Hutchings Final Exam- Fall 2007

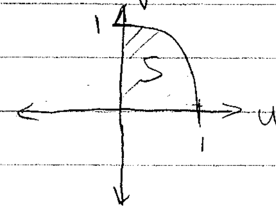
Problem 4

Calculate the double integral $\iint_D xy\, dA$ where D is the region in the plane defined by $x^4+y^4\le 1$ $x\ge 0$ $y\ge 0$

Hint: Use a change of variables to replace D with a simpler region

Let $u=x^2$ $v=y^2$. The new region S is the plane defined by $u^2+v^2\le 1$

$x=\sqrt{u}$ $y=\sqrt{v}$ $u\ge 0$ $v\ge 0$



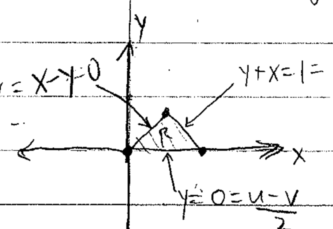
Jacobian: $\frac{\partial(x,y)}{\partial(u,v)} = \left| \begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{matrix} \right| = \left| \begin{matrix} \frac{1}{2\sqrt{u}} & 0 \\ 0 & \frac{1}{2\sqrt{v}} \end{matrix} \right| = \frac{1}{4\sqrt{u}\sqrt{v}}$

$$\iint_D xy\, dA = \iint_S \sqrt{u}\sqrt{v}\cdot \frac{1}{4\sqrt{u}\sqrt{v}}\, dA = \frac{1}{4}\iint_S dA = \frac{1}{4}(\text{Area } S) = \frac{1}{4}\left(\frac{\pi}{4}\right) = \boxed{\frac{\pi}{16}}$$

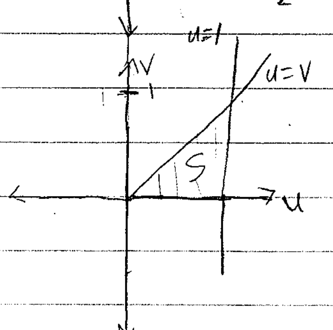
Hutchings Final Exam- Spring 2007

Problem 3 - Let R be the triangle with vertices $(0,0)$, $(1,0)$, and $(\frac{1}{2},\frac{1}{2})$. Evaluate the integral $\iint_R \frac{e^{x+y}}{x+y}\, dA$. Hint: Use the change of variables $u=x+y$ $v=x-y$

$x=\frac{u+v}{2}$ $y=\frac{u-v}{2}$



Jacobian: $\frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial(u,v)}{\partial(x,y)}^{-1} = \left| \begin{matrix} 1 & 1 \\ 1 & -1 \end{matrix} \right|^{-1} = |-2|^{-1} = \frac{1}{2}$



$$\iint_S \frac{e^u}{u}\cdot\frac{1}{2}\, dA = \frac{1}{2}\int_0^1\int_0^u \frac{e^u}{u}\, dv\, du$$

$$= \frac{1}{2}\int_0^1 \left[\frac{ve^u}{u}\right]_0^u du$$

$$= \frac{1}{2}\int_0^1 e^u\, du$$

$$= \frac{1}{2}\left[e^u\right]_0^1$$

$$= \frac{1}{2}(e^1-e^0)$$

$$= \frac{1}{2}(e-1) = \boxed{\frac{e-1}{2}}$$