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Final
Hutchings Fall 2003

1) Let $\vec{r}(t)$ be a parametrized curve in xy-plane satisfying $\vec{r}(0) = \langle 1, 2 \rangle$ and $\vec{r}\,'(0) = \langle 3, 4 \rangle$. Let $f(x,y) = e^{xy}$.
Calculate $\frac{d}{dt} f(\vec{r}(t)) \big|_{t=0}$

$f_x(x,y) = ye^{xy}$

$f_y(x,y) = xe^{xy}$

at $\vec{r}(0)$: $\begin{cases} f_x(1,2) = 2e^{1\cdot 2} = 2e^2 \\ f_y(1,2) = 1\cdot e^{1\cdot 2} = e^2 \end{cases}$

chain rule $\frac{d}{dt} f(\vec{r}(t))\big|_{t=0} = f'(\vec{r}(t))\cdot \vec{r}\,'(t)\big|_{t=0}$

$= (2e^2)(3) + (e^2)(4)$

$= 6e^2 + 4e^2$

$= 10e^2$

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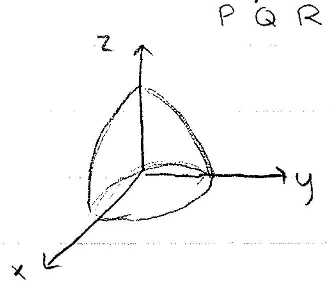
4) Calculate $\iint_S F\cdot dS$, where S is portion of paraboloid $z = 9 - x^2 - y^2$, $z \geq 0$ oriented upward

$F = \langle \underset{P}{x}, \underset{Q}{y}, \underset{R}{z} \rangle$



$\iint_S F\cdot dS = \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA$

$g(x,y) = z = 9 - x^2 - y^2$

$= \iint_D \left[-x(-2x) - y(-2y) + 9 - x^2 - y^2\right] dA$

$= \int_0^{2\pi}\int_0^3 (2x^2 + 2y^2 - x^2 - y^2 + 9)\, r\,dr\,d\theta$

$= \int_0^{2\pi}\int_0^3 (x^2 + y^2 + 9)\, r\,dr\,d\theta$

$= \int_0^{2\pi}\int_0^3 (r^3 + 9r)\, dr\,d\theta$

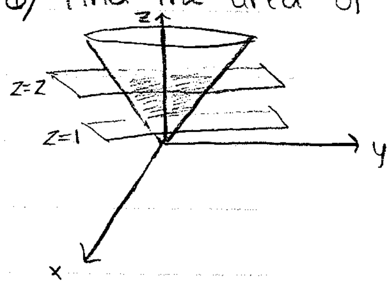
$= \int_0^{2\pi} \left[\frac{r^4}{4} + \frac{9r^2}{2}\right]_{r=0}^{r=3} d\theta$

$= \int_0^{2\pi} \left(\frac{81}{4} + \frac{81}{2}\right) d\theta$

$= 2\pi \cdot \frac{243}{4}$

$= \frac{243}{2}\pi$

6) Find the area of the part of the cone $z^2 = x^2+y^2$ between planes $z=1$, $z=2$



$$z = \sqrt{x^2+y^2}$$

$$z_x = \frac{x}{\sqrt{x^2+y^2}}, \quad z_y = \frac{y}{\sqrt{x^2+y^2}}$$

$$A = \int_0^{2\pi}\int_1^2 \sqrt{1+\left(\frac{x}{\sqrt{x^2+y^2}}\right)^2+\left(\frac{y}{\sqrt{x^2+y^2}}\right)^2}\, r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_1^2 \sqrt{1+\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}}\, r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_1^2 \sqrt{2}\, r\,dr\,d\theta$$

$$= \int_0^{2\pi} \sqrt{2}\cdot \frac{r^2}{2}\Big]_{r=1}^{r=2} d\theta$$

$$= 2\pi\cdot\sqrt{2}\left(2-\frac{1}{2}\right)$$

$$= 3\sqrt{2}\pi$$

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9) Either find, or prove it does not exist

(a) a function f on $\mathbb{R}^3$ such that $\nabla f = \langle y, x+z\cos y, \sin y\rangle$

If f exists, then $\operatorname{curl}(\nabla f) = 0$

$$\begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & x+z\cos y & \sin y \end{vmatrix} = (\cos y \cancel{-\cos y})i - (0-0)j + (1\cancel{-1})k = 0$$

Thus, there exists such a function.

If $f_y = x+z\cos y$, then $f = xy + z\sin y$

$f_x = y$ ✓

$f_z = \sin y$ ✓

$f = xy + z\sin y$

(b) a vector field F on $\mathbb{R}^3$ such that $\nabla\times F = \langle z, y, x\rangle$

If F exists, then $\operatorname{div}(\nabla\times F) = 0$

$$\operatorname{div}(\nabla\times F) = \frac{\partial}{\partial x}(z) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(x) = 1 \neq 0$$

No such field exists on $\mathbb{R}^3$