Hare: Final Exam 2003

12a. Stokes Theorem: $\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot d\vec{S}$

b. Use Stokes Theorem to evaluate $\iint_S \text{curl}\,F\cdot dS$ where $\vec{F}(x,y,z)=\langle yz, xz, xy\rangle$ and S is the paraboloid $z=4-x^2-y^2$ that lies above the plane $z=3$, oriented upwards.

At $z=3$: $3=4-x^2-y^2$

$1=x^2+y^2$

Parametrize: $\left.\begin{array}{l} x=\cos\theta \\ y=\sin\theta \\ z=3 \end{array}\right\}$ $\begin{array}{l} \vec{r}=\langle\cos\theta,\sin\theta,3\rangle \\ d\vec{r}=\langle-\sin\theta,\cos\theta,0\rangle\,d\theta \end{array}$

Stokes Theorem: $\iint_S \text{curl}\,\vec{F}\cdot d\vec{S} = \int_C \vec{F}\cdot d\vec{r}$

$= \int_C \langle yz, xz, xy\rangle\cdot d\vec{r}$

$= \int_C \langle 3\sin\theta, 3\cos\theta, \cos\theta\sin\theta\rangle\cdot\langle-\sin\theta,\cos\theta,0\rangle\,d\theta$

$= \int_0^{2\pi}(-3\sin^2\theta+3\cos^2\theta)\,d\theta$

$= \frac{3}{2}\int_0^{2\pi}[(\cancel{1}+\cos 2\theta)+\cancel{(1}+\cos 2\theta)]\,d\theta$

$= \frac{3}{\cancel{2}}\int_0^{2\pi}\cancel{2}\cos 2\theta\,d\theta$

$= \frac{3}{2}\sin 2\theta\Big|_0^{2\pi} = 0$