FINAL REVIEW - MATH 53

ANTON GERASCHENKO

Studying tips:

(0) Look at the midterm 1 and midterm 2 review sheets. You are responsible for that material too.
(1) Try doing the quizzes again (without looking at the solutions). Try doing the quizzes that other GSIs have posted.
(2) Look at the summaries in the appropriate sections of the worksheets. Try doing the *questions* in the worksheets; they take much less time than the problems, but still test your understanding.
(3) Look at the chapter reviews in the book. It doesn't take much time to do the concept checks and the True-False quizzes.
(4) Try doing the old midterms. Remember that you can submit a write-up of an old midterm problem for an extra credit point.
(5) This review sheet doesn't cover parameterizing curves or surfaces because I can't think of any way to summraize everything you should know. I recommend trying to come up with lots of different surfaces and trying to parameterize them. In particular, you should be quite good at parameterizing surfaces of revolution.

Vector Calculus:

- *Derivatives:* You've learned three types of derivatives. The notation is supposed to help you remember the formulas. You think of the operator ∇ as the "vector" $\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\rangle$.

(Gradient) You apply gradient to a function to get a vector field:

$$\nabla f := \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \langle f_x, f_y, f_z\rangle$$

The direction of the gradient is the direction of maximal increase of f. To get the rate of change of f in the direction of a unit vector $\mathbf{u}$, you take $D_{\mathbf{u}} f = \mathbf{u}\cdot\nabla f$. Perhaps the most useful fact about the gradient is that it is always normal to the level curves or level surfaces of f.

(Curl) You apply curl to a vector field to get another vector field:

$$\nabla \times \mathbf{F} := \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y\rangle$$

where $\mathbf{F} = \langle P, Q, R\rangle$. If you think of the vector field $\mathbf{F}$ as the flow of water, and you place a little paddle wheel at the point (x, y, z) with its axis pointing in the direction of a unit vector $\mathbf{n}$, then $(\nabla \times \mathbf{F}) \cdot \mathbf{n}$ measures how fast a little paddle wheel will turn.

To determine the curl of a vector field at a point from a picture, you should draw a little circle around the point of interest and ask, "does the vector field push me around this path?" If it does, then the curl is non-zero, and you can figure out the direction using the right hand rule.

(Divergence) You apply divergence to a vector field to get a function:

$$\nabla \cdot \mathbf{F} := \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = P_x + Q_y + R_z$$

The divergence measures how much the vector field $\mathbf{F}$ is exploding near a point. To determine the divergence of a vector field at a point from a picture, think of the vector field as the flow of water. Put a little blob around the point and ask, "does more water leave the blob than comes into it?" If more water leaves the blob than comes into it, then the divergence is positive; if more water comes into the blob than leaves it, then the divergence is negative.

You should know that

$$\nabla \times (\nabla f) = 0 \qquad \text{for any function } f\text{, and}$$
$$\nabla \cdot (\nabla \times \mathbf{F}) = 0 \qquad \text{for any vector field } \mathbf{F}.$$

In fact, if $\mathbf{G}$ is defined on all of space, the only way that you can have $\nabla \times \mathbf{G} = 0$ is if $\mathbf{G} = \nabla f$ for some function f, and the only way you can have $\nabla \cdot \mathbf{G} = \nabla \times \mathbf{F}$ for some $\mathbf{F}$. Thus, if you want to check if a vector field is conservative, the easiest way is to check if the curl is zero. If you want to check of a vector field is the curl of something, the easiest way is to check if the divergence is zero.

- *Line Integrals:* If you have a curve parameterized by t via

$$\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle,$$

then if you move for a small amount of time, the distance you travel is $|\mathbf{r}'(t)|\Delta t$, so the length element on the curve is

$$ds = |\mathbf{r}'(t)|\, dt.$$

If you wanted to compute the mass of a curve, given some linear density function $\lambda(x, y, z)$, you would integrate the function λ along the curve:

$$\text{Mass} = \int_C \lambda(x, y, z)\, ds = \int_t \lambda(x, y, z)|\mathbf{r}'(t)|\, dt.$$

Usually, you'd like to find the amount of work done by a vector field when you travel along the curve:

$$\int_C \underbrace{\mathbf{F} \cdot \mathbf{T}\, ds}_{\text{bit 'o work}} = \int_C \mathbf{F} \cdot \left(\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \right) |\mathbf{r}'(t)|\, dt$$
$$= \int_C \mathbf{F} \cdot \underbrace{\mathbf{r}'(t)\, dt}_{d\mathbf{r}}$$

Note that $d\mathbf{r} = \langle x'(t), y'(t), z'(t) \rangle\, dt$ can be written $\langle dx, dy, dz \rangle$. This is the form you see in Green's theorem:

$$\oint_C \langle P, Q \rangle \cdot d\mathbf{r} = \oint_C P\, dx + Q\, dy = \iint_D (Q_x - P_y)\, dA$$

- *Surface Integrals:* If you have a surface parameterized by u and v by

$$\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle,$$

then a little tangent parallelogram spanned by the vectors $\Delta u\,\mathbf{r}_u$ and $\Delta v\,\mathbf{r}_v$ has tiny area $|\mathbf{r}_u \times \mathbf{r}_v|\Delta u \Delta v$, so the area element on the surface is

$$dS = |\mathbf{r}_u \times \mathbf{r}_v| \underbrace{dA}_{du\,dv}.$$

If you want to compute the mass of the surface, given some area density function $\sigma(x,y,z)$, you would integrated the function σ on the surface:

$$\text{Mass} = \iint_S \sigma(x,y,z)\,dS = \iint_{u,v} \sigma(x,y,z)|\mathbf{r}_u \times \mathbf{r}_v|\,dA.$$

Usually, you'd like to find the total flux[1] of some vector field:

$$\iint_S \underbrace{(\mathbf{F}\cdot\mathbf{n})\,dS}_{\text{bit 'o flux}} = \iint_S \mathbf{F}\cdot\left(\frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}\right)|\mathbf{r}_u \times \mathbf{r}_v|\,dA$$

$$= \iint_S \mathbf{F}\cdot\underbrace{(\mathbf{r}_u \times \mathbf{r}_v)\,dA}_{d\mathbf{S}}$$

Special case: If your surface is the graph of a function $z = g(x,y)$, then you can let $u = x$ and $v = y$. When you do the computation, you get $\mathbf{r}_u \times \mathbf{r}_v = \langle -g_x, -g_y, 1\rangle$

- *Fundamental Theorems:* All of the main theorems of vector calculus state that if you integrate some kind of derivative of a function (or vector field) over a region, it is the same as integrating the original function (or vector field) over the boundary of the region.

  There are three main theorems (see page 795, §13.9 for the pictures):

Fundamental Theorem of Line Integrals	$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$
Stokes' Theorem	$\iint_S (\nabla \times \mathbf{F})\cdot d\mathbf{S} = \oint_{C=\partial S} \mathbf{F}\cdot d\mathbf{r}$
Divergence Theorem	$\iiint_E (\nabla \cdot \mathbf{F})\,dV = \iint_{S=\partial E} \mathbf{F}\cdot d\mathbf{S}$

As a special case of Stokes' Theorem, we get Green's Theorem. Say your surface S is in the xy-plane, and $\mathbf{F} = \langle P, Q, 0\rangle$, where P and Q do not depend on z. Then $d\mathbf{S} = \langle 0,0,1\rangle dA$ and $\nabla \times \mathbf{F} = \langle R_y - Q_z, P_z - R_x, Q_x - P_z\rangle = \langle 0, 0, Q_x - P_y\rangle$. So

$$\iint_S (Q_x - P_y)\,dA = \iint_S (\nabla \times \mathbf{F})\cdot d\mathbf{S} = \int_{C=\partial S} \mathbf{F}\cdot d\mathbf{r} = \int_C P\,dx + Q\,dy$$

[1]Flux measures how much a vector field flows through the surface. If your vector field tells you the flow of water in the bay, and your surface is a net, then the flux of the vector field through the surface measures how much water per unit time flows though the net.