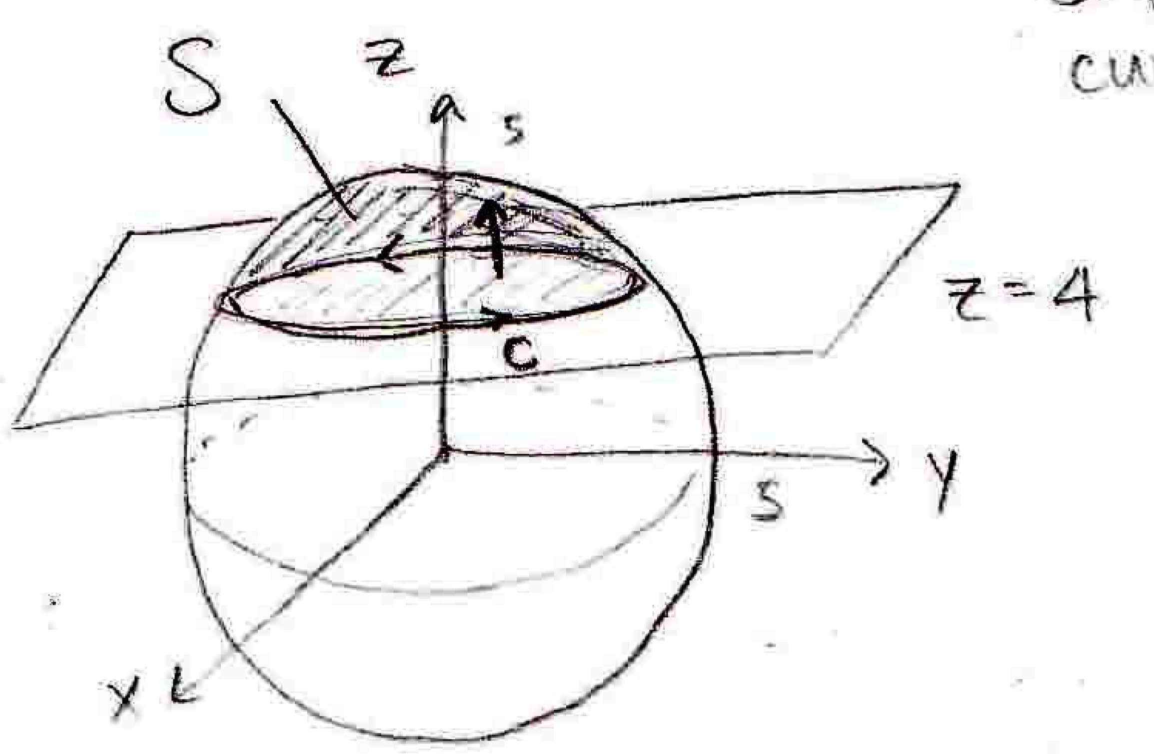
F03 Final Exam — Monica Chan

M. Hutchings, #12

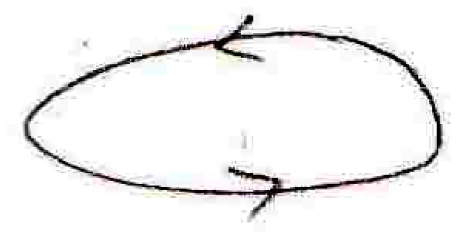
Let S be portion of sphere $x^2+y^2+z^2=25$ lying above the plane $z=4$, oriented by the upward pointing normal.

Calculate $\iint_S \underbrace{(\nabla\times\vec{F})}_{\text{curl}\vec{F}}\cdot d\vec{S}$ for $\vec{F}=\langle z^3-y^3,\ x^3-z^3,\ y^3-x^3\rangle$



$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot d\vec{S}$$

too hard, use line integral



C at $z=4$

$x^2+y^2=9$

$0\le t\le 2\pi$

$x=3\cos t$

$y=3\sin t$

$z=4$

$\vec{r}(t)=\langle 3\cos t,\ 3\sin t,\ 4\rangle$

$\vec{r}\,'(t)=\langle -3\sin t,\ 3\cos t,\ 0\rangle$

$\vec{F}(\vec{r}(t))=\langle 64-27\sin^3 t,\ 27\cos^3 t-64,\ 27(\sin^3 t-\cos^3 t)\rangle$

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt$$

$$= \int_0^{2\pi}(\underbrace{-192\sin t}_{=0} + 81\sin^4 t + 81\cos^4 t \underbrace{-192\cos t}_{=0})\,dt$$

$$= 81\int_0^{2\pi}(\sin^4 t+\cos^4 t)\,dt \longrightarrow (9\sin^2 t+9\cos^2 t)^2 = 81$$

$81\sin^4 t + 81\cos^4 t + 162\sin^2 t\cos^2 t = 81$

$81(\sin^4 t+\cos^4 t) = 81-162\sin^2 t\cos^2 t$

$$= \int_0^{2\pi}(81-162\sin^2 t\cos^2 t)\,dt$$

$\sin 2x = 2\sin x\cos x$

$\sin^2 2x = 4\sin^2 x\cos^2 x$

$$= \int_0^{2\pi}\left(81-\frac{162}{4}\sin^2 2t\right)dt$$

$\int_0^{2\pi}\sin^2 2t = \pi$

$$= 81t\,\Big|_0^{2\pi} - \frac{162}{4}(\pi)$$

$$= 162\pi - \frac{162\pi}{4}$$

$$\boxed{\int_C \vec{F}\cdot d\vec{r} = \frac{243}{2}\pi}$$