$z=(x-1)^2+(y-1)^2$

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Sec 111

① Find the maximum and maximum values of the function $f=(x-1)^2+(y-1)^2$ on the unit disc $x^2+y^2\le 1$.

↳ circular paraboloid with bottom at (1,1,0)

Sol'n: $\nabla f=\lambda\nabla g \rightarrow f=(x-1)^2+(y-1)^2,\ g=x^2+y^2$

$\langle 2(x-1), 2(y-1)\rangle = \lambda\langle 2x, 2y\rangle$

$$\left.\begin{aligned}2x-2&=\lambda 2x\\ 2y-2&=\lambda 2y\end{aligned}\right\}\div\ \ \frac{2x-2}{2y-2}=\frac{\lambda 2x}{\lambda 2y}\implies \frac{x-1}{y-1}=1\implies x-1=y-1\implies x=y$$

Plug into $x^2+y^2=1$:

$x^2+x^2=1$

$2x^2=1$

$x^2=\frac{1}{2}$

$x=\pm\frac{1}{\sqrt{2}}\implies y=\pm\frac{1}{\sqrt{2}}$

Thus critical pts are $\left(\frac{1}{\sqrt2},\frac{1}{\sqrt2}\right)$ and $\left(\frac{-1}{\sqrt2},\frac{-1}{\sqrt2}\right)$

$f\left(\frac{1}{\sqrt2},\frac{1}{\sqrt2}\right)=\left(\frac{1}{\sqrt2}-1\right)^2+\left(\frac{1}{\sqrt2}-1\right)^2=2\left(\frac{1}{2}-\frac{2}{\sqrt2}+1\right)$

$=1-\frac{4}{\sqrt2}+2=3-\frac{4}{\sqrt2}=\frac{6-4\sqrt2}{2}$

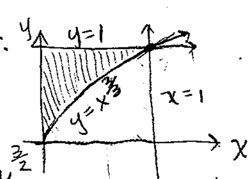
$f\left(\frac{-1}{\sqrt2},\frac{-1}{\sqrt2}\right)=\left(\frac{-1}{\sqrt2}-1\right)^2+\left(\frac{-1}{\sqrt2}-1\right)^2=2\left(\frac{1}{2}+\frac{2}{\sqrt2}+1\right)$

$=1+\frac{4}{\sqrt2}+2=3+\frac{4}{\sqrt2}=\frac{6+4\sqrt2}{2}$

$\therefore$
$$\boxed{\begin{aligned}&\text{Maximum value: } f\left(\tfrac{-1}{\sqrt2},\tfrac{-1}{\sqrt2}\right)=\frac{6+4\sqrt2}{2}\\ &\text{Minimum value: } f\left(\tfrac{1}{\sqrt2},\tfrac{1}{\sqrt2}\right)=\frac{6-4\sqrt2}{2}\end{aligned}}$$

② Calculate $\int_0^1\int_{x^{2/3}}^{1} x\cos(y^4)\,dy\,dx$

R:



$y=x^{\frac23}\implies x=y^{\frac32}$

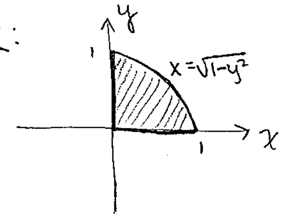
Switch order of integration

$$=\int_0^1\int_0^{y^{3/2}} x\cos(y^4)\,dx\,dy=\int_0^1 \frac{x^2}{2}\cos(y^4)\Big|_{x=0}^{x=y^{3/2}}dy=\int_0^1\frac{y^3}{2}\cos(y^4)\,dy$$

$$=\frac12\int_0^1 y^3\cos(y^4)\,dy=\frac18\int_0^1\cos u\,du=\frac18(\sin u)\Big|_{u=0}^{u=1}=\boxed{\frac18\sin 1}$$

let $u=y^4$
$du=4y^3\,dy$

③ Calculate $\int_0^1 \int_0^{\sqrt{1-y^2}} (x^2+y^2)^{2003}\, dx\, dy$

R: 

$x=\sqrt{1-y^2}$

$x^2=1-y^2$

$x^2+y^2=1$

Convert to polar

$$= \int_0^{\frac{\pi}{2}} \int_0^1 (r^2)^{2003} \cdot r\, dr\, d\theta = \int_0^{\frac{\pi}{2}} \int_0^1 r^{4007}\, dr\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{r^{4008}}{4008}\Big|_{r=0}^{r=1} d\theta = \int_0^{\frac{\pi}{2}} \frac{1}{4008}\, d\theta = \frac{\theta}{4008}\Big|_0^{\frac{\pi}{2}} = \boxed{\frac{\pi}{8016}}$$

④ Find the area of the region enclosed by the curve $x^2+xy+y^2=1$.
Hint: Use the substitution $x=u+v\sqrt{3}$, $y=u-v\sqrt{3}$

$$x^2+xy+y^2=1 \Rightarrow (u+v\sqrt{3})^2 + (u+v\sqrt{3})(u-v\sqrt{3}) + (u-v\sqrt{3})^2 = 1$$

$$u^2 + \cancel{2uv\sqrt{3}} + \cancel{3v^2} + u^2 - \cancel{3v^2} + u^2 - \cancel{2uv\sqrt{3}} + 3v^2 = 1$$

$$3u^2+3v^2=1 \Longrightarrow u^2+v^2=\frac{1}{3} \rightarrow \text{circle w/ } r=\frac{1}{\sqrt{3}}!$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & \sqrt{3} \\ 1 & -\sqrt{3} \end{vmatrix} = -\sqrt{3}-\sqrt{3} = -2\sqrt{3}$$

$$\iint_R 1\, dx\, dy = \iint_S 1 \cdot \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv = 2\sqrt{3} \cdot \text{area(circle)} = 2\sqrt{3} \cdot \left(\pi\left(\tfrac{1}{\sqrt{3}}\right)^2\right) = \boxed{\frac{2\sqrt{3}}{3}\pi}$$