Veronica Chin

Problem 4. Consider z as a function of x and y given implicitly by the relation

$$xz^3 - xy^2 - yz + z^2 = 1.$$

(a) Compute $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at $x = y = z = 1$. (5 points)

(b) Suppose that x and y are themselves functions of u and v given by $x = u/v$ and $y = uv$. Compute $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ at $u = v = z = 1$. (10 points)

(c) Set up a system of equations whose solutions include the triple (x, y, z) satisfying the given constraint, and also the additional constraint $x^2 + y^2 + z^2 = 3$, maximizing $f(x, y, z) = xyz$. Do not attempt to solve the system. (10 points)

(a) $xz^3 - xy^2 - yz + z^2 - 1 = 0$

$$\frac{\partial z}{\partial x} = \frac{-\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial z}} = \frac{-(z^3 - y^2)}{3xz^2 - y + 2z} = \boxed{\frac{y^2 + z^3}{3xz^2 - y + 2z}}$$

At $(1,1,1)$: $\frac{\partial z}{\partial x} = \frac{2}{3-1+2} = \boxed{\frac{1}{2}}$

$$\frac{\partial z}{\partial y} = \frac{-\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial z}} = \frac{-2xy + z}{3xz^2 - y + 2z} = \boxed{\frac{z - 2xy}{3xz^2 - y + 2z}}$$

$\frac{\partial z}{\partial y} = \frac{1-2}{1-1+2} = \boxed{\frac{-1}{2}}$

(b) $x = u/v$, $y = uv$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = \left(\frac{y^2+z^3}{3xz^2-y+2z}\right)\left(\frac{1}{v}\right) + \left(\frac{z-2xy}{3xz^2-y+2z}\right)(v) = \left(\tfrac{1}{2}\right)(1) + \left(\tfrac{-1}{2}\right)(1) = \boxed{0}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = \left(\frac{y^2+z^3}{3xz^2-y+2z}\right)\left(\frac{-u}{v^2}\right) + \left(\frac{z-2xy}{3xz^2-y+2z}\right)(u) = \left(\tfrac{1}{2}\right)(-1) + \left(\tfrac{-1}{2}\right)(1) = \boxed{-1}$$

(c) $f(x,y,z) = xyz$ and $x^2+y^2+z^2 = 3$

$g = x^2+y^2+z^2$

$\nabla f = \lambda \nabla g$

$\langle yz, xz, xy \rangle = \lambda \langle 2x, 2y, 2z \rangle$

$$\boxed{\begin{aligned} yz &= \lambda 2x \\ xz &= \lambda 2y \\ xy &= \lambda 2z \end{aligned}}$$

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Problem 5. Let $f(x,y) = y^3 - xy + x^2 - 2x + 2$, defined on the region R defined by the conditions $-1 \le x \le 1$ and $0 \le y \le 1$.

(a) Find all critical points of f within R. (10 points)

(b) Classify the critical points you found in (a) using the Second Derivative Test. (5 points)

(c) Find the absolute maximum and minimum of f on the region R. (10 points)

(a) $f(x,y) = y^3 - xy + x^2 - 2x + 2$

$f_x = -y + 2x - 2 = 0$

$f_y = 3y^2 - x = 0 \rightarrow x = 3y^2$

$-y + 6y^2 - 2 = 0$

$6y^2 - y - 2 = 0$

$(2y+1)(3y-2) = 0$

$y = -\frac{1}{2}$ or $y = \frac{2}{3} \Rightarrow x = \frac{3}{4}$ or $x = \frac{4}{3}$

$\boxed{\text{Crit. pts. within R: } \left(\frac{3}{4}, \frac{-1}{2}\right) \text{ and } \left(\frac{4}{3}, \frac{2}{3}\right)}$

(b) $D_{(\frac{3}{4},-\frac{1}{2})} = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ -1 & -3 \end{vmatrix} = -6-1 = -7 < 0 \rightarrow$ $\boxed{\left(\tfrac{3}{4}, -\tfrac{1}{2}\right) \text{ is neither a local max. nor a local min, "saddle point"}}$

$f_{xx} = 2$ $\quad f_{yx} = -1$

$f_{xy} = -1$ $\quad f_{yy} = 6y$

$\hookrightarrow$ for $\left(\frac{3}{4}, -\frac{1}{2}\right)$: -3

$\hookrightarrow$ for $\left(\frac{4}{3}, \frac{2}{3}\right) = 4$

$D_{(\frac{4}{3},\frac{2}{3})} = \begin{vmatrix} 2 & -1 \\ -1 & 4 \end{vmatrix} = 8 - 1 = 7 > 0$ and $f_{xx} = 2 > 0$

$\boxed{\left(\tfrac{4}{3}, \tfrac{2}{3}\right) \text{ is a Minimum}}$

(c) Check borders:

$f(-1,0) = 0 - 0 + 1 + 2 + 2 = 5$

$f(1,0) = 0 - 0 + 1 - 2 + 2 = 1$

$f(-1,1) = 1 + 1 + 1 + 2 + 2 = 7 \longrightarrow \boxed{(-1, 1, 7) \text{ abs. max}}$

$f(1,1) = 1 - 1 + 1 - 2 + 2 = 1$

$f\left(\frac{4}{3}, \frac{2}{3}\right) = \left(\frac{2}{3}\right)^3 - \left(\frac{4}{3}\right)\left(\frac{2}{3}\right) + \left(\frac{4}{3}\right)^2 - 2\left(\frac{4}{3}\right) + 2 = \frac{8}{27} - \frac{8}{9} + \frac{16}{9} - \frac{8}{3} + 2$

$= \frac{8}{27} - \frac{24}{27} + \frac{48}{27} - \frac{72}{27} + \frac{54}{27} = \frac{14}{27} \longrightarrow \boxed{\left(\tfrac{4}{3}, \tfrac{2}{3}, \tfrac{14}{27}\right) \text{ abs. min}}$

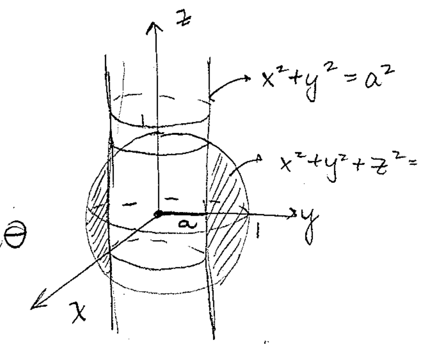
Problem 6. Consider a sphere of radius 1 centered at the origin. Remove from the sphere its intersection with an infinite cylinder of radius a whose axis is the z-axis. Let V be the remaining portion of the sphere.

(a) Set up a double integral in rectangular coordinates that computes the volume of V. (10 points)

(b) Set up a double integral in polar coordinates that computes the volume of V. (10 points)

(c) Compute the volume of V, as a function of a. (10 points)

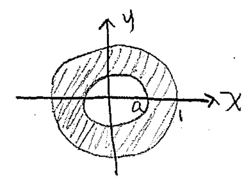
(a) $\int_{x=-1}^{1}\int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x,y)\,dy\,dx = \int_{x=-1}^{1}\int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1-x^2-y^2}\,dy\,dx$



(b) $\int_0^{2\pi}\int_a^1 r\cdot f(r\cos\theta, r\sin\theta)\,dr\,d\theta = \int_0^{2\pi}\int_a^1 r\sqrt{1-r^2}\,dr\,d\theta$

$= \int_a^1\int_0^{2\pi} r\sqrt{1-r^2}\,d\theta\,dr$

Intersection on xy plane:



(c) $V = \int_a^1\int_0^{2\pi} r\sqrt{1-r^2}\,d\theta\,dr = 2\pi\int_a^1 r\sqrt{1-r^2}\,dr$

let $u = 1-r^2$
$du = -2r\,dr$

$= -\pi\int_{1-a^2}^{0}\sqrt{u}\,du = \pi\int_0^{1-a^2}\sqrt{u}\,du$

$= \pi\left(\frac{2}{3}u^{\frac{3}{2}}\right)\Big|_{u=0}^{u=1-a^2} = \boxed{\frac{2\pi}{3}(1-a^2)^{\frac{3}{2}}}$