1,1,2)}$$
Use to get vectors $\vec{PQ}$ & $\vec{PR}$

$\left.\begin{array}{l}\vec{PQ} = \langle 1,0,1\rangle \\ \vec{PR} = \langle 1,1,2\rangle\end{array}\right\}$ Cross to get normal vector to plane

$$\vec{PQ} \times \vec{PR} = \vec{n} = \begin{vmatrix} i & j & k \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{vmatrix} = (-1)i - (2-1)j + (1)k$$

$$n = \langle -1,-1,1\rangle$$

Use as coefficients to get equation of plane, Plugging in $(0,0,0)$ as a point on the plane.

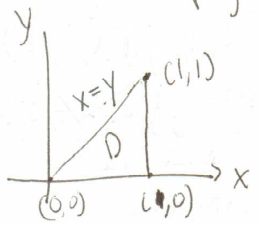
$$-1(x-0) - (y-0) + (z-0) = 0$$
$$-x-y+z=0$$
$$z = x+y \quad \text{equation of the surface } S \text{ where } 0 \le z \le 2,\ 0 \le y \le 1,\ 0 \le x \le 1$$

Because z is a function of x and y $z = g(x,y)$, we can use the equation $\iint_S F \cdot d\vec{S} = \iint_D \left(-A\frac{\partial g}{\partial x} - B\frac{\partial g}{\partial y} + C\right)$ where $F = \langle A,B,C\rangle$ knowing S is oriented upwards

$\left.\begin{array}{l} F = \langle 3,4,5\rangle \\ \frac{\partial g}{\partial x} = 1 \\ \frac{\partial g}{\partial y} = 1 \end{array}\right\}$ $\iint_S \langle 3,4,5\rangle \cdot d\vec{S} = \iint_D (-3(1) - 4(1) + 5)\, dA$

$$= \iint_D -2\, dA$$

D is the ~~prog~~ projection of S onto the xy plane



We can use this to calculate the parameters for integration

$0 \le x \le 1$
$0 \le y \le x$

$$\int_0^1 \int_0^x -2 \, dy \, dx = \int_0^1 -2x \, dx = -\frac{\cancel{2}x^2}{\cancel{2}} \Big|_{x=0}^{x=1} = \boxed{-1}$$

$$\boxed{\iint_S \langle 3, 4, 5 \rangle \cdot d\vec{S} = -1}$$